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by

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ABSTRACT. Rademacher theorem states that every Lipschitz function on the Euclidean space is differentiable almost everywhere, where "almost everywhere" refers to the Lebesgue measure. Our main result is an extension of this theorem where the Lebesgue measure is replaced by an arbitrary measure μ. In particular we show that the differentiability properties of Lipschitz functions at μ-almost every point are related to the decompositions of μ in terms of rectifiable one-dimensional measures. In the process we obtain a differentiability result for Lipschitz functions with respect to (measures associated to) k-dimensional normal currents, which we use to extend certain formulas involving normal currents and maps of class C^1 to Lipschitz maps.

KEYWORDS: Lipschitz functions, differentiability, Rademacher theorem, normal currents.

MSC (2010): 26B05, 49Q15, 26A27, 28A75, 46E35.

1. INTRODUCTION

The study of the differentiability properties of Lipschitz functions has a long story, and many facets. In recent years much attention has been devoted to the differentiability of Lipschitz functions on infinite dimensional Banach spaces (see the monograph by J. Lindenstrauss, D. Preiss and J. Tišer [16]) and on metric spaces (we just mention here the works by J. Cheeger [7], S. Keith [13] and D. Bate [5]), but at about the same time it became clear that even Lipschitz functions on $\mathbb{R}^n$ are not completely understood, and that Rademacher theorem, which states that every Lipschitz function on $\mathbb{R}^n$ is differentiable almost everywhere,[1] is not the end of story.

To this regard, the first fundamental contribution is arguably the paper [19], where Preiss proved, among other things, that there exist null sets E in $\mathbb{R}^2$ such that every Lipschitz function on $\mathbb{R}^2$ is differentiable at some point of E.[2] This result showed that Rademacher theorem is not sharp, in the sense that while the set of non-differentiability points of a Lipschitz function is always contained in a null set, the opposite inclusion does not always hold.

[1] We use terms "almost everywhere" and "null set" without further specification to mean "almost everywhere" and "null set" with respect to the Lebesgue measure. The same for "absolutely continuous measure" and "singular measure".

[2] The construction of such sets has been variously improved in recent years, cf. [9].

Note that this construction does not give a null sets E such that every Lipschitz map [3] is differentiable at some point of E, and indeed it was later proved by G. Alberti, M. Csörnyei and D. Preiss that every null set in $\mathbb{R}^2$ is contained in the non-differentiability set of some Lipschitz map $f : \mathbb{R}^2 \to \mathbb{R}^2$ (this result was announced in [2], [3] and appears in [4]). The situation in higher dimension has been clarified only very recently: first M. Csörnyei and P.W. Jones announced (see [8]) that every null set in $\mathbb{R}^n$ is contained in the non-differentiability set of a Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^n$, and then D. Preiss and G. Speight [20] proved that there exist null sets E in $\mathbb{R}^n$ such that every Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^m$ with $m < n$ is differentiable at some point of E.

In this paper we approach the differentiability of Lipschitz functions from a slightly different point of view. Consider again the statement of Rademacher theorem: the "almost everywhere" there refers to the Lebesgue measure, and clearly the statement remains true if we replace the Lebesgue measure with a measure μ which is absolutely continuous, but of course it fails if μ is an arbitrary singular measure.

However in many cases it is clear how to modify the statement to make it true. For example, if S is a k-dimensional surface of class C^1 contained in $\mathbb{R}^n$ and μ is the k-dimensional volume measure on S, then every Lipschitz function on $\mathbb{R}^n$ is differentiable at μ-a.e. $x \in S$ in all directions in the tangent space $\mathrm{Tan}(S, x)$. Furthermore this statement is optimal in the sense that there are Lipschitz functions f on $\mathbb{R}^n$ which, for every $x \in S$, are not differentiable at x in any direction which is not in $\mathrm{Tan}(S, x)$ (the obvious example is the distance function $f(x) := \mathrm{dist}(x, S)$).

We aim to prove a statement of similar nature for an arbitrary finite measure μ on $\mathbb{R}^n$. More precisely, we want to identify for μ-a.e. x the largest set of directions $V(\mu, x)$ such that every Lipschitz function on $\mathbb{R}^n$ is differentiable at μ-a.e. x in every direction in $V(\mu, x)$.

We begin with a simple observation: let μ be a measure on $\mathbb{R}^n$ that can be decomposed as [4]

$$\mu = \int_I \mu_t \, dt \tag{1.1}$$

where I is the interval $[0, 1]$ endowed with the Lebesgue measure dt, and each μ_t is the length measure on some rectifiable curve E_t, and assume in addition that there exists a vectorfield τ on $\mathbb{R}^n$ such that for a.e. t and μ_t-a.e. $x \in E_t$ the vector $\tau(x)$ is tangent to E_t at x. Then every Lipschitz function f on $\mathbb{R}^n$ is differentiable at x in the direction $\tau(x)$ for μ-a.e. x.

Indeed by applying Rademacher theorem to the Lipschitz function $f \circ \gamma_t$, where γ_t is a parametrization of E_t by arc-length, we obtain that f is differentiable at the point $\gamma(s)$ in the direction $\dot\gamma(s)$ for a.e. s, which means that f is differentiable

[3] As usual, in this paper we reserve the word "function" for real-valued maps.

[4] The meaning of formula (1.1) is that $\mu(E) = \int_I \mu_t(E)\, dt$ for every Borel set E; the precise definition of integral of a measure-valued map is given in §2.5.

at x in the direction $\tau(x)$ for μ_t-a.e. x and a.e. t, and by formula (1.1) "for μ_t-a.e. x and a.e. t" is equivalent to "for μ-a.e. x".

This observation suggests that the set of directions $V(\mu, x)$ we are looking for should be related to the set of all decompositions of μ, or of parts of μ, of the type considered in the previous paragraph. Accordingly, we give the following "provisional" definition: consider all possible families of measures $\{\mu_t\}$ such that the measure $\int_I \mu_t \, dt$ is absolutely continuous w.r.t. μ, and each μ_t is the restriction of the length measure to a subset E_t of a rectifiable curve, and for every $x \in E_t$ let $\mathrm{Tan}(E_t, x)$ be the tangent line to this curve at x (if it exists); let then $V(\mu, x)$ be the smallest linear subspace of $\mathbb{R}^n$ such that for every family $\{\mu_t\}$ as above there holds $\mathrm{Tan}(E_t, x) \subset V(\mu, x)$ for a.e. t. We call the map $x \mapsto V(\mu, x)$ the *decomposability bundle* of μ.

Even though this definition presents some flaws at close scrutiny, it should be sufficient to understand the following theorem, which is the main result of this paper; the "final" version of this definition is slightly different and more involved, and has been postponed to the next section (see §2.7).

1.1. Theorem. *Let μ be a finite measure on $\mathbb{R}^n$, and let $V(\mu, \cdot)$ be the decomposability bundle of μ (see §2.7). Then the following statements hold:*

(i) *Every Lipschitz function f on $\mathbb{R}^n$ is differentiable at μ-a.e. x with respect to the linear subspace $V(\mu, x)$, that is, there exists a linear function from $V(\mu, x)$ to $\mathbb{R}$, denoted by $d_V f(x)$, such that*

$$f(x+h) = f(x) + d_V f(x)\, h + o(h) \quad \text{for } h \in V(\mu, x).$$

(ii) *The previous statement is optimal in the sense that there exists a Lipschitz function f on $\mathbb{R}^n$ such that for μ-a.e. x and every $v \notin V(\mu, x)$ the derivative of f at x in the direction v does not exist.*

1.2. Remark. The differentiability part of this theorem, namely statement (i), applies also to Lipschitz maps $f : \mathbb{R}^n \to \mathbb{R}^m$ (because it applies to each component of f). Note that for the non-differentiability part we are able to give a real-valued example. In other words, the results on the differentiability of Lipschitz maps with respect to measures—at least those presented in this paper—are not sensitive to the dimension of the codomain of the maps, unlike the results concerning the differentiability at every point of a given set (see the discussion above).[5]

1.3. Relations with results in the literature. Even though it was never written down explicitly, the idea that the differentiability properties of Lipschitz functions w.r.t. a general measure μ are encoded in the decompositions of μ in terms of rectifiable measures was somewhat in the air (for example, it is clearly

[5] The difference between measures and sets is partly explained by the fact that given a Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^m$ which is non-differentiable μ-a.e., it is relatively easy to obtain a function $g : \mathbb{R}^n \to \mathbb{R}$ which is not differentiable μ-a.e. For instance, if f_i are the components of f, the linear combination $g := \sum \alpha_i f_i$ would do for (Lebesgue-) almost every choice of the vector of coefficients $(\alpha_1, \dots, \alpha_m)$. This statement, which incidentally is not difficult to prove, has no counterpart for sets.

assumed as a starting point in [5], where this idea is extended to the context of metric spaces to give a characterization of Lipschitz differentiability spaces).

Moreover the proof of Theorem 1.1(ii), namely the construction of a Lipschitz function which is not differentiable in any direction which is not in $V(\mu, x)$, is a simplified version of a construction given in [4]. Note that the original construction gives Lipschitz functions (actually maps) which are non-differentiable at any point of a given set, but if we need a function which is non-differentiable μ-a.e. we are allowed to discard μ-null sets, and this possibility makes room for significant simplifications.

Finally, let us mention that the notion of decomposability bundle is related to a notion of tangent space to measures given in [1] (see Remark 2.8(iii)).

1.4. Computation of the decomposability bundle. In certain cases the decomposability bundle $V(\mu, x)$ can be computed using Proposition 2.9. We just recall here that if μ is absolutely continuous w.r.t. the Lebesgue measure then $V(\mu, x) = \mathbb{R}^n$ for μ-a.e. x, and if μ is absolutely continuous w.r.t. the restriction of the Hausdorff measure $\mathscr{H}^k$ to a k-dimensional surface S of class C^1 (or even a k-rectifiable set S, cf. §2.4) then $V(\mu, x) = \mathrm{Tan}(S, x)$ for μ-a.e. x (Proposition 2.9(iii)). On the other hand, if μ is the canonical measure associated to well-known examples of self-similar fractals such as the snowflake curve and the Sierpiński carpet, then $V(\mu, x) = \{0\}$ for μ-a.e. x (see Remark 2.10).

1.5. Application to the theory of currents. In Section 3 we study the decomposability bundle of measures associated to k-dimensional normal currents. More precisely, given a normal current T, which we write as $T = \tau\mu$ where μ is a positive measure and τ is a k-vectorfield (see §3.2), we show that the linear subspace of $\mathbb{R}^n$ spanned by the k-vector $\tau(x)$ is contained in $V(\mu, x)$ for μ-a.e. x, and in particular $V(\mu, x)$ has dimension at least k for μ-a.e. x (see §3.9 and Theorem 3.12(i)). At the same time we obtain a direct proof[6] of the fact that a Lipschitz function f is differentiable at x w.r.t. the space spanned by $\tau(x)$ for μ-a.e. x (Theorem 3.12(ii)). Using this result we then give explicit formulas for the boundary of the interior product of a normal k-current and a Lipschitz h-form (Proposition 3.15) and for the push-forward of a normal k-current according to a Lipschitz map (Proposition 3.18)

1.6. On the validity of Rademacher theorem. It is natural to ask for which measures μ on $\mathbb{R}^n$ Rademacher theorem holds in the usual form, that is, every Lipschitz function (or map) on $\mathbb{R}^n$ is differentiable μ-a.e. Clearly the class of such measures contains all absolutely continuous measures, but does it contains any singular measure?

The answer turns out to be negative in every dimension n, because a singular measure μ is supported on a null set E, and for every null set E contained in $\mathbb{R}^n$ there exists a Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^n$ which is non-differentiable at every point of E. The latter result has been known for long time when $n = 1$ (see for

[6] That is, a proof which is not based on Theorem 1.1.

instance the construction in [26]); for $n = 2$ it has been given in [4], and then extended to every $n \geq 2$ by Csörnyei and Jones, as announced in [8].

1.7. Remark. (i) Theorem 1.1 shows that Rademacher theorem holds for a measure μ if and only if $V(\mu, x) = \mathbb{R}^n$ for μ-a.e. x, and allows us to rephrase the conclusions of the previous subsection as follows: if $V(\mu, x) = \mathbb{R}^n$ for μ-a.e. x then μ is absolutely continuous w.r.t. the Lebesgue measure.[7]

(ii) For $n = 1$ it is easy to show directly that $V(\mu, x) = \{0\}$ for μ-a.e. x: indeed μ is supported on a null set E, null sets in $\mathbb{R}$ are purely unrectifiable (§2.4), and in every dimension the decomposability bundle of a measure supported on purely unrectifiable set is trivial (Proposition 2.9(iv)).

(iii) For $n = 2$ the fact that $V(\mu, x) \neq \mathbb{R}^2$ for μ-a.e. x follows also from a result proved in [1] (see Remark 2.8(iv)).

1.8. Higher dimensional decompositions. For $k = 1, \ldots, n$ let $\mathscr{F}_k(\mathbb{R}^n)$ be the class of all measure μ on $\mathbb{R}^n$ which are absolutely continuous w.r.t. a measure of the form $\int_I \mu_t \, dt$ where each μ_t is the volume measure on a k-dimensional surface E_t of class C^1.[8] By Proposition 2.9(vi), for every μ in this class the decomposability bundle $V(\mu, x)$ has dimension at least k at μ-a.e. x and it is natural to ask whether the converse is true, namely that $\dim(V(\mu, x)) \geq k$ for μ-a.e. x implies that μ belongs to $\mathscr{F}_k(\mathbb{R}^n)$.

The answer is positive for $k = 1$ and $k = n$ (the case $k = 1$ is immediate, while the case $k = n$ follows from Remark 1.7(i)). Recently, A. Máthé proved in [17] that the answer is negative in all the other cases.

1.9. Differentiability of Sobolev functions. Since the continuous representatives of functions in the Sobolev space $W^{1,p}(\mathbb{R}^n)$ with $p > n$ are differentiable almost everywhere, it is natural to ask what differentiability result we have when the Lebesgue measure is replaced by a singular measure μ. In [4] an example is given of a continuous function in $W^{1,p}(\mathbb{R}^n)$ which is not differentiable in any direction at μ-a.e. point; it seems therefore that Theorem 1.1 admits no significant extension of to (first order) Sobolev space.

The rest of this paper is organized as follows: in Section 2 we give the precise definition of decomposability bundle and a few of its basic properties; in Section 3 we study the decomposability bundle of measures associated to normal currents, prove the corresponding differentiability results, and derive a few applications related to the theory of normal currents; sections 4 and 5 contain the proof of the main result (Theorem 1.1).

In order to make the structure of the proofs more transparent, we have postponed to an appendix (Section 6) the proofs of several technical lemmas used in the rest of the paper.

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[7] In view of Proposition 2.9(i) we can say (slightly) more: for every measure μ on $\mathbb{R}^n$ there holds $V(\mu, x) \neq \mathbb{R}^n$ for μ^s-a.e. x, where μ^s is the singular part of the measure μ w.r.t. the Lebesgue measure.

[8] Or, equivalently, each μ_t is the restriction of the k-dimensional Hausdorff measure $\mathscr{H}^k$ to a k-rectifiable set E_t (see §2.4).

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2. Decomposability bundle

We begin this section by recalling some definitions and notation used through the entire paper (more specific definitions and notation will be introduced when needed). We then give the definition of decomposability bundle $V(\mu,\cdot)$ of a measure μ (see §2.7) and prove a few basic properties (Proposition 2.9).

2.1. General notation. For the rest of this paper, sets and functions are tacitly assumed to be Borel measurable, and measures are always defined on the appropriate Borel σ-algebra. Unless we write otherwise, measures are positive and finite. We say that a measure μ on a space X is supported on the Borel set E if $|\mu|(X \setminus E) = 0$.[1]

We say that a map $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at the point $x \in \mathbb{R}^n$ w.r.t. a linear subspace V of $\mathbb{R}^n$ if there exists a linear map $L : V \to \mathbb{R}^m$ such that the following first-order Taylor expansion holds

$$f(x+h) = f(x) + Lh + o(h) \quad \text{for all } h \in V.$$

The linear map L is unique, it is called the derivative of f at x w.r.t. V and is denoted by $d_V f(x)$.[2]

We add below a list of frequently used notations (for the notations related to multilinear algebra and currents see §3.1):

$B(r)$	closed ball with center 0 and radius r in $\mathbb{R}^n$;
$B(x,r)$	closed ball with center x and radius r in $\mathbb{R}^n$;
$\mathrm{dist}(x,E)$	distance between the point x and the set E;
$v \cdot w$	scalar product of $v, w \in \mathbb{R}^n$;
$C(e,\alpha)$	convex closed cone in $\mathbb{R}^n$ with axis e and angle α (see §4.1);
1_E	characteristic function of a set E, defined on any ambient space and taking values 0 and 1;
$\mathrm{Gr}(\mathbb{R}^n)$	set of all linear subspaces of $\mathbb{R}^n$, that is, the union of the Grassmannians $\mathrm{Gr}(\mathbb{R}^n, k)$ with $k = 0, \dots, n$.

[1] Note that E does not need to be closed, and hence it may not contain the support of μ.

[2] If $V = \mathbb{R}^n$ then $d_V f(x)$ is the usual derivative, and is denoted by $df(x)$. Note that if $V = \{0\}$ the notion of differentiability makes sense, but is essentially void (every map is differentiable w.r.t. V at every point).

$d_{\mathrm{gr}}(V, V')$	distance between $V, V' \in \mathrm{Gr}(\mathbb{R}^n)$, defined as the maximum of $\delta(V, V')$ and $\delta(V', V)$, where $\delta(V, V')$ is the smallest number d such that for every $v \in V$ there exists $v' \in V'$ with $|v - v'| \leq d|v|$;[3]
$\langle L\,;\,v\rangle$	(also written Lv) action of a linear map L on a vector v; linear maps are always endowed with the operator norm, denoted by $|\cdot|$;
$D_v f(x)$	derivative of a map $f : \mathbb{R}^n \to \mathbb{R}^m$ in the direction v at a point x;
$df(x)$	derivative of a map $f : \mathbb{R}^n \to \mathbb{R}^m$ at a point x, viewed as a linear map from from $\mathbb{R}^n$ to $\mathbb{R}^m$;
$d_V f(x)$	derivative of a map $f : \mathbb{R}^n \to \mathbb{R}^m$ at a point x w.r.t. a subspace V (see at the beginning of this subsection);
$\mathrm{Tan}(S, x)$	tangent space to S at a point x, where S is a surface (submanifold) of class C^1 in $\mathbb{R}^n$ or a rectifiable set (see §2.4);
$\mathrm{Lip}(f)$	Lipschitz constant of a map f (between two metric spaces);
$\mathscr{L}^n$	Lebesgue measure on $\mathbb{R}^n$;
$\mathscr{H}^d$	d-dimensional Hausdorff measure (on any metric space X);
L^p	stands for $L^p(\mathbb{R}^n, \mathscr{L}^n)$; for the L^p space on a different measured space (X, μ) we use the notation $L^p(\mu)$;
$\|\cdot\|_p$	norm in $L^p = L^p(\mathbb{R}^n, \mathscr{L}^n)$; we use $\|\cdot\|_\infty$ also to denote the supremum norm of continuous functions;
$1_E\,\mu$	restriction of a measure μ to a set E;
$f_\#\,\mu$	push-forward of a measure μ on a space X according to a map $f : X \to X'$, that is, $[f_\#\,\mu](E) := \mu(f^{-1}(E))$ for every Borel set E contained in X';
$\lvert\mu\rvert$	total variation measure associated to the real- or vector-valued measure μ; thus μ can be written as $\mu = \rho\,\lvert\mu\rvert$ where the (real- or vector-valued) density ρ satisfies $\lvert\rho\rvert = 1$-a.e.
$\|\mu\|$	$:= \lvert\mu\rvert(X)$, mass of the measure μ;
$\lambda \ll \mu$	means that the measure λ is absolutely continuous w.r.t. μ, hence $\lambda = \rho\mu$ for a suitable density ρ;
$\int_I \mu_t\,dt$	integral of the measures μ_t with $t \in I$ with respect to a measure dt (see §2.5).

2.2. Essential span of a family of vectorfields. Given a measure μ on $\mathbb{R}^n$ and a family $\mathscr{G}$ of Borel maps from $\mathbb{R}^n$ to $\mathrm{Gr}(\mathbb{R}^n)$, we say that V is a minimal element of $\mathscr{G}$ if for every $V' \in \mathscr{G}$ there holds $V(x) \subset V'(x)$ for μ-a.e. x, and we say that $\mathscr{G}$ is closed under countable intersection if given a countable subfamily $\{V_i\} \subset \mathscr{G}$ the map V defined by $V(x) := \cap_i V_i(x)$ for every $x \in \mathbb{R}^n$ belongs to $\mathscr{G}$.

Let now $\mathscr{F}$ be a family of Borel vectorfields on $\mathbb{R}^n$, and let $\mathscr{G}$ be the class of all Borel maps $V : \mathbb{R}^n \to \mathrm{Gr}(\mathbb{R}^n)$ such that for every $\tau \in \mathscr{F}$ there holds

$$\tau(x) \subset V(x) \quad \text{for } \mu\text{-a.e. } x.$$

[3] Note that $d_{\mathrm{gr}}(V, V')$ agrees with the Hausdorff distance between the closed sets $V \cap B(1)$ and $V' \cap B(1)$; this shows that d_{gr} is indeed a metric on $\mathrm{Gr}(\mathbb{R}^n)$.

Since $\mathscr{G}$ is closed under countable intersection, by Lemma 2.3 below it admits a unique minimal element, which we call *μ-essential span* of $\mathscr{F}$.

2.3. Lemma. *Let $\mathscr{G}$ be a family of Borel maps from $\mathbb{R}^n$ to $\mathrm{Gr}(\mathbb{R}^n)$ which is closed under countable intersection. Then $\mathscr{G}$ admits a minimal element V, which is unique modulo equivalence μ-a.e.*[4]

Proof. Uniqueness follows immediately from minimality. To prove existence, set

$$\Phi(V) := \int_{\mathbb{R}^n} \dim(V(x))\, d\mu(x)$$

for every $V \in \mathscr{G}$, then take a sequence $\{V_i\}$ in $\mathscr{G}$ such that $\Phi(V_i)$ tends to the infimum of Φ over $\mathscr{G}$, and let V be the intersection of all V_i; thus V belongs to $\mathscr{G}$ and is a minimum of Φ over $\mathscr{G}$, and we claim that V is a minimal element of $\mathscr{G}$: if not, there would exists $V' \in \mathscr{G}$ such that $V''(x) := V(x) \cap V'(x)$ is strictly contained in $V(x)$ for all x in some set of positive measure, thus V'' belongs to $\mathscr{G}$ and $\Phi(V'') < \Phi(V)$. □

2.4. Rectifiable and unrectifiable sets. Given $k = 1, 2, \dots$ we say that a set E contained in $\mathbb{R}^n$ is k-rectifiable if it has finite $\mathscr{H}^k$ measure and it can be covered, except for a $\mathscr{H}^k$-null subset, by countably many images of Lipschitz maps from $\mathbb{R}^k$ to $\mathbb{R}^n$, or, equivalently, by countably many k-dimensional surfaces (submanifolds) of class C^1.[5]

In particular it is possible to define for $\mathscr{H}^k$-a.e. $x \in E$ an approximate tangent space $\mathrm{Tan}(E, x)$.[6] The tangent bundle is actually characterized by the property that for every k-dimensional surface S of class C^1 there holds

$$\mathrm{Tan}(E, x) = \mathrm{Tan}(S, x) \quad \text{for } \mathscr{H}^k\text{-a.e. } x \in E \cap S. \tag{2.1}$$

Finally we say that a set E in $\mathbb{R}^n$ is purely unrectifiable (or more precisely 1-purely unrectifiable) if $\mathscr{H}^1(E \cap S) = 0$ for every 1-rectifiable set S, or equivalently for every curve S of class C^1.

2.5. Integration of measures. Let I be a locally compact, separable metric space endowed with a finite measure dt, and for every $t \in I$ let μ_t be a measure on $\mathbb{R}^n$, possibly real- or vector-valued, such that:

(a) the function $t \mapsto \mu_t(E)$ is Borel for every Borel set E in $\mathbb{R}^n$;
(b) the function $t \mapsto \|\mu_t\|$ is Borel and the integral $\int_I \|\mu_t\|\, dt$ is finite.

Then we denote by $\int_I \mu_t\, dt$ the measure on $\mathbb{R}^n$ defined by

$$\Big[\int_I \mu_t\, dt\Big](E) := \int_I \mu_t(E)\, dt \quad \text{for every Borel set } E \text{ in } \mathbb{R}^n.$$

[4] In other words, every other minimal element V' satisfies $V(x) = V'(x)$ for μ-a.e. x.

[5] See for instance [18], §3.10 and Proposition 3.11, or [15], Definition 5.4.1 and Lemma 5.4.2. Note that the terminology and even the definition vary (slightly) depending on the author.

[6] See for example [15], Theorem 5.4.6, or [18], Proposition 3.12.

2.6. Remark. (i) Assumption (a) above is equivalent to say that $t \mapsto \mu_t$ is a Borel map from I to the space of finite (real- or vector-valued) measures on $\mathbb{R}^n$ endowed with the weak* topology.[7] Note that assumption (a) and the definition of mass imply that the function $t \mapsto \|\mu_t\|$ is Borel, thus the first part of assumption (b) is redundant.

(ii) Given I and dt as in the previous subsection, there always exists a Borel map $\phi : [0,1] \to I$ such that the push-forward according to ϕ of the Lebesgue measure agrees up to a constant factor with the measure dt (see for instance [6], Theorem A.3); therefore, by composing the map $t \mapsto \mu_t$ with ϕ, one can always assume that in the expression $\int_I \mu_t \, dt$, I is the interval $[0,1]$ and dt is (a multiple of) the Lebesgue measure.

2.7. Decomposability bundle. Let I be the interval $[0,1]$ endowed with the Lebesgue measure dt. Given a measure μ on $\mathbb{R}^n$ we denote by $\mathscr{F}_\mu$ the class of all families $\{\mu_t : t \in I\}$ such that

(a) each μ_t is the restriction of $\mathscr{H}^1$ to a 1-rectifiable set E_t;
(b) $\{\mu_t\}$ satisfies the assumptions (a) and (b) in §2.5;
(c) the measure $\int_I \mu_t \, dt$ is absolutely continuous w.r.t. μ.

Then we denote by $\mathscr{G}_\mu$ the class of all Borel maps $V : \mathbb{R}^n \to \mathrm{Gr}(\mathbb{R}^n)$ such that for every $\{\mu_t\} \in \mathscr{F}_\mu$ there holds

$$\mathrm{Tan}(E_t, x) \subset V(x) \quad \text{for } \mu_t\text{-a.e. } x \text{ and a.e. } t \in I.$$

Since $\mathscr{G}_\mu$ is closed under countable intersection, by Lemma 2.3 it admits a minimal element which is unique modulo equivalence μ-a.e. We call this map *decomposability bundle* of μ, and denote it by $x \mapsto V(\mu, x)$.

2.8. Remark. (i) In view of Remark 2.6(ii), nothing would change in the definition of decomposability bundle if we let I range among all locally compact, separable metric spaces, and dt range among all finite measures on I. We tacitly use this fact in the following.

(ii) This definition of the decomposability bundle differs from the one given in the Introduction in two respects: firstly, the minimality property that characterizes $V(\mu, \cdot)$ is now precisely stated, and secondly the sets E_t are now 1-rectifiable sets, and not just subsets of a rectifiable curve. This modification does not affect the definition, but it is convenient for technical reasons.

(iii) The decomposability bundle $V(\mu, x)$ is related to the bundle $E(\mu, x)$ introduced in [1], Definition 2.1. The latter is the μ-essential span of the family of vectorfields given by the Radon-Nikodým density of the measure derivatives Du with respect to μ, where u ranges over all BV functions on $\mathbb{R}^d$.[8]

It is not difficult to prove that $x \mapsto E(\mu, x)$ is the minimal element of the family of all maps $E : \mathbb{R}^n \to \mathrm{Gr}(\mathbb{R}^n)$ which satisfy the following property: for

[7] As dual of the appropriate Banach space of continuous functions on $\mathbb{R}^n$.

[8] Thus the property $E(\mu, x) \neq \{0\}$ for μ-a.e. x characterizes the measures μ such that $\mu \ll |Du|$ for some BV function u on $\mathbb{R}^n$.

every family of measures $\{\mu_t : t \in I\}$ such that $\mu \ll \int_I \mu_t \, dt$ and each μ_t is the restriction of $\mathscr{H}^{d-1}$ to a $(d-1)$-rectifiable set E_t, there holds

$$\mathrm{Tan}(E_t, x)^\perp \subset E(x) \quad \text{for } \mu_t\text{-a.e. } x \text{ and a.e. } t.$$

Thus for $n = 2$, $E(\mu, x)$ agrees with $V(\mu, x)$ rotated by 90° for μ-a.e. x.

(iv) It is proved in [1], Theorem 3.1, that if μ is a singular measure on $\mathbb{R}^2$ then $E(\mu, x) \neq \mathbb{R}^2$ for μ-a.e. x, and therefore the same holds for $V(\mu, x)$.

The next result collects some relevant properties of the decomposability bundle (see also Remark 1.7).

2.9. Proposition. *Let μ, μ' be measures on $\mathbb{R}^n$. Then the following statements hold:*

- (i) *(strong locality principle) if $\mu' \ll \mu$ then $V(\mu', x) = V(\mu, x)$ for μ'-a.e. x; more generally, if $1_E \, \mu' \ll \mu$ for some set E, then $V(\mu', x) = V(\mu, x)$ for μ'-a.e. $x \in E$;*
- (ii) *if μ is supported on a k-dimensional surface S of class C^1 then $V(\mu, x) \subset \mathrm{Tan}(S, x)$ for μ-a.e. x;*
- (iii) *if $\mu \ll 1_E \, \mathscr{H}^k$ where E is a k-rectifiable set, then $V(\mu, x) = \mathrm{Tan}(E, x)$ for μ-a.e. x; in particular if $\mu \ll \mathscr{L}^n$ then $V(\mu, x) = \mathbb{R}^n$ for μ-a.e. x;*
- (iv) *$V(\mu, x) = \{0\}$ for μ-a.e. x if and only if μ is supported on a purely unrectifiable set E;*

Moreover, given a family of measures $\{\lambda_s : s \in I\}$ as in §2.5,

- (v) *if $\int_I \lambda_s \, ds \ll \mu$ then $V(\lambda_s, x) \subset V(\mu, x)$ for λ_s-a.e. x and a.e. s;*
- (vi) *if $\mu \ll \int_I \lambda_s \, ds$ and λ_s is of the form $\lambda_s = 1_{E_t} \mathscr{H}^k$ where E_s is a k-rectifiable set for a.e. s, then $V(\mu, x)$ has dimension at least k for μ-a.e. x.*

2.10. Remark. (i) Many popular examples of self-similar fractals, including the von Koch snowflake curve, the Cantor set, and the so-called Cantor dust (a cartesian product of Cantor sets) are purely unrectifiable, and therefore every measure μ supported on them satisfies $V(\mu, x) = \{0\}$ for μ-a.e. x (Proposition 2.9(iv)).

(ii) The Sierpiński carpet is a self-similar fractal that contains many segments, and therefore is not purely unrectifiable. However, the *canonical* probability measure μ associated to this fractal is supported on a purely unrectifiable set, and therefore $V(\mu, x) = \{0\}$ for μ-a.e. x. The same occurs to other fractals of similar nature, such as the Sierpiński triangle and the Menger-Sierpiński sponge.

To prove Proposition 2.9 we need the following result, the proof of which is postponed to Section 6.

2.11. Lemma. *For every measure μ on $\mathbb{R}^n$ one of the following (mutually incompatible) alternatives holds:*

- (i) *μ is supported on a purely unrectifiable set E (see §2.4);*
- (ii) *there exists a nontrivial measure of the form $\mu' = \int_I \mu_t \, dt$ such that μ' is absolutely continuous w.r.t. μ and each μ_t is the restriction of $\mathscr{H}^1$ to some 1-rectifiable set E_t.*

Proof of Proposition 2.9. Through this proof $\{\mu_t : t \in I\}$ usually denotes a family in $\mathscr{F}_\mu$ (cf. §2.7), and in particular each measure μ_t is the restriction of $\mathscr{H}^1$ to some 1-rectifiable set E_t.

Statement (i). If $\mu' \ll \mu$ then $\mathscr{F}_{\mu'}$ is contained in $\mathscr{F}_\mu$, which implies that $V(\mu, \cdot) \in \mathscr{G}_{\mu'}$, and thus $V(\mu', x) \subset V(\mu, x)$ for μ'-a.e. x. To prove the opposite inclusion, take a Borel set F such that the restriction μ to F satisfies $1_F\mu \ll \mu'$;[9] then, given a family $\{\mu_t\} \in \mathscr{F}_\mu$, the family $\{\mu'_t\}$ of the restrictions $\mu'_t := 1_F\mu_t = 1_{E_t\cap F}\mathscr{H}^1$ belongs to $\mathscr{F}_{\mu'}$, which implies that $V(\mu', x) \subset \mathrm{Tan}(E_t, x)$ for μ_t-a.e. $x \in F$ and a.e. t. Hence the bundle V defined by $V(x) := V(\mu', x)$ for $x \in F$ and $V(x) := \mathbb{R}^n$ for $x \notin F$ belongs to $\mathscr{G}_\mu$, therefore $V(\mu, x) \subset V(x)$ for μ-a.e. x, which means that $V(\mu, x) \subset V(\mu', x)$ for μ-a.e. $x \in F$, that is, for μ'-a.e. x. The proof of the first part of statement (i) is thus complete.

By applying the first part of the statement (i) to the measures $1_E\mu'$ and μ, and then to the measures $1_E\mu'$ and μ' we obtain $V(1_E\mu', x) = V(\mu, x) = V(\mu', x)$ for μ'-a.e. $x \in E$, which proves the second part of statement (i).

Statement (ii). Given an arbitrary family $\{\mu_t\}$ in $\mathscr{F}_\mu$, we observe that $\int \mu_t\, dt \ll \mu$ and $\mu(\mathbb{R}^n \setminus S) = 0$ imply

$$0 = \int \mu_t(\mathbb{R}^n \setminus S)\, dt = \int \mathscr{H}^1(E_t \setminus S)\, dt\,,$$

which in turn implies that, for a.e. t, the set E_t is contained (up to an $\mathscr{H}^1$-null set) in S. Thus $\mathrm{Tan}(E_t, x) \subset \mathrm{Tan}(S, x)$ for μ_t-a.e. x, which means that $\mathrm{Tan}(S, \cdot)$ belongs to $\mathscr{G}_\mu$, and therefore $V(\mu, x) \subset \mathrm{Tan}(S, x)$ for μ-a.e. x.

Statement (iii). Using statement (i) and the definition of k-rectifiable set (see §2.4) we can reduce to the case $\mu = 1_E\mathscr{H}^k$ where E is a subset of a k-dimensional surface S of class C^1, and we can further assume that S is parametrized by a diffeomorphism $g : A \to S$ of class C^1, where A is a bounded open set in $\mathbb{R}^k$.

We set $E' := g^{-1}(E)$ and $\mu' := 1_{E'}\mathscr{L}^k$. Then we fix a nontrivial vector $e \in \mathbb{R}^k$, and for every t in the hyperplane $e^\perp$ we let E'_t be the intersection of the set E' with the line $\{t + he : h \in \mathbb{R}\}$, and set $\mu'_t := 1_{E'_t}\mathscr{H}^1$. By Fubini's theorem we have that $\mu' = \int \mu'_t\, dt$ where dt is the restriction of $\mathscr{H}^{k-1}$ to the projection of A onto $e^\perp$.

Next we set $E_t := g(E'_t)$ and $\mu_t := 1_{E_t}\mathscr{H}^1$. Thus each E_t is a 1-rectifiable set whose tangent space at $x = g(x')$ is spanned by the vector $\tau(x) := dg(x')\, e$. Moreover, taking into account that g is a diffeomorphism, we get that $\int \mu_t\, dt$ and μ are mutually absolutely continuous. Therefore $\tau(x) \in V(\mu, x)$ for μ_t-a.e. x and a.e. t, that is, for μ-a.e. x.

Finally, we take vectors $e_1, \dots, e_k$ that span $\mathbb{R}^k$, thus the corresponding vectorfields $\tau_i(x) := dg(x')\, e_i$ span $\mathrm{Tan}(S, x)$ for every x, and we conclude that $\mathrm{Tan}(E, x) = \mathrm{Tan}(S, x) \subset V(\mu, x)$ for μ-a.e. x. The opposite inclusion follows from statement (ii).

[9] For example, F is the set where the Radon-Nikodým density of μ' w.r.t. μ is strictly positive

Statement (iv). We prove the "if" part first. If μ is supported on a set E and $\{\mu_t\} \in \mathscr{F}_\mu$, then, arguing as in the proof of statement (ii), we obtain that for a.e. t the set E_t is contained in E up to an $\mathscr{H}^1$-null set. In particular, if E is purely unrectifiable we obtain that $\mathscr{H}^1(E_t) = 0$, that is, $\mu_t = 0$. Hence the trivial bundle $V(x) := \{0\}$ belongs to $\mathscr{G}_\mu$, and in particular $V(\mu, x) := \{0\}$ for μ-a.e. x.

The "only if" part follows from Lemma 2.11; indeed the alternative (ii) in this lemma is ruled out by the fact that $V(\mu, x) = \{0\}$ for μ-a.e. x.

Statement (v). Let I' be the set of all $s \in I$ for which $V(\mu, \cdot)$ does not belong to $\mathscr{G}_{\lambda_s}$; then statement (v) can be rephrased by saying that I' is negligible. Indeed for every $s \in I'$ we can find a family $\{\mu_{s,t} = 1_{E_{s,t}}\mathscr{H}^1 : t \in I\}$ in $\mathscr{F}_{\lambda_s}$ such that the inclusion

$$\mathrm{Tan}(E_{s,t}, x) \subset V(\mu, x) \quad \text{for } \mu_{s,t}\text{-a.e. } x \text{ and a.e. } t \tag{2.2}$$

does not hold for every $s \in I'$. On the other hand, since $\int \lambda_s \, ds \ll \mu$, we have that

$$\int_{I \times I'} \mu_{s,t} \, dt \, ds \ll \int_{I'} \lambda_s \, ds \ll \mu$$

and therefore the family $\{\mu_{s,t} : t \in I, s \in I'\}$ belongs to $\mathscr{F}_\mu$, which implies that (2.2) holds for a.e. $s \in I'$. Thus I' must have measure 0.[10]

Statement (vi). By statement (i) it suffices to prove the claim when $\mu = \int \lambda_s \, ds$. In this case statement (v) implies that $V(\mu, x)$ contains $V(\lambda_s, x)$ for λ_s-a.e. x and a.e. s, and $V(\lambda_s, x)$ has dimension k by statement (iii). Thus $V(\mu, x)$ has dimension at least k for λ_s-a.e. x and a.e. s, that is, for μ-a.e. x. □

3. Measures associated to normal currents

In the first part of this section we study the decomposability bundle of measures associated to 1-dimensional normal currents, and obtain a differentiability result for Lipschitz functions w.r.t. these measures (Theorem 3.5); we then use this result to prove a formula for the boundary of the product of a normal 1-current and a Lipschitz function (Proposition 3.6).

In the second part of the section we extend these results to normal currents with arbitrary dimensions (Theorem 3.12 and Proposition 3.15) and then prove a formula for the push-forward of a normal current according to a Lipschitz map (Proposition 3.18).

The results in the first part of this section (Theorem 3.5 and Proposition 3.6) play a key role in the proof of Theorem 1.1(i), while the second part of this section (starting from §3.9) is essentially independent from the rest of the paper, and it

[10] This proof is not correct as written, because the map $(s, t) \mapsto \mu_{s,t}$ is not necessarily Borel measurable in both variables (in the sense of Remark 2.6(i)). For a correct proof, the families $\{\mu_{s,t} : t \in I\}$ should be chosen for every $s \in I'$ in a measurable fashion, and this can be achieved by means of a suitable measurable selection theorem; since this statement is not essential for the rest of the paper we omit the details.

can be skipped by the reader who is not specifically interested in the theory of currents.

3.1. Notation related to currents. We list here the notation from multilinear algebra and the theory of currents that is used in this section (the numbers between brackets refer to the relevant subsections):

$\wedge_k(V)$	space of k-vectors in the vector space V;
$\wedge^k(V)$	space of k-covectors on the vector space V;[1]
$v \wedge w$	exterior product of the multi-vectors (or multi-covectors) v and w;
$\langle \alpha\,;\, v\rangle$	duality pairing of the k-covector α and the k-vector v, also written as $\langle v\,;\,\alpha\rangle$;
$\langle T\,;\,\omega\rangle$	duality pairing of the k-current T and the k-form ω (§3.2);
$d\omega$	exterior derivative of the k-form ω;
∂T	boundary of the current T (§3.2);
$\mathbb{M}(T)$	mass of the current T (§3.2);
$\mathrm{span}(v)$	span of the k-vector v (§3.9);
$v \llcorner \alpha$	interior product of the k-vector v and the h-covector α (§3.11);
$T \llcorner \omega$	interior product of the k-current T and the h-form ω (§3.11);
$f_\# T$	push-forward of the current T according to the map f (§3.17).

3.2. Currents and normal currents. We recall here the basic notions and terminology from the theory of currents; introductory presentations of this theory can be found for instance in [15], [18], and [23]; the most complete reference remains [11]. A *k-dimensional current* (or k-current) in $\mathbb{R}^n$ is a continuous linear functional on the space of k-forms on $\mathbb{R}^n$ which are smooth and compactly supported.

The boundary of a k-current T is the $(k-1)$-current ∂T defined by $\langle \partial T\,;\,\omega\rangle := \langle T\,;\,d\omega\rangle$ for every smooth and compactly supported $(k-1)$-form ω on $\mathbb{R}^n$. The *mass* of T, denoted by $\mathbb{M}(T)$, is the supremum of $\langle T\,;\,\omega\rangle$ over all forms ω such that $|\omega| \le 1$ everywhere. A current T is called *normal* if both T and ∂T have finite mass.

By Riesz theorem a current with finite mass can be represented as a finite measure with values in the space $\wedge_k(\mathbb{R}^n)$ of k-vectors in $\mathbb{R}^n$, and therefore it can be written in the form $T = \tau\mu$ where μ is a finite positive measure and τ is a k-vectorfield such that $\int |\tau| d\mu < +\infty$. In particular the action of T on a form ω is given by

$$\langle T\,;\,\omega\rangle = \int_{\mathbb{R}^n} \langle \omega(x)\,;\,\tau(x)\rangle\, d\mu(x)\,,$$

and the mass $\mathbb{M}(T)$ is the total mass of T as a measure, that is, $\mathbb{M}(T) = \int |\tau| d\mu$.[2]

[1] When $V = \mathbb{R}^n$ we endow $\wedge_k(V)$ and $\wedge^k(V)$ with the euclidean norms determined by the canonical basis of these spaces. Note however that the choice of the norms on these spaces has no relevant consequence in what follows.

[2] Since $\wedge_0(\mathbb{R}^n) = \mathbb{R}$, a 0-current with finite mass is a real-valued measure, and since $\wedge_1(\mathbb{R}^n) = \mathbb{R}^n$, a 1-current with finite mass is an $\mathbb{R}^n$-valued measure and can be written as $\tau\mu$ where μ is a

We say that μ is a measure *associated* to a current T with finite mass if T can be represented as $T = \tau\mu$ with $\tau(x) \neq 0$ for μ-a.e. x.

3.3. Integral currents. Let E be a k-rectifiable set. An *orientation* of E is a k-vectorfield τ on $\mathbb{R}^n$ such that the k-vector $\tau(x)$ is *simple*, has norm 1, and spans the approximate tangent space $\mathrm{Tan}(E, x)$ for $\mathscr{H}^k$-a.e. $x \in E$.[3] A *multiplicity* on E is any integer-valued function m such that $\int_E m \, d\mathscr{H}^k < +\infty$. For every choice of E, τ, m as above we denote by $[E, \tau, m]$ the k-current defined by $[E, \tau, m] := m\tau 1_E \, \mathscr{H}^k$, that is,

$$\big\langle [E, \tau, m] \, ; \, \omega \big\rangle := \int_E \langle \omega \, ; \, \tau \rangle \, m \, d\mathscr{H}^k \, .$$

Currents of this type are called *integer-multiplicity rectifiable currents*. A current T is called *integral* if both T and ∂T can be represented as integer-multiplicity rectifiable currents.

The next statement contains a decomposition for normal 1-currents which is strictly related to a decomposition given in [24], and plays a key role in the proof of Theorem 3.5 below.

3.4. Theorem (See [4]). *Let $T = \tau\mu$ be a normal 1-current with $|\tau(x)| = 1$ for μ-a.e. x, and let δ be a positive real number. Then:*

(i) *μ can be decomposed as $\mu = \int_I \mu_t \, dt$ (in the sense of §2.5) where each μ_t is the restriction of $\mathscr{H}^1$ to a 1-rectifiable set E_t; moreover $\mathrm{Tan}(E_t, x)$ is spanned by $\tau(x)$ for μ_t-a.e. x and a.e. t;*

(ii) *T can be decomposed as $T = \int_I T_t \, dt$ where each T_t is the integral current given by $T_t := [E_t, \tau, 1]$ (cf. §3.3); moreover*

$$\mathbb{M}(T) = \int_I \mathbb{M}(T_t) \, dt = \int_I \mathscr{H}^1(E_t) \, dt \, , \tag{3.1}$$

$$\mathbb{M}(\partial T) + \delta \geq \int_I \mathbb{M}(\partial T_t) \, dt \, . \tag{3.2}$$

3.5. Theorem. *Let $T = \tau\mu$ be a normal 1-current in $\mathbb{R}^n$ and f a Lipschitz function on $\mathbb{R}^n$. Then*

(i) *$\tau(x)$ belongs to $V(\mu, x)$ for μ-a.e. x;*

(ii) *f is differentiable at x in the direction $\tau(x)$ for μ-a.e. x.*

Proof. *First case: $|\tau(x)| = 1$ for μ-a.e. x.* Consider the decomposition of μ given in statement (i) of Theorem 3.4. For μ_t-a.e. x and a.e. t, the vector $\tau(x)$ belongs to $\mathrm{Tan}(E_t, x)$ which in turn is contained in $V(\mu, x)$ (recall §2.7), and this proves statement (i). Statement (ii) follows from the fact that given any

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positive measure and τ is a vectorfield. The boundary of a 1-current T agrees, up to a change of sign, with the (distributional) divergence of T.

[3] The span of a simple k-vector $v = v_1 \wedge \cdots \wedge v_k$ in $\mathbb{R}^n$ is the linear subspace of $\mathbb{R}^n$ generated by the factors $v_1, \dots, v_k$, see also §3.9.

1-rectifiable set E, then f is differentiable at x w.r.t. the tangent line $\mathrm{Tan}(E,x)$ for $\mathscr{H}^1$-a.e. x.[4]

Second case: $\tau(x) \neq 0$ for μ-a.e. x. We reduce to thew previous case by replacing μ with the measure $\mu' := |\tau|\,\mu$ (we use Proposition 2.9(i) to prove that $V(\mu,x) = V(\mu',x)$ for μ-a.e. x).

The general case. Let E be the set of all x where $\tau(x) \neq 0$. The assertions in statements (i) and (ii) are clearly true for μ-a.e. $x \notin E$, and therefore it suffices to prove them for μ-a.e. $x \in E$. In other words we can reduce to the previous case by replacing μ with the restriction $\mu' := 1_E\,\mu$ (we use Proposition 2.9(i) to prove that $V(\mu,x) = V(\mu',x)$ for μ-a.e. $x \in E$). □

3.6. Proposition. *Let $T = \tau\mu$ be a normal 1-current in $\mathbb{R}^n$ and f a bounded Lipschitz function on $\mathbb{R}^n$. Then $fT := f\tau\mu$ is a normal 1-current with boundary*

$$\partial(fT) = f\,\partial T + (D_\tau f)\mu\,. \tag{3.3}$$

3.7. Remark. (i) The directional derivative of f at the right-hand side of last formula exists for μ-a.e. x by Theorem 3.5(ii).

(ii) If f is a bounded function of class C^1 and $T = \tau\mu$ is a normal k-current, then it is easy to see that fT is a normal current and that formula (3.3) holds. However this formula cannot be seamlessly extended to the case where f is Lipschitz because the derivative df does not necessarily exists μ-a.e.

(iii) Proposition 3.6 is a special case of Proposition 3.15.

To prove Proposition 3.6 we need the following lemma, the proof of which is postponed to Section 6.

3.8. Lemma. *Let f be a Lipschitz function on $\mathbb{R}^n$, μ a measure on $\mathbb{R}^n$, and $V : \mathbb{R}^n \to \mathrm{Gr}(\mathbb{R}^n)$ Borel map such that f is differentiable at μ-a.e. x w.r.t. the linear subspace $V(x)$, the derivative being denoted by $d_V f(x)$. Then there exists a sequence of smooth functions $f_j : \mathbb{R}^n \to \mathbb{R}$ such that the following statements hold (as $j \to +\infty$):*

(i) *the functions f_j converge to f uniformly;*

(ii) $\mathrm{Lip}(f_j)$ *converge to* $\mathrm{Lip}(f)$*;*

(iii) $d_V f_j(x) \to d_V f(x)$ *for μ-a.e. x.*[5]

Proof of Proposition 3.6. Take any sequence of smooth functions f_j on $\mathbb{R}^d$ which converge pointwise to f and are uniformly bounded and uniformly Lipschitz. It is easy to see that the currents f_jT converge to the current fT (in the weak sense of currents). Moreover the masses of the current f_jT and the

[4] This property is an immediate consequence of Rademacher theorem for Lipschitz functions of one variable when E is the image of a Lipschitz path, and can be extended to a general 1-rectifiable set E using the fact that by definition (see 2.4) E can be covered, up to an $\mathscr{H}^1$-null subset, by countably many curves of class C^1 (here we also need property (2.1)).

[5] Here $d_V f_j(x)$ is the restriction of the linear function $df_j(x)$ to the subspace $V(x)$; as usual, convergence is intended in the sense of the operator norm for linear functions on V.

masses of their boundaries are uniformly bounded,[6] which implies that fT is also a normal current.

It remains to prove formula (3.3). To this end we consider the approximating functions f_j obtained by applying Lemma 3.8 to f with $V(x) := \mathrm{span}(\tau(x))$, and observe that property (iii) in that lemma can be rewritten as

$$\lim_{j\to+\infty} D_\tau f_j(x) = D_\tau f(x) \quad \text{for } \mu\text{-a.e. } x. \tag{3.4}$$

Now, since each f_j is smooth, we know that formula (3.3) holds with f_j in place of f, and we use (3.4) to pass to the limit as $j \to +\infty$ and recover formula (3.3) for f. □

In order to extend Theorem 3.5 and Proposition 3.6 to currents with arbitrary dimension, we need some additional notions.

3.9. Span of a multivector. Given a k-vector v in the vector space V, we denote by $\mathrm{span}(v)$ the smallest linear subspace W of V such that v belongs to $\wedge_k(W)$.[7] A few relevant properties of the span are given in the statement below, the proof of which is postponed to Section 6.

3.10. Proposition. *Given a k-vector v as above, the following statements hold:*

(i) *if $v = 0$ then $\mathrm{span}(v) = \{0\}$;*
(ii) *if $v \neq 0$ then $\mathrm{span}(v)$ has dimension at least k;*
(iii) *if $\mathrm{span}(v)$ has dimension k then v is simple, that is, $v = v_1 \wedge \cdots \wedge v_k$ with $v_1, \dots, v_k \in V$;*
(iv) *if $v \neq 0$ and $v = v_1 \wedge \cdots \wedge v_k$ with $v_i \in V$ then $\mathrm{span}(v)$ is the linear subspace of V generated by the vectors $v_1, \dots, v_k$;*
(v) *$\mathrm{span}(v)$ consists of all vectors of the form $v \llcorner \alpha$ with $\alpha \in \wedge^{k-1}(V)$.*

3.11. Interior product. Let h, k be integers with $0 \le h \le k$. Given a k-vector v and an h-covector α on V, the *interior product* $v \llcorner \alpha$ is the $(k-h)$-vector uniquely defined by the duality relation

$$\langle v \llcorner \alpha\,;\, \beta\rangle = \langle v\,;\, \alpha \wedge \beta\rangle \quad \text{for every } \beta \in \wedge^{k-h}(V).$$

Accordingly, given a k-current T in $\mathbb{R}^n$ and a smooth h-form ω on $\mathbb{R}^n$, the *interior product* $T \llcorner \omega$ is the $(k-h)$-current defined by

$$\langle T \llcorner \omega\,;\, \sigma\rangle = \langle T\,;\, \omega \wedge \sigma\rangle \tag{3.5}$$

[6] More precisely, taken a finite constant m such that $\|f_j\|_\infty \le m$ and $\mathrm{Lip}(f_j) \le m$, there holds $\mathbb{M}(f_j T) \le m\,\mathbb{M}(T)$ and $\mathbb{M}(\partial(f_j T)) \le m(\mathbb{M}(\partial T) + \mathbb{M}(T))$; the latter estimate follows from formula (3.3) for functions f of class C^1 (see Remark 3.7(ii)).

[7] Note that if W is a linear subspace of V then $\wedge_k(W)$ is canonically identified with a linear subspace of $\wedge_k(V)$ and accordingly $\wedge^k(W)$ is canonically identified with a quotient of $\wedge^k(W)$; moreover there holds $\wedge_k(W) \cap \wedge_k(W') = \wedge_k(W \cap W')$ for every W, W' subspaces of V, and therefore the definition of $\mathrm{span}(v)$ is well-posed.

for every smooth $(h-k)$-form σ with compact support on $\mathbb{R}^n$. In this case a simple computation gives[8]

$$\partial(T \llcorner \omega) = (-1)^h \big[(\partial T) \llcorner \omega - T \llcorner d\omega\big] . \tag{3.6}$$

Note that if T has finite mass and ω is bounded and continuous then formula (3.5) still makes sense, $T \llcorner \omega$ is a current with finite mass, and given a representation $T = \tau\mu$ there holds $T \llcorner \omega = (\tau \llcorner \omega)\,\mu$.

Along the same line, if T is a normal current and ω is of class C^1, bounded and with bounded derivative, then $T \llcorner \omega$ is a normal current and formula (3.6) holds.

3.12. Theorem. *Let $T = \tau\mu$ be a normal k-current on $\mathbb{R}^n$, and let f be a Lipschitz function on $\mathbb{R}^n$. Then*

(i) $\mathrm{span}(\tau(x))$ *is contained in $V(\mu, x)$ for μ-a.e. x;*

(ii) *f is differentiable at x w.r.t. the linear subspace* $\mathrm{span}(\tau(x))$ *for μ-a.e. x, and we denote by $d_\tau f(x)$ the corresponding derivative.*

To prove this theorem we need the following lemma, the proof of which is postponed to Section 6.

3.13. Lemma. *Let be given a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, a point $x \in \mathbb{R}^n$, and a linear function $\alpha : V \to \mathbb{R}$ where V is a linear subspace of $\mathbb{R}^n$. Assume moreover that the derivative of f at x in the direction v exists and is equal to $\langle \alpha\,;\, v\rangle$ for all v in a dense subset D of V.*

Then f is differentiable w.r.t. V at x and $d_V f(x) = \alpha$.

Proof of Theorem 3.12. Take a finite set $\{\alpha_j : j \in J\}$ that spans $\wedge^{k-1}(\mathbb{R}^n)$.

Given $\alpha \in \wedge^{k-1}(\mathbb{R}^n)$, then $T \llcorner \alpha = (\tau \llcorner \alpha)\mu$ is a normal 1-current (see §3.11), and therefore Theorem 3.5(i) implies that the vector $\tau(x) \llcorner \alpha$ belongs to $V(\mu, x)$ for μ-a.e. x. In particular there exists a μ-null set N such that

$$\tau(x) \llcorner \alpha_j \in V(\mu, x) \quad \text{for every } j \in J \text{ and every } x \in \mathbb{R}^n \setminus N. \tag{3.7}$$

Moreover the vectors $\tau(x) \llcorner \alpha_j$ span $\{\tau(x) \llcorner \alpha : \alpha \in \wedge^{k-1}(\mathbb{R}^n)\}$, which by Proposition 3.10(v) agrees with $\mathrm{span}(\tau(x))$. This fact and (3.7) imply that $\mathrm{span}(\tau(x))$ is contained in $V(\mu, x)$ for every $x \in \mathbb{R}^n \setminus N$, and statement (i) is proved.

Statement (ii) follows from statement (i) and Theorem 1.1(i). We think however that the following direct proof is worth mentioning.

Fix the Lipschitz function f. Given $\alpha \in \wedge^{k-1}(\mathbb{R}^n)$, by applying Theorem 3.5(ii) to the 1-current $T \llcorner \alpha$ we obtain that the derivative of f in the direction $\tau \llcorner \alpha$ exists at μ-a.e. x, and formula (3.3) yields

$$D_{\tau \llcorner \alpha} f \mu = \partial(f T \llcorner \alpha) - f\, \partial T \llcorner \alpha . \tag{3.8}$$

Take now a countable dense subset S of $\mathbb{R}$ that contains 0 and 1, and let D be the set of all linear combinations $\sum s_j \alpha_j$ with $s_j \in S$ for every $j \in J$, and α_j as

[8] Apply the definition of boundary and (3.5) and the formula $d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^h \omega \wedge d\sigma$.

above. Since D is countable, there exists a μ-null set N such that

$$D_{\tau \llcorner \alpha} f(x) \text{ exists for every } x \notin N \text{ and every } \alpha \in D.$$

Moreover given $\alpha \in D$ of the form $\alpha = \sum s_j \alpha_j$, using the linearity of the right-hand side of (3.9) in the variable α we obtain

$$D_{\tau \llcorner \alpha} f(x) = \sum_{j \in J} s_j \, D_{\tau \llcorner \alpha_j} f(x) \tag{3.9}$$

for μ-a.e. x. And again, since D is countable, we can find a μ-null set N' such that (3.9) holds for every $\alpha \in D$ and every $x \notin (N \cup N')$.

We conclude the proof by showing that f is differentiable w.r.t. $\operatorname{span}(\tau(x))$ at every point x outside the μ-null set $N \cup N'$.

Fix such a point x. Since the vectors $v_j := \tau(x) \llcorner \alpha_j$ generate $\operatorname{span}(\tau(x))$, we can choose a subset J' of J such that $\{v_j : j \in J'\}$ is a basis of $\operatorname{span}(\tau(x))$. Let then ω be the linear function on $\operatorname{span}(\tau(x))$ defined by [9]

$$\langle \omega \, ; \, v_j \rangle := D_{v_j} f(x) = D_{\tau \llcorner \alpha_j} f(x) \quad \text{for every } j \in J'.$$

Consider now a vector v of the form

$$v = \sum_{j \in J'} s_j v_j \quad \text{with } s_j \in S \text{ for every } j \in J'. \tag{3.10}$$

Then, setting $\alpha := \sum_{j \in J'} s_j \, \alpha_j$, we obtain [10]

$$D_v f(x) = D_{\tau \llcorner \alpha} f(x) = \sum_{j \in J'} s_j \, D_{\tau \llcorner \alpha_j} f(x) = \sum_{j \in J'} s_j \langle \omega \, ; \, v_j \rangle = \langle \omega \, ; \, v \rangle \, .$$

Finally we notice that the vectors v of the form (3.10) are dense in $\operatorname{span}(\tau(x))$, and therefore we can apply Lemma 3.13 and obtain that f is differentiable at x w.r.t. $\operatorname{span}(\tau(x))$ with derivative $d_V f(x) = \omega$. □

3.14. Exterior derivative of Lipschitz forms. If ω is a Lipschitz h-form on $\mathbb{R}^n$, the (pointwise) exterior derivative $d\omega(x)$ is defined for $\mathscr{L}^n$-a.e. x but in general not for μ-a.e. x. However, given a normal 1-current $T = \tau\mu$ on $\mathbb{R}^n$, the coefficients of ω w.r.t. any basis of $\wedge^h(\mathbb{R}^n)$ are Lipschitz functions, and by Theorem 3.12(ii) are differentiable at x w.r.t. the linear subspace $\operatorname{span}(\tau(x))$ for μ-a.e. x. Hence at any such x it is possible to define the exterior derivative of ω relative to $\operatorname{span}(\tau(x))$, which we denote by $d_\tau \omega(x)$.

The precise construction is the following: given a basis $\{\alpha_i\}$ of $\wedge^h(\mathbb{R}^n)$, we denote by ω_i the coefficients of ω w.r.t. this basis, so that $\omega(x) = \sum_i \omega_i(x) \, \alpha_i$ for every $x \in \mathbb{R}^n$. Then, given a point x such that the functions ω_i are all

[9] The directional derivatives in this formula exist because all α_j belong to D and $x \notin N$.

[10] The directional derivatives in this formula exist because α_j and α belong to D and $x \notin N$; the second equality follows from (3.9) and the fact that $x \notin N'$; the third equality follows from the definition of ω.

differentiable at x w.r.t. to $V := \mathrm{span}(\tau(x))$, we chose a basis $\{e_j\}$ of V and denote by $d_\tau\omega(x)$ the $(h+1)$-covector on V defined by

$$d_\tau\omega(x) := \sum_{i,j} D_{e_j}\omega_i(x)\, e_j^* \wedge \alpha_i$$

where $\{e_j^*\}$ is the dual basis associated to $\{e_j\}$.[11]

3.15. Proposition. *Let $T = \tau\mu$ be a normal k-current on $\mathbb{R}^n$ and ω a bounded Lipschitz h-form on $\mathbb{R}^n$ with $0 \le h < k$. Then $T \llcorner \omega = (\tau \llcorner \omega)\, \mu$ is a normal $(k-h)$-current with boundary*

$$\partial(T \llcorner \omega) = (-1)^h \big[(\partial T) \llcorner \omega - (\tau \llcorner d_\tau\omega)\, \mu\big]\,. \tag{3.11}$$

3.16. Remark. The exterior derivative $d_\tau\omega$ is given in §3.14; more precisely $d_\tau\omega(x)$ is a well-defined $(h+1)$-covector on $\mathrm{span}(\tau(x))$ for μ-a.e. x, and then $\tau(x) \llcorner d_\tau\omega(x)$ is a well-defined $(k-h-1)$-vector in $\mathrm{span}(\tau(x))$, and therefore also a $(k-h-1)$-vector in $\mathbb{R}^n$.

Proof. It is not difficult to show that $T \llcorner \omega$ is a normal current and satisfies formula (3.11) when ω is a bounded h-form of class C^1 (see §3.11). We then proceed as in the proof of Proposition 3.15 to extend these properties to the case where ω is bounded and Lipschitz. The key point here is to apply Lemma 3.8 with $V(x) := \mathrm{span}(\tau(x))$ to the coefficients of ω (w.r.t. some basis of $\wedge^h(\mathbb{R}^n)$) and construct a sequence of smooth h-forms ω_j which are uniformly bounded and uniformly Lipschitz, converge to ω uniformly, and satisfy

$$\lim_{j\to+\infty} \tau(x) \llcorner d\omega_j(x) = \tau(x) \llcorner d_\tau\omega(x) \quad \text{for } \mu\text{-a.e. } x. \qquad \square$$

We conclude this section by proving a formula for the push-forward of a normal current according to a Lipschitz map.

3.17. Push-forward of currents. Given a smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$ and a k-current T in $\mathbb{R}^n$ with compact support, the *push-forward* of T according to f is the k-current $f_\# T$ in $\mathbb{R}^m$ defined by

$$\langle f_\# T\,;\, \omega\rangle := \langle T\,;\, f^\#\omega\rangle$$

for every smooth k-form ω on $\mathbb{R}^m$, where $f^\#\omega$ is the pull-back of ω.[12]

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[11] That is, the basis of the dual space V^* defined by $\langle e_i^*\,;\, e_j\rangle = \delta_{ij}$ for every $1 \le i,j \le n$, where $\delta_{ij} := 1$ if $i = j$ and $\delta_{ij} := 0$ if $i \ne j$, as usual.

[12] Since T has compact support, $\langle T\,;\,\omega\rangle$ is well-defined for every smooth k-form σ on $\mathbb{R}^n$, even without compact support, and in particular it is defined for $\sigma := f^\#\omega$, see for instance [18], Section 4.3A. The assumption that T has compact support can be removed if one assumes that f is proper, see for instance [15], Section 7.4.2.

If in addition T has finite mass and we write it as $T = \tau\mu$, then, by the definition of pull-back of a form, we have [13]

$$\langle f_\# T\,;\,\omega\rangle = \int_{\mathbb{R}^n} \big\langle df(x)_\#\,\tau(x)\,;\,\omega(f(x))\big\rangle\, d\mu(x)\,. \tag{3.12}$$

Note that formula (3.12) can be naturally extended to maps f which are of class C^1. Note that $f_\# T$ has finite mass, and more precisely $\mathbb{M}(f_\# T) \leq L^k\,\mathbb{M}(T)$ where L is the Lipschitz constant of the restriction of f to the support of T.

The definition of push-forward can be further extended to the case where T is a normal current and f is a Lipschitz map, and $f_\# T$ is a normal current (see for instance [11], §4.1.14, or [15], Lemma 7.4.3).

Note that in this case formula (3.12) does not makes sense because the derivative $df(x)$ is not defined for μ-a.e. x. However, Theorem 3.12(ii) states that for μ-a.e. x the map f is differentiable at x w.r.t. $\mathrm{span}(\tau(x))$, and since τ is a k-vector in $\mathrm{span}(\tau(x))$, then $d_\tau f(x)_\#\,\tau(x)$ is a well-defined k-vector in $\mathbb{R}^m$ (see footnote 13).

We can indeed prove the following:

3.18. Proposition. *Let $T = \tau\mu$ be a normal k-current on $\mathbb{R}^n$ with compact support and let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map. Then for every continuous k-form ω on $\mathbb{R}^m$ there holds*

$$\langle f_\# T\,;\,\omega\rangle = \int_{\mathbb{R}^n} \big\langle d_\tau f(x)_\#\,\tau(x)\,;\,\omega(f(x))\big\rangle\, d\mu(x)\,. \tag{3.13}$$

Proof. The key point is that the push-forward $f_\# T$ is defined in such a way that the currents $(f_j)_\# T$ converge to $f_\# T$ weakly (or in the flat norm) for every sequence of smooth maps $f_j : \mathbb{R}^n \to \mathbb{R}^m$ that are uniformly Lipschitz and converge to f uniformly (see [11], §4.1.14).[14]

We now use Lemma 3.8 to choose the approximating maps f_j so that for μ-a.e. x the restrictions of the linear map $df_j(x)$ to $\mathrm{span}(\tau(x))$ converge to the linear map $d_\tau f(x)$, and in particular the k-vectors $df_j(x)_\#\tau(x)$ converge to $d_\tau f(x)_\#\tau(x)$.

Then for every smooth k-form ω with compact support on $\mathbb{R}^m$ we have [15]

$$\langle f_\# T\,;\,\omega\rangle = \lim_{j\to+\infty} \langle (f_j)_\# T\,;\,\omega\rangle$$

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[13] To every linear map L from the vector space V to the vector space W is associated a linear map from $\wedge_k(V)$ to $\wedge_k(W)$, denoted by $L_\#$, and characterized by the fact that $L_\#(v_1\wedge\cdots\wedge v_k) = Lv_1 \wedge\cdots\wedge Lv_k$ for every $v_1,\dots,v_k$ in V. The expression $df(x)_\#\,\tau(x)$ in formula (3.12) should be understood in this sense, and denotes a k-vector in $\mathbb{R}^m$.

[14] It is immediate that the currents $(f_j)_\# T$ and their boundaries have uniformly bounded mass and therefore converge up to subsequence to some limit normal current: the entire point about the definition of $f_\# T$ is exactly to show that this limit is unique and does not depend on the choice of the approximating maps f_j.

[15] The second equality follows from (3.12), the third one from Lebesgue's dominated convergence theorem using the domination $\big|\langle df_j(x)_\#\tau(x)\,;\,\omega(x)\rangle\big| \leq |df_j(x)|^k|\tau(x)| \leq L^k|\tau(x)|$ where L is a constant such that $\mathrm{Lip}(f_j) \leq L$ for every j (recall that τ belongs to $L^1(\mu)$).

$$= \lim_{j\to+\infty} \int_{\mathbb{R}^n} \big\langle df_j(x)_\# \tau(x)\,;\,\omega(x)\big\rangle\, d\mu(x)$$
$$= \int_{\mathbb{R}^n} \big\langle d_\tau f(x)_\# \tau(x)\,;\,\omega(x)\big\rangle\, d\mu(x)\,.$$

We have thus proved identity (3.13) for every ω which is smooth and compactly supported; and we extend it by density to every ω which is bounded and continuous. □

4. PROOF OF THEOREM 1.1(i)

To prove statement (i) of Theorem 1.1 we must show that, given a Lipschitz function f and a measure μ, the directional derivative $D_v f(x)$ exists for every v in $V(\mu,x)$ and μ-a.e. x, and it is linear in v.

While the existence of the directional derivative can be easily obtained in many ways, the linearity is more subtle. Our approach is based on a representation of the decomposability bundle $V(\mu,x)$ in terms of normal 1-currents given in Proposition 4.4. This result allows us to use formula (3.3) to characterize the directional derivatives of f, and the linearity of this formula yields the linearity of the directional derivatives.

In the rest of this section μ is a fixed measure on $\mathbb{R}^n$. We begin with a couple of definitions, then we give the characterization of the decomposability bundle in terms of normal 1-currents (Proposition 4.4) and finally we prove statement (i) of Theorem 1.1.

4.1. Cones and cone-null sets. Given a unit vector e in $\mathbb{R}^n$ and a real number $\alpha \in (0,\pi/2)$ we denote by $C(e,\alpha)$ the closed cone of axis e and angle α in $\mathbb{R}^n$, that is,

$$C(e,\alpha) := \big\{ v\in\mathbb{R}^n :\ v\cdot e \geq \cos\alpha\cdot|v| \big\}\,.$$

Given a cone $C = C(e,\alpha)$ in $\mathbb{R}^n$, we call *C-curve* any set of the form $\gamma(J)$ where J is a compact interval in $\mathbb{R}$ and $\gamma : J\to\mathbb{R}^n$ is a Lipschitz path such that

$$\dot\gamma(s) \in C \quad \text{for } \mathscr{L}^1\text{-a.e. } s\in J.$$

Following [4], we say that a set E in $\mathbb{R}^n$ is *C-null* if

$$\mathscr{H}^1\big(E\cap G\big) = 0 \quad \text{for every } C\text{-curve } G.$$

4.2. Remark. C-curves can be characterized as the sets $\gamma(J)$ where J is a compact interval in $\mathbb{R}$ and $\gamma : J\to\mathbb{R}^n$ satisfies

$$\mathrm{Lip}(\gamma)\leq 1 \text{ and } e\cdot\dot\gamma(s)\geq\cos\alpha \text{ for } \mathscr{L}^1\text{-a.e. } s\in J.$$

4.3. An auxiliary bundle. A 1-dimensional normal current T on $\mathbb{R}^n$ can be viewed as a finite measure with values in $\mathbb{R}^n$ (not necessarily abosolutely continuous w.r.t. μ), and therefore the Radon-Nikodým density of T w.r.t. μ, denoted by τ, is a vectorfield in $L^1(\mu;\mathbb{R}^n)$, and is characterized by the fact that T can be written as

$$T = \tau\mu + R$$

where R is a $\mathbb{R}^n$-valued measure singular w.r.t. μ.

We then denote by $\mathscr{F}'_\mu$ the set of all vectorfields $\tau \in L^1(\mu;\mathbb{R}^n)$ that can be obtained as the Radon-Nikodým density w.r.t. μ of some 1-dimensional normal current T, and we denote $x \mapsto V'(\mu,x)$ the μ-essential span of the family $\mathscr{F}'_\mu$ (see §2.2).[1]

4.4. Proposition. *Let $V'(\mu,x)$ be as above and let $V(\mu,x)$ be the decomposability bundle of μ. Then $V'(\mu,x) = V(\mu,x)$ for μ-a.e. x.*

The proof of this result relies on the following key lemma.

4.5. Lemma. *Let $C = C(e,\alpha)$ be a closed convex cone in $\mathbb{R}^n$ (cf. 4.1), let $\mathrm{Int}(C)$ be the interior of C, and let σ be a non-trivial measure on $\mathbb{R}^n$ which admits a decomposition $\sigma = \int_I \sigma_t \, dt$ as in §2.5, where each σ_t is the restriction of $\mathscr{H}^1$ to a 1-rectifiable set E_t such that the approximate tangent line $\mathrm{Tan}(E_t,x)$ is spanned by a vector in $\mathrm{Int}(C)$ for $\mathscr{H}^1$-a.e. $x \in E_t$.*

Then there exists a normal 1-current T which, when viewed as an $\mathbb{R}^n$-valued measure, takes values in C (and therefore its Radon-Nikodým density w.r.t. any positive measure takes values in C as well), and its total variation measure $|T|$ is not singular w.r.t. σ.

Proof. The idea of the proof is quite simple: for every t we choose a C-curve G_t such that $\mathscr{H}^1(E_t \cap G_t) > 0$, denote by T_t the 1-current associated to G_t, and then set $T := \int T_t \, dt$. The actual proof, however, is rather involved, one of the reasons being that the curves G_t should be chosen in a Borel measurable fashion w.r.t. t in order to give sense to the integral $\int T_t \, dt$.

We denote by λ the measure on the space I (denoted so far by dt), and define the following objects:

- $\mathscr{X}$ set of all paths $\gamma : J := [0,1] \to \mathbb{R}^n$ such that $\mathrm{Lip}(\gamma) \le 1$ and $\dot\gamma(s) \in C$ for $\mathscr{L}^1$-a.e. $s \in J$ (thus $\gamma(J)$ is a C-curve, cf. §4.1); we endow $\mathscr{X}$ with the supremum distance;
- I' set of all $t \in I$ such that $\mathscr{H}^1(E_t) > 0$, or equivalently $\sigma_t \neq 0$; note that I' is Borel because the map $t \mapsto \sigma_t$ is Borel (cf. Remark 2.6(i)), and $\lambda(I') > 0$ because σ is non-trivial.

The rest of the proof is divided in several steps.

Step 1. For every $t \in I'$ there exists $\gamma \in \mathscr{X}$ such that

$$\mathscr{H}^1(E_t \cap \gamma(J)) = \sigma_t(\gamma(J)) > 0 \,. \tag{4.1}$$

Since the set E_t is rectifiable and $\mathscr{H}^1(E_t) > 0$, we can find a curve G of class C^1 such that $\mathscr{H}^1(E_t \cap G) > 0$. Let then x_0 be a density point of $E_t \cap G$ in G such that $\mathrm{Tan}(E_t,x_0)$ agrees with $\mathrm{Tan}(G,x_0)$ and is spanned by a vector in $\mathrm{Int}(C)$. Then $\mathrm{Tan}(G,x)$ is spanned by a vector in $\mathrm{Int}(C)$ for all x in a subarc G' of G that

[1] To be precise, these Radon-Nikodým densities belong to $L^1(\mu;\mathbb{R}^n)$, and therefore are not vectorfields but μ-equivalence classes of vectorfields; note however that the μ-essential span is actually well-defined for families of μ-equivalence classes of vectorfields.

contains x_0, and $\mathscr{H}^1(E_t \cap G') > 0$. We thus take as γ a suitable parametrization of G'.

Step 2. The set F of all $(t,\gamma) \in I \times \mathscr{X}$ such that (4.1) holds is Borel.

Let $\mathscr{K}$ be space of compact subsets of $\mathbb{R}^n$ endowed with the Hausdorff distance, and let $\mathscr{M}^+$ be the space of finite, positive Borel measures on $\mathbb{R}^n$ endowed with the weak* topology. It is not difficult to show that $(\sigma, K) \mapsto \sigma(K)$ is an upper semicontinuous function on $\mathscr{M}^+ \times \mathscr{K}$,[2] and that $\gamma \mapsto \gamma(J)$ is a continuous map from $\mathscr{X}$ to $\mathscr{K}$. Therefore, recalling that $t \mapsto \sigma_t$ is a Borel map from I to $\mathscr{M}^+$ (recall §2.5), we have that the function $f : I \times \mathscr{X} \to \mathbb{R}$ defined by $f(t,\gamma) := \sigma_t(\gamma(J))$ is Borel, and then F, being the set where f is strictly positive, is Borel too.

Step 3. For every $t \in I'$ we can choose $\gamma_t \in \mathscr{X}$ so that (4.1) holds and $t \mapsto \gamma_t$ agrees with a Borel map on some Borel subset I'' of I' with $\lambda(I'') = \lambda(I')$.

The set F defined in Step 2 is a Borel subset of $I \times \mathscr{X}$, and by Step 1 its projection on I contains I'. Thus, by the von Neumann-Aumann measurable selection theorem (see for instance [25], Theorem 5.5.2), it is possible to choose $\gamma_t \in \mathscr{X}$ for every $t \in I'$ so that (t, γ_t) belongs to F (that is, γ_t satisfies (4.1)) and the map $t \mapsto \gamma_t$ is *universally measurable*. This implies that there exists a Borel set I'' contained in I' such that $\lambda(I' \setminus I'') = 0$ and the restriction of $t \mapsto \gamma_t$ to I'' is Borel.

Step 4. Construction of the normal 1-current T.

We let T be the integral (over $t \in I''$) of the 1-currents canonically associated to the paths γ_t,[3] that is,

$$\langle T \,;\, \omega \rangle := \int_{I''} \left[\int_J \langle \omega(\gamma_t(s)) \,;\, \dot\gamma_t(s) \rangle \, ds \right] dt \tag{4.2}$$

for every smooth 1-form ω on $\mathbb{R}^n$ with compact support.[4] A simple computation shows that

$$\langle \partial T \,;\, \varphi \rangle = \langle T \,;\, d\varphi \rangle = \int_{I''} \big[\varphi(\gamma_t(1)) - \varphi(\gamma_t(0)) \big] \, dt \tag{4.3}$$

for every smooth 0-form (or function) φ on $\mathbb{R}^n$ with compact support. It follows immediately from (4.2) and (4.3) that both T and ∂T have finite mass, and therefore T is normal.

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[2] For instance, one can use that $\sigma(K) = \inf_{\varepsilon > 0} \langle \sigma \,;\, \varphi_{K,\varepsilon} \rangle$, where $\varphi_{K,\varepsilon}(x)$ is the positive part of $1 - \mathrm{dist}(x,K)/\varepsilon$, and observe that for every $\varepsilon > 0$ the function $(\sigma, K) \mapsto \langle \sigma \,;\, \varphi_{K,\varepsilon} \rangle$ is continuous (because it is continuous in σ and uniformly Lipschitz in K).

[3] Namely the push-forward according to γ_t of the 1-current $[J, e, 1]$, where e is the standard orientation of $\mathbb{R}$ (cf. §3.3).

[4] Here and in the rest of this proof we write ds for $d\mathscr{L}^1(s)$. The integral in this formula is well-defined because $t \mapsto \gamma_t$ is a Borel map from I'' to $\mathscr{X}$ (Step 3), and then $t \mapsto \dot\gamma_t$ is a Borel map from I'' to $L^1(J; \mathbb{R}^n)$.

Step 5. The measure T takes values in the cone C.
Formula (4.2) implies that for every Borel set E in $\mathbb{R}^n$

$$T(E) = \int_{I''} \int_{\gamma_t^{-1}(E)} \dot\gamma_t(s)\, ds\, dt\,, \tag{4.4}$$

and since $\dot\gamma_t(s)$ belongs to the closed convex cone C, so does $T(E)$.

Step 6. The measures σ and $|T|$ are not mutually singular.
Since I'' is contained in I, for every Borel set E in $\mathbb{R}^n$ we have that

$$\sigma(E) = \int_I \sigma_t(E)\, dt \geq \int_{I''} \sigma_t(E)\, dt\,. \tag{4.5}$$

Let $m := \cos\alpha$; then for every Borel set E in $\mathbb{R}^n$ we have[5]

$$\begin{aligned} |T|(E) \geq T(E)\cdot e &= \int_{I''} \left[\int_{\gamma_t^{-1}(E)} \dot\gamma_t(s)\cdot e\, ds \right] dt \\ &\geq m \int_{I''} \left[\int_{\gamma_t^{-1}(E)} |\dot\gamma_t(s)|\, ds \right] dt \\ &\geq m \int_{I''} \mathscr{H}^1(\gamma_t(J)\cap E)\, dt = m \int_{I''} \sigma'_t(E)\, dt\,, \end{aligned} \tag{4.6}$$

where σ'_t is the restriction of $\mathscr{H}^1$ to $\gamma_t(J)$.

We argue now by contradiction, and assume that σ and $|T|$ are mutually singular. Then there exist disjoint Borel sets F, F' such that σ is supported on F and $|T|$ is supported on F'. Then (4.5) and (4.6) imply that for λ-a.e. $t \in I''$ the measures σ_t and σ'_t are supported on F and F', respectively, and therefore they are mutually singular. On the other hand σ_t and σ'_t are the restrictions of $\mathscr{H}^1$ to E_t and $\gamma_t(J)$ respectively, and since $\mathscr{H}^1(E_t \cap \gamma_t(J)) > 0$ by the choice of γ_t, these measures are never mutually singular. We thus have a contradiction (because $\lambda(I'') > 0$). □

Proof of Proposition 4.4. This proof is divided in two parts.

Part 1. $V'(\mu, x)$ is contained in $V(\mu, x)$ for μ-a.e. x.

By the definition of $V'(\mu, x)$ and μ-essential span (see §4.3 and §2.2) we must show that for every normal 1-current T with Radon-Nikodým density τ w.r.t. μ there holds $\tau(x) \in V(\mu, x)$ for μ-a.e. x.

Take indeed a measure μ_s singular w.r.t. μ such that T is absolutely continuous w.r.t. $\mu' := \mu + \mu_s$ and write $T = \tau'\mu'$; then

(i) $\tau(x) = \tau'(x)$ for μ-a.e. x by the uniqueness of Radon-Nikodým density;
(ii) $\tau'(x) \in V(\mu', x)$ for μ'-a.e. x by Theorem 3.5(i);
(iii) $V(\mu', x) = V(\mu, x)$ for μ-a.e. x by Proposition 2.9(i).

[5] The first equality follows from (4.4); the second inequality follows from the fact that $\dot\gamma_t(s)$ belongs to $C = C(e, \alpha)$ and that every $v \in C$ satisfies $v \cdot e \geq m|v|$; the last inequality follows from the area formula.

These statements together imply that $\tau(x) \in V(\mu, x)$ for μ-a.e. x.

Part 2. $V(\mu, x)$ is contained in $V'(\mu, x)$ for μ-a.e. x.

By the definition of $V(\mu, x)$ (see §2.7), we must show that given a measure μ' of the form $\mu' = \int_I \mu_t \, dt$ such that $\mu' \ll \mu$ and each μ_t is the restriction of $\mathscr{H}^1$ to a 1-rectifiable set E_t, then

$$\operatorname{Tan}(E_t, x) \subset V'(\mu, x) \quad \text{for } \mu_t\text{-a.e. } x \text{ and a.e. } t. \tag{4.7}$$

We argue by contradiction, and assume that (4.7) does not hold.

We will use the following objects:

- p projection from $I \times \mathbb{R}^n$ onto $\mathbb{R}^n$;
- μ'' measure on $I \times \mathbb{R}^n$ given by $\mu'' := \int_I (\delta_t \times \mu_t) \, dt$ where δ_t is the Dirac mass at t; thus the push-forward of μ'' according to p is μ';
- D set of all $(t, x) \in I \times \mathbb{R}^n$ where $\operatorname{Tan}(E_t, x)$ exists;
- $\mathscr{M}^+_{\text{loc}}$ space of locally finite, positive Borel measures on $\mathbb{R}^n$ endowed with the weak* topology (induced by the action on the space of continuous functions with compact support in $\mathbb{R}^n$);
- $\mathscr{K}$ set of all measure on $\mathbb{R}^n$ of the form $1_V \mathscr{H}^1$ with V a line in $\mathbb{R}^n$;
- $\sigma_{x,r}$ measure on $\mathbb{R}^n$ defined by $\sigma_{x,r}(E) := \frac{1}{r}\sigma\big(x + rE\big)$ for every measure σ on $\mathbb{R}^n$, $x \in \mathbb{R}^n$ and $r > 0$.
- $\mathscr{F}$ family of cones $C = C(e, \alpha)$ with e ranging in a given countable dense subset of the unit sphere in $\mathbb{R}^n$ and α ranging in a given countable dense subset of $(0, \pi/2)$.

The rest of the proof is divided in three steps.

Step 1. D is a Borel subset of $I \times \mathbb{R}$ and $(t, x) \mapsto \operatorname{Tan}(E_t, x)$ is a Borel map from D to $\operatorname{Gr}(\mathbb{R}^n)$.

We only sketch the proof of this claim. The key point is that a 1-rectifiable set E has approximate tangent line V at x if and only if the measures $\sigma_{x,r}$ associated to $\sigma := 1_E \mathscr{H}^1$ converge as $r \to 0$ to $1_V \mathscr{H}^1$ in $\mathscr{M}^+_{\text{loc}}$.

Since $(\sigma, x, r) \mapsto \sigma_{x,r}$ is a continuous map from $\mathscr{M}^+_{\text{loc}} \times \mathbb{R}^n \times (0, 1]$ to $\mathscr{M}^+_{\text{loc}}$, the set D' of all $(\sigma, x) \in \mathscr{M}^+_{\text{loc}} \times \mathbb{R}^n$ such that the limit $\sigma_x := \lim_{r \to 0} \sigma_{x,r}$ exists is Borel, and $(\sigma, x) \mapsto \sigma_x$ is a Borel map from D' to $\mathscr{M}^+_{\text{loc}}$. Thus the set D'' of all $(\sigma, x) \in D'$ such that σ_x belongs to the closed set $\mathscr{K}$ is also Borel.

This fact and the Borel measurability of $t \mapsto \sigma_t$ imply that the D is Borel.

To prove that $(t, x) \mapsto \operatorname{Tan}(E_t, x)$ is Borel one uses the Borel measurability of the map $(\sigma, x) \in D'' \mapsto \sigma_x \in \mathscr{K}$ and the continuity of the map that to each measure in $\mathscr{K}$ associates the supporting line in $\operatorname{Gr}(\mathbb{R}^n)$.

Step 2. There exists a cone $C = C(e, \alpha)$ and a subset F of D with $\mu''(F) > 0$ such that for every $(t, x) \in F$ there holds $V'(\mu, x) \cap C = \{0\}$, and $\operatorname{Tan}(E_t, x)$ is spanned by a vector in $\operatorname{Int}(C)$.

Let F' be the set of all $(t,x) \in D$ such that $\mathrm{Tan}(E_t,x)$ is not contained in $V'(\mu,x)$.[6] The assumption that (4.7) does not hold means that $\mu''(F') > 0$.

Now, for every $(t,x) \in F'$ there exists a cone C' that separates $\mathrm{Tan}(E_t,x)$ and $V'(\mu,x)$, in the sense that $\mathrm{Tan}(E_t,x)$ is spanned by a vector in $\mathrm{Int}(C')$, and $V'(\mu,x) \cap C' = \{0\}$.

Moreover we can take the separating cone C' in the family $\mathscr{F}$ defined above, and since $\mathscr{F}$ is countable, there exists at least one cone $C \in \mathscr{F}$ with the following property: the set F of all $(x,t) \in F'$ such that $\mathrm{Tan}(E_t,x)$ and $V'(\mu,x)$ are separated by C satisfies $\mu''(F) > 0$.[7]

Step 3. Completion of the proof.

We denote by σ the push-forward of the restriction of μ'' to F according to the projection p. Thus σ is non-trivial, $\sigma \ll \mu' \ll \mu$, and $\sigma = \int_I \sigma_t \, dt$ where each σ_t is the restriction of $\mathscr{H}^1$ to $E'_t := E_t \cap F_t$, and F_t is the set of all x such that $(t,x) \in F$.

Moreover $\mathrm{Tan}(E'_t,x)$, whenever it exists, agrees with $\mathrm{Tan}(E_t,x)$, and by the choice of F it is spanned by a vector in $\mathrm{Int}(C)$. Thus we can apply Lemma 4.5 to σ and obtain a normal 1-current T such that σ and $|T|$ are not mutually singular and the Radon-Nikodým density of T w.r.t. μ, denoted by τ, satisfies $\tau(x) \in C$ for μ-a.e. x.

Now, since $|T|$ and σ are not mutually singular, the set of all points x where $\tau(x) \neq 0$ is not σ-null, and recalling the definition of $V'(\mu,x)$ (cf. §4.3) we obtain that the set of all x where $V'(\mu,x)$ intersects $C \setminus \{0\}$ is not σ-null.

On the other hand, by Step 2 we have that $V'(\mu,x) \cap C = \{0\}$ for σ-a.e. x. We thus have a contradiction. □

Proof of Theorem 1.1(i). Fix the Lipschitz function f and let $\mathscr{F}'_\mu$ be the subspace of $L^1(\mu;\mathbb{R}^n)$ defined in §4.3.

Given a vectorfield $\tau \in \mathscr{F}'_\mu$, consider a normal 1-current T associated to τ, namely a current whose Radon-Nikodým density w.r.t. μ is τ. By applying Theorem 3.5(ii) and Proposition 3.6 to T and f, we obtain that the derivative of f in the direction τ exists for μ-a.e. x and satisfies

$$D_\tau f \, \mu = \partial(fT) - f \, \partial T \,. \tag{4.8}$$

Take now a countable subset $\{\tau_j : j \in J\}$ which is dense in $\mathscr{F}'_\mu$, and for every $j \in J$ let T_j be a current associated to τ_j as above. Take moreover a countable dense subset S of $\mathbb{R}$ that contains 0 and 1, and let D be the set of all linear combinations $\tau = \sum s_j \tau_j$ with $s_j \in S$ for every $j \in J$ and $s_j = 0$ for all j except finitely many.

Since D is countable there exists a μ-null set N such that the directional derivative $D_\tau f(x)$ exists for every $x \notin N$ and every $\tau \in D$. Moreover, given

[6] Using that the map $(t,x) \mapsto \mathrm{Tan}(E_t,x)$ is Borel (Step 1) it is not difficult to show that F' is Borel. The same argument apply to the set F below.

[7] The fact that F is Borel follows essentially from the Borel measurability of the set D and of the maps $(t,x) \mapsto \mathrm{Tan}(E_t,x)$ and $x \mapsto V'(\mu,x)$ (cf. Step 1).

$\tau \in D$ of the form $\tau = \sum s_j \tau_j$, if we set $T := \sum s_j T_j$ and use the linearity of the right-hand side of (4.8) we get

$$D_\tau f(x) = \sum_{j \in J} s_j \, D_{\tau_j} f(x) \tag{4.9}$$

for μ-a.e. $x \notin N$. And again, since D is countable, we can find a μ-null set N' such that (4.9) holds for every $x \notin N \cup N'$ and every $\tau \in D$.

Finally, since $\{\tau_j : j \in J\}$ is dense in $\mathscr{F}'_\mu$ and $V'(\mu, \cdot)$ is the essential span of $\mathscr{F}'_\mu$, it is easy to show that $\{\tau_j(x) : j \in J\}$ is dense in $V'(\mu, x)$ for μ-a.e. x. Since moreover $V'(\mu, x) = V(\mu, x)$ for μ-a.e. x (Proposition 4.4), there exists a μ-null set N'' such that $\{\tau_j(x) : j \in J\}$ is dense in $V(\mu, x)$ for all $x \notin N''$.

We conclude the proof by showing that f is differentiable w.r.t. $V(\mu, x)$ at every point x outside the μ-null set $N \cup N' \cup N''$.

Fix such a point x. Since $x \notin N''$, the vectors $\tau_j(x)$ are dense in $V(\mu, x)$, and therefore we can find a finite subset J' of J such that $\{\tau_j(x) : j \in J'\}$ is a basis of $V(\mu, x)$. Let then ω be the linear function on $V(\mu, x)$ defined by [8]

$$\langle \omega \, ; \, \tau_j(x) \rangle := D_{\tau_j} f(x) \quad \text{for every } j \in J'.$$

Consider now a vector v of the form

$$v = \sum_{j \in J'} s_j \, \tau_j(x) \quad \text{with } s_j \in S \text{ for every } j \in J'. \tag{4.10}$$

Then, setting $\tau := \sum_{j \in J'} s_j \tau_j \in D$, we get [9]

$$D_v f(x) = D_\tau f(x) = \sum_{j \in J'} s_j \, D_{\tau_j} f(x) = \sum_{j \in J'} s_j \langle \omega \, ; \, \tau_j(x) \rangle = \langle \omega \, ; \, v \rangle \, .$$

Finally we notice that the vectors v of the form (4.10) are dense in $V(\mu, x)$, and therefore we can apply Lemma 3.13 and obtain that f is differentiable at x w.r.t. $V(\mu, x)$ with derivative $d_V f(x) = \omega$. □

5. Proof of Theorem 1.1(ii)

Statement (ii) of Theorem 1.1 is an immediate consequence of a more precise statement proved in Theorem 5.4 below. To obtain this theorem we begin by proving a statement of similar nature under more restrictive assumptions (Proposition 5.2).

We begin this section with a few definitions, then we give the main statements, and then the proofs.

[8] The directional derivatives in this formula exist because $\tau_j \in D$ and $x \notin N$.

[9] The directional derivatives in this formula exist because $\tau, \tau_j \in D$ and $x \notin N$; the second equality follows from (4.9) and the fact that $x \notin N'$; the third equality follows from the definition of ω.

Through this section μ is a measure on $\mathbb{R}^n$. Given a function f on $\mathbb{R}^n$, a point $x \in \mathbb{R}^n$ and a vector $v \in \mathbb{R}^n$, we consider the upper and lower (one-sided) directional derivatives

$$D_v^+ f(x) := \limsup_{h\to 0^+} \frac{f(x+hv)-f(x)}{h} \,,$$
$$D_v^- f(x) := \liminf_{h\to 0^+} \frac{f(x+hv)-f(x)}{h} \,.$$

5.1. The set E and the space X. For the rest of this section E is a Borel set in $\mathbb{R}^n$ with the following property: there exist an integer d with $0 < d \le n$, and continuous vectorfields $e_1, \dots, e_n$ on $\mathbb{R}^n$ such that

- $e_1(x), \dots, e_n(x)$ form an orthonormal basis of $\mathbb{R}^n$ for every $x \in \mathbb{R}^n$;
- $e_1(x), \dots, e_d(x)$ span $V(\mu, x)^\perp$ for every $x \in E$.

We write D_j for the directional derivative D_{e_j}, and denote by X the space of all Lipschitz functions f on $\mathbb{R}^n$ such that

$$\big|D_j f(x)\big| \le 1 \quad \text{for } \mathscr{L}^n\text{-a.e. } x \text{ and every } j = 1, \dots, n,$$

endowed X with the supremum distance. It is then easy to show that X is a complete metric space.

5.2. Proposition. *Given a vector $v \in \mathbb{R}^n$, let N_v be the set of all $f \in X$ such that for μ-a.e. $x \in E$ there holds*

$$D_v^+ f(x) - D_v^- f(x) \ge \frac{\operatorname{dist}(v, V(\mu,x))}{3\sqrt{d}} \,. \tag{5.1}$$

Then N_v is residual in X, and in particular it is dense.[1]

5.3. Proposition. *Let N be the set of all $f \in X$ such that, for μ-a.e. $x \in E$, inequality (5.1) holds for every $v \in \mathbb{R}^n$. Then N is residual in X, and in particular it is dense in X.*

5.4. Theorem. *There exists a Lipschitz function f on $\mathbb{R}^n$ such that, for μ-a.e. $x \in \mathbb{R}^n$, there holds $D_v^+ f(x) - D_v^- f(x) > 0$ for every $v \notin V(\mu, x)$.*

5.5. Remark. (i) The function f in Theorem 5.4 is not differentiable at x in the direction v for every $v \notin V(\mu, x)$ and for μ-a.e. x; this proves statement (i) of Theorem 1.1.

(ii) In Propositions 5.2 and 5.3 the class of non-differentiable functions under consideration is proved to be residual, and not just nonempty; from this point of view both statements are stronger than Theorem 5.4, which we could not frame as a residuality result.

(iii) Note that $N_{cv} = N_v$ for every $v \in \mathbb{R}^n$ and every $c > 0$ (because both terms in inequality (5.1) are 1-homogeneous in v), and therefore it suffices to prove Proposition 5.2 under the additional assumption $|v| = 1$.

[1] Recall that a set in a topological space is residual if it contains the a countable intersection of open dense sets; by Baire Theorem a residual set in a complete metric space is dense.

Proposition 5.3 and Theorem 5.4 follow easily from Proposition 5.2, which is therefore the key result in the whole section. To prove Proposition 5.2 we follow a general strategy for the proof of residuality results devised by B. Kirchheim in [14]. The starting point is the following definition.

5.6. The operators $T^{\pm}_{\sigma,\sigma'}$ and U_σ. From now till the end of the proof of Lemma 5.11 we fix a vector v in $\mathbb{R}^n$ with $|v| = 1$, and for every $x \in \mathbb{R}^n$ we set

$$d_v(x) := \mathrm{dist}(v, V(\mu, x)) .$$

For every $\sigma > \sigma' \geq 0$, and every function $f : \mathbb{R}^n \to \mathbb{R}$ we consider the functions $T^{\pm}_{\sigma,\sigma'} f$ and $U_\sigma f$ defined as follows for every $x \in \mathbb{R}^n$:

$$\begin{aligned} T^{+}_{\sigma,\sigma'} f(x) &:= \sup_{\sigma' < h \leq \sigma} \frac{f(x+hv) - f(x)}{h} , \\ T^{-}_{\sigma,\sigma'} f(x) &:= \inf_{\sigma' < h \leq \sigma} \frac{f(x+hv) - f(x)}{h} , \\ U_\sigma f(x) &:= T^{+}_{\sigma,0} f(x) - T^{-}_{\sigma,0} f(x) . \end{aligned}$$

One readily checks that $T^{+}_{\sigma,0} f(x)$ and $T^{-}_{\sigma,0} f(x)$ are respectively increasing and decreasing in σ, and therefore $U_\sigma f(x)$ is increasing in σ. Moreover

$$D^{+}_v f(x) - D^{-}_v f(x) = \inf_{\sigma>0} \big(U_\sigma f(x)\big) = \inf_{m=1,2,\dots} \big(U_{1/m} f(x)\big) . \tag{5.2}$$

Finally we notice that $\frac{1}{h}(f(x+hv) - f(x))$ and $D_v f(x)$ (if it exists) are both smaller than $T^{+}_{\sigma,0} f(x)$ and larger than $T^{-}_{\sigma,0} f(x)$ if $h \leq \sigma$, which yields the following useful estimate:

$$U_\sigma f(x) \geq \left| D_v f(x) - \frac{f(x+hv) - f(x)}{h} \right| \quad \text{for every } 0 < h \leq \sigma. \tag{5.3}$$

Proof of Proposition 5.2. By Lemma 5.7 below U_σ is of Baire class 1 as a map from X to $L^1(\mu)$ for every $\sigma > 0$,[2] and by Lemma 5.11 we have

$$U_\sigma f(x) \geq \frac{d_v(x)}{3\sqrt{d}} \quad \text{for } \mu\text{-a.e. } x \in E \tag{5.4}$$

whenever f is a continuity point of U_σ. Since the points of continuity of a map of Baire class 1 are residual (see [12], Theorem 24.14), it follows that the class $N_{v,\sigma}$ of all $f \in X$ which satisfy (5.4) is residual in X. But then also the intersection of all $N_{v,1/m}$ with m a positive integer is residual, and (5.2) implies that this intersection agrees with N_v. □

5.7. Lemma. *For every $\sigma > \sigma' \geq 0$, the maps $T^{\pm}_{\sigma,\sigma'}$ take X into $L^1(\mu)$, are continuous for $\sigma' > 0$, and are of Baire class 1 for $\sigma' = 0$. Consequently U_σ is a map from X to $L^1(\mu)$ of Baire class 1.*

[2] For the definition of maps of Baire class 1 between two metrizable spaces see [12], Definition 24.1. We just recall here that this class contains (but in general does not agrees with) the class of all pointwise limit of a sequences of continuous maps.

Proof. The functions $T^+_{\sigma,\sigma'}f$ belong to $L^1(\mu)$ for every $\sigma > \sigma' \geq 0$ and every $f \in X$ because they are bounded, and more precisely

$$\left|T^+_{\sigma,\sigma'}f(x)\right| \leq \mathrm{Lip}(f) \quad \text{for every } x \in \mathbb{R}^n. \tag{5.5}$$

Concerning the continuity of $T^+_{\sigma,\sigma'}$, one readily checks that given $\sigma' > 0$ and $f, f' \in X$ there holds

$$\left|T^+_{\sigma,\sigma'}f'(x) - T^+_{\sigma,\sigma'}f(x)\right| \leq \frac{2}{\sigma'}\|f' - f\|_\infty \quad \text{for every } x \in \mathbb{R}^n,$$

and therefore

$$\left\|T^+_{\sigma,\sigma'}f' - T^+_{\sigma,\sigma'}f\right\|_{L^1(\mu)} \leq \frac{2}{\sigma'}\|\mu\|\,\|f' - f\|_\infty\,.$$

To prove that $T^+_{\sigma,0}$ is of of Baire class 1 it suffices to notice that it is the pointwise limit of the continuous maps $T^+_{\sigma,\sigma'}$ as $\sigma' \to 0$. Indeed, it follows from the definition that, as σ' tends to 0, $T^+_{\sigma,\sigma'}f(x)$ converges to $T^+_{\sigma,0}f(x)$ for every $f \in X$ and every $x \in \mathbb{R}^n$, and then $T^+_{\sigma,\sigma'}f$ converges to $T^+_{\sigma,0}f$ in $L^1(\mu)$ by the dominated convergence theorem (a domination is given by estimate (5.5)).

The rest of the statement can be proved in a similar way. □

5.8. Lemma. *Let ε, σ be positive real numbers, f a function in X, and E' a Borel subset of E. Then there exist a smooth function $f'' \in X$ and a compact set K contained in E' such that*

- (i) $\|f'' - f\|_\infty \leq 2\varepsilon$;
- (ii) $\mu(K) \geq \mu(E')/(4d)$;
- (iii) $U_\sigma f''(x) \geq d_v(x)/(3\sqrt{d})$ *for every* $x \in K$.

This is the key step in the proof of Proposition 5.2; for the proof we need the following results, the proofs of which are postponed to Section 6.

5.9. Proposition. *Let be given a Borel set F in $\mathbb{R}^n$ and a cone $C = C(e, \alpha)$ in $\mathbb{R}^n$ such that*

$$V(\mu, x) \cap C = \{0\} \quad \text{for } \mu\text{-a.e. } x \in F.$$

Then there exists a C-null set F' contained in F such that $\mu(F') = \mu(F)$.

5.10. Lemma. *Let be given a closed ball $B = B(\bar{x}, r)$ in $\mathbb{R}^n$, a cone $C = C(e, \alpha)$ in $\mathbb{R}^n$, and a C-null compact set K contained in B. Then for every $\varepsilon > 0$ and every $r' > r$ there exists a smooth function $g : \mathbb{R}^n \to \mathbb{R}$ such that*

- (i) $\|g\|_\infty \leq \varepsilon$ *and the support of g is contained in* $B' := B(\bar{x}, r')$;

and for every $x \in B'$,

- (ii) $-\varepsilon \leq D_e g(x) \leq 1$, *and* $D_e g(x) = 1$ *if* $x \in K \cap B$;
- (iii) $|d_W g(x)| \leq 2/\tan\alpha$ *where* $W := e^\perp$.[3]

Proof of Lemma 5.8. The idea is to take a smooth function f' close to f, and then modify it into a function f'' so to get $U_\sigma f''(x)$ large enough for sufficiently many $x \in E'$. This modification will be obtained by adding to f' a

[3] Here $d_V g(x)$ is the derivative of g at x w.r.t. V (see §2.1), and $|d_V g(x)|$ is its operator norm.

finite number of smooth "perturbations" with small supremum norms and small, disjoint supports, but with large derivatives in at least one direction.

Step 1. We take a smooth function f' on $\mathbb{R}^n$ such that

(a) $\|f' - f\|_\infty \le \varepsilon$;

(b) $\|D_j f'\|_\infty < 1$ *for* $j = 1, \dots, n$, *and in particular* f' *belongs to* X *(cf. §5.1).*

We take $s > 0$ such that $s\|f\|_\infty < \varepsilon$ and set

$$f' := (1-s) f * \rho_t$$

where $\rho_t(x) = t^{-n}\rho(x/t)$ and ρ is a mollifier with compact support.

Using that f is uniformly continuos one can easily check that f' converges uniformly to $(1-s)f$ as $t \to 0$, then $\|f' - f\|_\infty$ converges to $s\|f\|_\infty < \varepsilon$, which implies that (a) holds for t small enough.

Since the vectorfield e_j that defines the partial derivative D_j is continuous, it is not difficult to show that $\|D_j f'\|_\infty$ converges as $t \to 0$ to $(1-s)\|D_j f\|_\infty < 1$ (recall that $\|D_j f\|_\infty \le 1$ because $f \in X$) and therefore also (b) holds for t small enough.

Step 2. Construction of the set E'_k.

For every integer k with $1 < k \le d$ we set

$$E'_k := \left\{ x \in E' \, : \, |v \cdot e_k(x)| \ge \frac{d_v(x)}{\sqrt{d}} \right\}. \tag{5.6}$$

Now, for every $x \in E$ we have that $V(\mu, x)^\perp$ is spanned by the orthonormal basis $e_1(x), \dots, e_d(x)$ (see §5.1) and therefore

$$d_v(x) = \mathrm{dist}(v, V(\mu, x)) = \left[\sum_{k=1}^{d} (v \cdot e_k(x))^2 \right]^{1/2} \le \sqrt{d} \sup_{1 \le k \le d} |v \cdot e_k(x)| \, .$$

This implies that every $x \in E'$ must belong to E'_k for at least one k, that is, the sets E'_k cover E'. In particular there exists at least one value of k such that

$$\mu(E'_k) \ge \frac{\mu(E')}{d} \, . \tag{5.7}$$

For the rest of the proof k is assigned this specific value.

For the next four steps we fix a point $\bar{x} \in E'_k$ and positive number r, r' such that

$$0 < r < \sigma/3 \, , \quad r < r' \le 2r \, . \tag{5.8}$$

Step 3. Construction of the sets $E_{\bar{x},r}$.

Let $\alpha(\bar{x}, r)$ be the supremum of the angle between $V(\mu, x)$ and $V(\mu, \bar{x})$ as x varies in $E \cap B(\bar{x}, r)$.[4] Since $V(\mu, x)$ is continuous in $x \in E$ (cf. §5.1), we have that

$$\alpha(\bar{x}, r) \to 0 \quad \text{as } r \to 0. \tag{5.9}$$

[4] The angle between two d-dimensional planes V and V' in $\mathbb{R}^n$ is $\arcsin(d_{\mathrm{gr}}(V, V'))$, where $d_{\mathrm{gr}}(V, V')$ is defined in §2.2.

Moreover the cone

$$C(\bar{x}, r) := C\big(e_k(\bar{x}), \pi/2 - 2\,\alpha(\bar{x}, r)\big)$$

satisfies

$$C(\bar{x}, r) \cap V(\mu, x) = \{0\} \quad \text{for all } x \in E \cap B(\bar{x}, r),$$

and since E'_k is contained in E, we can apply Proposition 5.9 to the set $F := E'_k \cap B(\bar{x}, r)$ and find a $C(\bar{x}, r)$-null set F' contained in F that $\mu(F') = \mu(F)$. Then we can take a compact set $K_{\bar{x},r}$ contained in F' such that

$$\mu(K_{\bar{x},r}) \geq \frac{1}{2}\mu(F') = \frac{1}{2}\mu(E'_k \cap B(\bar{x}, r))\,. \tag{5.10}$$

Note that $K_{\bar{x},r}$ is $C(\bar{x}, r)$-null because it is contained in F'.

Step 4. Construction of the perturbations $\bar{g}_{\bar{x},r,r'}$.
We set

$$\varepsilon' := \min\big\{\varepsilon,\, r(r' - r),\, 1 - \|D_j f'\|_\infty \text{ with } j = 1, \dots, n\big\}$$

Since $K_{\bar{x},r}$ is $C(\bar{x}, r)$-null and $\varepsilon' > 0$,[5] we can use Lemma 5.10 to find a function $g_{\bar{x},r,r'}$ such that

(c) $\|g_{\bar{x},r,r'}\|_\infty \leq \varepsilon'$ and the support of $g_{\bar{x},r,r'}$ is contained in $B(\bar{x}, r')$;

and for every $x \in B(\bar{x}, r')$,

(d) $-\varepsilon' \leq D_e g_{\bar{x},r,r'}(x) \leq 1$ and $D_e g_{\bar{x},r,r'}(x) = 1$ if $x \in K_{\bar{x},r}$, where $e := e_k(\bar{x})$;
(e) $|d_W g_{\bar{x},r,r'}(x)| \leq 2\tan\big(2\,\alpha(\bar{x}, r)\big)$ where $W := e^\perp = e_k(\bar{x})^\perp$.

Finally we set

$$\bar{g}_{\bar{x},r,r'} := \pm\frac{1}{2} g_{\bar{x},r,r'}$$

where $\pm$ means $+$ if $D_k f'(\bar{x}) \leq 0$ and $-$ otherwise.

Step 5. There exists $r_0 = r_0(\bar{x}) > 0$ such that for $r < r_0$ there holds

$$U_\sigma(f' + \bar{g}_{\bar{x},r,r'})(x) \geq \frac{d_v(x)}{3\sqrt{d}} \quad \textit{for every } x \in K_{\bar{x},r}. \tag{5.11}$$

In the following, given a quantity m depending on $\bar{x}, r, r'$ and $x \in B(\bar{x}, r)$, write $m = o(1)$ to mean that, for every $\bar{x}$, m tends to 0 as $r \to 0$, uniformly in all remaining variables.[6] To simplify the notation, we write g and $\bar{g}$ for $g_{\bar{x},r,r'}$ and $\bar{g}_{\bar{x},r,r'}$.

For every $x \in K_{\bar{x},r} \subset B(\bar{x}, r)$ we take $h = h(x) > 0$ such that $x + hv$ belongs to $\partial B(\bar{x}, r')$. Then, taking into account that $|v| = 1$ and (5.8), we have

$$r' - r \leq h \leq r + r' \leq 3r \leq \sigma\,.$$

[5] The number ε' is strictly positive because $r' > r$ and because of statement (b) in Step 1.
[6] In other words, for every $\bar{x}$ and every $\varepsilon > 0$ there exists $\bar{r} > 0$ such that $|m| \leq \varepsilon$ if $r \leq \bar{r}$.

We can then apply estimate (5.3) to the function $f'' := f' + \bar g$, and taking into account that $\bar g = \pm\frac{1}{2} g$ and $g(x+hv) = 0$,[7] we get

$$\begin{aligned} U_\sigma f''(x) &\geq \Big| D_v f''(x) - \frac{f''(x+hv) - f''(x)}{h} \Big| \\ &= \Big| D_v \bar g(x) + D_v f'(x) - \frac{f'(x+hv) - f'(x)}{h} + \frac{\bar g(x)}{h} \Big| \\ &\geq \frac{1}{2} \big| D_v g(x) \big| - \Big| D_v f'(x) - \frac{f'(x+hv) - f'(x)}{h} \Big| - \frac{|g(x)|}{2h} . \end{aligned} \tag{5.12}$$

Since f' is of class C^1, we clearly have

$$\Big| D_v f'(x) - \frac{f'(x+hv) - f'(x)}{h} \Big| = o(1) . \tag{5.13}$$

Using statement (c) in Step 4, the inequality $r' - r < h$ given above, and the choice of ε', we get

$$\frac{|g(x)|}{h} \leq \frac{\varepsilon'}{r' - r} \leq r = o(1) . \tag{5.14}$$

Finally, to estimate $|D_v g(x)|$ we decompose v as $v = (v\cdot e)e + w$ with $e := e_k(\bar x)$ and $w \in W := e^\perp$. Then

$$D_v g(x) = (v \cdot e)\, D_e g(x) + \langle d_W g(x)\, ;\, w\rangle$$

and therefore[8]

$$\begin{aligned} |D_v g(x)| &\geq |v\cdot e|\, |D_e g(x)| - |d_W g(x)| \\ &\geq |v \cdot e| - 2\tan\big(2\,\bar\alpha(\bar x, r)\big) \\ &\geq |v\cdot e_k(x)| - |e_k(x) - e_k(\bar x)| - 2\tan\big(2\,\bar\alpha(\bar x, r)\big) \\ &\geq |v\cdot e_k(x)| - o(1) \geq d_v(x)/\sqrt{d} - o(1) . \end{aligned} \tag{5.15}$$

Putting estimates (5.12), (5.13), (5.14), (5.15) together we get

$$U_\sigma(f' + \bar g)(x) = U_\sigma f''(x) \geq \frac{d_v(x)}{2\sqrt{d}} - o(1)$$

which clearly implies the claim in Step 5.

Step 6. There exists $r_1 = r_1(\bar x) > 0$ such that $f' + \bar g_{\bar x, r, r'} \in X$ if $r < r_1$.

Since $\bar g$ is supported in $B(\bar x, r')$ and f' belongs to X (Step 1), to prove that $f' + \bar g$ belongs to X it suffices to show that

$$\big| D_j(f' + \bar g_{\bar x, r, r'})(x) \big| \leq 1 \quad \textit{for every } x \in B(\bar x, r') \textit{ and } j = 1, \dots, n. \tag{5.16}$$

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[7] The support of g is contained in $B(\bar x, r')$ by statement (c) in Step 4.

[8] The second inequality follows from statements (d) and (e) in Step 4 and the fact that $x \in K_{\bar x, r}$; for the third one we used that $|v| = 1$ and $e = e_k(\bar x)$; the fourth follows from (5.9) and the fact $e_k(x)$ is continuous in x, and the last inequality follows from (5.6) and the fact that $x \in K_{\bar x, r} \subset E'_k$.

We begin with the case $j = k$. Recalling the definition of D_k (in §5.1) and the identities $\bar{g} = \pm\frac{1}{2}g$, $e = e_k(\bar{x})$, we obtain

$$D_k\bar{g}(x) = D_e\bar{g}(x) + \langle d\bar{g}(x)\,;\, e(x) - e\rangle = \pm\frac{1}{2}D_e g(x) + o(1)\,,$$
$$D_k f'(x) = D_k f'(\bar{x}) + \big(D_k f'(x) - D_k f'(\bar{x})\big) = D_k f'(\bar{x}) + o(1)\,,$$

and therefore

$$\big|D_k(f' + \bar{g})(x)\big| = \Big|D_k f'(\bar{x}) \pm \frac{1}{2}D_e g(x)\Big| + o(1)\,. \tag{5.17}$$

Recall now that $-\varepsilon' \leq D_e g(x) \leq 1$ (statement (d) above), that the sign $\pm$ means $+$ when $D_k f'(\bar{x}) \leq 0$ and $-$ otherwise, and that we chose ε' so that $\|D_k f'\|_\infty \leq 1 - \varepsilon'$. Using these facts we get the estimate

$$\Big|D_k f'(\bar{x}) \pm \frac{1}{2}D_e g(x)\Big| \leq 1 - \varepsilon'/2\,,$$

which, together with (5.17), clearly implies that (5.16) holds for r small enough.

To prove (5.16) for $j \neq k$ is actually simpler: recall indeed that $\|D_j f'\|_\infty < 1$ (statement (b) above) and note that [9]

$$\begin{aligned}|D_j\bar{g}(x)| &\leq \big|\langle d\bar{g}(x)\,;\, e_j(\bar{x})\rangle\big| + \big|\langle d\bar{g}(x)\,;\, e_j(x) - e_j(\bar{x})\rangle\big| \\ &\leq \tan\big(2\,\alpha(\bar{x}, r)\big) + |d\bar{g}(x)|\,|e_j(x) - e_j(\bar{x})| = o(1)\,.\end{aligned}$$

Step 7. Construction of the function f'' and the set K.

We consider the family $\mathscr{G}$ of all closed balls $B(\bar{x}, r)$ with $\bar{x} \in E'_k$ and $r > 0$ such that the conclusions of Step 5 and Step 6 hold (that is, r smaller than $r_0(\bar{x})$ and $r_1(\bar{x})$). By a standard corollary of Besicovitch covering theorem (see for example [15], Proposition 4.2.13) we can extract from $\mathscr{G}$ finitely many disjoint balls $B_i = B(\bar{x}_i, r_i)$ such that

$$\sum_i \mu(E'_k \cap B_i) \geq \frac{1}{2}\mu(E'_k)\,. \tag{5.18}$$

Since the balls $B_i = B(\bar{x}_i, r_i)$ are closed and disjoint, for every i we can find $r'_i > r_i$ such that the enlarged balls $B'_i := B(\bar{x}_i, r'_i)$ are still disjoint. Finally, for every i we set $\bar{g}_i := \bar{g}_{\bar{x}_i, r_i, r'_i}$, $K_i := K_{\bar{x}_i, r_i}$, and

$$f'' := f' + \sum_i \bar{g}_i\,, \quad K := \bigcup_i K_i\,.$$

We now check that f'' and K satisfy all requirements.

The function f'' is smooth because so are f' and $\bar{g}_i$, and the set K is compact because so are the sets K_i.

Note that the supports of the functions $\bar{g}_i$ are disjoint (because they are contained in the balls B'_i), and therefore at every point $x \in \mathbb{R}^n$ the derivative of f'' agrees either with the derivative of f' or with that of $f' + \bar{g}_i$ for some i. Recalling

[9] For the second inequality we use statement (e) in Step 4 and the fact that $\bar{g} = \pm\frac{1}{2}g$.

the definition of X in §5.1, we then deduce that f'' belongs to X by the fact that f' belongs to X (Step 1) and $f' + \bar{g}_i$ belongs to X for every i (Step 6).

Statement (i), namely that $\|f'' - f\| \le 2\varepsilon$, follows from statements (a) in Step 1 and (c) in Step 4, and the fact that the functions g_i have disjoint supports.

Statement (ii), namely that $\mu(K) \ge \mu(E')/(4d)$, follows from estimates (5.10), (5.18), and (5.7).

Consider now $x \in K_i$ for some i. By Step 5, $U_\sigma(f' + \bar{g}_i)(x) \ge d_v(x)/(3\sqrt{d})$. Moreover the proof of this estimates involves only the restriction of $f' + \bar{g}_i$ to the ball B'_i, where $f' + \bar{g}_i$ agrees with f''. Thus the same estimates holds for $U_\sigma f''(x)$ as well, which means that statement (iii) holds. □

5.11. Lemma. *Take $f \in X$ and $\sigma > 0$. If U_σ is continuous at f (as a map from X to $L^1(\mu)$ then (5.4) holds.*

Proof. We assume that (5.4) fails and prove that U_σ is not continuous at f. Indeed, if (5.4) does not hold, we can find a set E' contained in E with $\mu(E') > 0$ and $\delta > 0$ such that

$$U_\sigma f(x) \le \frac{d_v(x)}{3\sqrt{d}} - \delta \quad \text{for every } x \in E'.$$

Then we use Lemma 5.8 to construct a sequence of smooth functions $f_h \in X$ and of compact sets K_h contained in E' such that $f_h \to f$ uniformly as $h \to +\infty$, and for every h there holds $\mu(K_h) \ge \mu(E')/(4d)$ and

$$U_\sigma f_h(x) \ge \frac{d_v(x)}{3\sqrt{d}} \quad \text{for every } x \in K_h.$$

Thus $U_\sigma f_h$ does not converge to $U_\sigma f$ in the $L^1(\mu)$-norm, and more precisely

$$\big\|U_\sigma f_h - U_\sigma f\big\|_{L^1(\mu)} \ge \int_{K_h} \big|U_\sigma f_h - U_\sigma f\big|\, d\mu \ge \delta\mu(K_h) \ge \frac{\delta}{4d}\mu(E')\,. \qquad \square$$

Proof of Proposition 5.3. Let D be a countable dense subset of $\mathbb{R}^n$, and let N' be the intersection of all sets N_v defined in Proposition 5.2 with $v \in D$. By Proposition 5.2 the sets N_v are residual in X, and then also N' is residual.

Let now be given $f \in N'$. One readily checks that for μ-a.e. $x \in E$ inequality (5.1) holds for every $v \in D$, and we deduce that it actually holds for every $v \in \mathbb{R}^n$ using the fact that both sides of (5.1) are continuous in v (and D is dense in R^n); notice indeed that the directional upper and lower derivatives $D_v^\pm f(x)$ are Lipschitz in v (with the same Lipschitz constant as f).

We have thus proved that f belongs to N, thus N contains N', and therefore it is residual. □

For the proof of Theorem 5.4 we need the following lemma, the proof of which is postponed to Section 6.

5.12. Lemma. *Let f be a Lipschitz function on $\mathbb{R}^n$, K a compact set in $\mathbb{R}^n$, and ϕ an increasing, strictly positive function on $(0, +\infty)$. Then for every $\varepsilon > 0$ there exists a Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$ such that*

(i) *g agrees with f on K and is smooth in $\mathbb{R}^n \setminus K$;*
(ii) *$|g(x) - f(x)| \le \phi(\mathrm{dist}(x, K))$ for every $x \in \mathbb{R}^n$;*
(iii) *$\mathrm{Lip}(g) \le \mathrm{Lip}(f) + \varepsilon$.*

Proof of Theorem 5.4. The idea is simple: we cover $\mathbb{R}^n$ with a countable family of pairwise disjoint sets E_i which satisfy the assumption in §5.1, then we use Proposition 5.3 to find functions f_i which satisfy (5.1) for every v and μ-a.e. $x \in E_i$, and we regularize these functions out of the set E_i using Lemma 5.12; finally we take as f a weighted sum of these modified functions.

For every $x \in \mathbb{R}^n$ let $d(x)$ be the dimension of $V(\mu, x)^\perp$, and let F_0 be the set of all x such that $d(x) > 0$.

Step 1. For every (Borel) set F contained in F_0 with $\mu(F) > 0$ there exists a compact set $E \subset F$ with $\mu(E) > 0$ which satisfies the assumption in §5.1.

We use the von Neumann-Aumann measurable selection theorem (see [25], Theorem 5.5.2) to choose $e_1, \dots, e_n$ Borel vectorfields on $\mathbb{R}^n$ and F' is a Borel set that covers μ-almost all of F in such a way that

(a) $e_1(x), \dots, e_n(x)$ form an orthonormal basis of $\mathbb{R}^n$ for every $x \in \mathbb{R}^n$;
(b) $e_1(x), \dots, e_{d(x)}(x)$ span $V(\mu, x)^\perp$ for every $x \in F'$.

Then we use Lusin's theorem to find a compact set $E \subset F'$ with $\mu(E) > 0$ such that the restrictions of the function d and the vectorfields e_j to E are continuous; thus d is locally constant on E, and possibly replacing E with a smaller subset we can further assume that d is constant on E and that the restrictions of each e_j to E takes values in the ball $B_j := B(e_j(\bar{x}), \delta)$ for some $\bar{x} \in E$ and some (small) $\delta > 0$.

To conclude the proof we modify the vectorfields e_j in the complement of E so to they become continuous on the whole $\mathbb{R}^n$ and still satisfy assumption (a) above. This last step is achieved by first extending the restriction of each e_j to E to a continuous map from $\mathbb{R}^n$ to B_j (using Tietze extension theorem) and then applying the Gram-Schmidt orthonormalization process to the resulting vectorfields (note that if δ is small enough these vectorfields are linearly independent at every point).

Step 2. There exist a countable collection (E_i) of pairwise disjoint compact sets that satisfy the assumption in §5.1 and $\mu(E_i) > 0$, and their union contains μ-a.e. point.

Let $\mathscr{G}$ be the class of all countable collections (E_i) that satisfy all requirements with the possible exception of the last one (the union contains μ-a.e. point). The class $\mathscr{G}$ is nonempty and admits an element which is maximal with respect to inclusion. Using Step 1 it is easy to prove that this maximal element must also satisfy the last requirement.

Step 3. Construction of the functions f_i and g_i.

We take (E_i) as in Step 2, and for every i we use Proposition 5.3 to find a Lipschitz function f_i with $\mathrm{Lip}(f_i) \le 1$ such that for μ-a.e. $x \in E_i$

$$D_v^+ f_i(x) - D_v^- f_i(x) > 0 \quad \text{for every } v \notin V(\mu, x). \tag{5.19}$$

Next we apply Lemma 5.12 to each f_i to find a Lipschitz function g_i with $\mathrm{Lip}(g_i) \le 2$ which agrees with f_i on E_i, is smooth on $\mathbb{R}^n \setminus E_i$, and satisfies

$$|g_i(x) - f_i(x)| \le \big(\mathrm{dist}(x, E_i)\big)^2 \quad \text{for every } x \in \mathbb{R}^n.$$

This implies in particular that for every $x \in E_i$ and every $v \in \mathbb{R}^n$ there holds

$$g(x + hv) = f(x + hv) + O(|h|^2) \quad \text{for every } h \in \mathbb{R},$$

which yields $D_v^\pm g(x) = D_v^\pm f(x)$; then (5.19) implies that for μ-a.e. $x \in E_i$

$$D_v^+ g_i(x) - D_v^- g_i(x) > 0 \quad \text{for every } v \notin V(\mu, x). \tag{5.20}$$

Step 4. Construction of the function f.
We finally set

$$f(x) := \sum_i \frac{g_i(x)}{2^i} \quad \text{for every } x \in \mathbb{R}^n.$$

The function f is clearly Lipschitz with $\mathrm{Lip}(f) \le 4$ (because $\mathrm{Lip}(g_i) \le 2$ for every i), and we claim that for μ-a.e. x there holds $D_v^+ f_i(x) - D_v^- f_i(x) > 0$ for every $v \notin V(\mu, x)$. Taking into account (5.20) and the fact that the union of the sets E_i contains μ-a.e. x, it suffices to prove that for every i and every $x \in E_i$ the function

$$\hat{g}_i := \sum_{j \ne i} \frac{g_j(x)}{2^j}$$

is differentiable at x. If the sum is finite this is an immediate consequence of the fact that the functions g_j are differentiable at x for every $j \ne i$.[10] If the sum is infinite, this is an immediate consequence of the fact that for every $\delta > 0$ the function $\hat{g}_i$ can be written as $\hat{g}_i = g_i' + g_i''$ where g_i' differentiable at x and $\mathrm{Lip}(g_i'') \le \delta$.[11] □

6. Appendix: proofs of technical results

In this appendix we collect the proofs of several technical results used in the previous sections. We begin with the proof of Proposition 3.10.

Proof of Proposition 3.10. Statement (i) is immediate, while statements (ii) and (iii) are consequence of the following general facts, respectively: if $\dim(W) < k$ then every k-vector in W is null, and if $\dim(W) = k$ then every k-vector in W is simple.

To prove statement (iv), we denote by W the linear subspace of V generated by $v_1, \dots, v_k$. Clearly $\mathrm{span}(v)$ is contained in W; moreover $\mathrm{span}(v)$ has dimension at least k by statement (ii) while W has dimension at most k; therefore $\mathrm{span}(v)$ and W agree and have both dimension k.

[10] Recall that g_j is smooth out of the set E_j, which does not intersect E_i, and then $x \notin E_j$.

[11] Let g_i' and g_i'' be the sums of g_j over all $j \ne i$ such that $j \le j_0$ and $j > j_0$, respectively. For j_0 large enough there holds $\mathrm{Lip}(g_i'') \le \sum_{j > j_0} \mathrm{Lip}(g_i)/2^j \le \delta$.

To prove statement (v), we need some additional notation. Let $n := \dim(V)$, let $\{e_i : i = 1, \dots, n\}$ be a basis of V, and let $\{e_i^* : i = 1, \dots, n\}$ be the corresponding dual basis (see footnote 11 in Section 3). For every integer k with $0 < k \le n$ we denote by $I(n,k)$ the set of all multi-indexes $\mathbf{i} = (i_1, \dots, i_k)$ whose coordinates are integers and satisfy $1 \le i_1 < i_2 < \cdots < i_k \le n$, and for every such $\mathbf{i}$ we set $e_{\mathbf{i}} := e_{i_1} \wedge \cdots \wedge e_{i_k}$ and $e_{\mathbf{i}}^* := e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*$.[1]

It is then easy to check that for every $\mathbf{i} \in I(n,k)$ and $\mathbf{j} \in I(n,k-1)$ there holds

$$e_{\mathbf{i}} \llcorner e_{\mathbf{j}}^* = \begin{cases} (-1)^{h-1} e_{i_h} & \text{if } \mathbf{j} = (i_1, \dots, i_{h-1}, i_{h+1}, \dots, i_k) \\ & \text{with } h = 1, \dots, k, \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

Let now W denote the set of all $v \llcorner \alpha$ with $\alpha \in \wedge^{k-1}(V)$. The proof of statement (v), namely that $\mathrm{span}(v) = W$, is divided in two steps.

Step 1. W is contained in $\mathrm{span}(v)$.

We must show that $\tau \llcorner \alpha \in \mathrm{span}(v)$ for ever y $\alpha \in \wedge^{k-1}(V)$.

We let $k' := \dim(\mathrm{span}(v))$ and choose the basis $\{e_i : i = 1, \dots, n\}$ of V so that $\{e_i : i = 1, \dots, k'\}$ is a basis of $\mathrm{span}(v)$. By definition τ is a k-vector in $\mathrm{span}(v)$, and therefore it can be written as a linear combination of the $e_{\mathbf{i}}$ with $\mathbf{i} \in I(n,k)$ such that $i_k \le k'$, while α can be written as a linear combination of $e_{\mathbf{j}}^*$ with $\mathbf{j} \in I(n,k-1)$.

Thus $\tau \llcorner \alpha$ is a linear combination of the vectors $e_{\mathbf{i}} \llcorner e_{\mathbf{j}}^*$ with $\mathbf{i}, \mathbf{j}$ as above, and therefore it suffices to prove that every such vector belongs to $\mathrm{span}(v)$. To this end, notice that (6.1) implies that $e_{\mathbf{i}} \llcorner e_{\mathbf{j}}^*$ is either 0 or e_i for some $i \le i_k$, and therefore it belongs $\mathrm{span}(v)$ whenever $i_k \le k'$.

Step 2. $\mathrm{span}(v)$ *is contained in W, that is, v is a k-vector in W.*

We let now $k' := \dim(W)$ and choose the basis $\{e_i : i = 1, \dots, n\}$ of V so that $\{e_i : i = 1, \dots, k'\}$ is a basis of W. We must show that v is a linear combination of the $e_{\mathbf{i}}$ with $\mathbf{i} \in I(n,k)$ such that $i_k \le k'$, or equivalently that $\langle v\,;\, e_{\mathbf{j}}^* \rangle = 0$ for all $\mathbf{j} \in I(n,k)$ such that $j_k > k'$.

Observe now that each of these $e_{\mathbf{j}}^*$ can be written as $e_{\mathbf{j}'}^* \wedge e_j^*$ with $\mathbf{j}' \in I(n,k-1)$ and $j > k'$. Therefore

$$\langle v\,;\, e_{\mathbf{j}}^* \rangle = \langle v\,;\, e_{\mathbf{j}'}^* \wedge e_j^* \rangle = \langle v \llcorner e_{\mathbf{j}'}^*\,;\, e_j^* \rangle = 0$$

because $v \llcorner e_{\mathbf{j}'}^*$ belongs by definition to W, and $\langle w\,;\, e_j^* \rangle = 0$ for every $w \in W$ and every $j \ge k'$ by the choice of the basis. □

Next we prove Lemma 2.11 and Proposition 5.9. Both are obtained starting from Rainwater's lemma below.

6.1. Rainwater's Lemma. (See [21] or [22], Lemma 9.4.3). *Let X be a compact metric space, $\mathscr{F}$ a family of probability measures on X which is convex*

[1] Recall that $\{e_{\mathbf{i}} : \mathbf{i} \in I(n,k)\}$ is a basis of $\wedge_k(V)$ while $\{e_{\mathbf{i}}^* : \mathbf{i} \in I(n,k)\}$ is the corresponding dual basis of $\wedge^k(V)$.

and weak compact, and μ a measure on X which is singular with respect to every $\lambda \in \mathscr{F}$. Then μ is supported on a Borel set E which is λ-null for every $\lambda \in \mathscr{F}$.*

For our purposes we need the following variant of Rainwater's lemma:

6.2. Corollary. *Let X be a compact metric space and $\mathscr{F}$ a weak* compact family of probability measures on X. Then for every measure μ on X one of the following (mutually incompatible) alternatives holds:*

(i) *μ is supported on a Borel set E which is λ-null for every $\lambda \in \mathscr{F}$;*

(ii) *there exists a probability measure σ supported on $\mathscr{F}$ and a Borel set E such that the measure*

$$\int_{\lambda \in \mathscr{F}} (1_E \, \lambda) \, d\sigma(\lambda)$$

is nontrivial and absolutely continuous w.r.t. μ.[2]

Proof. We denote by $P(\mathscr{F})$ the space of probability measures on the compact space $\mathscr{F}$, and for every $\sigma \in P(\mathscr{F})$ we denote by $[\sigma]$ the corresponding average of the elements of $\mathscr{F}$, that is, the measure on X given by

$$[\sigma] := \int_{\lambda \in \mathscr{F}} \lambda \, d\sigma(\lambda) \, .$$

We claim that the class $\mathscr{F}'$ of all $[\sigma]$ with $\sigma \in P(\mathscr{F})$ is convex and compact (w.r.t. the weak* topology of measures on X). Convexity is indeed obvious, and compactness follows from the compactness of the space $P(\mathscr{F})$ (endowed with the weak* topology of measures on $\mathscr{F}$) and the continuity of the map $\sigma \mapsto [\sigma]$, which in turn follows from the identity $\langle [\sigma] \, ; \varphi \rangle = \langle \sigma \, ; \hat{\varphi} \rangle$ where φ is any continuous function on X and $\hat{\varphi}$ is the continuous function on $\mathscr{F}$ defined by $\hat{\varphi}(\lambda) := \langle \lambda \, ; \varphi \rangle$.

There are now two possibilities: either μ is singular with respect to all measures in $\mathscr{F}'$ or not.

In the first case Theorem 6.1 implies that μ is supported on a set E which is null w.r.t. all measures in $\mathscr{F}'$, and therefore also w.r.t. all measures in $\mathscr{F}$ (because $\mathscr{F}$ is contained in $\mathscr{F}'$). Thus (i) holds.

In the second case there exists $\sigma \in P(\mathscr{F})$ such that μ is not singular with respect to $[\sigma]$, and therefore by the Lebesgue-Radon-Nikodým theorem there exists a set E such that the restriction of $[\sigma]$ to E is nontrivial and absolutely continuous w.r.t. μ. Thus (ii) holds with this σ and this E. □

6.3. Lemma. *Let $C = C(e, \alpha)$ be a cone in $\mathbb{R}^n$ with axis e and angle α (cf. §4.1). Then, for every measure μ on $\mathbb{R}^n$, one of the following (mutually incompatible) alternatives holds:*

(i) *μ is supported on a Borel set E which is C-null (see §4.1);*

(ii) *there exists a nontrivial measure of the form $\mu' = \int_I \mu_t \, dt$ such that μ' is absolutely continuous w.r.t. μ, each μ_t is the restriction of $\mathscr{H}^1$ to some 1-rectifiable set E_t, and*

$$\mathrm{Tan}(E_t, x) \subset (C \cup (-C)) \quad \textit{for } \mu_t\textit{-a.e. } x \textit{ and a.e. } t.$$

[2] It is easy to check that this measure is well-defined in the sense of §2.5.

Proof. The idea is to apply Corollary 6.2 to the measure μ and a sequence of suitably chosen families $\mathscr{F}_k$ of probability measures.

Step 1. Construction of the families $\mathscr{F}_k$.

Given $k = 1, 2, \dots$, we define the following objects:

- $\mathscr{G}_k$ set of all paths γ from $[0,1]$ to the closed ball $B(k)$ such that $\mathrm{Lip}(\gamma) \le 1$ and $\dot\gamma(s) \cdot e \ge \cos\alpha$ for $\mathscr{L}^1$-a.e. $s \in [0,1]$;
- G_γ $:= \gamma([0,1])$, image of the path $\gamma \in \mathscr{G}_k$;
- μ_γ restriction of $\mathscr{H}^1$ to the curve G_γ;
- λ_γ push-forward according γ of the Lebesgue measure on $[0,1]$;
- $\mathscr{F}_k$ set of all λ_γ with $\gamma \in \mathscr{G}_k$.

One easily check that each G_γ is a C-curve (see §4.1) contained in B_k, and λ_γ is a probability measure supported on G_γ such that

$$\mu_\gamma \le \lambda_\gamma \le \frac{1}{\cos\alpha}\,\mu_\gamma\,. \tag{6.2}$$

In particular $\mathscr{F}_k$ is a subset of the space $P(B_k)$ of probability measures on B_k.

Step 2. Each $\mathscr{F}_k$ is a weak compact subset of $P(B_k)$.*

This is a consequence of the following statements:

(a) the space $\mathscr{G}_k$ endowed with the supremum distance is compact;

(b) $\mathscr{F}_k$ is the image of $\mathscr{G}_k$ according to the map $\gamma \mapsto \lambda_\gamma$ and this map is continuous (as a map from $\mathscr{G}_k$ to $P(B_k)$ endowed with the weak* topology).

Statement (a) follows from the usual compactness for the class of all paths $\gamma : [0,1] \to B(k)$ with $\mathrm{Lip}(\gamma) \le 1$ and the fact that we can rewrite the second constraint in the definition of $\mathscr{G}_k$ as

$$(\gamma(s') - \gamma(s)) \cdot e \ge \cos\alpha\,(s' - s) \quad \text{for every } s, s' \text{ with } 0 \le s \le s' \le 1,$$

which is clearly closed with respect to uniform convergence. To prove statement (b) we observe that for every $\gamma \in \mathscr{G}_k$ and every continuous test function $\varphi : B_k \to \mathbb{R}$ there holds

$$\langle \lambda_\gamma\,;\,\varphi\rangle = \int_{B_k} \varphi\, d\lambda_\gamma = \int_{[0,1]} \varphi(\gamma(s))\, d\mathscr{L}^1(s)\,,$$

and therefore the function $\gamma \mapsto \langle \lambda_\gamma\,;\,\varphi\rangle$ is continuous on $\mathscr{G}_k$.

Step 3. Completion of the proof.

For every $k = 1, 2, \dots$ we apply Corollary 6.2 to the measure μ_k given by the restriction of μ to B_k, and family $\mathscr{F}_k$, which by Step 2 is a weak* compact subset of $P(B_k)$.

There are now two possibilities: either there exists k such that statement (ii) of Corollary 6.2 holds, or statement (i) of Corollary 6.2 holds for every k.

In the first case there exists a probability measure σ on the space $\mathscr{G}_k$ and a Borel set E such that the measure

$$\int_{\mathscr{G}_k} (1_E\,\lambda_\gamma)\, d\sigma(\gamma)$$

is nontrivial and absolutely continuous w.r.t. μ_k, and therefore also w.r.t. μ. Then, using (6.2) we obtain that also the measure

$$\mu' := \int_{\mathscr{G}_k} (1_E \, \mu_\gamma) \, d\sigma(\gamma)$$

is nontrivial and absolutely continuous w.r.t. μ, and since each measure $1_E \, \mu_\gamma$ is the restriction of $\mathscr{H}^1$ to a subset of the C-curve G_γ, we have that μ' satisfies all the requirements in statement (ii), which therefore holds true.

In the second case we obtain that for every k the measure μ_k is supported on a set E_k contained in B_k which is null w.r.t. all measures in $\mathscr{F}_k$, and using the first inequality in (6.2)) we obtain that

$$\mathscr{H}^1(E_k \cap G_\gamma) = 0 \quad \text{for every } \gamma \in \mathscr{G}_k. \tag{6.3}$$

Now notice that intersection of every C-curve G with B_k can be covered by finitely many curves G_γ with $\gamma \in \mathscr{G}_k$ (use Remark 4.2), and therefore (6.3) implies $\mathscr{H}^1(E_k \cap G) = 0$. We have thus proved that E_k is C-null.

We then let E be the union of all E_k and observe that E is C-null, too, and μ is supported on E. Thus (i) holds. □

Proof of Lemma 2.11. We choose a finite family of cones $\{C_i\}$ (in the sense of §4.1) the interiors of which cover $\mathbb{R}^n \setminus \{0\}$, and then apply Lemma 6.3 to μ and to each C_i. There are now two possibilities: either there exists i such that statement (ii) of Lemma 6.3 holds, or statement (i) of Lemma 6.3 holds for every i.

In the first case we immediately obtain that statement (ii) holds. In the second case, for every i there exists a set E_i which supports μ and is C_i-null. We then let E be the intersection of all E_i and claim that E satisfies the requirements in statement (i), which therefore holds true.

It is indeed obvious that E supports μ. Concerning the unrectifiability of E, note that since the interiors of the cones C_i cover $\mathbb{R}^n \setminus \{0\}$, we can cover every curve G of class C^1 in $\mathbb{R}^n$ by countably many sub-arcs G_j, each one contained in a C_i-curve for some i. Therefore $\mathscr{H}^1(E \cap G_j) = 0$ because E is C_i-null. Hence $\mathscr{H}^1(E \cap G) = 0$, and we have proved that E is purely unrectifiable (cf. §2.4). □

Proof of Proposition 5.9. Let $\tilde{\mu}$ be the restriction of μ to the set F; thus $V(\tilde{\mu}, x) = V(\mu, x)$ for $\tilde{\mu}$-a.e. x by Proposition 2.9(i), and in particular

$$V(\tilde{\mu}, x) \cap C = \{0\} \quad \text{for } \tilde{\mu}\text{-a.e. } x. \tag{6.4}$$

We must prove that $\tilde{\mu}$ is supported on a C-null set, and for this it suffices to apply Lemma 6.3 (to the measure $\tilde{\mu}$ and the cone C) and show that, of the two alternatives given in that statetment, only alternative (i) is viable. Indeed the definition of the decomposability bundle in §2.7 and (6.4) imply that for every family $\{\mu_t : t \in I\}$ in $\mathscr{F}_{\tilde{\mu}}$ there holds $\mathrm{Tan}(E_t, x) \cap C = \{0\}$ for μ_t-a.e. x and a.e. t, and this contradicts alternative (ii). □

We now prove lemma 5.10. The proof relies on the following proposition, which is a simplified version of a result contained in [4] (we include a proof for the sake of completeness).

6.4. Proposition. *Let be given a measure μ on $\mathbb{R}^n$, a cone $C = C(e, \alpha)$ in $\mathbb{R}^n$, and a C-null compact set K in $\mathbb{R}^n$. Then for every $\varepsilon > 0$ there exists a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ such that, for every $x \in \mathbb{R}^n$,*

(i) $0 \le f(x) \le \varepsilon$;
(ii) $0 \le D_e f(x) \le 1$ *and* $D_e f(x) = 1$ *if* $x \in K$;
(iii) $|d_V f(x)| \le 1/\tan\alpha$ *where* $V := e^\perp$.[3]

Proof. We first construct a Lipschitz function g that satisfies statements (i), (ii) and (iii) with K replaced by a suitable open set A that contains K, and then we regularize g by convolution to obtain f.

Step 1. There exists an open set A such that $K \subset A$ and (cf. §4.1)

$$\mathscr{H}^1\big(A \cap G\big) \le \varepsilon \quad \text{for every } C\text{-curve } G. \tag{6.5}$$

More precisely, we claim that there exists $\delta > 0$ such that $\mathscr{H}^1\big(K_\delta \cap G\big) \le \varepsilon$ for every C-curve G, where K_δ is the set of all x such that $\operatorname{dist}(x, K) \le \delta$, and then it suffices to take A equal to the interior of K_δ.

We argue by contradiction: if the claim does not hold, then for every $\delta > 0$ there exists a C-curve G_δ such that $\mathscr{H}^1\big(K_\delta \cap G_\delta\big) \ge \varepsilon$. Let now R be the line in $\mathbb{R}^n$ spanned by e and let J' be a compact interval such that the segment $J := \{te : t \in J'\} \subset R$ contains the projections of K on R. We can then assume that the projections of each G_δ on R agrees with J. Moreover, the fact that G_δ is a C-curve means that it can be parametrized by a Lipschitz path $\gamma_\delta : J' \to \mathbb{R}^n$ of the form

$$\gamma_\delta(s) = se + \eta_\delta(s) \quad \text{with } \eta_\delta(s) \in V := e^\perp \text{ for every } s \in J',$$

where $\eta_\delta : J' \to V$ is Lipschitz and satisfies

$$|\dot\eta_\delta(s)| \le \tan\alpha \quad \text{for a.e. } s \in J'.$$

Finally we set $K'_\delta := \gamma_\delta^{-1}(K_\delta) = \gamma_\delta^{-1}(K_\delta \cap G_\delta)$.

Possibly passing to a subsequence, we can assume that, as $\delta \to 0$, the maps η_δ converge uniformly to some Lipschitz map $\eta_0 : J' \to V$, and the compact sets K'_δ converge to some compact set $K'_0 \subset J'$ in the Hausdorff distance.

Therefore the paths γ_δ converge to γ_0 given by $\gamma_0(s) := se + \eta_0(s)$, the set $G_0 := \gamma_0(J')$ is a C-curve, and finally $K \cap G_0$ contains $K_0 := \gamma_0(K'_0)$. We prove next that K_0 has positive length, which contradicts the fact that K is a C-curve.

[3] Here $d_V f(x)$ is the derivative of h at x w.r.t. V (see §2.1), and $|d_V f(x)|$ is its operator norm.

Indeed [4]

$$\mathscr{H}^1(K_0) \geq \mathscr{L}^1(K'_0) \geq \limsup_{\delta\to 0} \mathscr{L}^1(K'_\delta)$$
$$\geq \limsup_{\delta\to 0} \big(\cos\alpha\, \mathscr{H}^1(K_\delta \cap G_\delta)\big) \geq \varepsilon \cos\alpha > 0\,.$$

Step 2. Construction of g.

For every $x \in \mathbb{R}^n$ we denote by $\mathscr{G}_x$ the class of all C-curves $G = \gamma([a, b])$ whose end-point $x_G := \gamma(b)$ is of the form $x_G = x + se$ for some $s \geq 0$, and we set

$$g(x) := \sup_{G\in\mathscr{G}_x} \big(\mathscr{H}^1(A \cap G) - |x_G - x|\big)$$

Starting from the definition one can readily checks that the following properties hold for every $x \in \mathbb{R}^n$:

(a) $0 \leq g(x) \leq \varepsilon$ (here we use (6.5));
(b) $g(x) \leq g(x + se) \leq g(x) + s$ for every $s > 0$, and if the segment $[x, x + se]$ is contained in A then $g(x + se) = g(x) + s$;
(c) $|g(x + v) - g(v)| \leq |v|/ \tan\alpha$ for every $v \in V := e^\perp$;

Statements (b) and (c) imply that g is Lipschitz and

(b') $0 \leq D_e g(x) \leq 1$ for $\mathscr{L}^n$-a.e. x and $D_e g(x) = 1$ for $\mathscr{L}^n$-a.e. $x \in A$;
(c') $|d_V g(x)| \leq 1/ \tan\alpha$ for $\mathscr{L}^n$-a.e. x.

Step 3. Construction of f.

We take r so that $0 < r < \mathrm{dist}(K, \mathbb{R}^n \setminus A)$ and set $f := g * \rho$ where ρ is a mollifier with support contained in the ball $B(r)$. Then statements (i), (ii) and (iii) follow from statements (a), (b') and (c'), respectively. □

Proof of Lemma 5.10. We fix for the time being $\varepsilon' > 0$ and take a smooth function f that satisfies statements (i), (ii) and (iii) in Proposition 6.4 with ε' instead of ε. Then we set

$$g := \varphi f$$

where $\varphi : \mathbb{R}^n \to [0, 1]$ is a smooth cut-off function such that $\varphi(x) = 1$ for $x \in B(\bar{x}, r)$ and $\varphi(x) = 0$ for $x \in B(\bar{x}, r')$. Thus g satisfies statement (i) of Lemma 5.10.

Moreover we can choose φ so that $|d\varphi(x)| \leq 2/(r' - r)$ for every x; then

$$dg = \varphi\, df + f\, d\varphi \quad \text{and} \quad \|f\, d\varphi\|_\infty \leq \frac{2\varepsilon'}{r' - r}\,,$$

and therefore g satisfies statements (ii) and (iii) of Lemma 5.10 if we take ε' so that $2\varepsilon'/(r' - r)$ is smaller than ε and $1/ \tan\alpha$. □

We conclude this section with the proofs of Lemmas 3.8, 3.13, and 5.12. We begin with a definition and a couple of auxiliary lemmas.

[4] The second inequality follows from the upper semicontinuity of the Lebesgue measure w.r.t. the Hausdorff convergence of compact sets; the third inequality follows from the fact that $\mathscr{H}^1(\gamma_0(E)) \leq \mathscr{L}^1(E)/\cos\alpha$ for every set $E \subset J'$, which in turn follows from the fact that $|\dot{\eta}_0(s)| \leq \tan\alpha$ for a.e. s.

6.5. Deviation from linearity. Given a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, a point $x \in \mathbb{R}^n$, a linear subspace V of $\mathbb{R}^n$, a linear function $\alpha : V \to \mathbb{R}$, and $\delta > 0$, we set

$$m(f, x, V, \alpha, \delta) := \inf_{h\in V,\, 0<|h|\leq\delta} \frac{|f(x+h) - f(x) - \alpha h|}{|h|} \,.$$

Thus $m(f, x, V, \alpha, \delta)$ measures the deviation of f from the linear function α around x. In particular we have that f is differentiable at x w.r.t. V with derivative $d_V f(x) = \alpha$ if an only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $m(f, x, V, \alpha, \delta) \leq \varepsilon$.[5]

6.6. Lemma. *Let be given f, x and V as above, W and W' linear subspaces of V, α and α' linear functions on V. Then, setting $m := m(f, x, W, \alpha, \delta)$, $m' := m(f, x, W', \alpha', \delta)$, we have*

$$m \leq m' + |\alpha' - \alpha| + (L + |\alpha'|)d \,,$$

where $L := \mathrm{Lip}(f)$, $d := d_{\mathrm{gr}}(W, W')$, and d_{gr} is the distance in $\mathrm{Gr}(\mathbb{R}^n)$.[6]

Proof. We fix $h \in W$ with $|h| \leq \delta$, and denote by h' the orthogonal projection of h on W'. Then, taking into account the definition of d_{gr}, we have

$$|h'| \leq |h| \leq \delta \quad \text{and} \quad |h - h'| \leq d|h| \,. \tag{6.6}$$

Now, writing $f(x + h) - f(x) - \alpha h$ as $I + II + III + IV$ with

$$\begin{aligned} I &:= f(x + h) - f(x + h') \\ II &:= f(x + h') - f(x) - \alpha' h' \\ III &:= \alpha' h' - \alpha' h \\ IV &:= \alpha' h - \alpha h \end{aligned}$$

and using the estimates[7]

$$\begin{aligned} |I| &\leq L|h - h'| \leq Ld|h| \\ |II| &\leq m'|h'| \leq m'|h| \\ |III| &\leq |\alpha'|\,|h' - h| \leq d|\alpha'|\,|h| \\ |IV| &\leq |\alpha' - \alpha|\,|h| \end{aligned}$$

we obtain

$$|f(x + h) - f(x) - \alpha h| \leq \big[m' + |\alpha' - \alpha| + (L + |\alpha'|)d\big]\,|h| \,,$$

which implies the desired estimate. □

[5] Since m is increasing in δ, this is equivalent to say that m tends to 0 as $\delta \to 0$.

[6] As usual, we write $|\beta|$ for the operator norm of a linear function $\beta : V \to \mathbb{R}$.

[7] The first and third estimates follows from the second inequality in (6.6) and the fact that f is Lipschitz, the second one follows from the definition of m' and the first inequality in (6.6).

6.7. Lemma. *Let f be a Lipschitz function on $\mathbb{R}^n$, μ a measure on $\mathbb{R}^n$, and $V : \mathbb{R}^n \to \mathrm{Gr}(\mathbb{R}^n)$ Borel map such that f is differentiable at μ-a.e. x w.r.t. $V(x)$, the derivative being denoted by $d_V f(x)$. Then for every $\varepsilon > 0$ there exist a compact set K in $\mathbb{R}^n$ and a function $g : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that:*

(i) $\mu(\mathbb{R}^n \setminus K) \leq \varepsilon$;
(ii) $\|g - f\|_\infty \leq \varepsilon$;
(iii) $\mathrm{Lip}(g) \leq \mathrm{Lip}(f) + \varepsilon$;
(iv) $|d_V g(x) - d_V f(x)| \leq \varepsilon$ *for every* $x \in K$.

6.8. Remark. In the special case where $V(x)$ does not depend on x we can take $g := f * \rho$ with a suitable function ρ with integral equal to 1. More precisely, when $n = 2$ and V is the line $\mathbb{R} \times \{0\}$, we can take as ρ the characteristic function of the rectangle $[-r, r] \times [-\bar{r}, \bar{r}]$, renormalized so to have integral equal to 1, with $\bar{r} = r^2$ and r sufficiently small. This is the idea behind Step 5 in the proof below.

Proof. We set $L := \mathrm{Lip}(f)$ and denote by E be the set of all $x \in \mathbb{R}^n$ where f is differentiable w.r.t. $V(x)$.

For every x in E we extend the linear function $d_V f(x) : V(x) \to \mathbb{R}$ to a linear function $\alpha(x) : \mathbb{R}^n \to \mathbb{R}$ by setting $\alpha(x)\, h := 0$ for every $h \in V(x)^\perp$; thus $|\alpha(x)| = |d_V f(x)| \leq L$. Note that both E and $\alpha : E \to (\mathbb{R}^n)^*$ are Borel measurable (we omit the verification).

The rest of the proof is divided in several steps.

Step 1. There exist $\delta > 0$ and finitely many pairwise disjoint compact sets K_i with the following properties:

(a) $\mu(\mathbb{R}^n \setminus K) \leq \varepsilon$ *where K is the union of all K_i (thus statement (i) holds);*

and for every i,

(b) $d_{\mathrm{gr}}(V(x), V(x')) \leq \varepsilon/L$ *for every* $x, x' \in K_i$;
(c) $|\alpha(x) - \alpha(x')| \leq \varepsilon$ *for every* $x, x' \in K_i$;
(d) $m(f, x, V(x), \alpha(x), \delta) \leq \varepsilon$ *for every* $x \in K_i$.

For every $x \in E$ the function f is differentiable w.r.t. $V(x)$ with derivative $\alpha(x)$, and therefore there exists $\delta > 0$, depending on x, such that the estimate in (d) holds (cf. §6.5). Since moreover $\mu(\mathbb{R}^n \setminus E) = 0$, we can find a subset E' of E such that $\mu(\mathbb{R}^n \setminus E') \leq \varepsilon/2$ and the estimate in (d) holds with the same δ for all $x \in E'$. This value of δ is the one we choose.

Next we partition E' into finite a number N of Borel sets E_i such that the oscillations of the maps $x \mapsto V(x)$ and $x \mapsto \alpha(x)$ on each E_i are less that ε/L and ε, respectively. Finally for every i we take a compact set K_i contained in E_i such that $\mu(E_i \setminus K_i) \leq \varepsilon/(2N)$. It is now easy to check that statements (a–d) hold.

Step 2. For every i we choose $x_i \in K_i$ and set $V_i := V(x_i)$ and $\alpha_i := \alpha(x_i)$. Then for every $x \in K_i$ there holds $m(f, x, V_i, \alpha_i, \delta) \leq 4\varepsilon$.

We obtain this estimate by applying Lemma 6.6 together with the estimates in statements (b), (c) and (d) and the fact that $|\alpha(x)| \leq L$.

Step 3. Given i and $h \in \mathbb{R}^n$ we write $h = h' + h''$ with $h' \in V_i$ and $h'' \in V_i^\perp$. Then for every $x \in K_i$ and every h with $|h'| \le \delta$ there holds

$$|f(x+h) - f(x) - \alpha_i h'| \le 4\varepsilon |h'| + L|h''| \,. \tag{6.7}$$

The estimate in Step 2 yields $|f(x+h') - f(x) - \alpha_i h'| \le 4\varepsilon|h'|$, and using that $|f(x+h) - f(x+h')| \le L|h''|$ we obtain (6.7).

Step 4. Let ρ be a positive function on $\mathbb{R}^n$ with integral 1 and support contained in the ball $B(r)$. Then $f * \rho$ is a function of class C^1 that satisfies

(e) $\|f - f * \rho\|_\infty \le Lr$;

(f) $\|d(f * \rho)\|_\infty \le L$.

Statement (e) is obtained by a simple computation taking into account that f is Lipschitz and that the support of ρ is contained in $B(r)$.

The distributional derivative $d(f * \rho) = df * \rho$, being the convolution of an L^∞ and an L^1 function, is bounded and continuous, which means that $f * \rho$ is of class C^1 and has bounded derivative. Moreover $\|d(f*\rho)\|_\infty \le \|df\|_\infty \|\rho\|_1 = L$, and statement (f) is proved.

Step 5. For every i and every $r > 0$ there exists a positive function ρ_i with integral 1 and support contained in $B(r)$ such that $f_i := f * \rho_i$ satisfies the following property: for every $x \in K_i$ the restriction of the linear function $df_i(x) - \alpha_i$ to the subspace V_i has norm at most $M\varepsilon$, where the constant M depends only on n.

We assume that $k := \dim(V_i) > 0$, otherwise there is nothing to prove. We then take $r' > 0$ and denote by B' the ball with center 0 and radius r' contained in V_i, and by B'' the ball with center 0 and radius $r'' := \varepsilon r'/L$ contained in $V_i^\perp$. We then identify $\mathbb{R}^n$ with the product $V_i \times V_i^\perp$ and set

$$\rho_i := c\, 1_{B' \times B''} \quad \text{with } c := \frac{1}{\mathscr{L}^k(B')\, \mathscr{L}^{n-k}(B'')} \,.$$

We claim that if $r' \le \delta/2$ then $f_i := f * \rho_i$ satisfies

$$\big|f_i(x+h) - f_i(x) - \alpha_i h\big| \le M\varepsilon |h| \tag{6.8}$$

for every $x \in K_i$, every $h \in V_i$ with $|h| \le r'$, and a suitable M. This inequality shows that f_i has the property required in Step 5.

We fix x and h as above. A simple computation yields

$$f_i(x+h) - f_i(x) - \alpha_i h = \int_{\mathbb{R}^n} e(z)\, \big(\rho_i(h - z) - \rho_i(-z)\big)\, d\mathscr{L}^n(z) \tag{6.9}$$

where $e(z) := f(x+z) - f(x) - \alpha_i z$ for every $z \in \mathbb{R}^n$. We observe now that estimate (6.7) yields

$$|e(z)| \le 4\varepsilon |z'| + L|z''| \tag{6.10}$$

for every $z \in \mathbb{R}^n$ such that $|z'| \le \delta$, where z' and z'' come from the decomposition $z = z' + z''$ with $z' \in V_i$ and $z'' \in V_i^\perp$ (cf. Step 3). Therefore, in order to use (6.10) to estimate the integral in (6.9), we must check that $|z'| \le \delta$ for every z

such that $\rho_i(h-z) - \rho_i(-z) \neq 0$. Indeed, taking into account the definition of ρ_i and the fact that h belongs to V_i, we obtain

$$\rho_i(h-z) - \rho_i(-z) = \begin{cases} \pm c & \text{if } z' \in (B'+h)\triangle B' \text{ and } z'' \in B'', \\ 0 & \text{otherwise,} \end{cases} \tag{6.11}$$

and therefore if $\rho_i(h-z) - \rho_i(-z) \neq 0$ then z' belongs to the symmetric difference $(B'+h)\triangle B'$; in particular $|z'| \leq r' + |h| \leq 2r' \leq \delta$, as required.

Then, denoting by c_h the volume of the unit ball in $\mathbb{R}^h$ for $h = 0, 1, \dots,$ we obtain[8]

$$\begin{aligned} \big|f_i(x+h) - f_i(x) - \alpha_i h\big| & \\ &\leq \int_{\mathbb{R}^n} \big[4\varepsilon|z'| + L|z''|\big] \, \big|\rho_i(h-z) - \rho_i(-z)\big| \, d\mathscr{L}^n(z) \\ &\leq \big[8\varepsilon r' + Lr''\big] \int_{\mathbb{R}^n} |\rho_i(h-z) - \rho_i(-z)| \, d\mathscr{L}^n(z) \\ &\leq 9\varepsilon r \, \frac{\mathscr{L}^k\big((B'+h)\triangle B'\big)}{\mathscr{L}^k(B')} \leq \frac{18\, c_{k-1}}{c_k} \varepsilon |h| \, . \end{aligned}$$

We have thus proved (6.8) with M equal to the maximum of $18\, c_{k-1}/c_k$ over all $k = 1, \dots, n$.

Step 6. Take M and f_i as in Step 5. Then for every $x \in K_i$ there holds

$$|d_V f_i(x) - d_V f(x)| \leq (M+3)\varepsilon \, . \tag{6.12}$$

Taking into account §6.5 and the fact that $d_V f(x)$ agrees with $\alpha(x)$ on $V(x)$ we rewrite (6.12) as

$$m(df_i(x), 0, V(x), \alpha(x), 1) \leq (M+3)\varepsilon \, , \tag{6.13}$$

and the estimate in Step 5 as

$$m(df_i(x), 0, V_i, \alpha_i, 1) \leq M\varepsilon \, . \tag{6.14}$$

We then obtain (6.13) from (6.14) by applying Lemma 6.6 together with the following estimates: $d_{\mathrm{gr}}(V(x), V_i) \leq \varepsilon/L$ (statement (b)), $|\alpha(x) - \alpha_i| \leq \varepsilon$ (statement (c)), and $|\alpha_i| \leq L$.

Step 7. Construction of the function g.

Since the compact sets K_i are pairwise disjoint, there exists a smooth partition of unity $\{\sigma_i\}$ of $\mathbb{R}^n$ such that the functions σ_i take the value 1 on some neighbourhood of K_i, and are constant outside some compact set.[9] Thus the derivatives

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[8] The first inequality follows from (6.9) and (6.10), for the second one we use that $|z'| \leq 2r'$ and $|z''| \leq r''$ whenever $\rho_i(h-z) - \rho_i(-z) \neq 0$ (cf. (6.11)); for the third one we use that $r'' = \varepsilon r'/L$, formula (6.11), and the definition of c; for the fourth one we use that the volume of B' is $c_k(r')^k$ and the volume of $(B'+h)\triangle B'$ is at most $2c_{k-1}(r')^{k-1}|h|$.

[9] Take as $\{\sigma_i\}$ a smooth partition of unity of $\mathbb{R}^n$ subject to the open cover $\{A_i\}$ constructed as follows: U is a bounded open set that contains all K_i, C_1 be the union of K_1 and the complement of U, and $C_i := K_i$ for every $i > 1$, and finally A_i is the complement of the union of all K'_j with $j \neq i$.

$d\sigma_i$ have compact support and therefore are bounded, and

$$m := \max\Big\{1;\, \sum_i \|d\sigma_i\|_\infty\Big\}$$

is a finite number. Now we take $f_i = f * \rho_i$ as in Step 5, where ρ_i supported in the ball $B(r)$ with $r := \varepsilon/(mL)$, and set

$$g := \sum_i \sigma_i f_i \,.$$

The function g is clearly of class C^1. We prove next that g satisfies statements (ii), (iii) and (iv).

Note that statement (e) and the choice of r and m yield

$$|f_i(x) - f(x)| \le Lr = \frac{\varepsilon}{m} \le \varepsilon \quad \text{for every } x \in \mathbb{R}^d, \tag{6.15}$$

and since $g(x)$ is a convex combination of the numbers $f_i(x)$, it must satisfies $|g(x) - f(x)| \le \varepsilon$ as well, which proves statement (ii).

Given $x \in K$, take i such that $x \in K_i$, and note that $g = f_i$ on the neighbourhood of K_i where $\sigma_i = 1$; hence (6.12) becomes $|d_V g(x) - d_V f(x)| \le (M+3)\varepsilon$, which is the inequality in statement (iv) with $(M+3)\varepsilon$ instead of ε...

It remains to prove statement (iii), namely that that $|dg(x)| \le L+\varepsilon$ for every x. This estimate is an immediate consequence of the identity [10]

$$dg(x) = \sum_i \sigma_i(x)\, df_i(x) + \sum_i (f_i(x) - f(x))\, d\sigma_i(x)\,,$$

and the inequalities $|df_i(x)| \le L$ (statement (f)), $|f_i(x) - f(x)| \le \varepsilon/m$ (by (6.15)), and $\sum_i |d\sigma_i(x)| \le m$ (by the choice of m). □

Proof of Lemma 3.8. We first construct a sequence of approximating functions f_n of class C^1 that satisfy requirements (i), (ii) and (iii) using Lemma 6.7 above, and then regularize these functions by convolution to make them smooth. □

Proof of Lemma 3.13. As pointed out in §6.5, to prove that the Lipschitz function f is differentiable w.r.t. V at x and $d_V f(x) = \alpha$ it suffices to show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$m(f, x, V, \alpha, \delta) \le \varepsilon\,. \tag{6.16}$$

We fix $\varepsilon > 0$ and set $L := \mathrm{Lip}(f)$. Since D is dense in V we can choose a finite family $\mathscr{D}$ of lines generated by vectors in D such that for every line W contained in V there exists $W' \in \mathscr{D}$ with

$$d_{\mathrm{gr}}(W, W') \le \frac{\varepsilon}{2(L + |\alpha|)}\,. \tag{6.17}$$

[10] Here we use that $\sum_i d\sigma_i(x) = 0$, cf. footnote 15.

By the assumption on D, f is differentiable at x w.r.t. W' with derivative α for every line $W' \in \mathscr{D}$, and therefore there exists $\delta > 0$ such that

$$m(f, x, W', \alpha, \delta) \leq \frac{\varepsilon}{2}, \tag{6.18}$$

moreover, since $\mathscr{D}$ is finite we can assume that δ does not depend on W'.

Fix now a line W contained in V and take $W' \in \mathscr{D}$ such that (6.17) holds. Then Lemma 6.6 and inequality (6.18) imply

$$m(f, x, W, \alpha, \delta) \leq m(f, x, W', \alpha, \delta) + (L + |\alpha|)\, d_{\mathrm{gr}}(W, W') \leq \varepsilon\,.$$

We then obtain (6.16) using the fact that $m(f, x, V, \alpha, \delta)$ agrees with the supremum of $m(f, x, W, \alpha, \delta)$ over all lines W contained in V. □

Proof of lemma 5.12. We let $L := \mathrm{Lip}(f)$ and for every $k = 1, 2, \dots$ we set [11]

$$A_k := \left\{ x \in \mathbb{R}^n : \ \frac{1}{k+1} < \mathrm{dist}(x, K) < \frac{1}{k-1} \right\}.$$

Then $\{A_k\}$ is an open cover of the open set $A := \mathbb{R}^n \setminus K$, and therefore we can take a smooth partition of unity $\{\sigma_k\}$ of A subject to this cover.[12]

Next we choose a decreasing sequence of positive real numbers r_k such that for every k there holds [13]

$$L\, \|d\sigma_k\|_\infty r_k \leq 2^{-k}\varepsilon \quad \text{and} \quad L r_k \leq \phi\Big(\frac{1}{k+1}\Big)\,, \tag{6.19}$$

and a sequence of positive mollifiers ρ_k with support contained in the ball $B(r_k)$. Finally we set

$$g := f + \sum_{k=1}^{+\infty} \sigma_k (f * \rho_k - f)\,. \tag{6.20}$$

To prove statement (i) note first that g agrees with f on K because every $x \in K$ does not belong to A_k for every k, and then $\sigma_k(x) = 0$. On the other hand g is well-defined and smooth on the open set A because the functions in the sum in (6.20) are locally null for all but finitely many indexes k (recall footnote 12) and g can be rewritten as

$$g = \sum_{k=1}^{+\infty} \sigma_k (f * \rho_k)\,.$$

Let us now prove statement (ii). Since the support of ρ_k is contained in the ball $B(r_k)$ and f has Lipschitz constant L, a simple computation shows that for every $x \in \mathbb{R}^n$ there holds

$$|f * \rho_k(x) - f(x)| \leq L r_k\,. \tag{6.21}$$

[11] For $k = 1$ we convene that $1/0 = +\infty$.

[12] This means that each σ_k is a non-negative smooth function on A with support contained in A_k and that every $x \in A$ admits a neighbourhood where σ_k vanishes for all but finitely many k, and $\sum_k \sigma_k(x) = 1$.

[13] Each $\|d\sigma_k\|_\infty$ is finite because $d\sigma_k$ is continuous and compactly supported in A.

Therefore, given $x \in \mathbb{R}^n \setminus K$ and denoting by $k(x)$ the smallest k such that $x \in A_k$, we have [14]

$$\begin{aligned} |g(x) - f(x)| &\le \sum_{k \ge k(x)} \sigma_k(x) \, |f * \rho_k(x) - f(x)| \\ &\le \sum_{k \ge k(x)} \sigma_k(x) \, L r_k \\ &\le L r_{k(x)} \le \phi\Big(\frac{1}{k(x)+1}\Big) \le \phi(\mathrm{dist}(x, K)) \, . \end{aligned}$$

We conclude the proof by showing that g is Lipschitz and satisfies statement (iii). For every $h = 1, 2, \dots$ set

$$g_h := f + \sum_{k=1}^{h} \sigma_k (f * \rho_k - f) \, .$$

Since the functions g_h are Lipschitz and converge pointwise to g as $h \to +\infty$, it suffices to show that $\mathrm{Lip}(g_h) \le L + \varepsilon$ for every h, or equivalently that the distributional derivatives dg_h satisfies

$$\|dg_h\|_\infty \le L + \varepsilon \, . \tag{6.22}$$

Let h be fixed for the rest of the proof. We can write g_h as

$$g = \sum_{k=0}^{h} \sigma_k f_k$$

where we have set $\sigma_0 := 1 - (\sigma_1 + \dots + \sigma_h)$, $f_0 := f$, and $f_k := f * \rho_k$ for $0 < k \le h$. Then [15]

$$dg_h = \sum_{k=0}^{h} \sigma_k \, df_k + f_k \, d\sigma_k = \sum_{k=0}^{h} \sigma_k \, df_k + \sum_{k=1}^{h} (f_k - f) \, d\sigma_k \, . \tag{6.23}$$

Observe now that $df_k = df * \rho_k$ where df is the distributional derivative of f, and then $\|df_k\|_\infty \le \|df\|_\infty \|\rho_k\|_1 \le L$; hence the second sum in line (6.23) is a (pointwise) convex combinations of functions with L^∞-norm at most L, and therefore its L^∞-norm is at most L as well. Thus it remains to show that the L^∞-norm of the third sum in line (6.23) is at most ε, and indeed [16]

$$\Big\| \sum_{k=1}^{h} (f_k - f) \, d\sigma_k \Big\|_\infty \le \sum_{k=1}^{h} \|f_k - f\|_\infty \, \|d\sigma_k\|_\infty \le \sum_{k=1}^{h} L r_k \|d\sigma_k\|_\infty \le \varepsilon \, . \qquad \square$$

[14] For the first inequality we use that $x \notin A_k$ (and then $\sigma_k(x) = 0$) for $k < k(x)$; the second inequality follows from (6.21); the third one follows from the fact that the sum of all $\sigma_k(x)$ is 1 and $r_k(x) \ge r_k$ for every $k \ge k(x)$; the fourth one follows from the second inequality in (6.19), the fifth one from the fact that x belongs to $A_{k(x)}$ (and the definition of the sets A_k).

[15] For the second equality we use that $d\sigma_0 + \dots + d\sigma_h = 0$, which is obtained by deriving the identity $\sigma_0 + \dots + \sigma_h = 1$.

[16] The second inequality follows from the fact that $f_k = f * \rho_k$ and (6.21), the third one from the first inequality in (6.19).

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