

Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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by

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Preprint no.: 82

2014



Inequality Detecting Entanglement for Arbitrary Bipartite Systems

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Abstract

Based on the generators of $SU(n)$ we present inequalities for detecting quantum entanglement for $2 \otimes d$ and $M \otimes N$ systems. These inequalities provide a sufficient condition of entanglement for bipartite mixed states and give rise to an experimental way of entanglement detection.

I. INTRODUCTION

Quantum entanglement has played very important roles in quantum information processing such as quantum teleportation, quantum cryptography, quantum dense coding and parallel computing [1–3]. One of the important problems in the theory of quantum entanglement is to detect the quantum entanglement by measuring some suitable quantum mechanical observables. The Bell inequalities can be used to detect perfectly the entanglement of pure bipartite states [4–8]. Besides Bell inequalities, the entanglement witness are also useful in experimental detection of quantum entanglement for mixed states [9–15]. For bipartite mixed states, a necessary and sufficient inequality has been derived for detecting entanglement of two-qubit states [16]. The inequality in [17] is both necessary and sufficient in detecting entanglement of qubit-qutrit states, and necessary for qubit-qudit states. In Ref. [18] an inequality detecting entanglement for arbitrary dimensional bipartite states has been presented.

In stead of particular construction of the quantum mechanical observables in [17, 18], in this paper we use directly the generators of $SU(n)$ and present new inequalities for detecting entanglement of arbitrary dimensional bipartite mixed states. These inequalities give a necessary condition of separability for general mixed states. Any violation of the inequalities implies quantum entanglement. The paper is organized in the following way. In Sec. 2, based on Pauli matrices and the generators of $SU(d)$, we present inequalities for

detecting entanglement of $2 \otimes d$ systems. In Sec. 3, we present inequalities for detecting entanglement of $M \otimes N$ systems. Conclusions are given in Sec. 4.

II. INEQUALITIES DETECTING ENTANGLEMENT FOR $2 \otimes d$ SYSTEMS

Let H be n -dimensional vector space with computational basis $\{|i\rangle\}_{i=1}^n$. The generalized Gell-Mann matrices (GGM) are the generators of $SU(n)$ defined by [19]:

i) $\frac{n(n-1)}{2}$ symmetric GGM

$$\lambda_s^{jk} = |j\rangle\langle k| + |k\rangle\langle j|, \quad 1 \leq j < k \leq n; \quad (1)$$

ii) $\frac{n(n-1)}{2}$ antisymmetric GGM

$$\lambda_\alpha^{jk} = -i|j\rangle\langle k| + i|k\rangle\langle j|, \quad 1 \leq j < k \leq n; \quad (2)$$

iii) $(n-1)$ diagonal GGM

$$\lambda^l = \sqrt{\frac{2}{l(l+1)}} \left(\sum_{j=1}^l |j\rangle\langle j| - l|l+1\rangle\langle l+1| \right), \quad 1 \leq l \leq n-1. \quad (3)$$

In total we have $n^2 - 1$ GGM which are Hermitian and traceless. The operator $|j\rangle\langle k|$ with $j, k = 1, \dots, n$ can be also expressed in terms of GGM [19]

$$|j\rangle\langle k| = \begin{cases} \frac{1}{2}(\lambda_s^{jk} + i\lambda_\alpha^{jk}), & \text{for } j < k; \\ \frac{1}{2}(\lambda_s^{kj} - i\lambda_\alpha^{kj}), & \text{for } j > k; \\ -\sqrt{\frac{j-1}{2j}}\lambda^{j-1} + \sum_{m=0}^{n-j-1} \frac{1}{\sqrt{2(j+m)(j+m+1)}}\lambda^{j+m} + \frac{1}{n}I, & \text{for } j=k. \end{cases} \quad (4)$$

As any 2×2 matrix can be expanded according to the Pauli matrices plus identity, for the case of $n = 2$, operators $|j\rangle\langle k|$, $j, k = 1, 2$, can be written as

$$\begin{aligned} |1\rangle\langle 1| &= \frac{1}{2}(I_2 + \sigma_1), & |2\rangle\langle 2| &= \frac{1}{2}(I_2 - \sigma_1), \\ |1\rangle\langle 2| &= \frac{1}{2}(\sigma_2 + i\sigma_3), & |2\rangle\langle 1| &= \frac{1}{2}(\sigma_2 - i\sigma_3), \end{aligned} \quad (5)$$

where σ_i ($i = 1, 2, 3$) are the Pauli matrices. For the case of $n = d$, $|j\rangle\langle k|$, $j, k = 1, 2$, can be

written as

$$\begin{aligned}
|1\rangle\langle 1| &= \sum_{m=0}^{d-2} \frac{1}{\sqrt{2(m+1)(m+2)}} \lambda^{m+1} + \frac{1}{d} I, \\
|2\rangle\langle 2| &= -\frac{1}{2} \lambda^1 + \sum_{m=0}^{d-3} \frac{1}{\sqrt{2(m+2)(m+3)}} \lambda^{m+2} + \frac{1}{d} I, \\
|1\rangle\langle 2| &= \frac{1}{2} (\lambda_s^{12} + i \lambda_\alpha^{12}), \\
|2\rangle\langle 1| &= \frac{1}{2} (\lambda_s^{12} - i \lambda_\alpha^{12}).
\end{aligned} \tag{6}$$

Next we construct quantum mechanical operators for bipartite $2 \otimes d$ systems A and B . Set

$$\begin{aligned}
\widehat{Y}_1 &= \frac{1}{2} (\sigma_2 + i \sigma_3) \otimes \frac{1}{2} (\lambda_s^{12} - i \lambda_\alpha^{12}) + \frac{1}{2} (\sigma_2 - i \sigma_3) \otimes \frac{1}{2} (\lambda_s^{12} + i \lambda_\alpha^{12}), \\
\widehat{Y}_2 &= \frac{1}{2} (I_2 + \sigma_1) \otimes \left(\sum_{m=0}^{d-2} \frac{1}{\sqrt{2(m+1)(m+2)}} \lambda^{m+1} + \frac{1}{d} I_d \right) \\
&\quad - \frac{1}{2} (I_2 - \sigma_1) \otimes \left(-\frac{1}{2} \lambda^1 + \sum_{m=0}^{d-3} \frac{1}{\sqrt{2(m+2)(m+3)}} \lambda^{m+2} + \frac{1}{d} I_d \right), \\
\widehat{Y}_3 &= \frac{1}{2} (I_2 + \sigma_1) \otimes \left(\sum_{m=0}^{d-2} \frac{1}{\sqrt{2(m+1)(m+2)}} \lambda^{m+1} + \frac{1}{d} I_d \right) \\
&\quad + \frac{1}{2} (I_2 - \sigma_1) \otimes \left(-\frac{1}{2} \lambda^1 + \sum_{m=0}^{d-3} \frac{1}{\sqrt{2(m+2)(m+3)}} \lambda^{m+2} + \frac{1}{d} I_d \right).
\end{aligned} \tag{7}$$

Denote $Y_i = \text{Tr}(\rho(U \otimes V) \widehat{Y}_i (U \otimes V)^\dagger)$, $i = 1, 2, 3$, where U and V are unitary transformations on systems A and B , respectively. We have the following theorem:

Theorem 1: *Any separable state $\rho \in H_2 \otimes H_d$ obeys the following inequality*

$$Y_3^2 \geq Y_1^2 + Y_2^2. \tag{8}$$

[Proof] First we prove that the inequality holds for product states. If ρ is separable, its partial transposed matrix ρ^{T_B} is non-negative, i.e. $\text{Tr}(\rho^{T_B} P_{AB}) \geq 0$, where P_{AB} is an arbitrary projector to $2 \otimes 2$ subsystems. Or more generally $\text{Tr}[\rho^{T_B} (U^A \otimes U^B) P_{AB} (U^A \otimes U^B)^\dagger] \geq 0$, where U^A and U^B are local unitary operators. Any $2 \otimes 2$ pure state has the

Schmidt decomposition,

$$|\phi\rangle = \sin\theta|1\rangle_A \otimes |1\rangle_B + \cos\theta|2\rangle_A \otimes |2\rangle_B. \quad (9)$$

Hence P_{AB} can be written as

$$P_{AB} = (U \otimes V)|\phi\rangle\langle\phi|(U \otimes V)^\dagger, \quad (10)$$

where U and V are unitary operators. We have

$$\text{Tr}[\rho^{T_B}(U^A \otimes U^B)P_{AB}(U^A \otimes U^B)^\dagger] \quad (11)$$

$$= \text{Tr}\{\rho[(U^A \otimes U^B)(U \otimes V)|\phi\rangle\langle\phi|(U \otimes V)^\dagger(U^A \otimes U^B)^\dagger]^{T_B}\}$$

$$= \text{Tr}\{\rho(U^A U \otimes (U^B V)^\dagger)^{T_B}(|\phi\rangle\langle\phi|)^{T_B}(U^A U)^\dagger \otimes (U^B V)^{T_B}\}$$

$$= \text{Tr}[\rho(U \otimes V)(|\phi\rangle\langle\phi|)^{T_B}(U \otimes V)^\dagger], \quad (12)$$

where the last equation is obtained by choosing $U^A = I$ and $U^B = (VV^{T_B})^\dagger$. Therefore we get the following inequality,

$$\text{Tr}[\rho^{T_B}(U^A \otimes U^B)P_{AB}(U^A \otimes U^B)^\dagger] = \text{Tr}[\rho(U \otimes V)(|\phi\rangle\langle\phi|)^{T_B}(U \otimes V)^\dagger] \geq 0. \quad (13)$$

Using Eqs. (5), (6), (7) and (9), we have

$$\begin{aligned} (|\phi\rangle\langle\phi|)^{T_B} &= \sin^2\theta|11\rangle\langle 11| + \frac{1}{2}\sin 2\theta(|12\rangle\langle 21| + |21\rangle\langle 12|) + \cos^2\theta|22\rangle\langle 22| \\ &= \frac{1}{2}(\widehat{Y}_1 \sin 2\theta - \widehat{Y}_2 \cos 2\theta + \widehat{Y}_3). \end{aligned} \quad (14)$$

Substituting Eq. (14) into inequality (13) and setting $t = \tan\theta$, we obtain

$$(Y_2 + Y_3)t^2 + 2Y_1t + (Y_3 - Y_2) \geq 0. \quad (15)$$

Since $Y_2 + Y_3 = 2\text{Tr}(\rho(U \otimes V)|11\rangle\langle 11|(U \otimes V)^\dagger) \geq 0$ and (15) is valid for any t , we get

$$Y_1^2 + Y_2^2 - Y_3^2 \leq 0. \quad (16)$$

Therefore the inequality $Y_3^2 \geq Y_1^2 + Y_2^2$ holds for any product states.

It has been proved that if the inequality $a_i^2 \geq b_i^2 + c_i^2$ holds for arbitrary real numbers b_i and c_i and non-negative a_i , $i = 1, \dots, n$, then

$$\left(\sum_{i=1}^n p_i a_i\right)^2 \geq \left(\sum_{i=1}^n p_i b_i\right)^2 + \left(\sum_{i=1}^n p_i c_i\right)^2, \quad (17)$$

for $0 \leq p_i \leq 1$ and $\sum_{i=1}^n p_i = 1$ [17]. For general separable mixed states

$$\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|, \quad 0 \leq p_i \leq 1, \quad \sum_i p_i = 1, \quad (18)$$

where $|\phi_i\rangle$ are all product states, using (17) one can verify that any mixed separable state ρ also obeys the inequality (8). \square

Remark 1: If ρ is separable, the inequality $\text{Tr}[\rho^{T_B}(U^A \otimes U^B)P_{AB}(U^A \otimes U^B)^\dagger] \geq 0$ is valid for any local unitary operators U^A and U^B . In proving the theorem, we have chosen $U^A = I$ and $U^B = (VV^{T_B})^\dagger$, so that we can use the mean values of the set of quantum mechanical observables $(U \otimes V)\widehat{Y}_i(U \otimes V)^\dagger$ to detect quantum entanglement.

Remark 2: The operators in (7) are so introduced in terms of the quantum mechanical observables: the Pauli matrices and the $SU(n)$ generators. In fact, due to the direct relations between $|j\rangle\langle k|$ and the generators of $SU(n)$, for bipartite quantum systems $\widehat{Y}_1, \widehat{Y}_2$ and \widehat{Y}_3 can be simply written as $\widehat{Y}_1 = |12\rangle\langle 21| + |21\rangle\langle 12|$, $\widehat{Y}_2 = |1\bar{1}\rangle\langle 11| - |22\rangle\langle 22|$, $\widehat{Y}_3 = |11\rangle\langle 11| + |22\rangle\langle 22|$.

To show the advantage of our inequality, let us consider the following examples.

Example 1: The two-qubit Werner state is given by [20],

$$W(a) = a|\Psi^-\rangle\langle\Psi^-| + \frac{1-a}{4}I_4. \quad (19)$$

where $0 \leq a \leq 1$, I_4 denotes the 4×4 identity matrix. $|\Psi^-\rangle$ is the maximally entangled two-qubit state,

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|1\rangle_A|2\rangle_B - |2\rangle_A|1\rangle_B). \quad (20)$$

$W(a)$ is separable for $0 \leq a \leq \frac{1}{3}$ and entangled for $\frac{1}{3} < a \leq 1$.

We may simply choose $U = I_2$ and $V = I_2$, then

$$Y_1^2 + Y_2^2 - Y_3^2 = \frac{1+a}{4}(3a-1) > 0.$$

Therefore $W(a)$ is entangled for $\frac{1}{3} < a \leq 1$. Our inequalities can detect all the entanglement.

It is noted that the inequalities constructed in Ref. [17] can detect the entanglement of $W(a)$ only for $1 \leq a < \frac{-2+4\sqrt{5}}{19} \approx 0.37$ when the same U and V are used.

Example 2: Let us consider the $2 \otimes 3$ mixed state [17],

$$\rho = a|\Psi^+\rangle\langle\Psi^+| + \frac{1-a}{6}I_6, \quad (21)$$

where $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle)$. This state is entangled if and only if $a > \frac{1}{4}$. If we choose $U = \cos p (|1\rangle\langle 1| + |2\rangle\langle 2|) + \sin p (|1\rangle\langle 2| - |2\rangle\langle 1|)$ and $V = I_3$, then

$$F \equiv Y_1^2 + Y_2^2 - Y_3^2 = \frac{1 + 2a}{9}(6a \sin^2 p - 2a - 1).$$

For $p = \frac{\pi}{2}$, we have $Y_1^2 + Y_2^2 - Y_3^2 > 0$. Therefore ρ is entangled for $a > \frac{1}{4}$. Our inequalities can detect all the entanglement in ρ , see Fig. 1.

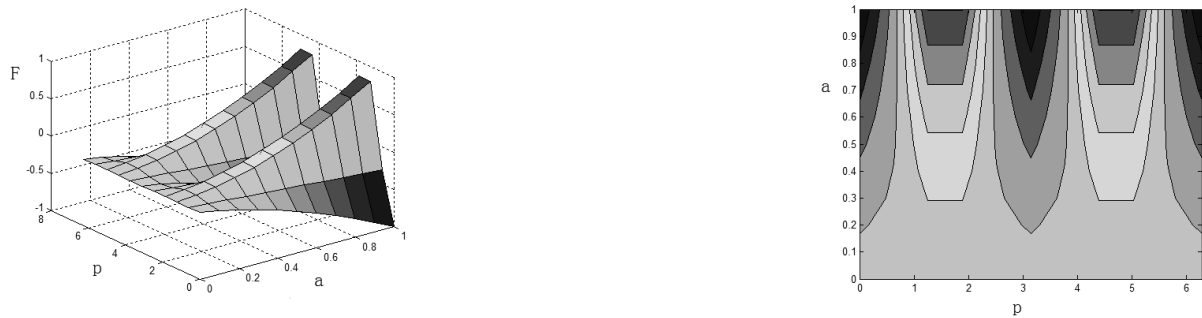


FIG. 1: $U = \cos p (|1\rangle\langle 1| + |2\rangle\langle 2|) + \sin p (|1\rangle\langle 2| - |2\rangle\langle 1|)$, $V = I_3$. Left figure: F with respect to p and a . Right figure: contour plot of the left figure.

III. INEQUALITIES DETECTING ENTANGLEMENT FOR $M \otimes N$ SYSTEMS

Now we consider general $M \otimes N$ systems. Let H_M, H_N be M, N -dimensional vector spaces for systems A and B , respectively. For $j < k$, set

$$\begin{aligned}
\widehat{Y}_1 &= \frac{1}{2}(\lambda_s^{jk} + i\lambda_\alpha^{jk})_A \otimes \frac{1}{2}(\lambda_s^{jk} - i\lambda_\alpha^{jk})_B + \frac{1}{2}(\lambda_s^{jk} - i\lambda_\alpha^{jk})_A \otimes \frac{1}{2}(\lambda_s^{jk} + i\lambda_\alpha^{jk})_B, \\
\widehat{Y}_2 &= \left(-\sqrt{\frac{j-1}{2j}}\lambda^{j-1} + \sum_{m=0}^{M-j-1} \frac{1}{\sqrt{2(j+m)(j+m+1)}}\lambda^{j+m} + \frac{1}{M}I \right) \otimes \left(-\sqrt{\frac{j-1}{2j}}\lambda^{j-1} \right. \\
&\quad \left. + \sum_{m=0}^{N-j-1} \frac{1}{\sqrt{2(j+m)(j+m+1)}}\lambda^{j+m} + \frac{1}{N}I \right) \\
&\quad - \left[\left(-\sqrt{\frac{k-1}{2k}}\lambda^{k-1} + \sum_{m=0}^{M-k-1} \frac{1}{\sqrt{2(k+m)(k+m+1)}}\lambda^{k+m} + \frac{1}{M}I \right) \otimes \left(-\sqrt{\frac{k-1}{2k}}\lambda^{k-1} \right. \right. \\
&\quad \left. \left. + \sum_{m=0}^{N-k-1} \frac{1}{\sqrt{2(k+m)(k+m+1)}}\lambda^{k+m} + \frac{1}{N}I \right) \right], \\
\widehat{Y}_3 &= \left(-\sqrt{\frac{j-1}{2j}}\lambda^{j-1} + \sum_{m=0}^{M-j-1} \frac{1}{\sqrt{2(j+m)(j+m+1)}}\lambda^{j+m} + \frac{1}{M}I \right) \otimes \left(-\sqrt{\frac{j-1}{2j}}\lambda^{j-1} \right. \\
&\quad \left. + \sum_{m=0}^{N-j-1} \frac{1}{\sqrt{2(j+m)(j+m+1)}}\lambda^{j+m} + \frac{1}{N}I \right) \\
&\quad + \left(-\sqrt{\frac{k-1}{2k}}\lambda^{k-1} + \sum_{m=0}^{M-k-1} \frac{1}{\sqrt{2(k+m)(k+m+1)}}\lambda^{k+m} + \frac{1}{M}I \right) \otimes \left(-\sqrt{\frac{k-1}{2k}}\lambda^{k-1} \right. \\
&\quad \left. + \sum_{m=0}^{N-k-1} \frac{1}{\sqrt{2(k+m)(k+m+1)}}\lambda^{k+m} + \frac{1}{N}I \right), \tag{22}
\end{aligned}$$

and $Y_i = \text{Tr}[\rho(U \otimes V)\widehat{Y}_i(U \otimes V)^\dagger]$ with $i = 1, 2, 3$, where U and V are local unitary transformations on systems A and B , respectively. We have the following theorem:

Theorem 2: Any separable state $\rho \in H_M \otimes H_N$ obeys the following inequality

$$Y_3^2 \geq Y_1^2 + Y_2^2. \tag{23}$$

[Proof] Any product states can be written as

$$|\xi\rangle = \sum_{i=1}^M \sum_{l=1}^N a_i b_l |il\rangle, \tag{24}$$

with $\sum_{i=1}^M |a_i|^2 = \sum_{l=1}^N |b_l|^2 = 1$. Using Eqs. (22) and (24), we have

$$\begin{aligned} Y_3^2 - Y_1^2 - Y_2^2 &= 4[|a_j a_k b_j b_k|^2 - \text{Re}^2(a_j a_k^* b_j^* b_k)] \\ &= 4(a_j b_k a_k^* b_j^*)(a_j b_k a_k^* b_j^*)^* - \text{Re}^2(a_j b_k a_k^* b_j^*) \geq 0. \end{aligned} \quad (25)$$

Therefore $Y_3^2 \geq Y_1^2 + Y_2^2$ holds for any product states. Using the similar methods in proving Theorem 1, we have that the inequality also holds for general separable mixed states. \square

To show the usefulness of our inequality, let us consider the Horodecki's $3 \otimes 3$ state:

$$\sigma_\alpha = \frac{2}{7}|\psi^+\rangle\langle\psi^+| + \frac{\alpha}{7}\sigma_+ + \frac{5-\alpha}{7}\sigma_-, \quad (26)$$

where $\sigma_+ = \frac{1}{3}(|12\rangle\langle 12| + |23\rangle\langle 23| + |31\rangle\langle 31|)$, $\sigma_- = \frac{1}{3}(|21\rangle\langle 21| + |32\rangle\langle 32| + |13\rangle\langle 13|)$, $|\psi^+\rangle = \frac{1}{\sqrt{3}}(|11\rangle + |22\rangle + |33\rangle)$. σ_α is separable for $2 \leq \alpha \leq 3$, bound entangled for $3 < \alpha \leq 4$, and free entangled for $4 < \alpha \leq 5$ [21]. If we choose $U = \cos p (|1\rangle\langle 1| + |2\rangle\langle 2|) + \sin p (|1\rangle\langle 2| - |2\rangle\langle 1|)$ and $V = I_3$, then

$$F = Y_1^2 + Y_2^2 - Y_3^2 = \left(\frac{2}{21}\right)^2 [(\alpha^2 - 5\alpha + 10) \sin^4 p - 2 \sin^2 p - 4].$$

For $p = \frac{\pi}{2}$, we have $Y_1^2 + Y_2^2 - Y_3^2 > 0$. Therefore σ_α is entangled for $\alpha > 4$.

In Ref. [18] an inequality for detecting entanglement of arbitrary dimensional bipartite systems has been presented, which can also detect the entanglement of (26) for $\alpha > 4$. The quantum mechanical observables in our inequalities are constructed systematically according to $SU(n)$ generators, in contrast to artificial constructions of observable operators in Ref. [18]. The mean values can be easily calculated. In fact, due to the direct relations between $|j\rangle\langle k|$ and the generators of $SU(n)$, for bipartite quantum systems \widehat{Y}_1 , \widehat{Y}_2 and \widehat{Y}_3 can be simply written as $\widehat{Y}_1 = |jk\rangle\langle kj| + |kj\rangle\langle jk|$, $\widehat{Y}_2 = |jj\rangle\langle jj| - |kk\rangle\langle kk|$, $\widehat{Y}_3 = |jj\rangle\langle jj| + |kk\rangle\langle kk|$, where $j = 1, 2, \dots, M$, $k = 1, 2, \dots, N$ and $j < k$.

IV. CONCLUSIONS

We have presented inequalities for detecting quantum entanglement of $2 \otimes d$ and $M \otimes N$ systems. These inequalities give necessary conditions of separability for mixed states. Since these inequalities are given by quantum mechanical observables, namely, Hermitian operators, they supply experimental ways of detecting entanglement by measuring the mean

values of these local observables. For example, the state defined in Eq. (21) violates the inequalities for quantum mechanical observables given by choosing $U = |1\rangle\langle 2| - |2\rangle\langle 1|$ and $V = I$. Any violation of the inequalities implies that the quantum states are entangled. As for examples, it has been shown that our inequalities can detect entanglement well when the measurement operators are suitably chosen. Our inequalities are complementary to the existing ones. As the inequalities are directly given by the $SU(n)$ generators acting on local subsystems, the approach can be readily generalized to deal with the detection of entanglement for multipartite systems.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (11101017 and 11275131), Beijing Natural Science Foundation Program and Scientific Research Key Program of Beijing Municipal Commission of Education (KZ201210028032).

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