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A MAXIMUM PRINCIPLE FOR GENERALIZATIONS OF HARMONIC MAPS IN HERMITIAN, AFFINE, WEYL, AND FINSLER GEOMETRY

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Abstract. In this note we prove that the maximum principle of Jäger-Kaul for harmonic maps holds for a more general class of maps, V-harmonic maps. This includes Hermitian harmonic maps [23], Weyl harmonic maps [25], affine harmonic maps [21] and Finsler maps from a Finsler manifold into a Riemannian manifold. With this maximum principle we establish the existence of V-harmonic maps into regular balls.

1. Introduction

The maximum principle plays a crucial role in partial differential equations, especially in elliptic equations. While the maximum principle naturally applies to scalar equation, sometimes one can also derive a geometric version for a system of equations. An important example is the result of Jäger and Kaul for harmonic maps into a regular ball in a Riemannian manifold, which we are now going to formulate.

Let (M, g) be a closed Riemannian manifold with boundary ∂M and (X, h) a complete manifold without boundary. Let $B_R(p) := \{q \in X : d(p, q) \leq R\}$, with $d(\cdot, \cdot)$ the distance function on X, be a regular ball in X, that is, disjoint from the cut-locus of its center p and of radius $R < \frac{\pi}{2\sqrt{\kappa}}$, where $\kappa = \max\{0, \sup_{B_R(p)} K_X\}$ and $\sup_{B_R(p)} K_X$ is an upper bound of the sectional curvature K of X on $B_R(p)$.

Theorem A. ([15]) *Let u_1, $u_2 \in C^0(M, X)$ be two harmonic maps into a regular ball $B_R(p)$. Then the function $\Theta : M \to \mathbb{R}$ defined by*

$$\Theta(x) = \frac{q_\kappa(d(u_1(x), u_2(x)))}{\cos(\sqrt{\kappa}d(p, u_1(x))) \cos(\sqrt{\kappa}d(p, u_2(x)))} \tag{1}$$

satisfies the maximum principle, namely

$$\max_M \Theta \leq \max_{\partial M} \Theta.$$

This maximum principle has many applications in the study of harmonic maps and it is optimal in the sense that Theorem A does not hold when $R = \frac{\pi}{2\sqrt{\kappa}}$. See examples in [14].

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Recently, geometric considerations lead to several interesting generalizations of harmonic maps. The abstract principle is always the same. One considers a geometric structure with an invariant second order elliptic operator, that is, some kind of Laplacian. With this operator, one can then define not only corresponding generalized harmonic functions, but also analogues of harmonic maps into Riemannian manifolds. One important example is given by the Hermitian harmonic maps introduced by Jost-Yau in [23]. These maps provide a useful tool for studying non-Kähler Hermitian manifolds. For recent work on Hermitian harmonic maps see [3], [35], [8] and [29]. We discuss this class of maps in Section 4. In a similar vein, Weyl harmonic maps from a manifold with a Weyl strucuture into a Riemannian manifold were proposed in [25]. Interesting applications of Weyl harmonic maps have been obtained in [25] and [26]. Likewise, affine harmonic maps mapping from an affine manifold into a Riemannian manifold as a new tool for studying affine structures have been introduced in [21, 22]. In another direction, harmonic maps from a Finsler manifold into a Riemannian manifold have been studied in [1], [10], [36], [37] and [42]. The precise definitions of these generalizations will be given in Section 4.

It is therefore desirable and probably necessary for further progress on such maps to have some systematic analytical framework that applies to those kinds of elliptic systems.

A fundamental difficulty arises from the fact that these systems, in contrast to harmonic mappings between Riemannian manifolds, often cannot be derived from a variational principle, essentially because the aforementioned elliptic operator is not of divergence form (the Finsler harmonic maps are an exception). Therefore, many of the analytical tools developed for harmonic mappings are no longer available for the study of these generalizations. In fact, even the heat flow method encounters difficulties in the absence of a variational structure, see [23] for an example where no Hermitian harmonic map exists even when the target has nonpositive sectional curvature. Of course, in the elliptic theory, besides the variational methods, we also have the powerful maximum principle techniques.

It therefore is a natural question whether there exists a maximum principle for such generalized harmonic maps. In this note we shall derive such a maximum principle, even for a more general class of maps which we shall now introduce. This also has the advantage of unifying the above mentioned examples within a natural geometric framework.

Let V be a smooth vector field on M. We call a map $u : M \to X$ a *V-harmonic maq* if u satisfies

$$\tau(u) + du(V) = 0, \tag{2}$$

where $\tau(u) = \mathrm{tr}Ddu$ is the tension field of the map u. Since du is a section of $T^*M \otimes u^{-1}TX$, $du(V)$ is a section of the pull-back bundle $u^{-1}TX$. The case $V = 0$, of course, corresponds to the ordinary harmonic maps, and so, this is a natural generalization.

In local coordinates $\{x^\alpha\}$ on M and $\{y^i\}$ on X respectively, Equation (2) reads:

$$\Delta_M u^i + \Gamma^i_{jk} \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} g^{\alpha\beta} + V^\alpha \frac{\partial u^i}{\partial x^\alpha} = 0, \tag{3}$$

where Δ_M is the Laplacian on (M, g), Γ^i_{jk} stands for the Christoffel symbols of (X, h), and $V := V^\alpha \frac{\partial}{\partial x^\alpha}$. This is a second order elliptic system on the manifold (M, g).

In the special case where V is a gradient vector field, i.e., $V = \nabla f$ for some function $f : M \to \mathbb{R}$, then (2) is

$$\tau(u) + du(\nabla f) = 0, \tag{4}$$

which is the Euler-Lagrange equation for

$$\int_M |\nabla u|^2 e^f dv(g).$$

Hence in this case, (4) is of divergence form. In fact, it can be rewritten as

$$e^{-f}\mathrm{tr}\nabla(e^f du) = 0. \tag{5}$$

A solution of (4) is called an f-harmonic map, and this goes back at least to Lichnerowicz [30]. There are many interesting examples of f-harmonic maps. For example, a self-similar solution of the harmonic heat flow is an f-harmonic map with a special f, see [32]. Harmonic maps from a Riemannian measure space are also f-harmonic. For a recent study of f-harmonic maps, see [32, 27, 28, 31, 45].

In general, however, Equation (2) is not in a divergence form. All generalized harmonic maps mentioned above are V-harmonic maps for various vector fields V, which have geometric meanings. In the Hermitian harmonic maps discussed later in Section 4.1, V given in (39) measures how far an Hermitian metric being Kähler. In the Weyl geometry in Section 4.2 V measures how far a Weyl connection being a Levi-Civita connection. In the affine geometry in Section 4.3 V measures how far a flat connection being a Levi-Civita connection and in the Finsler harmonic map in Section 4.4 the corresponding V measures how far a Finsler metric being Riemmaniann metric. Moreover, these examples, with the exception of the last one, i.e., harmonic maps from a Finsler manifold, are in general not in divergence form. The harmonic maps from a Finsler manifold into a Riemannian manifold are in fact f-harmonic maps. See Section 4 below.

The main result of this note now is that the maximum principle of Jäger-Kaul is true for V-harmonic maps.

Theorem 1. *Let $u_1,\ u_2 \in C^0(M,X)$ be two V-harmonic maps into a regular ball $B_R(p)$. Then the function $\Theta : M \to \mathbb{R}$ defined by (1) satisfies the maximum principle,*

$$\max_M \Theta \leq \max_{\partial M} \Theta.$$

For the heat flow of V-harmonic maps, the analogue result also holds. Precisely, for $T > 0$, we set

$$M_T := M \times [0,T],$$

and denote the parabolic boundary of M_T by

$$\partial_p M_T := (M \times \{0\}) \cup (\partial M \times [0,T]).$$

Consider the heat flow of V-harmonic maps

$$\partial_t u = \tau(u) + du(V). \tag{6}$$

We have

Theorem 2. *Let $u_1,\ u_2 \in C^0(M,X)$ be two solutions of the heat flow equation (6) for V-harmonic maps into a regular ball $B_R(p)$. Then the function $\Theta : M_T \to \mathbb{R}$ defined by (1) with M replaced by M_T satisfies the maximum principle:*

$$\max_{M_T} \Theta \leq \max_{\partial_p M_T} \Theta.$$

A direct application of the maximum principle, Theorem 2, is the existence of the V-harmonic maps into a regular ball. We shall use the continuity method of von Wahl [43, 44], which relies heavily on the maximum principle, to prove Theorem 3.

Theorem 3. *Let $M, X, V, B_R(p)$ be as in Theorem 2. Suppose $u_0 \in H^{2,q}(M, X)$, $q > m := \dim M$, with $u_0(M) \subset B_R(p)$. Then the Dirichlet problem:*

$$\left\{ \begin{array}{rcl} \tau(u) + du(V) & = & 0, \\ u|_{\partial M} & = & u_0|_{\partial M}, \end{array} \right. \tag{7}$$

admits a unique solution $u \in H^{2,q}(M, X)$ such that $u(M) \subset B_R(p)$.

We would like to mention that in [4], such a strategy is also applied to obtain a general existence result for the Dirichlet problem of Dirac-harmonic maps (see [5]), which satisfy a system of equations consisting of a second order elliptic system and a first order elliptic system.

For the study of ordinary harmonic maps, see for example [6, 17, 18, 19, 33, 38, 46].

2. The maximum principle

We first remark that for a C^0 V-harmonic map u one can show that $u \in C^\infty(M) \cap C^0(\overline{M})$. Our proof builds upon the ideas of Jäger and Kaul [15]. We consider the following operator

$$\mathcal{L}_V := \mathcal{L} + e^{-\Phi} V,$$

where $\mathcal{L}$ was introduced in [15] which is defined by

$$\mathcal{L}(\cdot) = \operatorname{div}(e^{-\Phi} \nabla \cdot).$$

Here Φ is defined by

$$\Phi = \sum_{i=1}^{2} \omega \circ \psi_i,$$

where ψ_i $(i = 1, 2)$ is given by

$$\psi_i(x) = q_\kappa(d(p, u_i(x))), \quad \psi(x) = q_\kappa(d(u_1(x), u_2(x))) \tag{8}$$

and $\omega : [0, \kappa^{-1}) \to \mathbb{R}$ is given by

$$\omega(t) = -\log(1 - \kappa t).$$

Here $q_\kappa : \mathbb{R} \to \mathbb{R}_+$ is defined by

$$q_\kappa(t) = \left\{ \begin{array}{ll} \dfrac{1}{2} t^2, & \text{if } \kappa = 0, \\ \dfrac{1}{\kappa}(1 - \cos \sqrt{\kappa} t), & \text{if } \kappa > 0 \end{array} \right.$$

and u_1 and u_2 are the two V-harmonic maps we are considering. Set

$$\Omega_0 := \{x \in M \mid u_1(x) = u_2(x)\}.$$

Let Δ_V be the operator on C^2 functions defined by

$$\Delta_V = \Delta + V.$$

Lemma 1. *Let ψ_i $(i=1,2)$, ψ and ω be defined by (8). Let u_1, $u_2 \in C^0(M,X)$ be two V-harmonic maps into a regular ball $B_R(p)$. We have*

$$\begin{aligned} \Delta_V \psi &\geq \begin{cases} \displaystyle\sum_{i=1}^2 |\nabla u_i|^2 & \text{on } \Omega_0, \\ \displaystyle\frac{1}{2\psi}|\nabla \psi|^2 - \kappa\psi \sum_{i=1}^2 |\nabla u_i|^2 & \text{on } M\backslash\Omega_0, \end{cases} \\ \Delta_V \psi_i &\geq (1-\kappa\psi_i)|\nabla u_i|^2. \end{aligned} \tag{9}$$

Proof. As in [15], we define $U, U_1, U_2 : M \to X \times X$ by

$$U(x) = (u_1(x), u_2(x)), \quad U_i(x) = (p, u_i(x)), \quad i=1,2.$$

On $X \times X$ we consider the product metric

$$\langle (v_1, v_2), (w_1, w_2)\rangle = h(v_1, v_2) + h(w_1, w_2),$$

for $(v_1, v_2), (w_1, w_2) \in T_{y_1}X \times T_{y_2}X$. It is easy to see that U, U_1 and U_2 are V-harmonic maps. Set

$$Q_\kappa(\cdot,\cdot) := q_\kappa(d(\cdot,\cdot)).$$

Then the functions ψ, ψ_i $(i=1,2)$ are written as

$$\psi = Q_\kappa \circ U, \quad \psi_i = Q_\kappa \circ U_i.$$

It is clear that

$$\begin{aligned} |\nabla\psi|^2 &= \sum_\alpha \langle \nabla_\alpha Q_\kappa \circ U, \partial_\alpha U\rangle^2 \\ |\nabla\psi_i|^2 &= \sum_\alpha \langle \nabla_\alpha Q_\kappa \circ U_i, \partial_\alpha U_i\rangle^2. \end{aligned}$$

Since U and U_i $(i=1,2)$ are V-harmonic maps, we have

$$\begin{aligned} \Delta_V \psi &= \Delta_V(Q_\kappa \circ U) = \sum_\alpha \nabla^2 Q_\kappa(\partial_\alpha U, \partial_\alpha U) \\ &\quad + \langle \nabla Q_\kappa \circ U, \tau(U)\rangle + \langle \nabla Q_\kappa \circ U, dU(V)\rangle \\ &= \sum_\alpha \nabla^2 Q_\kappa(U)(\partial_\alpha U, \partial_\alpha U). \end{aligned} \tag{10}$$

Similarly, we have

$$\Delta_V \psi_i = \sum_\alpha \nabla^2 Q_\kappa(U_i)(\partial_\alpha U_i, \partial_\alpha U_i), \quad i=1,2. \tag{11}$$

Now the Lemma follows from the following Lemma. ∎

Lemma 2. *The Hessian of Q_κ satisfies*

$$\nabla^2 Q_\kappa(y)(v,v) \geq \begin{cases} |v|^2, & \text{if } y_1 = y_2 \\ \dfrac{\langle \nabla Q_\kappa(y), v\rangle^2}{2Q_\kappa(y)} - \kappa Q_\kappa(y)|v|^2, & \text{if } y_1 \neq y_2, \end{cases}$$

for any $v \in T_y(X \times X)$*,* $y = (y_1, y_2) \in X \times X$*, and*

$$\nabla^2 Q_\kappa(v,v) \geq (1 - \kappa Q_k(y))|v|^2,$$

if $v = (0, v_2)$ *or* $v = (v_1, 0)$*.*

Proof. This is Lemma 3 in [15]. ∎

It is clear that this Lemma is the crucial point of the argument of Jäger and Kaul. Now we give the

Proof of Theorem 1. First we have

$$\begin{aligned}\mathcal{L}_V(e^\Phi \psi) &= \operatorname{div}(\nabla\psi + \psi\nabla\Phi) + V(\psi) + \psi V(\Phi) \\ &= \Delta_V \psi + \psi \Delta_V \Phi + \langle \nabla\psi, \nabla\Phi \rangle.\end{aligned}$$

On Ω_0 we have $\psi = 0$ and $\nabla\psi = 0$, and hence

$$\mathcal{L}_V(e^\Phi \psi) = \Delta_V \psi \geq 0,$$

in view of (9). By using Young's inequality and noticing that $\omega'' = \omega'^2$ we have on $M \backslash \Omega_0$

$$\begin{aligned}\mathcal{L}_V(e^\Phi \psi) &= \Delta_V \psi + \psi \sum_{i=1}^{2} (\omega'' \circ \psi_i)|\nabla \psi_i|^2 \\ &\quad + \psi \sum_{i=1}^{2} (\omega' \circ \psi_i) \Delta_V \psi_i + \sum_{i=1}^{2} (\omega' \circ \psi_i) \langle \nabla\psi, \nabla\psi_i \rangle \qquad (12) \\ &\geq \Delta_V \psi - \frac{|\nabla\psi|^2}{2\psi} + \psi \sum_{i=1}^{2} (\omega' \circ \psi_i) \Delta_V \psi_i.\end{aligned}$$

Since $\omega'(\psi_i) = \frac{\kappa}{1 - \kappa\phi_i}$, from (9) we have

$$\psi \sum_{i=1}^{2} \omega'(\psi_i) \Delta_V \psi_i \geq \kappa\psi \sum_{i=1}^{2} |\nabla u_i|^2. \qquad (13)$$

(12) and (13), together with (9), imply that $\mathcal{L}_V(e^\Phi \psi) \geq 0$. Applying the ordinary maximum principle we obtain

$$\max_M e^\Phi \psi \leq \max_{\partial M} e^\Phi \psi.$$

∎

For the case of heat flows, we first have:

Lemma 3. *Let* ψ_i $(i = 1, 2)$*,* ψ *and* ω *be defined by (8). Let* $u_1,\ u_2 \in C^0(M_T, X)$ *be two solutions of the heat flow equation (6) for* V*-harmonic maps into a regular ball* $B_R(p)$*. We* *have*

$$
(14)\qquad (\Delta_V-\partial_t)\psi \;\geq\; \begin{cases} \displaystyle\sum_{i=1}^{2}|\nabla u_i|^2 & \text{on } \Omega_0,\\[2mm] \displaystyle\frac{1}{2\psi}|\nabla\psi|^2-\kappa\psi\sum_{i=1}^{2}|\nabla u_i|^2 & \text{on } M_T\backslash\Omega_0,\end{cases}
$$

$$
(\Delta_V-\partial_t)\psi_i \;\geq\; (1-\kappa\psi_i)|\nabla u_i|^2.
$$

where $\Omega_0 := \{(x,t)\in M_T|u_1(x,t)=u_2(x,t)\}$.

Proof. The proof is the same as that of Lemma 1, except that in (10) and (11), one uses the heat flow equation (6) instead of the V-harmonic map equation (2). ■

With this Lemma one can readily prove Theorem 2.

Proof of Theorem 2. We consider a parabolic operator as follows:

$$
\amalg_V := \mathcal{L}_V - e^{-\Phi}\partial_t = \mathcal{L} + e^{-\Phi}V - e^{-\Phi}\partial_t.
$$

As in the proof of Theorem 1, by using (14) we can deduce that $\amalg_V\Theta \geq 0$ on M_T. From the ordinary maximum principle for parabolic operators, we have

$$
\max_{M_T}\Theta \leq \max_{\partial_p M_T}\Theta.
$$

■

3. Existence results

As an application of the maximum principle proved in the previous section, we shall now prove existence results for Dirichlet problems for V-harmonic maps. Here we follow the ideas developed by von Wahl in [43, 44] closely.

First we need some known results for linear parabolic equations. For completeness we list them in the following lemma. For the proof, see [44], pp.139-140, Lemma I.2 for (1) and pp.137-138 for (2):

Lemma 4. *(c.f.* [44]*) (1) Assume for $q>2$ that*

$$
f\in \bigcap_{0<\widetilde{T}<+\infty} L^q((0,\widetilde{T}),L^q(M)),\quad \varphi,h\in H^{2,q}(M),\quad \varphi-h\in H_0^{2,q}(M),
$$

and let $w\in L^q((0,T),H^{2,q}(M))$ with $w'\equiv\partial_t w\in L^q((0,T),L^q(M))$ be the solution of

$$
w'-\Delta w+f=0,\quad w-h\in L^q((0,T),H_0^{2,q}(M)),\quad w(0)=\varphi,\quad 0<T<+\infty.
$$

Then we have

$$\int_0^T \|w'\|^q_{L^q(M)}dt + \int_0^T \|w(t)\|^q_{2,q}dt$$
$$\le C(M,q)\left(\int_0^T \|f(t)\|^q_{L^q(M)} + \int_0^T \|w(t)\|^q_{L^q(M)} + \int_0^T \|h\|^q_{2,q}dt + \|\varphi\|^q_{2,q}\right), \tag{15}$$

$$\int_{\widetilde{T}-\frac{1}{3}}^{\widetilde{T}+\frac{4}{3}} \xi^q(t)\|w'(t)\|^q_{L^q(M)}dt + \int_{\widetilde{T}-\frac{1}{3}}^{\widetilde{T}+\frac{4}{3}} \xi^q(t)\|w(t)\|^q_{2,q}dt$$
$$\le C(q,M,\xi)\left(\int_{\widetilde{T}-\frac{1}{3}}^{\widetilde{T}+\frac{4}{3}} \xi^q(t)\|f(t)\|^q_{L^q(M)}dt + \int_{\widetilde{T}-\frac{1}{3}}^{\widetilde{T}+\frac{4}{3}} \xi^q(t)\|h\|^q_{2,q}dt\right.$$
$$\left. + \int_{\widetilde{T}-\frac{1}{3}}^{\widetilde{T}+\frac{4}{3}} \|w(t)\|^q_{L^q(M)}dt\right), \tag{16}$$

where $\frac{1}{3} < \widetilde{T} < T - \frac{4}{3}$, $\xi \ge 0$, $\xi \in C_0^{0,1}(\widetilde{T} - \frac{1}{3}, \widetilde{T} + \frac{4}{3})$, *and* $C(\cdots)$ *stands for a positive constant depending only on the entries in the brackets.*

(2) Let $\mathbf{v} : (a,b) \to (H^{2,q}(M))^n$ *be a map with* $q > m$ *and denote*

$$\|\mathbf{v}\|^{1,2,q}_{(a,b)} := \left(\int_a^b \|\mathbf{v}'\|^q_{(L^q(M))^n}dt + \int_a^b \|\mathbf{v}\|^q_{(H^{2,q}(M))^n}dt\right)^{1/q}.$$

Suppose $\mathbf{v}$ *solves*

$$\mathbf{v}' - \Delta\mathbf{v} + \mathbf{F}(x,\mathbf{v},\nabla\mathbf{v}) = \mathbf{0} \tag{17}$$

with $\|\mathbf{v}\|^{1,2,q}_{(a,b)} \le C_1 < +\infty$ *and* $\|\mathbf{F}\|^{0,q}_{(a,b)} := \left(\int_a^b \|\mathbf{F}(x,\mathbf{v},\nabla\mathbf{v})\|^q_{L^q(M)}dt\right)^{1/q} \le C_2 < +\infty$, *then there is an* $\alpha > 0$ *such that*

$$\|\mathbf{v}\|^{1+\alpha}_{(a,b)} := \|\mathbf{v}\|_{L^\infty((a,b),C^{1+\alpha}(\bar{M}))} \le C, \tag{18}$$

where $C > 0$ *is a constant depending only on* q, M, C_1 *and* C_2.

∎

Proof of Theorem 3. For the regular ball $B_R(p)$ choose normal coordinates $\{y^i\}_{i=1,2,\cdots,n}$ centered at p. We only consider maps from M into $B_R(p)$. Such a map can be seen as a vector-valued function

$$u := (u^1, \cdots, u^n) \in (H^{2,q}(M))^n.$$

In this representation, a V-harmonic map $u \to B_R(p) \subset X$ satisfies the elliptic system:

$$\Delta u^i + \Gamma^i_{jk}(u)u^j_\alpha u^k_\beta g^{\alpha\beta} + V^\alpha u^i_\alpha = 0, \quad i = 1,2,\cdots,n.$$

For simplicity of notation, we write it in a concise form

$$\Delta u + \Gamma(du,du) + du(V) = 0. \tag{19}$$

Now we consider the initial-boundary problem for the heat flow of V-harmonic maps:

$$\begin{cases} \partial_t u &= \Delta u + \Gamma(du, du) + du(V), \\ u - u_0 &\in H_0^{2,q}(M, X), \quad u(0) = u_0, \\ u(M \times [0, T]) &\subset B_R(p). \end{cases} \tag{20}$$

Step 1. Global existence of flow (20).

Consider the following initial boundary problem with parameter $\lambda \in [0, 1]$ and $T > 0$:

$$P_T^\lambda(u_0) : \quad \begin{cases} \partial_t u &= \Delta u + \Gamma(du, du) + du(V), \\ u - \lambda u_0 &\in (H_0^{2,q}(M))^n, \\ u(0) &= \lambda u_0, \\ u(M \times [0, T]) &\subset B_R(p). \end{cases} \tag{21}$$

Let Λ_T be the set of all $\lambda \in [0, 1]$ such that $P_T^\lambda(u_0)$ admits a solution $u_\lambda \in L^q((0, T), (H^{2,q}(M))^n)$ and $u'_\lambda \in L^q((0, T), (L^q(M))^n)$. Furthermore, let Λ^* be the set of all $\lambda \in \Lambda_T$ such that $[0, \lambda] \subset \Lambda_T, \forall T > 0$ and

(i) $\|u_\tau\|^{1,2,q}_{(0,T)} \leq C(q, M, V, X, R, u_0)(T + 1)^{1/q}$, $0 \leq \tau \leq \lambda$, $\forall T > 0$;
(ii) $\|u_\tau\|^{1,2,q}_{(\widetilde{T}, \widetilde{T}+1)} \leq C(q, M, V, X, R, u_0)$, $0 \leq \tau \leq \lambda$, $\frac{1}{3} < \widetilde{T} < T - \frac{4}{3}$.

It is clear that $0 \in \Lambda^*$. We claim that

there exists $\delta_0 > 0$ such that if $\lambda_1 \in \Lambda^*$ and $\lambda_2 - \lambda_1 \leq \delta_0$, then $\lambda_2 \in \Lambda^*$.

Since $0 \in \Lambda^*$, applying this claim $[\frac{1}{\delta_0} + 1]$ times we have that $1 \in \Lambda^*$, i.e. the heat flow (20) has a global solution with estimates (i) and (ii).

We divide the proof of the claim into 2 substeps.

Substep 1. Suppose $\lambda_1 < \lambda_2$ with $\lambda_1 \in \Lambda^*$ and $\lambda_2 \in \Lambda_T$. Since $u_{\lambda_2} - u_{\lambda_1}$ satisfies

$$\partial_t(u_{\lambda_1} - u_{\lambda_2}) = \Delta(u_{\lambda_1} - u_{\lambda_2}) + \Gamma(du_{\lambda_1}, du_{\lambda_1}) - \Gamma(du_{\lambda_2}, du_{\lambda_2}) + du_{\lambda_1}(V) - du_{\lambda_2}(V),$$

by using Lemma 4 we have the following key estimates by using the maximum principle and an extended Sobolev inequality

$$\begin{aligned}
\|u_{\lambda_2}-u_{\lambda_1}\|^{1,2,q}_{(0,T)} &\le C(q,M,V,X)\Big(\int_0^T \||\nabla u_{\lambda_2}|^2\|^q_{L^q}dt+\int_0^T \||\nabla u_{\lambda_1}|^2\|^q_{L^q}dt \\
&\quad +\int_0^T \|u_{\lambda_2}-u_{\lambda_1}\|^q_{L^q}dt+\int_0^T (\|\nabla u_{\lambda_2}\|^q_{L^q}+\|\nabla u_{\lambda_1}\|^q_{L^q})dt \\
&\quad +|\lambda_2-\lambda_1|(T+1)\|u_0\|^q_{L^q}dt\Big)^{1/q} \\
&\le C(q,M,V,X)\Big(\int_0^T \||\nabla u_{\lambda_2}-\nabla u_{\lambda_1}|^2\|^q_{L^q}dt+\int_0^T \||\nabla u_{\lambda_1}|^2\|^q_{L^q}dt \\
&\quad +TR^q+T+(T+1)\|u_0\|^q_{L^q}\Big)^{1/q} \\
&\le C(q,M,V,X)\Big(\int_0^T \|u_{\lambda_2}-u_{\lambda_1}\|^q_{2,q}\|u_{\lambda_2}-u_{\lambda_1}\|^q_{L^\infty(M)}dt \\
&\quad +\int_0^T \|u_{\lambda_1}\|^q_{2,q}\|u_{\lambda_1}\|^q_{L^\infty(M)}dt+TR^q+T+(T+1)\|u_0\|^q_{L^q}\Big)^{1/q} \\
&\le C(q,M,V,X)|\lambda_2-\lambda_1|\cdot\|u_0\|_{L^\infty}\|u_{\lambda_2}-u_{\lambda_1}\|^{1,2,q}_{(0,T)} \\
&\quad +C(q,M,V,X)((\|u_{\lambda_1}\|^{1,2,q}_{(0,T)}R)^q+TR^q+T+(T+1)\|u_0\|^q_{L^q})^{1/q}, \qquad (22)
\end{aligned}$$

where in the third inequality, we have used the extended Sobolev inequality (see Theorem A in Appendix), and in the fourth inequality we have used the maximum principle, Theorem 2. We choose $\delta_1(q,M,V,X,u_0)>0$ small such that $C(q,M,V,X)\delta_1\cdot\|u_0\|_{L^\infty}=\frac{1}{2}$. From (22), it is clear that if $|\lambda_2-\lambda_1|\le\delta_1$, then $\|u_{\lambda_2}-u_{\lambda_1}\|^{1,2,q}_{(0,T)}$, and hence $\|u_{\lambda_2}\|^{1,2,q}_{(0,T)}$ satisfies:

$$\|u_{\lambda_2}\|^{1,2,q}_{(0,T)}\le C(q,M,V,X,u_0,R,\lambda_1)(T+1)^{1/q}. \tag{23}$$

Similarly, using (16) we have a small constant $\delta_2(p,M,V,X,u_0)>0$ independent of λ_1 such that if $|\lambda_2-\lambda_1|\le\delta_2$, then

$$\|u_{\lambda_2}\|^{1,2,q}_{(\widetilde{T},\widetilde{T}+1)}\le C(q,M,V,X,u_0,R,\lambda_1),\quad \frac{1}{3}<\widetilde{T}<T-\frac{4}{3}. \tag{24}$$

Set $\delta_0=\min(\delta_1,\delta_2)$. If $|\lambda_2-\lambda_1|\le\delta_0$, we have both (23) and (24). This proves that if $\lambda_1\in\Lambda^*$, then

$$[0,\lambda_1+\delta_0]\cap\{\lambda\,|\,[0,\lambda]\subset\Lambda_T,\,\forall T\}\subset\Lambda^*. \tag{25}$$

Substep 2 We show in this Substep that $[0,\lambda_1+\delta_0]\subset\Lambda^*$, the Claim.

This Substep follows from the proof of Substep 1 as follows. Let $\lambda_2\in(\lambda_1,\lambda_1+\delta_0]$. By standard parabolic theory, there exists a maximal existence time $T^*>0$ for $P^{\lambda_2}_{T^*}(u_0)$ such that if $T^*<\infty$, then the solution $u: M\times[0,T^*)\to X$ satisfies

$$\lim_{t\to T^*}\|u\|_{C^1(M)}(t)=\infty. \tag{26}$$

We claim that $T^*=\infty$. By contradiction assume that $T^*<\infty$. As in Substep 1, we can show that for any $T\in(0,T^*)$

$$\|u_{\lambda_2}\|^{1,2,q}_{(0,T)}\le C(q,M,V,X,u_0,R,\lambda_1)(T+1)^{1/q}\le C(q,M,V,X,u_0,R,\lambda_1)(T^*+1)^{1/q}. \tag{27}$$

This implies, by Lemma 4 (2) and standard parabolic theory again, that

$$\sup_{t\in[0,T]} \|u_{\lambda_2}\|_{C^1(M)} \le C(q, M, V, X, u_0, R, \lambda_1, T^*), \tag{28}$$

which does not depend on T, a contradiction to (26). Therefore $T^* = \infty$, which, together with Substep 1, proves the Claim.

Therefore, we conclude that $\Lambda^* = [0, 1]$. In particular, we have a solution u for $P^1_T(u_0)$ for any $T > 0$ with estimates

$$\|u\|^{1,2,q}_{(0,T)} \le C(q, M, V, X, u_0, R)(T + 1)^{1/q}, \tag{29}$$

$$\|u\|^{1,2,q}_{(\widetilde{T},\widetilde{T}+1)} \le C(q, M, V, X, u_0, R), \quad \frac{1}{3} < \widetilde{T} < T - \frac{4}{3}. \tag{30}$$

From (29), (30) and Lemma 4 (2), we then obtain

$$\|u(t, \cdot)\|_{1+\alpha} \le C(q, M, V, X, u_0, R), \quad \forall t \in (0, +\infty) \tag{31}$$

for some $\alpha > 0$. Consequently, by the parabolic regularity theory, we have the uniform estimate:

$$\|u\|_{C^{1+\alpha,2+\alpha}(M)} \le C. \tag{32}$$

Step 2. *(Sub-)convergence to a V-harmonic map.*

For $u_1(t, x) = u(t, x)$, $u_2(t, x) = u(t + \sigma, x)$, $\sigma > 0$, $\forall(x, t) \in M \times (0, +\infty)$, as in the proof of Theorem 2, the function Θ satisfies

$$\begin{cases} (\Delta - \partial_t)(\dfrac{\Theta}{\sigma^2}) + \langle V - \nabla\Phi, \nabla(\dfrac{\Theta}{\sigma^2})\rangle \ge 0, \\ \Theta|_{\partial M} = 0. \end{cases} \tag{33}$$

For such a functions $\frac{\Theta}{\sigma^2}$, one can deduce by the ordinary maximum principle that (see e.g. [44], pp.178-179)

$$\frac{\Theta}{\sigma^2} \le C(t - t_0)^{-k}, \quad \forall t \ge t_0 \tag{34}$$

for any positive integer k and some $t_0 > 0$. From the definition of the function Θ, by letting $\sigma \to 0$, we have

$$|u_t| \to 0, \quad \text{as } t \to +\infty \tag{35}$$

pointwise.

From this and (32), one can readily show that u subconverges to a V-harmonic map u_∞ satisfying (7) and $u_\infty(M) \subset B_R(p)$. This completes the proof of Theorem 3. ∎

With the Schauder and higher regularity estimates one can improve Theorem 3 to

Theorem 4. *Let $M, X, V, B_R(p)$ be as in Theorem 1. Let u_0 be a continuous map with $u_0(M) \subset B_R(p)$. Then the Dirichlet problem:*

$$\begin{cases} \tau(u) + du(V) = 0, \\ u|_{\partial M} = u_0|_{\partial M}, \end{cases} \tag{36}$$

admits a unique solution $u \in C^\infty(M) \cap C^0(\overline{M})$ such that $u(M) \subset B_R(p)$.

4. Applications

4.1. **Hermitian harmonic maps.** Let M be a complex manifold with an Hermitian metric $g = g_{\alpha\bar{\beta}} dz^\alpha d\bar{z}\beta$, and let X be a Riemmanian manifold with a metric h. An *Hermitan harmonic map*, introduced in [23], is a map from M into X satisfying

$$g^{\alpha\bar{\beta}}\left(\frac{\partial^2 u^i}{\partial z^\alpha \partial \bar{z}^\beta} + \Gamma^i_{jk}\frac{\partial u^j}{\partial z^\alpha}\frac{\partial u^k}{\partial \bar{z}^\beta}\right) = 0. \tag{37}$$

When (M, g) is Kähler, then an Hermitan harmonic map from M is just a usual harmonic map. In general, this is not the case, however, and this can be seen as follows.

Let J be the almost complex structure, and let e_A, $A = 1, \cdots, m, m+1, \cdots, 2m$ be a local basis of M, such that $e_{n+\alpha} = Je_\alpha$ for $\alpha = 1, \cdots, m$. Let ∇ be the Levi-Civita connection. Consider the complexified tangent space $TM \otimes \mathbb{C}$ of TM. A basis is $\{E_\alpha, E_{\bar{\alpha}}\}$ with $E_\alpha = \frac{1}{2}(e_\alpha + \sqrt{-1}e_{m+\alpha})$ and $E_{\bar{\alpha}} = \frac{1}{2}(e_\alpha - \sqrt{-1}e_{m+\alpha})$. Choose a torsion free connection $\widetilde{\nabla}$ compatible with the holomorphic structure. Let Δ be the Laplacian with respect to g and

$$\widetilde{\Delta} = h^{\alpha\beta}\frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta}.$$

Then [35]

$$(\widetilde{\Delta} - \Delta)f = (\widetilde{\nabla}_{e_A} e_A - \nabla_{e_A} e_A)f = \langle V, \nabla f\rangle.$$

From this one can write (37) as

$$\tau(u) + du(V) = 0, \tag{38}$$

where V is defined by

$$V = \widetilde{\nabla}_{e_A} e_A - \nabla_{e_A} e_A. \tag{39}$$

V consists of the first derivatives of J. Hence $V = 0$, if M is Kähler. When both the domain M and the target X is a Kähler manifold, then a holomorphic map is harmonic, but this is not necessarily true when M is not Kähler. However, holomorphic maps are Hermitian harmonic. Therefore, in order to find holomorphic maps, one should naturally study Hermitian harmonic maps. This is the motivation for Hermitian harmonic maps. In [23], the existence of a solution for the Dirichlet problem for Hermitian harmonic maps from a compact Hermitian manifold with boundary is obtained if the target X is a complete manifold of nonpositive sectional curvature, and if a topological nontriviality condition is satisfied. J. Y. Chen [3] generalized this result to a nonpositively curved target with a suitable boundary. L. Ni [35] and Grunau-Kühnel [8] obtained an existence results for Hermitian harmonic maps from complete, noncompact Hermitian manifolds into nonpositively curved targets. See also [41]. Recently Li-Zhang [29] proved the existence of Hermitian harmonic maps into a regular ball with radius $R < \frac{\arccos \frac{2}{\sqrt{5}}}{\sqrt{\kappa}}$. Applying our main results, we have

Corollary 1. *Let M be a compact Hermitian manifold with non-empty boundary ∂M and X a complete Riemannian manifold with sectional curvature bounded from above by $\kappa > 0$. Let $u_0 : M \to X$ be a continuous map with $u_0(M) \subset B_R$, a regular ball with radius $R < \frac{\pi}{2\sqrt{\kappa}}$. Then there exists a unique Hermitian harmonic map $u : M \to B_R \subset X$ with $u = u_0$ on ∂M.*

This result is optimal, as mentioned above. For a special class of Hermitian harmonic maps, the Hermitian harmonic morphisms, see [34].

4.2. **Weyl harmonic maps.** The Weyl harmonic maps have been introduced recently by Kokarev [25].

Let (M, c) be a conformal manifold with $c = [g]$, the conformal class of g, and (X, h) a Riemannian manifold. A Weyl structure on (M, c) is a torsion-free linear connection ∇^W preserving the conformal structure c, in the sense that there exists a 1-form Θ such that $\nabla^W g = \Theta \otimes g$ for any $g \in c$. Equivalently, ∇^W is defined by

$$\nabla^W_X Y = \nabla_X Y - \frac{1}{2}\Theta(X)Y - \frac{1}{2}\Theta(Y)X + \frac{1}{2}g(X,Y)\Theta^\sharp, \quad \forall X, Y \in TM,$$

where ∇ is the Levi-Civita connection and $\Theta^\sharp$ the vector field dual to Θ with respect to g. The 2-form $d\Theta$ does not depend on $g \in c$. A Weyl structure is called closed (resp. exact) if Θ is closed (resp. exact). Let $u : M \to X$ be a smooth map. From the connection ∇^W and the Levi-Civita connection ∇^h on X, one can define a connection $\widetilde{\nabla}$ on the bundle $T^*M \otimes u^{-1}TX$ and the second fundamental form

$$Ddu(X,Y) = \widetilde{\nabla}_X(du)Y := du(\nabla^W_X Y) - \nabla^h_{du(X)} du(Y).$$

A map $u : M \to X$ is called a Weyl harmonic maps if the trace of the second fundamental vanishes, i.e.,

$$\tau^W := \mathrm{tr} D = 0. \tag{40}$$

A direct computation shows that it is equivalent to

$$\tau^W = \tau - \frac{n-2}{2} du(\Theta^\sharp) = 0. \tag{41}$$

Hence, if the dimension of M is two, then (41) is just the harmonic equation. It is clear that a Weyl harmonic map is also a V-harmonic map with $V = -\frac{n-2}{2}\Theta^\sharp$. In [25] the existence of Weyl harmonic maps into a non-positively curved target was obtained with the method of [23]. Interesting applications on the rigidity were presented in [25] and [26].

As a direct consequence of our main results we have

Corollary 2. *Let (M, g) be a compact manifold with non-empty boundary ∂M and X a complete Riemannian manifold with sectional curvature bounded from above by $\kappa \geq 0$. Let M be endowed with a Weyl connection preserving the conformal class of g. Let $u_0 : M \to X$ be a continuous map with $u_0(M) \subset B_R$, a regular ball with radius $R < \frac{\pi}{2\sqrt{\kappa}}$. Then there exists a unique Weyl harmonic map $u : M \to B_R \subset X$ with $u = u_0$ on ∂M.*

4.3. **Affine harmonic maps.** An *affine manifold* is a smooth manifold M^m admitting an atlas of charts such that any coordinate transformation is a Euclidean affine transformation on the Euclidean space $\mathbb{R}^m$. On such a manifold M, one has a natural global flat and torsion-free connection ∇. Suppose that M is an m-dimensional affine manifold and (X, h) an n-dimensional Riemannian manifold. Choose local coordinates $\{x^\alpha\}_{\alpha=1,\cdots,m}$ and $\{y^i\}_{i=1,\cdots,n}$ on M and X respectively. Let $u : M \to X$ be a smooth map. Then the flat connection ∇ on M and the Levi-Civita connection on X induce a connection on the bundle $T^*M \otimes u^{-1}TX$, which in turn induces the second fundamental form Ddu of u as before.

Given a metric $g = (g_{\alpha\beta})$ on M. Then we have the following elliptic system of second order:

$$\sigma(u) \equiv \mathrm{Tr}_g Ddu = 0. \tag{42}$$

A solution of (42) is called an *affine harmonic maq*. These maps were introduced in [21] and [22]. They proved the existence results for affine harmonic maps in given homotopy classes from a compact affine manifold M to another compact Riemannian manifold N with nonpositive sectional curvature, under a topological condition.

In local coordinates, (42) reads

$$g^{\alpha\beta}\left(\frac{\partial^2 u^i}{\partial x^\alpha \partial x^\beta} + \Gamma^i_{jk}(u)\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\beta}\right) = 0, \quad i = 1, 2, \cdots, n. \tag{43}$$

Regarding (M, g) as a Riemannian manifold, we have the usual tension field $\tau(u)$ of u defined by

$$\tau^i(u) \equiv g^{\alpha\beta}\left(\frac{\partial^2 u^i}{\partial x^\alpha \partial x^\beta} - \Gamma^\gamma_{\alpha\beta}\frac{\partial u^i}{\partial x^\gamma} + \Gamma^i_{jk}(u)\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\beta}\right), \quad i = 1, 2, \cdots, n, \tag{44}$$

where $\Gamma^\gamma_{\alpha\beta}$ and Γ^i_{jk} stands for the Christoffel symbols of the Levi-Civita connections of (M, g) and (X, h) respectively. Define a vector field on M by

$$V := \Gamma^\gamma_{\alpha\beta} g^{\alpha\beta}\frac{\partial}{\partial x^\gamma}.$$

Then Equation (42) can be written as

$$\tau(u) + du(V) = 0, \tag{45}$$

which means that an affine harmonic map can be regarded as a V-harmonic map. Therefore, by our previous results, we immediately have:

Corollary 3. *Let M be a compact affine manifold with non-empty boundary ∂M and equipped with a metric g, and let X be a complete Riemannian manifold with sectional curvature bounded from above by a constant $\kappa \geq 0$. Let $u_0 : M \to X$ be a continuous map with $u_0(M) \subset B_R$, a regular ball with radius $R < \frac{\pi}{2\sqrt{\kappa}}$. Then there exists a unique affine harmonic map $u : M \to B_R \subset X$ with $u = u_0$ on ∂M.*

4.4. **Harmonic maps from Finsler manifolds into Riemannian manifolds.** In recent years, the geometry and analysis of Finsler manifolds have been intensively studied. Harmonic maps from Finsler manifolds into Riemannian manifolds were introduced in [1] and [36]. The two definitions are slightly different, due to different choices of the volume form on the Finsler manifold. Here we follow the definition of [36]. In [37], Mo-Yang proved the existence of harmonic maps from a Finsler manifold into a nonpositively curved manifolds by using the heat flow method. In [42], von der Mosel and Winklmann generalized results of Giaquinta-Hildebrandt and Hildebrandt-Jost-Widman from Riemannian to Finsler domains. Among other results they gave an existence result for harmonic maps with the image of the boundary data contained in a regular ball of radius $R < \frac{\pi}{2\sqrt{\kappa}}$. We will see that such results can also be proved from the viewpoint of V-harmonic maps, or f-harmonic maps in this case.

Let M be an m-dimensional manifold and (X, h) an n-dimensional Riemannian manifold. Choose local coordinates $\{x^\alpha\}_{\alpha=1,\cdots,m}$ and $\{z^i\}_{i=1,\cdots,n}$ on M and X respectively. For any $y \in T_xM$, one can write $y = y^\alpha \frac{\partial}{\partial x^\alpha}$. This gives local coordinates $\{(x^\alpha, y^\alpha)\}_{\alpha=1,2,\cdots,m}$ in the tangent bundle TM of M. A *Finsler structure F on M* is a function $F : TM \to [0, +\infty)$ satisfying $F \in C^\infty(TM \setminus 0)$ and $F(x, ty) = tF(x, y)$, $\forall t > 0$, $\forall (x, y) \in TM$. Denote by $g_{\alpha\beta}(x,y) := \frac{1}{2}(F^2)_{y^\alpha y^\beta}$ the *fundamental tensor*. The Sasaki metric G on $TM \setminus 0$ induces a Riemannian metric G_{SM} with a volume form $d\mathrm{vol}_{SM}$ on the sphere bundle $SM := \{(x,[y]) := (x,ty) | t > 0, (x,y) \in TM \setminus 0\}$. Consider a map $u : (M,F) \to (N,h)$, the energy density of u is given by $e(u) : SM \to [0,+\infty)$:

$$e(u)(x,[y]) := \frac{1}{2} g^{\alpha\beta}(x,y) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} h_{ij}(u),$$

where $(g^{\alpha\beta}) := (g_{\alpha\beta})^{-1}$. The energy functional of u is then defined as

$$E(u) := \frac{1}{\mathrm{vol}(S^{m-1})} \int_{SM} e(u) d\mathrm{vol}_{SM},$$

where $\mathrm{vol}(S^{m-1})$ is the volume of the stand sphere S^{m-1}.

Choose an orthonormal frame $\{e_\alpha\}_{\alpha=1,2,\cdots,m}$ on the Riemannian vector bundle $(\pi^* TM, g)$ and its dual frame $\{\omega_\alpha\}_{\alpha=1,2,\cdots,m}$ with $\omega_m = \omega$. If we denote $\omega_\alpha = v_{\alpha\beta} dx^\beta$, then $\det(v_{\alpha\beta}) = \sqrt{\det(g_{\alpha\beta})}$. Taking the exterior derivatives of the forms ω_α then yields the Chern connection forms $\{\omega_{\alpha\beta}\}$. This gives rise to the volume form of (SM, G_{SM}):

$$d\mathrm{vol}_{SM} := \omega_1 \wedge \cdots \wedge \omega_m \wedge \omega_{m1} \wedge \cdots \wedge \omega_{m,m-1}.$$

Setting

$$\Omega := \omega_{m1} \wedge \cdots \wedge \omega_{m,m-1} \quad mod \quad dx^\alpha, \qquad dx := dx^1 \wedge \cdots \wedge dx^m,$$

we may write the energy functional as follows:

$$E(u) = \frac{1}{\mathrm{vol}(S^{m-1})} \int_M dx \int_{S_x M} e(u) \sqrt{\det(g_{\alpha\beta})} \Omega.$$

Denote

$$S^{\alpha\beta}(x) := \frac{\int_{S_x M} g^{\alpha\beta}(x,y) \sqrt{\det(g_{\gamma\xi}(x,y))} \Omega}{\mathrm{vol}(S^{m-1}) \sigma(x)},$$

where $\sigma(x) := \int_{S_x M} \sqrt{\det(g_{\gamma\xi}(x,y))} \Omega$, then (c.f. [37], [1])

$$E(u) = \frac{1}{2} \int_M S^{\alpha\beta}(x) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} h_{ij}(u(x)) \sigma(x) dx.$$

Regarding u as a map from the Riemannian manifold $(M, (S_{\alpha\beta}))$ into (X,h), where $(S_{\alpha\beta}) := (S^{\alpha\beta})^{-1}$, then its energy density is $\tilde{e}(u) := \frac{1}{2} S^{\alpha\beta}(x) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} h_{ij}(u)$. Setting $d\mathrm{vol}_S := \sqrt{\det(S_{\alpha\beta})} dx$ and $f := \ln \frac{\sigma}{\sqrt{\det(S_{\alpha\beta})}}$, one can rewrite the energy functional as

$$E(u) = \int_M \tilde{e}(u) e^f d\mathrm{vol}_S,$$

which means that a harmonic map $u : (M,F) \to (X,h)$ is a f-harmonic map, therefore u must satisfy

$$\tau(u) + du(V) = 0$$

with $V := \nabla f$, namely, u is a V-harmonic map from $(M, (S_{\alpha\beta}))$ to (X,h). Thus, in contrast to the previous examples, here we do have an underlying variational structure.

Applying our previous results, we immediately have the following

Corollary 4 ([42], [37]). *Let (M, F) be a compact Finsler manifold with non-empty boundary ∂M and X a complete Riemannian manifold with sectional curvature bounded from above by a constant $\kappa \geq 0$. Let $u_0 : M \to X$ be a continuous map with $u_0(M) \subset B_R$, a regular ball with radius $R < \frac{\pi}{2\sqrt{\kappa}}$. Then there exists a unique harmonic map $u : (M, F) \to B_R \subset X$ with $u = u_0$ on ∂M.*

5. Appendix

For convenience of the readers, we recall the extend Sobolev inequality in this Appendix.

Theorem A. *(c.f. [7], p.27, Theorem 10.1.) Let Ω be a bounded domain in $\mathbb{R}^m$ with $\partial\Omega \in C^k$, and let u be any function in $W^{k,r}(\Omega) \cap L^q(\Omega)$, $1 \leq r, q \leq \infty$. For any integer j, $0 \leq j < k$, and for any number a in the interval $j/k \leq a \leq 1$, set*

$$\frac{1}{p} = \frac{j}{m} + a(\frac{1}{r} - \frac{k}{m}) + (1-a)\frac{1}{q}.$$

If $k - j - m/r$ is not a nonnegative integer, then

$$\|D^j u\|_{\Omega}^{0,p} \leq C(\|u\|_{\Omega}^{k,r})^a (\|u\|_{\Omega}^{0,q})^{1-a}. \tag{46}$$

If $k - j - m/r$ is a nonnegative integer, then (46) holds for $a = j/k$. The constant C depends only on Ω, r, q, k, j, a.

References

[1] P. Centore, Finsler Laplacians and minimal-energy maps, *Internat. J. Math.* **11** (2000), 1–13.

[2] D. M. J. Calderbank and H. Pedersen, Einstein-Weyl geometry, in Essays on Einstein Manifolds, Surveys in Differential Geometry, **VI**, Int. Press, Boston, MA, 1999.

[3] J. Y. Chen, A boundary value problem for Hermitian harmonic maps and applications. Proc. Amer. Math. Soc. **124** (1996), 2853–2862.

[4] Q. Chen, J. Jost and G. Wang, The maximum principle and the Dirichlet problem for Dirac-harmonic maps, *Cal. Var. P.D.E.*, DOI10.1007/s00526-012-0512-5, 2012.

[5] Q. Chen, J. Jost, J.Y. Li and G. Wang, Dirac-harmonic maps, *Math. Z.* **254** (2006), 409–432.

[6] J. Eells and L., Lemaire, A report on harmonic maps, Bull. London Math. Soc. **10** (1978), 1–68.

[7] A. Friedman, *Partial differential equations*, Holt, Rinihart and Winston, Inc., New York, Chicago, San Francisco, Atlanta, Dallas, Montreal, Toronto, London, Sydney, 1969.

[8] H.-C. Grunau, M. Kühnel, On the existence of Hermitian-harmonic maps from complete Hermitian to complete Riemannian manifolds, *Math. Z.* **249** (2005), 297–327.

[9] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd Edition. Springer: Berlin, Heidelberg, New York, Tokyo, 1983.

[10] J. W. Han, Y. B. Shen, Harmonic maps from complex Finsler manifolds, *Pacific J. Math.* **236** (2008), 341–356.

[11] R. Hamilton, Harmonic maps of manifolds with boundary, *Lecture Notes in Mathematics*, **471**, Springer, Berlin, Heidelberg, New York (1975).

[12] P. Hartman, On homotopic harmonic maps, *Can. J. Math.* **19** (1967), 673–687.

[13] S. Hildebrandt, J. Jost, K.O. Widman, Harmonic mappings and minimal submanifolds, *Inv. math.* **62** (1980), 269–298

[14] S. Hildebrandt, H. Kaul, K.O. Widman, An existence theorem for harmonic mappings of Riemannian manifolds, *Acta Math. 138* (1977), 1–16

[15] W. Jäger and H. Kaul, Uniqueness and stability of harmonic maps and their Jacobi fields, *Manusc. Math.* **28** (1979), 269–291.

[16] W. Jäger and H. Kaul, Uniqueness of harmonic mappings and of solutions of elliptic equations on Riemannian manifolds, *Math. Ann.* **240** (1979), 231–250.
[17] J. Jost, *Harmonic mappings between Riemannian manifolds*, ANU-Press, Canberra, 1984
[18] J. Jost, *Riemannian geometry and geometric analysis*, Sixth edition. Universitext. Springer, 2011.
[19] J. Jost, Two-dimensional geometric variational problems, *Wiley,* 1991.
[20] J. Jost, Existence proofs for harmonic mappings with the help of a maximum principle, *Math. Z.* **184** (1983), 489–496.
[21] J. Jost, F.M. Simsir, Affine harmonic maps *Analysis (Munich)* **29** (2009), 185–197.
[22] J. Jost, F.M. Simsir, Non-divergence harmonic maps, *Harmonic maps and differential geometry,* 231–238, Contemp. Math., 542, Amer. Math. Soc., Providence, RI, 2011.
[23] J. Jost, S.T. Yau, A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorem in Hermitian geometry, *Acta Math.* **170** (1993) 221–254.
[24] H. Karcher, Riemannian center of mass and mollifier smoothing, *Comm. Pure Appl. Math.* **30** (1977), 509–541.
[25] G. Kokarev, On pseudo-harmonic maps in conformal geometry, *Proc. London Math. Soc.* **(3) 99** (2009), 168–194.
[26] G. Kokarev and D. Kotschick, Fibrations and fundamental groups of Kähler-Weyl manifolds, *Proc. Amer. Math. Soc.* **138** (2010), 997–1010.
[27] J. Y. Li, M. Wang, Liouville theorems for self-similar solutions of heat flows, J. Eur. Math. Soc., **11** (2009), 207–221
[28] J. Y. Li, X. R. Zhu, Non existence of quasi-harmonic spheres, Calc. Var. **37**(2010) 441–460.
[29] Z.Y. Li, X. Zhang, Hermitian harmonic maps into convex balls, *Canad. Math. Bull.* **50** (2007), 113–122.
[30] A. Lichnerowicz, Applications harmoniques et variétés Kähleriennes. *Rend. Sem. Mat. Fis. Milano,* **39** (1969) 186–195
[31] Y. X. Li, Y. D. Wang, Bubbling location for F-harmonic maps and inhomogeneous Landau-Lifshitz equations, *Comment. Math. Helv.,* **81** (2006), 433–448,
[32] F. H. Lin, C. Y. Wang, Harmonic and quasi-harmonic spheres, *Comm. Anal. Geom.,* **7** (1999) 397–429
[33] F- H. Lin,C. Y. Wang, The analysis of harmonic maps and their heat flows. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008. xii+267 pp.
[34] E. Loubeau, Hermitian harmonic maps, *Beiträge Algebra Geom.,* **40** (1999), 1–14.
[35] L. Ni, Hermitian harmonic maps from complete Hermitian manifolds to complete Riemannian manifolds, *Math. Z.* **232** (1999), 331–355.
[36] X. Mo, Harmonic maps from Finsler manifolds *Illinois J. Math.* **45** (2001), 1331–1345.
[37] X. Mo, Y. Y. Yang, The existence of harmonic maps from Finsler manifolds to Riemannian manifolds. *Sci. China Ser.* **A 48** (2005), 115–130
[38] R. Schoen and S.T. Yau, Lectures on harmonic maps, *Conference Proceedings and Lecture Notes in Geometry and Topology, II,* International Press, Cambridge, MA, 1997.
[39] J. Sacks and K. Uhlenbeck, The existence of minimal immersions of 2-spheres, *Ann. Math.* **113** (1981), 1–24.
[40] A. Tachikawa, Partial regularity results up to the boundary for harmonic maps into a Finsler manifold. *Ann. Inst. H. PoincaréAnal. Non Linéaire,* **26** (2009), 1953–1970,
[41] D. Ting, Hermitian harmonic maps from complete manifolds into convex balls. *Nonlinear Anal.,* **72** (2010), 3457– 3462.
[42] H., von der Mosel, Winklmann, Sven On weakly harmonic maps from Finsler to Riemannian manifolds, *Ann. Inst. H. Poincaré Anal. Non Linéaire,* **26** (2009), 39–57.
[43] W. von Wahl, The continuity or stability method for nonlinear elliptic and parabolic equations and systems, *Proceedings of the Second International Conference on Partial Differential Equations (Italian) (Milan, 1992). Rend. Sem. Mat. Fis. Milano* 62 (1992), 157–183.
[44] W. von Wahl, Wolf Klassische Lösbarkeit im Grossen fr nichtlineare parabolische Systeme und das Verhalten der Lösungen für $t \to +\infty$, *Nachrichten der Akademie der Wissenschaften in Göttingen II: Mathematisch-Physikalische Klasse* 5(1981), 131–177.
[45] G. Wang and Deliang Xu, Harmonic maps from smooth measure spaces, Internat. J. Math. **23** (2012), no. 9, 1250095.

[46] Y. L. Xin, Geometry of Harmonic Maps, Progr. Nonlinear Differential Equations Appl., vol 23, Birkhaüser, 1996.

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