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Preprint no.: 75 2014

EXISTENCE AND LIOUVILLE THEOREMS FOR V-HARMONIC MAPS FROM COMPLETE MANIFOLDS

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ABSTRACT. We establish existence and uniqueness theorems for V-harmonic maps from complete noncompact manifolds. This class of maps includes Hermitian harmonic maps, Weyl harmonic maps, affine harmonic maps and Finsler maps from a Finsler manifold into a Riemannian manifold. We also obtain a Liouville type theorem for V-harmonic maps. In addition, we prove a V-Laplacian comparison theorem under the Bakry-Emery Ricci condition.

Keywords and phrases: V-harmonic map, noncompact manifold, existence, Liouville theorem, V-Laplacian comparison theorem.
MSC 2000: 58E20, 53C27.

1. INTRODUCTION

Let (M, g) and (X, h) be Riemannian manifolds. Given a smooth vector field V on M, we call a map $u : M \to X$ a *V-harmonic map* if u satisfies

$$\tau_V(u) := \tau(u) + du(V) = 0, \tag{1.1}$$

where $\tau(u) = \text{tr}Ddu$ is the tension field of the map u. This is a generalization of the usual harmonic map.

The notion of V-harmonic maps was introduced in [5]. It includes the Hermitian harmonic maps introduced and studied in [17], the Weyl harmonic maps from a Weyl manifold into a Riemannian manifold [18], the affine harmonic maps mapping from an affine manifold into a Riemannian manifold [15], [16], and harmonic maps from a Finsler manifold into a Riemannian manifold [2], [12], [30], [33] and [31], see [5] for explanation of these relations. Another interesting special case is when V is a gradient vector field, i.e., $V = \nabla f$ for some function $f : M \to \mathbb{R}$, then (1.1) takes the form

$$\tau_f(u) := \tau(u) + du(\nabla f) = 0, \tag{1.2}$$

which is the Euler-Lagrange equation of

$$\int_M |\nabla u|^2 e^f dv(g).$$

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no. 267087. The research of QC is also partially supported by NSFC (No.11171259) and RFDP of China. HQ is partially supported by NSFC (No.11126189), PDFC and FRFCU of China. The authors thank the Max Planck Institute for Mathematics in the Sciences for good working conditions when this work was carried out.

In this case, (1.2) is of divergence form. A solution of (1.2) is called an *f-harmonic map* [19]. They are harmonic maps from the smooth metric measure space $(M, g, e^f dv_g)$. A self-similar solution of the harmonic heat flow is an f-harmonic map with a special f, see [27]. For recent studies of f-harmonic maps, see [27, 24, 25, 26, 35].

To express V-harmonic maps more explicitly, we let $\{x^\alpha\}$ and $\{y^i\}$ be local coordinates on M and X respectively. On X we choose the Levi-Civita connection ∇, and on M we choose the torsion-free connection ${}^V\nabla$ whose Christoffel symbols are given by ${}^V\Gamma^\sigma_{\alpha\beta} := \Gamma^\sigma_{\alpha\beta} - \frac{1}{n} g_{\alpha\beta} V^\sigma$, where $n := \dim M$. Correspondingly, we denote the standard Beltrami-Laplacian on M by Δ and the Laplacian of the torsion-free connection ${}^V\nabla$ by Δ_V. Define the second fundamental form of the map u with respect to these connections by

$$\nabla du(X, Y) = \nabla_Y du(X) - du({}^V\nabla_X Y).$$

The torsion-free assumption makes $\nabla du(\cdot, \cdot)$ symmetric. And the corresponding tension field of the map u is then given by

$$\begin{aligned}
\tau_V(u) =& \tau_V(u)^k \frac{\partial}{\partial y^k} = g^{\alpha\beta} \nabla du \left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} \right) \\
=& g^{\alpha\beta} \left(\frac{\partial^2 u^k}{\partial x^\alpha \partial x^\beta} + \Gamma^k_{ji} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} - {}^V\Gamma^\sigma_{\alpha\beta} \frac{\partial u^k}{\partial x^\sigma} \right) \frac{\partial}{\partial y^k} \\
=& \left(g^{\alpha\beta} \frac{\partial^2 u^k}{\partial x^\alpha \partial x^\beta} + g^{\alpha\beta} \Gamma^k_{ji} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} - g^{\alpha\beta} \Gamma^\sigma_{\alpha\beta} \frac{\partial u^k}{\partial x^\sigma} + V^\sigma \frac{\partial u^k}{\partial x^\sigma} \right) \frac{\partial}{\partial y^k}.
\end{aligned}$$

In particular, if X is the real line $\mathbb{R}$, i.e., u is a function on M, then

$$\begin{aligned}
\tau_V(u) =& g^{\alpha\beta} \frac{\partial^2 u}{\partial x^\alpha \partial x^\beta} - g^{\alpha\beta} \Gamma^\sigma_{\alpha\beta} \frac{\partial u}{\partial x^\sigma} + V^\sigma \frac{\partial u}{\partial x^\sigma} \\
=& \Delta u + \langle V, \nabla u \rangle = \Delta_V u.
\end{aligned}$$

We say that the elliptic differential operator Δ_V has *positive first eigenvalue* $\lambda_V(M)$ if for any smooth function ϕ on M with compact support set, holds

$$\int_M (-\Delta_V \phi)\phi \geq \lambda_V(M) \int_M \phi^2. \tag{1.3}$$

The operator Δ_V includes interesting examples, for instance, the operator

$$\mathfrak{L} u := \Delta u - \frac{1}{2} \langle x, \nabla u \rangle = e^{\frac{|x|^2}{4}} \operatorname{div}(e^{-\frac{|x|^2}{4}} \nabla u) \tag{1.4}$$

introduced by Colding-Minicozzi in [6] for self-shrinkers M^n in the Euclidean space $\mathbb{R}^{n+m}$, where x is the position vector of a point in $M \subset \mathbb{R}^{n+m}$. We recall that M^n is called a *self-shrinker in* $\mathbb{R}^{n+m}$ if it satisfies

$$H = -\frac{x^N}{2}.$$

Here H is the mean curvature vector of M in $\mathbb{R}^{n+m}$, x^N is the normal part of x.

As another example, the f-harmonic functions have many interesting properties comparable to the usual harmonic functions (c.f. [21], [32]), and they are naturally related to the Bakry-Emery Ricci tensor:

$$\mathrm{Ric}_f := \mathrm{Ric} - \mathrm{Hess}(f).$$

More generally, we will see in Lemma 1 below that V-harmonic maps are closely related to the following tensor:

$$\mathrm{Ric}_V := \mathrm{Ric} - \frac{1}{2} L_V g,$$

where L_V stands for the Lie derivative with respect to the vector field V. In recent years, the Bakry-Emery Ricci tensor has been studied in various contexts (see e.g. [29], [21], [36]). We also recall that a Riemannian manifold (M, g) satisfying $\mathrm{Ric}_V = \rho g$ for some constant ρ is called a *Ricci soliton* (see e.g. [1]). When $V = \nabla f$, namely, $\mathrm{Ric}_f = \rho g$ the Ricci soliton is called a *gradient steady (shrinking, expanding resp.) soliton* if the constant ρ is zero (positive, negative resp.).

In general, the elliptic system (1.1) has no variational structure and is not of a divergence form. Therefore many useful tools developed in the theory of harmonic maps are no longer available, and the analysis here is more difficult. In [5], by combining a maximum principle of Jäger-Kaul type [13] with the continuity method, the authors established an existence and uniqueness theorem for V-harmonic maps from compact Riemannian manifolds with boundary into a regular ball. Here, and in the sequel, a regular ball $B_R(p)$ is a distance ball in X that is disjoint from the cut-locus of its center p and with radius $R < \frac{\pi}{2\sqrt{\kappa}}$, where $\kappa = \max\{0, \sup_{B_R(p)} K_X\}$ and $\sup_{B_R(p)} K_X$ is an upper bound of the sectional curvature K of X on $B_R(p)$

Theorem A (c.f. [5]). *Let (M, g) be a compact Riemannian manifold with boundary ∂M and (X, h) a complete Riemannian manifold without boundary. Let $B_R(p)$ be a regular ball in X.*

Let $u_0 : M \to X$ be a continuous map with $u_0(M) \subset B_R(p)$. Then the Dirichlet problem:

$$\left\{ \begin{array}{rcl} \tau(u) + du(V) &=& 0, \\ u|_{\partial M} &=& u_0|_{\partial M}, \end{array} \right. \tag{1.5}$$

admits a unique solution $u \in C^\infty(M, X) \cap C^0(\overline{M}, X)$ such that $u(M) \subset B_R(p)$.

In this paper, based on Theorem A, we shall prove the existence of V-harmonic maps from complete noncompact Riemannian manifolds by the compact exhaustion method. In Section 2 we consider the case where the target manifold is a complete Riemannian manifold with nonpositive sectional curvature. For the usual harmonic maps from noncompact complete manifolds into nonpositively curved target manifolds, Li-Tam [23] proved existence theorems via the heat-flow method. Ding-Wang [7] improved their results by using the elliptic method. Later, J.Y.Li [20] reproved Ding-Wang's results by the heat-flow method again. In [34], L.Ni established the existence and uniqueness theorems for Hermitian harmonic maps under the positive spectrum condition on the holomorphic Laplace operator. In fact, [34] also improves on methods for the usual harmonic maps from noncompact complete manifolds. Grunau-Kühnel [10] introduced an invertibility condition (see Section 2) on the holomorphic Laplace operator and established the corresponding result. In the same line, we obtain the following existence and uniqueness theorem for V-harmonic maps which gives a uniform generalization of the above mentioned results.

We first introduce some notations. The energy density of u is

$$e(u) = g^{\alpha\beta}h_{ij}(u(x))\frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^j}{\partial x^\beta}.$$

For $\mu > 0$, we denote

$$C^0_\mu(M) := \{f : M \to \mathbb{R} : f \text{ is continuous and there exists } x_0 \in M \text{ and a constant } C = C(f) \text{ such that } |f(x)| \leq C(1+\rho(x,x_0))^{-\mu}\}$$

Assumption (A) (c.f. [10]) *There exists positive numbers $\mu, \mu' > 0$ such that for every $f \in C^0_\mu(M)$, there exists a solution $v \in C^0_{\mu'}(M)$ of*

$$-\Delta_V v = f \quad in \quad M.$$

Let us introduce the following conditions:

Definition 1. *Let V be a vector field on M, and $u_0 : (M,g) \to (X,h)$ be a map.*

Condition 1. The first eigenvalue $\lambda_V(M)$ for Δ_V is positive, $\|V\|_{L^\infty(M)} < \infty$ and $|\tau_V(u_0)| \in L^{2p}(M)$ for some $p \geq 1$;

Condition 2. The Assumption (A) is satisfied with positive numbers $\mu, \mu' > 0$ and $|\tau_V(u_0)| \in C^0_\mu(M)$.

Now we state the following

Theorem 1. *Let (M,g) be a complete noncompact Riemannian manifold, (X,h) a complete Riemannian manifold with nonpositive sectional curvature. Let u_0 be a smooth map from M to X.*

If either one of the conditions in Definition 1 is satisfied, then there exists a unique V-harmonic map $u : M \to X$ that is homotopic to u_0.

Furthermore, when condition 1 is satisfied, we have $\rho \in L^{2p}(M)$; when condition 2 is satisfied, we have $\rho \in C^0_{\mu'}(M)$, here ρ is the homotopy distance between u and u_0 on X.

Recall that for a gradient steady Ricci soliton (M,g,f), i.e., $\mathrm{Ric}_f \equiv 0$, there is a positive constant a such that [32]

$$|\nabla f|^2 + S = a^2, \quad \Delta f + S = 0, \quad S \geq 0, \tag{1.6}$$

here S is the scalar curvature of M. Munteanu-Wang [32] proved that Δ_f has positive first eigenvalue: $\lambda_f(M) = \frac{a^2}{4}$. Combining this with Theorem 1, we have

Corollary 1. *Let (M, g, f) be a gradient steady Ricci soliton, (X, h) be a complete Riemannian manifold with nonpositive sectional curvature, If there is a positive number $p \geq 1$ such that u_0 has bounded $\|\tau_f(u_0)\|_{L^{2p}}$, then there exists a unique f-harmonic map $u : M \to X$ such that u is homotopic to u_0.*

Remark We would like to mention that Ding-Xin [8] derived for any complete embeddedself-shrinker M^n in $\mathbb{R}^{n+1}$ an estimate for the first eigenvalue, in particular, it is positive. In view of (1.4), one may obtain a similar corollary as above.

Now we assume that the target manifold X is a complete Riemannian manifold with sectional curvature bounded above by a positive constant, and consider maps from complete noncompact manifolds into regular balls $B_R(p) \subset X$. Recall that for the usual harmonic maps, various Liouville theorems were established by S.Y.Cheng [3], H.Choi [4] etc. In Section 3, we prove the following Liouville theorem for V-harmonic maps:

Theorem 2. *Let (M, g) be a complete noncompact Riemannian manifold with*

$$\mathrm{Ric_V} := \mathrm{Ric}^M - \frac{1}{2} L_V g \geq -A,$$

where $A \geq 0$ is a constant, Ric^M is the Ricci curvature of M and L_V is the Lie derivative. Let (X, h) be a complete Riemannian manifold with sectional curvature bounded above by a positive constant κ. Let $u : M \to X$ be a V-harmonic map such that $u(M) \subset B_R(p)$, where $B_R(p)$ is a regular ball in X, i.e., disjoint from the cut-locus of p and $R < \frac{\pi}{2\sqrt{\kappa}}$. If V satisfies

$$\langle V, \nabla r \rangle \leq v(r) \tag{1.7}$$

for some nondecreasing function $v(\cdot)$ satisfying $\lim\limits_{r \to +\infty} \frac{|v(r)|}{r} = 0$, where r denotes the distance function on M from a fixed point $\tilde{p} \in M$, then $e(u)$ is bounded by a constant depending only on A, κ and R. Furthermore, if $A = 0$, namely,

$$\mathrm{Ric}^M \geq \frac{1}{2} L_V g,$$

then u must be a constant map.

Remark This result can be applied to gradient Ricci solitons, self-shrinkers in Euclidean spaces, quasi-harmonic maps etc. For instance, from Theorem 2 we immediately have:

Corollary 2. *Let (M, g, f) be a gradient steady Ricci soliton. If u is a bounded f-harmonic function on M, then u must be a constant.*

Proof. We note that in this case $V = \nabla f$, and by (1.6) we have $|V| \leq a$. ∎

Recently, Li-Wang [24] proved Liouville theorems for quasi-harmonic maps, that is, for maps $u : (\mathbb{R}^n, g_0) \to (X, h)$ satisfying

$$\tau(u) - \frac{1}{2} \langle x, \nabla u \rangle = 0.$$

Let $V=-\frac{1}{2}x$, then u is a V-harmonic map. It is easy to verify that

$$\mathrm{Ric}_V=\mathrm{Ric}-\frac{1}{2}L_Vg_0=\frac{1}{2}g_0>0$$

and $\langle V,\nabla r\rangle=-\frac{r}{2}$. So Theorem 2 is applicable to quasi-harmonic maps and gives an extension of the Liouville theorems in [24].

The method to derive the above Liouville theorem is to establish gradient estimates. For the usual harmonic maps, this kind of results were obtained by S.Y.Cheng [3], H.Choi [4], J.Y.Li [20]. The classical work in this direction goes back to the seminal paper [37] of S.T.Yau for harmonic functions on complete manifolds, see [22] for more details. These gradient estimates are based on the Laplacian comparison theorem. In our case, we need to establish a V-Laplacian comparison theorem under the Bakry-Emery Ricci condition.

Theorem 3. (V-Laplacian comparison theorem) *Let (M,g) be a complete Riemannian manifold, V a smooth vector field on M. Fix a point x_0 in M, and let r be the distance function on M from x_0 and $\gamma:[0,r]\to M$ a unit speed minimal geodesic from x_0 to the considered point. Suppose that*

$$\mathrm{Ric}_V:=\mathrm{Ric}^M-\frac{1}{2}L_Vg\geq(n-1)K, \tag{1.8}$$

where K is a constant. Then (assume $r\leq\frac{\pi}{2\sqrt{K}}$ when $K>0$)

$$\Delta_V r\leq(n-1)\frac{s_K'}{s_K}+\frac{1}{s_K^2(r)}\int_0^r s_K(2t)\langle V,\dot\gamma\rangle(t)dt. \tag{1.9}$$

In particular, if V satisfies

$$\langle V,\nabla r\rangle\leq v(r) \tag{1.10}$$

for some nondecreasing function $v(\cdot)$, then

$$\Delta_V r\leq(n-1)\frac{s_K'}{s_K}+v(r). \tag{1.11}$$

Here s_K denotes the unique solution of $s_K''+Ks_K=0,\quad s_K(0)=0,\quad s_K'(0)=1$. That is,

$$s_K(t)=\begin{cases}\frac{\sin\sqrt{K}t}{\sqrt{K}} & \text{if}\quad K>0,\\ t & \text{if}\quad K=0,\\ \frac{\sinh\sqrt{-K}t}{\sqrt{-K}} & \text{if}\quad K<0.\end{cases} \tag{1.12}$$

Remark Let V be the gradient of some function $f:M\to\mathbb{R}$, Wei-Wylie [36] proved a Laplacian comparison theorem and gave some interesting applications. Their result (Theorem 1.1 (a) in [36]) corresponds to the case $V=-\nabla f$ and $\langle V,\nabla r\rangle\leq a$ (constant) in our Theorem 3. In this case, from (1.11) we have $\Delta_f r\leq(n-1)\frac{s_K'}{s_K}+a$.

The gradient estimates for V-harmonic maps will also be a key tool for proving the existence theorem in Section 4, where we prove the existence of V-harmonic maps from complete Riemannian manifolds into regular balls by using the compact exhaustion procedure. Moreover, under the conditioin that the first eigenvalue for Δ_V is positive or the Assumption (A) in Section 2, the solution of (1.1) is unique. Precisely, we have

Theorem 4. *Let (M, g) be a complete noncompact Riemannian manifold, (X, h) be a complete Riemannian manifold with sectional curvature bounded above by a positive constant κ. Let $B_R(p)$ be a regular ball in X, i.e., disjoint from the cut-locus of p and $R < \frac{\pi}{2\sqrt{\kappa}}$. Let $u_0 : M \to X$ be a continuous map with $u_0(M) \subset B_R(p)$.*

If either one of the conditions in Definition 1 is satisfied, then there exists a unique V-harmonic map $u \in C^\infty(M, X)$ homotopic to u_0 such that $u(M) \subset B_R(p)$ and $(1 - \cos\sqrt{\kappa}\rho(u, u_0)) \in L^{2p}(M)$ in condition 1 ($(1 - \cos\sqrt{\kappa}\rho(u, u_0)) \in C^0_{\mu'}(M)$ in condition 2), where ρ is the homotopy distance between u and u_0 on X.

Since Hermitian harmonic maps constitute a special case of V-harmonic maps, our results extends the previously mentioned results on Hermitian harmonic maps. For instance, applying Theorem 4 to this case, we have:

Corollary 3. *Let (M, g) be a complete Hermitian manifold satisfying either condition 1 or condition 2 in Definition 1, let (X, h) be a complete Riemannian manifold with sectional curvature bounded above by a positive constant κ. Let $B_R(p)$ be a regular ball in X.*

Let u_0 be a smooth map with $u_0(M) \subset B_R(p)$. Then there exists a unique Hermitian harmonic map $u \in C^\infty(M, X)$ homotopic to u_0 such that $u(M) \subset B_R(p)$.

Remark Li-Zhang [28] and Dong [9] respectively proved the existence and uniqueness of Hermitian harmonic maps from a complete Hermitian manifold into a regular ball with radius $R < \frac{\arccos\frac{2\sqrt{5}}{5}}{\sqrt{\kappa}}$. Note that $\frac{\arccos\frac{2}{\sqrt{5}}}{\sqrt{\kappa}} < \frac{\pi}{6\sqrt{\kappa}}$. The condition $R < \frac{\pi}{2\sqrt{\kappa}}$ in the above result is optimal.

In [15] and [16], Jost-Simsir established existence theorems for affine harmonic maps from compact manifolds without boundary. Our Theorem 4 gives an extension of those results to affine harmonic maps from noncompact manifolds:

Corollary 4. *Let (M, g) be a complete noncompact affine manifold satisfying either condition 1 or condition 2 in Definition 1, let (X, h) be a complete Riemannian manifold with sectional curvature bounded above by a positive constant κ. Let $B_R(p)$ be a regular ball in X. Let u_0 be a smooth map with $u_0(M) \subset B_R(p)$. Then there exists a unique affine harmonic map $u \in C^\infty(M, X)$ homotopic to u_0 such that $u(M) \subset B_R(p)$.*

We also show that there exists no nontrivial V-harmonic map from compact Riemannian manifolds without boundary into regular balls:

Theorem 5. *Let (M, g) be a compact Riemannian manifold without boundary and (X, h) be a complete Riemannian manifold with sectional curvature bounded above by a positive constant κ. Let $u : M \to X$ be a V-harmonic map such that $u(M) \subset B_R(p)$, where $B_R(p)$ is a regular ball. Then u must be a constant map.*

Acknowledgement. We would like to thank Yuanlong Xin and Guofang Wang for interesting conversations.

2. V-harmonic maps from complete manifolds to manifolds with nonpositive curvature

In this section we prove existence and uniqueness theorems for V-harmonic maps from complete noncompact manifolds into nonpositively curved target manifolds.

Before proving the theorems, we first establish the following Bochner formula:

Lemma 1. *Let (M, g) and (X, h) be Riemannian manifolds. Suppose u is a V-harmonic map from M to X, then*

$$\begin{aligned}\frac{1}{2}\Delta_V e(u) &= |\nabla du|^2 + \sum_{\alpha=1}^{n}\langle du(\mathrm{Ric_V}(e_\alpha)), du(e_\alpha)\rangle \\ &\quad - \sum_{\alpha,\beta=1}^{n} R^X(du(e_\alpha), du(e_\beta), du(e_\alpha), du(e_\beta)),\end{aligned} \tag{2.1}$$

where $\{e_\alpha\}$ is a local orthonormal frame in M. In particular, if (M, g) is a compact Riemannian manifold, (X, h) has nonpositive sectional curvature, then the energy density e(u) satisfies

$$\Delta_V e(u) \geq 2\,|\nabla du|^2 - C_1 e(u), \tag{2.2}$$

here C_1 is a positive constant depending only on the bounds for the Ricci curvature of M and $\|V\|_{C^1(M)}$.

Proof. In order to prove (2.1) and (2.2), it will be convenient to calculate in normal coordinates at the points x and $u(x)$, i.e. $g_{\alpha\beta}(x) = \delta_{\alpha\beta}$ and $h_{ij} = \delta_{ij}$ and all Christoffel symbols vanish at x and $u(x)$.

We first write the equation for V-harmonic maps in local coordinates

$$g^{\alpha\beta}\frac{\partial^2 u^i}{\partial x^\alpha \partial x^\beta} - g^{\alpha\beta}\Gamma^\eta_{\alpha\beta}\frac{\partial u^i}{\partial x^\eta} + g^{\alpha\beta}\Gamma^i_{jk}\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\beta} + V^\alpha\frac{\partial u^i}{\partial x^\alpha} = 0. \tag{2.3}$$

Differentiating this equation w.r.t. x^ε (for simplicity, in the sequel, we denote partial derivatives by subscripts)

$$\begin{aligned}u^i_{x^\alpha x^\alpha x^\varepsilon} =& \frac{1}{2}\left(g_{\eta\alpha,\alpha\varepsilon} + g_{\eta\alpha,\alpha\varepsilon} - g_{\alpha\alpha,\eta\varepsilon}\right)u^i_{x^\eta} \\ &- \frac{1}{2}\left(h_{ij,kl} + h_{ik,jl} - h_{jk,il}\right)u^l_{x^\varepsilon}u^j_{x^\alpha}u^k_{x^\alpha} \\ &- V^\alpha_{x^\varepsilon}u^i_{x^\alpha} - V^\alpha u^i_{x^\alpha x^\varepsilon},\end{aligned} \tag{2.4}$$

we then obtain

$$
\begin{aligned}
\frac{1}{2}\Delta_V e(u) =&\frac{1}{2}\left(\frac{\partial^2}{\partial x^\sigma \partial x^\sigma}+V^\alpha\frac{\partial}{\partial x^\alpha}\right)\left(g^{\alpha\beta}h_{ij}(u(x))\frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^j}{\partial x^\beta}\right)\\
=&\frac{1}{2}[(g_{\eta\sigma,\sigma\alpha}+g_{\eta\sigma,\sigma\alpha}-g_{\alpha\eta,\sigma\sigma}-g_{\sigma\sigma,\eta\alpha})\,u^i_{x^\alpha}u^i_{x^\eta}\\
&-(h_{ij,kl}+h_{ik,jl}-h_{jk,il}-h_{il,jk})\,u^i_{x^\alpha}u^j_{x^\sigma}u^k_{x^\sigma}u^l_{x^\alpha}\\
&-2V^\sigma_{x^\alpha}u^i_{x^\sigma}u^i_{x^\alpha}+2\left(u^i_{x^\alpha x^\sigma}\right)^2]\\
=&\mathrm{Ric}^M_{\eta\alpha}u^i_{x^\alpha}u^i_{x^\eta}-R^X_{iklj}u^i_{x^\alpha}u^j_{x^\sigma}u^k_{x^\sigma}u^l_{x^\alpha}+|\nabla du|^2-V^\sigma_{x^\alpha}u^i_{x^\sigma}u^i_{x^\alpha}\\
=&|\nabla du|^2+(\mathrm{Ric}_V)_{\eta\alpha}u^i_{x^\alpha}u^i_{x^\eta}-R^X_{iklj}u^i_{x^\alpha}u^j_{x^\sigma}u^k_{x^\sigma}u^l_{x^\alpha}.
\end{aligned}
\tag{2.5}
$$

This proves (2.1).

Suppose M is compact, V is the smooth vector field on M and the sectional curvature of X is nonpositive, we then have

$$\Delta_V e(u) \geq 2|\nabla du|^2 - C_1 e(u).$$

■

Lemma 2. *(c.f. [34]) Let X be a Riemannian manifold with nonpositive sectional curvature.*
(1) Let u_1 and u_2 be two smooth maps from M into X and let $U(x,s)$ be the geodesic homotopy between them. If we denote by ρ the homotopy distance between u_1 and u_2, then

$$\triangle_V \rho^2 \geq 2\int_0^1 \left\|\nabla dU\left(\frac{\partial}{\partial s}\right)\right\|^2 ds - 2\rho\left(|\tau_V(u_1)|+|\tau_V(u_2)|\right). \tag{2.6}$$

In particular, we have

$$\Delta_V \rho \geq -\left(|\tau_V(u_1)|+|\tau_V(u_2)|\right). \tag{2.7}$$

(2) Let $\{u_i : M \to X\}$ be a sequence of V-harmonic maps, assume the homotopy distances between u_i and u_j are uniformly bounded: $\rho_{ij} \leq \tilde{C}$, then on any geodesic ball $B_0(r)$ the energies of u_i are uniformly bounded:

$$\int_{B_0(r)} e(u_i) \leq C.$$

(3) Assume $\lambda_V(M) > 0$. Suppose $f : M \to \mathbb{R}$ satisfies

$$\Delta_V f \geq -h, \tag{2.8}$$

where $h \geq 0$ is a function on M, then for any cut-off function φ and $p \geq 1$, we have

$$p\int_M f^{2p-1}\varphi^2 h + \int_M f^{2p}|\nabla\varphi|^2 - \int_M \langle V, \nabla\varphi\rangle f^{2p}\varphi \geq \lambda_V(M)\int_M f^{2p}\varphi^2. \tag{2.9}$$

If f is supported in $\Omega \subset M$, then

$$p\int_\Omega f^{2p-1}h \geq \lambda_V(M)\int_\Omega f^{2p}. \tag{2.10}$$

Proof. By the same proof as of Lemma 3.4. and p.344-345 in [34], one gets (1) and (2).

For (3), multiplying (2.8) by $f^{2p-1}\varphi^2$ and integrating by parts, we have

$$\begin{aligned} -(2p-1)\int_M f^{2p-2}\varphi^2|\nabla f|^2 - 2\int_M f^{2p-1}\varphi\langle\nabla\varphi,\nabla f\rangle &+ \int_M f^{2p-1}\varphi^2\langle V,\nabla f\rangle \\ &\geq -\int_M f^{2p-1}\varphi^2 h. \end{aligned} \tag{2.11}$$

Using $\int_M(-\Delta_V\phi)\phi \geq \lambda_V(M)\int_M \phi^2$ with $\phi = f^p\varphi$ yields

$$\begin{aligned} p^2\int_M f^{2p-2}\varphi^2|\nabla f|^2 &+ 2p\int_M f^{2p-1}\varphi\langle\nabla\varphi,\nabla f\rangle + \int_M f^{2p}|\nabla\varphi|^2 \\ &- p\int_M f^{2p-1}\varphi^2\langle V,\nabla f\rangle - \int_M f^{2p}\varphi\langle V,\nabla\varphi\rangle \geq \lambda_V(M)\int_M f^{2p}\varphi^2. \end{aligned} \tag{2.12}$$

Multiplying (2.11) by p and then adding it to (2.12) gives the desired inequality (2.9). In the same way (2.10) follows. ∎

Proof of Theorem 1. The basic ideas will be as in [34] and [10].

Let $\{\Omega_i\}$ be a compact exhaustion of M. By Theorem A we have $\{u_i\}$ which solve the Dirichlet problem

$$\begin{cases} \tau_V(u_i) = 0, \\ u_i|_{\partial\Omega_i} = u_0|_{\partial\Omega_i}, \\ u_i \quad \text{homotopic} \quad \text{to} \quad u_0 \quad \text{rel.} \quad \partial\Omega_i. \end{cases} \tag{2.13}$$

Let ρ_i be the homotopy distance between u_i and u_0, and ρ_{ij} be the homotopy distance between u_i and u_j. By Lemma 2, we have

$$\Delta_V\rho_i \geq -|\tau_V(u_0)|.$$

Step 1. ρ_{ij} is uniformly bounded over any compact domain $K \subset M$.

Case 1) When condition 1 is satisfied.

From the above inequality and the assumption that $\lambda_V(M) > 0$, and use (2.10) for $f = \rho_i$, $h = |\tau_V(u_0)|$, we can conclude that

$$\int_{\Omega_i}\rho_i^{2p} \leq \left(\frac{p}{\lambda_V(M)}\right)^{2p}\int_{\Omega_i}|\tau_V(u_0)|^{2p}. \tag{2.14}$$

Then by $\rho_{ij} \leq \rho_i + \rho_j$ we have

$$\|\rho_{ij}\|_{L^{2p}(\Omega_j)} \leq C_2, \tag{2.15}$$

where the constant C_2 is independent of i and j. From Lemma 2, we have

$$\Delta_V\rho_{ij} \geq 0. \tag{2.16}$$

Therefore by the local maximum principle, Theorem 9.20 in [11], one can conclude that for any $p > 0$, $B_r(\tilde{p}) \subset \Omega_i$,

$$\sup_{B_r(\tilde{p})}\rho_{ij} \leq C_3\,\|\rho_{ij}\|_{L^{2p}}, \tag{2.17}$$

where $B_r(\tilde{p})$ is a geodesic ball centered at $\tilde{p}$ with radius $r > 0$ on M and C_3 is a positive constant depending only on the geometry of M and independent of ρ_{ij}. Hence ρ_{ij} is uniformly bounded over any compact domain $K \subset M$ by some constant C_4 independent of i and j.

Case 2) When condition 2 is satisfied. Since $|\tau_V(u_0)| \in C^0_\mu(M)$, we can find a function $v \in C^0_{\mu'}(M)$, such that

$$\Delta_V v = -|\tau_V(u_0)| \quad in \quad M.$$

The above inequality and the definition of $C^0_{\mu'}(M)$, together with the maximum principle, imply that $v \geq 0$. From (2.7), we have

$$\Delta_V \rho_i \geq -|\tau_V(u_0)|.$$

Then by the maximum principle, we have

$$0 \leq \rho_i \leq v \tag{2.18}$$

on Ω_i. Hence $0 \leq \rho_{ij} \leq 2v$, it follows that ρ_{ij} is uniformly bounded on any compact set $K \subset M$.

Step 2. The uniform bound for the energy densities $e(u_i)$.

In order to prove the existence of the V-harmonic map, we only need to get a uniform bound for the $e(u_i)$.

Case 1) When condition 1 is satisfied.

For any compact set $K \subset M$, there exists i_0, such that $K \subset \Omega_i$ for $i > i_0$. Then by Lemma 1, we have

$$\Delta_V e(u_i) \geq -C_1 e(u_i), \tag{2.19}$$

i.e.,

$$\Delta e(u_i) + \langle V, \nabla e(u_i)\rangle \geq -C_1 e(u_i).$$

Hence by the local maximum principle, Theorem 9.20 in [11], we can reduce the pointwise estimate of $e(u_i)$ to an integral estimate of $e(u_i)$.

On the other hand, by Lemma 2 and the fact ρ_{ij} is uniformly bounded, we have

$$\int_{B_r(\tilde{p})} e(u_i) \leq C_4,$$

where C_4 is a positive constant independent of i. Thus, we have $e(u_i) \leq C_5$ for some positive constant C_5 which is independent of i.

Case 2) When condition 2 is satisfied. The same argument as in case 1 applies to deduce that $e(u_i)$ are uniformly bounded on any compact set K.

Then the existence of V-harmonic maps u follows from the standard diagonal sequence argument. In addition, when condition 1 is satisfied, we conclude from (2.14) that $\rho \in L^{2p}(M)$. When condition 2 is satisfied, we conclude from (2.18) that $0 \leq \rho \leq v$ and hence $\rho \in C^0_{\mu'}(M)$.

Step 3. The uniqueness of V-harmonic maps.

Let u_1,u_2 be two V-harmonic maps from our construction (see step 2). By Lemma 2 we have

$$\Delta_V \rho(u_1, u_2) \geq 0. \tag{2.20}$$

Case 1) When condition 1 is satisfied.

Choose a cut-off function φ supported in $B_{2r}(\tilde{p})$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in $B_r(\tilde{p})$, and $|\nabla\varphi| \leq C_0/r$. In (2.9), letting $f = \rho$, $h \equiv 0$, and φ as above, we can derive that

$$\lambda_V(M) \int_{B_r(\tilde{p})} \rho^{2p} \leq \frac{C_0^2}{r^2} \int_{B_{2r}(\tilde{p})} \rho^{2p} + \frac{C_0}{r} \|V\|_{L^\infty(M)} \int_{B_{2r}(\tilde{p})} \rho^{2p}. \tag{2.21}$$

Then letting $r \to \infty$, we have $\rho = 0$, that is $u_1 = u_2$.

Case 2) When condition 2 is satisfied.

From (2.18) we have $\rho(u_i, u_0) \in C^0_{\mu'}(M)$, $i = 1, 2$, consequently,

$$0 \leq \rho(u_1, u_2) \leq \rho(u_1, u_0) + \rho(u_2, u_0) \in C^0_{\mu'}(M).$$

Hence for any $\varepsilon > 0$ outside a sufficiently large ball $B_R(\tilde{p})$ around an arbitrary $\tilde{p} \in M$, we have

$$\rho(u_1, u_2) \leq \varepsilon.$$

From (2.20) and using the maximum principle, we get $\rho(u_1, u_2) \leq \varepsilon$ on all of M for any $\varepsilon > 0$ and hence $\rho(u_1, u_2) = 0$. This implies $u_1 = u_2$.

■

3. A Liouville theorem for V-harmonic maps

In this section, we shall derive a gradient estimate for V-harmonic maps and then establish a Liouville type theorem. Essentially it is a combination of the Bochner formula Lemma 1 for V-harmonic maps and the V-Laplacian comparison Theorem 3 with the gradient estimate method.

Let us first prove the V-Laplacian comparison theorem.

Proof of Theorem 3. For any $x \in M$, let $\gamma : [0, r] \to M$ be a minimal unit speed geodesic with $\gamma(0) = x_0$ and $\gamma(r) = x$. Choose a local orthonormal frame $\{e_\alpha\}$ near x such that $e_1 = \dot{\gamma}(x)$, by parallel translation along γ we have a frame $\{e_\alpha(t)\}$. Let $J_\alpha(t)$

be the Jacobi field with $J_\alpha(0) = 0,\ J_\alpha(r) = e_\alpha,\ \alpha = 2, \cdots, n$. Then

$$\begin{aligned}\Delta r(x) &= \sum_{\alpha=1}^{n}(e_\alpha e_\alpha r - \nabla_{e_\alpha} e_\alpha r)\\ &= -\sum_{\alpha=2}^{n}\nabla_{e_\alpha} e_\alpha r\\ &= \sum_{\alpha=2}^{n}\int_0^r \frac{d}{dt}\langle \nabla_{J_\alpha}\dot\gamma, J_\alpha\rangle dt\\ &= \sum_{\alpha=2}^{n}\int_0^r (|\dot J_\alpha|^2 - \langle R(\dot\gamma, J_\alpha)\dot\gamma, J_\alpha\rangle)dt.\end{aligned}$$

For any piecewise smooth function $f(\cdot)$ on $[0, r]$ with $f(0) = 0,\ f(r) = 1$, let $X_\alpha(t) := f(t)e_\alpha(t),\ \alpha = 2, \cdots, n$. Then by the basic index lemma, we have

$$\begin{aligned}\Delta r(x) &\le \sum_{\alpha=2}^{n}\int_0^r (|\dot X_\alpha|^2 - \langle R(\dot\gamma, X_\alpha)\dot\gamma, X_\alpha\rangle)dt\\ &= \int_0^r [(n-1)f'^2 - f^2\mathrm{Ric}(\dot\gamma,\dot\gamma)]dt.\end{aligned}$$

Noting that

$$\mathrm{Ric}_V(\dot\gamma,\dot\gamma) = \mathrm{Ric}(\dot\gamma,\dot\gamma) - \frac{1}{2}L_V g(\dot\gamma,\dot\gamma) = \mathrm{Ric}(\dot\gamma,\dot\gamma) - \langle \dot V, \dot\gamma\rangle,$$

we have

$$\begin{aligned}\Delta_V r(x) &\le \int_0^r [(n-1)f'^2 - f^2\mathrm{Ric}_V(\dot\gamma,\dot\gamma)]dt + \int_0^r (f^2)'\langle V,\dot\gamma\rangle dt\\ &\le (n-1)\int_0^r (f'^2 - Kf^2)dt + \int_0^r (f^2)'\langle V,\dot\gamma\rangle dt. \qquad (3.1)\end{aligned}$$

If we choose $f(t) = \frac{s_K(t)}{s_K(r)}$ (assume $r \le \pi/2\sqrt{K}$ when $K > 0$), then f satisfies

$$f'' + Kf = 0, \quad f(0) = 0, \quad f(r) = 1.$$

Using these in (3.1), we conclude that

$$\begin{aligned}\Delta_V r(x) &\le (n-1)(ff')|_0^r + \int_0^r \frac{(s_K^2(t))'}{s_K^2(r)}\langle V,\dot\gamma\rangle(t)dt.\\ &= (n-1)\frac{s_K'(r)}{s_K(r)} + \frac{1}{s_K^2(r)}\int_0^r s_K(2t)\langle V,\dot\gamma\rangle(t)dt. \qquad (3.2)\end{aligned}$$

If V satisfies

$$\langle V, \nabla r\rangle \le v(r)$$

for some nondecreasing function $v(\cdot)$, then from (3.2) we immediately have

$$\Delta_V r(x) \le (n-1)\frac{s_K'(r)}{s_K(r)} + v(r).$$

■

Proof of Theorem 2. Multiplying the metric tensor by a suitable constant we may assume that the upper bound of the sectional curvature of X is 1.

Let r, ρ be the respective distance functions on M and X from some fixed points $\tilde{p} \in M, p \in X$. Let $B_a(\tilde{p})$ be a geodesic ball of radius a around $\tilde{p}$. Define $\varphi = 1 - \cos\rho$. Then

$$\mathrm{Hess}^X(\varphi) \geq (\cos\rho)h. \tag{3.3}$$

Since $R < \frac{\pi}{2}$, we can choose a constant b such that $\varphi(R) < b < 1$ on $B_R(p)$. Define $f : B_a(\tilde{p}) \to \mathbb{R}$ by

$$f = \frac{\left(a^2 - r^2\right)^2 e(u)}{\left(b - \varphi \circ u\right)^2},$$

Since $f|_{\partial B_a(\tilde{p})} = 0$, f achieves an absolute maximum in the interior of $B_a(\tilde{p})$, say $f \leq f(q)$, for some q inside $B_a(\tilde{p})$. By using the support function technique, we may assume that f is smooth near q. We may also assume $e(u)|_q \neq 0$. Then from

$$\begin{aligned} \nabla f(q) &= 0, \\ \Delta_V f(q) &\leq 0, \end{aligned}$$

we obtain at q:

$$-\frac{2\nabla r^2}{a^2 - r^2} + \frac{\nabla e(u)}{e(u)} + \frac{2\nabla(\varphi \circ u)}{b - \varphi \circ u} = 0, \tag{3.4}$$

$$-\frac{2\Delta_V r^2}{a^2 - r^2} - \frac{2\left|\nabla r^2\right|^2}{\left(a^2 - r^2\right)^2} + \frac{\Delta_V e(u)}{e(u)} - \frac{\left|\nabla e(u)\right|^2}{e(u)^2} + \frac{2\Delta_V(\varphi \circ u)}{b - \varphi \circ u} + \frac{2\left|\nabla(\varphi \circ u)\right|^2}{\left(b - \varphi \circ u\right)^2} \leq 0. \tag{3.5}$$

From (2.1), we know that

$$\Delta_V e(u) \geq 2\left(u^j_{x^\alpha x^\sigma}\right)^2 - 2e(u)^2 - 2Ae(u). \tag{3.6}$$

Direct calculation gives that

$$\left|\nabla e(u)\right|^2 = 4u^j_{x^\alpha}u^k_{x^\beta}u^j_{x^\alpha x^\sigma}u^k_{x^\beta x^\sigma} \leq 4e(u)\left(u^j_{x^\alpha x^\sigma}\right)^2, \tag{3.7}$$

$$\left|\nabla(\varphi \circ u)\right|^2 = \varphi_{u^j}\varphi_{u^k}u^j_{x^\alpha}u^k_{x^\alpha} \leq \left|d\varphi\right|^2 e(u) \leq e(u). \tag{3.8}$$

It follows from (3.6) and (3.7) that

$$\frac{\Delta_V e(u)}{e(u)} \geq \frac{\left|\nabla e(u)\right|^2}{2e(u)^2} - 2e(u) - 2A. \tag{3.9}$$

From (3.4), we obtain

$$\frac{\left|\nabla e(u)\right|^2}{e(u)^2} \leq \frac{4\left|\nabla r^2\right|^2}{\left(a^2 - r^2\right)^2} + \frac{8\left|\nabla r^2\right|\left|\nabla(\varphi \circ u)\right|}{\left(a^2 - r^2\right)\left(b - \varphi \circ u\right)} + \frac{4\left|\nabla(\varphi \circ u)\right|^2}{\left(b - \varphi \circ u\right)^2}. \tag{3.10}$$

To estimate $\Delta(\varphi\circ u)$, by (3.3),

$$\begin{aligned}\Delta(\varphi\circ u) =&g^{\alpha\beta}\mathrm{Hess}(\varphi\circ u)\left(\frac{\partial}{\partial x^\alpha},\frac{\partial}{\partial x^\beta}\right)=g^{\alpha\beta}B_{\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}}(\varphi\circ u)\\ =&g^{\alpha\beta}\left(B_{u_*\frac{\partial}{\partial x^\alpha}u_*\frac{\partial}{\partial x^\beta}}(\varphi)+d\varphi\left(B_{\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}}(u)\right)\right)\\ =&g^{\alpha\beta}u^j_{x^\alpha}u^k_{x^\beta}B_{\frac{\partial}{\partial u^j}\frac{\partial}{\partial u^k}}(\varphi)+d\varphi(\tau(u))\\ =&g^{\alpha\beta}u^j_{x^\alpha}u^k_{x^\beta}\mathrm{Hess}^X(\varphi)\left(\frac{\partial}{\partial u^j},\frac{\partial}{\partial u^k}\right)+d\varphi(-du(V))\\ \geq&(\cos\rho)e(u)+d\varphi(-du(V)).\end{aligned}\tag{3.11}$$

Namely,

$$\Delta_V(\varphi\circ u)\geq(\cos\rho)e(u).\tag{3.12}$$

Since $\mathrm{Ric}^M-\frac{1}{2}L_Vg\geq -A$, by the V-Laplacian comparison Theorem 3, we have

$$\Delta_V r\leq\sqrt{(n-1)A}\coth\sqrt{\frac{A}{n-1}}r+v(r).\tag{3.13}$$

For $x\in B_a(\tilde p)$, $r\leq a$, hence there exists a constant $C_6>0$ depending on A such that

$$\Delta_V r^2\leq C_6(1+r)+2rv(a).\tag{3.14}$$

Substituting (3.8), (3.9), (3.10), (3.12) and (3.14) into (3.5), we have

$$\begin{aligned}&-\frac{C_6(1+r)+2r|v(a)|}{a^2-r^2}-\frac{8r^2}{(a^2-r^2)^2}-A\\ &+\left(\frac{\cos\rho}{b-\varphi\circ u}-1\right)e(u)-\frac{4r}{(a^2-r^2)(b-\varphi\circ u)}\sqrt{e(u)}\\ &\leq 0.\end{aligned}\tag{3.15}$$

Note the elementary fact that if $ax^2-bx-c\leq 0$ with a,b,c all positive, then

$$x\leq\max\{2b/a,2\sqrt{c/a}\}.$$

It is also easy to see that there is a constant $C_7>0$ such that $\frac{\cos\rho}{b-\varphi\circ u}-1>C_7$. Therefore, at the point q,

$$\begin{aligned}e(u)\leq\max&\left\{\frac{64r^2}{C_7^2(a^2-r^2)^2(b-\varphi\circ u)^2},\frac{4C_6(1+r)+8r|v(a)|}{C_7(a^2-r^2)}+\frac{4A}{C_7}\right.\\ &\left.+\frac{32r^2}{C_7\left(a^2-r^2\right)^2}\right\}.\end{aligned}\tag{3.16}$$

From this we can derive the upper bound of f, and it is easy to conclude that at every point of $B_{a/2}(\tilde p)$, we have:

$$e(u)\leq C(A+\frac{1}{a^2}+\frac{1+a(1+|v(a)|)}{a^2}).\tag{3.17}$$

Here $C>0$ is a constant that depends only on R,κ.

For any fixed x and letting $a \to \infty$ in (3.17), we have $e(u) \le CA$. If $A = 0$, then $e(u) \equiv 0$, that is, u must be constant.

■

4. V-harmonic maps from complete manifolds into regular balls

In this section, we prove the existence and uniqueness of V-harmonic maps from complete noncompact manifolds into regular balls in a complete Riemannian manifold with sectional curvature bounded above by a positive constant.

Proof of Theorem 4. We can always assume that the upper bound of the sectional curvature of X is 1. Let $\{\Omega_i\}$ be a compact exhaustion of M. By Theorem A we have a sequence of maps $\{u_i\}$ which solve the Dirichlet problem

$$\left\{ \begin{array}{l} \tau_V(u_i) = 0, \\ u_i|_{\partial\Omega_i} = u_0|_{\partial\Omega_i}, \\ u_i \quad \text{homotopic} \quad \text{to} \quad u_0 \quad \text{rel.} \quad \partial\Omega_i, \end{array} \right. \tag{4.1}$$

where $u_i \in C^\infty(\Omega_i, X) \cap C(\bar{\Omega}_i, X)$ and $u_i(\Omega_i) \subset B_R(p)$.

For any compact set $K \subset M$, there exists an integer $i_0 > 0$, such that $K \subset \Omega_i$ for $i > i_0$. Then by (2.1),

$$\Delta_V e(u_i) \ge 2\left|\nabla du_i\right|^2 - 2e(u_i)^2 - C_1 e(u_i), \tag{4.2}$$

where C_1 is a positive constant depending only on the bounds for the Ricci curvature of K and $\|V\|_{C^1(K)}$.

Since K is compact, we can then find finite geodesic balls $\{B_{a_j}(\tilde{p}_j)\}_{j=1}^{k_0} \subset M$, such that $\bigcup_{j=1}^{k_0} B_{a_j}(\tilde{p}_j) \supset K$. Hence for any $q \in K$, there exists a geodesic ball, say $B_{a_{j_0}}(\tilde{p}_{j_0})$ $(1 \le j_0 \le k_0)$, such that $q \in B_{a_{j_0}}(\tilde{p}_{j_0})$. Then we conclude as in the proof of Theorem 2, and derive

$$e(u_i)(q) \le \frac{1}{2} C_1 C_{8j_0}.$$

Hence

$$\sup_K e(u_i) \le \max_{1 \le j_0 \le k_0} \left\{ \frac{1}{2} C_1 C_{8j_0} \right\} =: C_9,$$

where C_8, C_9 are positive constants independent of i. Then using the standard elliptic estimate, we can get the $C^{2,\alpha}$-estimate on K, and the upper bound is independent of u_i. By the diagonal sequence method, we then obtain a smooth map u which is a solution of (1.1) on M. Obviously, $u(M) \subset B_R(p)$.

We shall prove the uniqueness of the V-harmonic map. Suppose v_1, v_2 are two smooth maps from M into the regular ball $B_R(p)$ and they are in the same homotopy class. Let

$Q(\cdot,\cdot) := 1-\cos\rho(\cdot,\cdot)$ and

$$\begin{aligned}\psi(x) =& Q(v_1(x), v_2(x))\\ \psi_i(x) =& Q(v_i(x), p) \quad i=1,2\\ \Phi(x) =& \sum_{i=1}^{2} \omega\circ\psi_i(x), \quad \text{where } \ \omega(t) = -\log(1-t).\end{aligned}$$

Define $U, U_1, U_2 : M \to X\times X$ by

$$U(x) = (v_1(x), v_2(x)), \quad U_i(x) = (p, v_i(x)), \quad i=1,2.$$

Choose the local coordinates $\{x^\alpha\}$ in M such that $g_{\alpha\beta} = \delta_{\alpha\beta}$ at the considered point. It is easy to see that

$$|\nabla\psi|^2 = \sum_\alpha \langle \nabla_\alpha Q\circ U, \partial_\alpha U\rangle^2, \qquad |\nabla\psi_i|^2 = \sum_\alpha \langle \nabla_\alpha Q\circ U_i, \partial_\alpha U_i\rangle^2.$$

Furthermore,

$$\begin{aligned}\Delta_V\psi \;\; = \;\; & \Delta_V(Q\circ U) = \sum_\alpha \nabla^2 Q(\partial_\alpha U, \partial_\alpha U)\\ & + \langle\nabla Q\circ U, \tau(U)\rangle + \langle \nabla Q\circ U, dU(V)\rangle.\end{aligned}$$

Similarly, we have

$$\Delta_V\psi_i = \textstyle\sum_\alpha \nabla^2 Q(\partial_\alpha U_i, \partial_\alpha U_i) + \langle \nabla Q\circ U_i, \tau(U_i)\rangle + \langle\nabla Q\circ U_i, dU_i(V)\rangle, \quad i=1,2.$$

By Lemma 3 in [13], the Hessian of Q satisfies

$$\nabla^2 Q(v,v) \ge \begin{cases} |v|^2, & \text{if } y_1 = y_2\\ \dfrac{\langle \nabla Q(y), v\rangle^2}{2Q(y)} - Q(y)|v|^2, & \text{if } y_1 \ne y_2,\end{cases}$$

for any $v\in T_y(X\times X)$, $y=(y_1,y_2)\in X\times X$, and

$$\nabla^2 Q(v,v) \ge (1-Q(y))|v|^2,$$

if $v=(0,v_2)$ or $v=(v_1,0)$. Therefore, we have

$$\begin{aligned}\Delta_V\psi \ge& \frac{|\nabla\psi|^2}{2\psi} - \psi\sum_{i=1}^{2} e(v_i) + \langle \nabla Q, \tau_V(v_1)\oplus\tau_V(v_2)\rangle\,,\\ \Delta_V\psi_i \ge& (1-\psi_i)e(v_i) + \langle\nabla Q, \tau_V(v_i)\oplus 0\rangle\,,\\ \Delta_V\Phi \ge& \sum_{i=1}^{2}\left\{\omega''(\psi_i)|\nabla\psi_i|^2 + \omega'(\psi_i)\left[(1-\psi_i)e(v_i) + \langle\nabla Q, \tau_V(v_i)\oplus 0\rangle\right]\right\}.\end{aligned}$$

It follows that

$$\begin{aligned} e^{-\Phi}\Delta_V e^{\Phi}\psi =& \Delta_V\psi + \psi\Delta_V\Phi + \psi\left|\nabla\Phi\right|^2 + 2\left\langle\nabla\Phi, \nabla\psi\right\rangle \\ \geq& \frac{|\nabla\psi|^2}{2\psi} + \psi\sum_{i=1}^{2}\left[\omega'(\psi_i)(1-\psi_i) - 1\right]e(v_i) + \psi\sum_{i=1}^{2}\omega''(\psi_i)|\nabla\psi_i|^2 \\ &+ \left\langle\nabla Q, \tau_V(v_1)\oplus\tau_V(v_2)\right\rangle + \psi\sum_{i=1}^{2}\omega'(\psi_i)\left\langle\nabla Q, \tau_V(v_i)\oplus 0\right\rangle \\ &+ \psi\left|\nabla\Phi\right|^2 + 2\left\langle\nabla\Phi, \nabla\psi\right\rangle. \end{aligned} \tag{4.3}$$

Since

$$\omega'(\psi_i) = \frac{1}{1-\psi_i} \in (1, \sec R), \quad \omega'' = \omega'^2,$$

$$\left|2\left\langle\nabla\Phi, \nabla\psi\right\rangle\right| \leq \frac{|\nabla\psi|^2}{2\psi} + 2\psi|\nabla\Phi|^2,$$

the inequality (4.3) implies that

$$e^{-\Phi}\Delta_V e^{\Phi}\psi \geq -C_{10}\left(|\tau_V(v_1)| + |\tau_V(v_2)|\right),$$

namely

$$\Delta_V e^{\Phi}\psi \geq -C_{11}\left(|\tau_V(v_1)| + |\tau_V(v_2)|\right), \tag{4.4}$$

where C_{11} is a positive constant independent of i.

Case 1). When the condition 1 is satisfied, we then set $v_1 = u_i, v_2 = u_0$ in (4.4), and derive

$$\Delta_V F \geq -C_{11}|\tau_V(u_0)|, \tag{4.5}$$

where $F = e^{\Phi}\psi$. From (4.5) and the assumption that $\lambda_V(M) > 0$, similar as (2.14) in Step 1 in the proof of Theorem 1, we can conclude that

$$\|F\|_{L^{2p}(M)} \leq \frac{C_{11}p}{\lambda_V(M)}\left\|\tau_V(u_0)\right\|_{L^{2p}(M)}.$$

It follows that

$$\left\|1 - \cos\rho(u_i, u_0)\right\|_{L^{2p}(M)} \leq \frac{C_{11}p}{\lambda_V(M)}\left\|\tau_V(u_0)\right\|_{L^{2p}(M)} =: C_{12} < \infty,$$

where C_{12} is a positive constant independent of i. Hence if u is a solution of (1.1) from our construction, then

$$\left\|1 - \cos\rho(u, u_0)\right\|_{L^{2p}(M)} \leq C_{12}. \tag{4.6}$$

Assume that u_1 and u_2 both are solutions of (1.1) from our construction, which are in the same homotopy class and both satisfying (4.6). Hence we have

$$\left\|1 - \cos\rho(u_1, u_2)\right\|_{L^{2p}(M)} \leq C_{13}.$$

It follows from the above inequality that $G := e^{\Phi}(1-\cos\rho(u_1,u_2))$ satisfies $\|G\|_{L^{2p}(M)} \leq C_{14}$. From formula (4.4), we have

$$\Delta_V G \geq 0. \tag{4.7}$$

Then using Lemma 2 as (2.21) in Step 3 in the proof of Theorem 1, we have

$$\frac{C_0}{r^2}\int_{B_{2r}(\tilde{p})} G^{2p} + \frac{C_0}{r}\|V\|_{L^\infty(M)}\int_{B_{2r}(\tilde{p})} G^{2p} \geq \lambda_V(M)\int_{B_r(\tilde{p})} G^{2p}.$$

Letting $r \to \infty$ and since $\|G\|_{L^{2p}(M)} \leq C_{14}$, we obtain $G = 0$, namely $u_1 = u_2$.

Case 2). When the condition 2 is satisfied, since $|\tau_V(u_0)| \in C^0_\mu(M)$, we can find a function $v \in C^0_{\mu'}(M)$, such that

$$\Delta_V v = -C_{11}|\tau_V(u_0)| \quad in \quad M.$$

The definition of $C^0_{\mu'}(M)$ together with the strong maximum principle implies that $v \geq 0$. Then by (4.5) and the maximum principle, we have

$$1 - \cos\rho(u_i, u_0) \leq v$$

on Ω_i. Hence if u is a solution of (1.1) from our construction, then

$$1 - \cos\rho(u, u_0) \leq v$$

on M. This implies that

$$1 - \cos\rho(u, u_0) \in C^0_{\mu'}(M). \tag{4.8}$$

If u_1 and u_2 both are solutions of (1.1) from our construction and both satisfy (4.8), then we have

$$0 \leq 1 - \cos\rho(u_1, u_2) \leq 4\left\{1 - \cos\rho(u_1, u_0) + 1 - \cos\rho(u_2, u_0)\right\} \in C^0_{\mu'}(M).$$

Hence for any $\varepsilon > 0$ outside a sufficiently large ball $B_r(\tilde{p})$ around an arbitrary $\tilde{p} \in M$, we have

$$1 - \cos\rho(u_1, u_2) \leq \left(\cos^2 R\right)\varepsilon.$$

On the other hand, by (4.4), we derive

$$\Delta_V\left(e^{\Phi}(1 - \cos\rho(u_1, u_2))\right) \geq 0.$$

Then using the maximum principle, we get $1 - \cos\rho(u_1, u_2) \leq \varepsilon$ on all of M for any $\varepsilon > 0$ and hence $1 - \cos\rho(u_1, u_2) = 0$. This implies $u_1 = u_2$.

■

Finally, we give the proof of Theorem 5.

Proof. Without loss of generality, we assume that $\kappa = 1$. Let $f(x) = Q(u(x), p)$, we have

$$\Delta_V f(x) \geq e(u)\cos\rho(u(x), p) \geq 0$$

Since M is compact and without boundary, then f must be constant, namely $\rho(u(x), p) =$ const. We conclude that $u(M) \subset S_r(p)$, where $S_r(p)$ denotes a geodesic sphere of radius r centered at p. Let $x_0 \in M$, and $u(x_0) \in S_r(p)$. By Proposition 2.4.1 in [14], we can join $u(x_0)$ with p by a unique shortest geodesic arc. On this geodesic arc, we can choose a point $\hat{p}$ which is different from p such that we can find another geodesic ball $B_{\hat{r}}(\hat{p})$ satisfying $B_r(p) \subset B_{\hat{r}}(\hat{p}) \subset B_R(p)$. Let $\hat{f}(x) = Q(u(x), \hat{p})$, we also have $\Delta_V \hat{f}(x) \geq 0$, using the maximum principle again, we get $\rho(u(x), \hat{p}) = \tilde{r} = \text{const.}$, for any $x \in M$, i.e., $u(M) \subset S_{\tilde{r}}(\hat{p})$. On the other hand, since $x_0 \in M$, then we have $\tilde{r} = \rho(u(x_0), \hat{p}) = r - \rho(p, \hat{p})$. Obviously, $u(x_0) \in S_r(p) \cap S_{\tilde{r}}(\hat{p})$, suppose there exists a different $y_1 \neq u(x_0)$, such that $y_1 \in S_r(p) \cap S_{\tilde{r}}(\hat{p})$, then $\tilde{r} = \rho(y_1, \hat{p}) > r - \rho(p, \hat{p})$. This yields a contradiction. Hence $S_r(p) \cap S_{\tilde{r}}(\hat{p}) = \{u(x_0)\}$, namely, $u(M) \subset S_r(p) \cap S_{\tilde{r}}(\hat{p}) = \{u(x_0)\}$, thus $u(M) = u(x_0)$.

■

Final remarks. As one can verify that the Gauss map of a self-shrinker in $\mathbb{R}^n$ is a V-harmonic map, the results on V-harmonic maps may be applied to submanifold geometry. Besides, in view of Lemma 1, Theorem 2 and Theorem 3, V-harmonic maps are naturally related to Ricci solitons and Bakry-Emery type Ricci curvature conditions. We expect further geometric applications of V-harmonic maps.

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