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by

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# One-way deficit of $SU(2)$ invariant states

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## Abstract

We calculate analytically the one-way deficit for systems consisting spin- $j$  and spin- $\frac{1}{2}$  subsystems with  $SU(2)$  symmetry. Comparing our results with the quantum discord of  $SU(2)$  invariant states, we show that the one-way deficit is equal to the quantum discord for half-integer  $j$ , and is larger than the quantum discord for integer  $j$ . Moreover, we also compare the one-way deficit with entanglement of formation. The quantum entanglement tends to zero as  $j$  increases, while the one-way deficit can remain significantly large.

## I. INTRODUCTION

Quantum entanglement has played significant roles in the field of quantum information and quantum computation such as super-dense coding [1], teleportation [2], quantum cryptography [3], remote-state preparation [4]. Quantum correlations other than quantum entanglement have also attracted much attention recently [5–11]. Among of them, quantum discord introduced by Oliver and Zurek and independently by Henderson and Vedral [6] is a widely accepted quantity. Quantum discord, a measure which quantifies the discrepancy between the quantum mutual information and the maximal classical information, can be present in separable mixed quantum states. Following this discovery, much work has been done in investigating the properties and behavior of quantum discord in various systems. Since complicated optimization procedure involved in calculating quantum discord, one has no analytical formulae of quantum discord even for two-qubit quantum systems [12–17].

Other significant nonclassical correlations besides entanglement and quantum discord, for example, the quantum deficit [18, 19], measurement-induced disturbance [12], symmetric discord [20, 21], relative entropy of discord and dissonance [22], geometric discord [23, 24], and continuous-variable discord [25, 26] have been studied recently. The work deficit [18] proposed to characterize quantum correlations in terms of entropy production and work extraction by Oppenheim *et al* is the first operational approach to connect the quantum correlations theory with quantum thermodynamics. Recently, Alexander Streltsov *et al* [27, 28] reveals that the one-way deficit plays an important role in quantum correlations as a resource for the distribution of entanglement. The one-way deficit of a bipartite quantum state  $\rho$  is defined by [29]:

$$\Delta^{\rightarrow}(\rho) = \min_{\{\Pi_k\}} S\left(\sum_i \Pi_k \rho \Pi_k\right) - S(\rho), \quad (1)$$

where  $\{\Pi_k\}$  is the projective measurements and  $S$  is the von Neumann entropy.

The one-way deficit and quantum discord have similar minimum form but they are different kinds of quantum correlations. We have obtained analytical formula of one-way deficit for some well known states such as Bell-diagonal states [30]. In this paper, we endeavored to calculate the one-way deficit of bipartite  $SU(2)$  invariant states consisting of a spin- $j$  and a spin- $\frac{1}{2}$  subsystems.

$SU(2)$ -invariant density matrices of two spins  $\vec{S}_1$  and  $\vec{S}_2$  are defined to be invariant under  $U_1 \otimes U_2$ ,  $U_1 \otimes U_2 \rho U_1^\dagger \otimes U_2^\dagger = \rho$ , where  $U_a = \exp(i\vec{\eta} \cdot \vec{S}_a)$ ,  $a \in \{1, 2\}$ , are the usual

rotation operator representation of  $SU(2)$  with real parameter  $\vec{\eta}$  and  $\hbar = 1$  [31, 32]. Those  $SU(2)$ -invariant states  $\rho$  commute with all the components of the total spin  $\vec{J} = \vec{S}_1 + \vec{S}_2$ . In real physical systems,  $SU(2)$ -invariant states arise from thermal equilibrium states of spin systems described by  $SU(2)$  invariant Hamiltonian [33]. The state space structure and entanglement of  $SO(3)$ -invariant bipartite quantum systems have been analyzed in the literature [34, 35]. For  $SU(2)$  invariant quantum spin systems, negativity is shown to be necessary and sufficient for separability [31, 32], and the relative entropy of entanglement has been analytically calculated [36] for  $(2j + 1) \otimes 2$  and  $(2j + 1) \otimes 3$  dimensional systems. Furthermore, the entanglement of formation (EoF), I-concurrence, I-tangle and convex-roof-extended negativity of the  $SU(2)$ -invariant states of a spin- $j$  and spin- $\frac{1}{2}$  [37] have been analytically calculated by using the approach in [38]. Quantum discord for  $SU(2)$ -invariant states composed of spin- $j$  and spin- $\frac{1}{2}$  systems has been analytically calculated in [39].

As an  $SU(2)$ -invariant state commutes with all the components of  $\vec{J}$ ,  $\rho$  has the general form [31],

$$\rho = \sum_{J=|S_1-S_2|}^{S_1+S_2} \frac{A(J)}{2J+1} \sum_{J^z=-J}^J |J, J^z\rangle_{00} \langle J, J^z|, \quad (2)$$

where the constants  $A(J) \geq 0$ ,  $\sum_J A(J) = 1$ ,  $|J, J^z\rangle_0$  denotes a state of total spin  $J$  and  $z$ -component  $J^z$ . We consider the case with  $\vec{S}_1$  of arbitrary length  $S$  and  $\vec{S}_2$  of length  $\frac{1}{2}$  [31]. Let  $S = j$ ,  $J^z = m$ . A general  $SU(2)$ -invariant density matrix has the form,

$$\rho^{ab} = \frac{F}{2j} \sum_{m=-j+\frac{1}{2}}^{j-1/2} |j - \frac{1}{2}, m\rangle \langle j - \frac{1}{2}, m| + \frac{1-F}{2(j+1)} \sum_{m=-j-\frac{1}{2}}^{j+1/2} |j + \frac{1}{2}, m\rangle \langle j + \frac{1}{2}, m|, \quad (3)$$

where  $F \in [0, 1]$  is a function of temperature in thermal equilibrium.  $\rho^{ab}$  is a  $(2j + 1) \otimes 2$  bipartite state. It has two eigenvalues  $\lambda_1 = F/2j$  and  $\lambda_2 = (1 - F)/(2j + 2)$  with degeneracies  $2j$  and  $2j + 2$ , respectively [39]. The entropy of  $\rho^{ab}$  is given by

$$S(\rho^{ab}) = -F \log\left(\frac{F}{2j}\right) - (1 - F) \log\left(\frac{1 - F}{2j + 2}\right). \quad (4)$$

As the eigenstates of the total spin can be given by the Clebsch-Gordon coefficients [40] in coupling a spin- $j$  to spin- $\frac{1}{2}$ ,

$$|j \pm \frac{1}{2}, m\rangle = \pm \sqrt{\frac{j + \frac{1}{2} \pm m}{2j + 1}} |j, m - \frac{1}{2}\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle + \sqrt{\frac{j + \frac{1}{2} \mp m}{2j + 1}} |j, m + \frac{1}{2}\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle, \quad (5)$$

the density matrix (3) can be written in product basis form [39],

$$\begin{aligned}
\rho^{ab} = & \frac{F}{2j} \sum_{m=-j+\frac{1}{2}}^{j-\frac{1}{2}} (a_-^2 |m - \frac{1}{2}\rangle \langle m - \frac{1}{2}| \otimes |\frac{1}{2}\rangle \langle \frac{1}{2}| \\
& + a_- b_- (|m - \frac{1}{2}\rangle \langle m + \frac{1}{2}| \otimes |\frac{1}{2}\rangle \langle -\frac{1}{2}| \\
& + |m + \frac{1}{2}\rangle \langle m - \frac{1}{2}| \otimes |-\frac{1}{2}\rangle \langle \frac{1}{2}|) \\
& + b_-^2 |m + \frac{1}{2}\rangle \langle m + \frac{1}{2}| \otimes |-\frac{1}{2}\rangle \langle -\frac{1}{2}|) \\
& + \frac{1-F}{2(j+1)} \sum_{m=-j-\frac{1}{2}}^{j+\frac{1}{2}} (a_+^2 |m - \frac{1}{2}\rangle \langle m - \frac{1}{2}| \otimes |\frac{1}{2}\rangle \langle \frac{1}{2}| \\
& + a_+ b_+ (|m - \frac{1}{2}\rangle \langle m + \frac{1}{2}| \otimes |\frac{1}{2}\rangle \langle -\frac{1}{2}| \\
& + |m + \frac{1}{2}\rangle \langle m - \frac{1}{2}| \otimes |-\frac{1}{2}\rangle \langle \frac{1}{2}|) \\
& + b_+^2 |m + \frac{1}{2}\rangle \langle m + \frac{1}{2}| \otimes |-\frac{1}{2}\rangle \langle -\frac{1}{2}|),
\end{aligned}$$

where  $a_{\pm} = \pm \sqrt{\frac{j+\frac{1}{2}\pm m}{2j+1}}$  and  $b_{\pm} = \sqrt{\frac{j+\frac{1}{2}\mp m}{2j+1}}$ .

When  $j = \frac{1}{2}$ , the state  $\rho^{ab}$  turns out to be the  $2 \otimes 2$  Werner state:

$$\rho = (1-c) \frac{I}{4} + c |\psi^-\rangle \langle \psi^-|, \quad c = \frac{4F-1}{3}, \quad (6)$$

with  $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ .

## II. MAIN RESULT

Any von Neumann measurement on the spin- $\frac{1}{2}$  subsystem can be written as [12]:

$$B_k = V \Pi_k V^\dagger, \quad k = 0, 1, \quad (7)$$

where  $\Pi_k = |k\rangle \langle k|$ ,  $|k\rangle$  is the computational basis,  $V = tI + i\vec{y} \cdot \vec{\sigma} \in SU(2)$ ,  $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$  are Pauli matrices,  $t$  and  $\vec{y}$  are real,  $t^2 + y_1^2 + y_2^2 + y_3^2 = 1$ . Set

$$\begin{aligned}
M = & \sum_{m=-j}^j (z_3 \frac{m(2Fj+F-j)}{j(j+1)(2j+1)} |m\rangle \langle m| \\
& + (z_1 + iz_2) \frac{\sqrt{j(j+1)-m(m+1)}(2Fj+F-j)}{2j(j+1)(2j+1)} |m\rangle \langle m+1| \\
& + (z_1 - iz_2) \frac{\sqrt{j(j+1)-m(m+1)}(2Fj+F-j)}{2j(j+1)(2j+1)} |m+1\rangle \langle m|).
\end{aligned}$$

After the measurement, one has the ensemble of post-measurement states  $\{\rho_k, p_k\}$  with  $p_0 = p_1 = \frac{1}{2}$  and the corresponding post-measurement states [39],

$$\begin{aligned}
\rho_0 &= \left[ \frac{1}{2j+1} \sum_{m=-j}^j |m\rangle\langle m| - \sum_{m=-j}^j \left( z_3 \frac{m(2Fj+F-j)}{j(j+1)(2j+1)} |m\rangle\langle m| \right. \right. \\
&\quad + (z_1 + iz_2) \frac{\sqrt{j(j+1)-m(m+1)}(2Fj+F-j)}{2j(j+1)(2j+1)} |m\rangle\langle m+1| \\
&\quad \left. \left. + (z_1 - iz_2) \frac{\sqrt{j(j+1)-m(m+1)}(2Fj+F-j)}{2j(j+1)(2j+1)} |m+1\rangle\langle m| \right) \right] \otimes V\Pi_0 V^\dagger, \\
&= \left[ \frac{1}{2j+1} I - M \right] \otimes V\Pi_0 V^\dagger,
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
\rho_1 &= \left[ \frac{1}{2j+1} \sum_{m=-j}^j |m\rangle\langle m| + \sum_{m=-j}^j \left( z_3 \frac{m(2Fj+F-j)}{j(j+1)(2j+1)} |m\rangle\langle m| \right. \right. \\
&\quad + (z_1 + iz_2) \frac{\sqrt{j(j+1)-m(m+1)}(2Fj+F-j)}{2j(j+1)(2j+1)} |m\rangle\langle m+1| \\
&\quad \left. \left. + (z_1 - iz_2) \frac{\sqrt{j(j+1)-m(m+1)}(2Fj+F-j)}{2j(j+1)(2j+1)} |m+1\rangle\langle m| \right) \right] \otimes V\Pi_1 V^\dagger, \\
&= \left[ \frac{1}{2j+1} I + M \right] \otimes V\Pi_1 V^\dagger,
\end{aligned} \tag{9}$$

where  $z_1 = 2(-ty_2 + y_1y_3)$ ,  $z_2 = 2(ty_1 + y_2y_3)$ ,  $z_3 = t^2 + y_3^2 - y_1^2 - y_2^2$  with  $z_1^2 + z_2^2 + z_3^2 = 1$ .

The eigenvalues of the post-measurement states [39] are the same:

$$\lambda_n^\pm = \frac{1}{2j+1} \pm \frac{j-n}{j(j+1)(2j+1)} |(F(2j+1)-j)|, \tag{10}$$

where  $n = 0, \dots, [j]$ ,  $[j]$  denotes the largest integer that is less or equal to  $j$ . Obviously the eigenvalues are independent of the measurement. Due to this fact, analytical expression for quantum discord of  $\rho^{ab}$  has been obtained in [39],

$$\begin{aligned}
D(\rho^{ab}) &= F \log_2 \left( \frac{F}{2j} \right) + (1-F) \log_2 \left( \frac{1-F}{2j+2} \right) + 1 \\
&\quad - \sum_{n=0}^{[j]} \left( \frac{1}{2j+1} + \frac{j-n}{j(j+1)(2j+1)} |(F(2j+1)-j)| \right) \\
&\quad \cdot \log_2 \left( \frac{1}{2j+1} + \frac{j-n}{j(j+1)(2j+1)} |(F(2j+1)-j)| \right) \\
&\quad - \sum_{n=0}^{[j]} \left( \frac{1}{2j+1} - \frac{j-n}{j(j+1)(2j+1)} |(F(2j+1)-j)| \right) \\
&\quad \cdot \log_2 \left( \frac{1}{2j+1} - \frac{j-n}{j(j+1)(2j+1)} |(F(2j+1)-j)| \right).
\end{aligned} \tag{11}$$

To evaluate the one-way deficit of  $\rho^{ab}$ , we calculate the eigenvalues of  $\sum_i \Pi_k \rho^{ab} \Pi_k = p_0 \rho_0 + p_1 \rho_1$ . From (8) and (9), since  $\frac{1}{2j+1}I - M$  commutes with  $\frac{1}{2j+1}I + M$ , by using (19) and (20) in [30], we have the eigenvalues of  $\sum_i \Pi_k \rho^{ab} \Pi_k$ ,

$$\bar{\lambda}_n^\pm = \frac{1}{2} \lambda_n^\pm, \quad (12)$$

with each algebraic multiplicity two. As the eigenvalues do not depend on the measurement parameters, the minimum of the entropy of the post-measurement states do not require any optimization over the projective measurements,

$$\begin{aligned} \min_{\{\Pi_k\}} S\left(\sum_i \Pi_k \rho^{ab} \Pi_k\right) &= - \sum_{n=0}^{\lfloor j \rfloor} \left( \frac{1}{(2j+1)} \pm \frac{j-n}{j(j+1)(2j+1)} |(F(2j+1) - j)| \right) \\ &\quad \cdot \log\left( \frac{1}{2(2j+1)} \pm \frac{j-n}{2j(j+1)(2j+1)} |(F(2j+1) - j)| \right). \end{aligned} \quad (13)$$



From Eqs. (1), (4) and (13), the one-way deficit of the state is given by

$$\begin{aligned}
\Delta^{\rightarrow}(\rho^{ab}) &= \min_{\{\Pi_k\}} S\left(\sum_i \Pi_k \rho^{ab} \Pi_k\right) - S(\rho^{ab}) \\
&= F \log\left(\frac{F}{2j}\right) + (1-F) \log\left(\frac{1-F}{2j+2}\right) \\
&\quad - \sum_{n=0}^{\lfloor j \rfloor} \left( \frac{1}{(2j+1)} + \frac{j-n}{j(j+1)(2j+1)} |(F(2j+1)-j)| \right) \\
&\quad \cdot \log \frac{1}{2} \left( \frac{1}{(2j+1)} + \frac{j-n}{j(j+1)(2j+1)} |(F(2j+1)-j)| \right) \\
&\quad - \sum_{n=0}^{\lfloor j \rfloor} \left( \frac{1}{(2j+1)} - \frac{j-n}{j(j+1)(2j+1)} |(F(2j+1)-j)| \right) \\
&\quad \cdot \log \frac{1}{2} \left( \frac{1}{(2j+1)} - \frac{j-n}{j(j+1)(2j+1)} |(F(2j+1)-j)| \right) \\
&= F \log_2\left(\frac{F}{2j}\right) + (1-F) \log_2\left(\frac{1-F}{2j+2}\right) + \frac{2}{2j+1} (\lfloor j \rfloor + 1) \\
&\quad - \sum_{n=0}^{\lfloor j \rfloor} \left( \frac{1}{2j+1} + \frac{j-n}{j(j+1)(2j+1)} |(F(2j+1)-j)| \right) \\
&\quad \cdot \log_2 \left( \frac{1}{2j+1} + \frac{j-n}{j(j+1)(2j+1)} |(F(2j+1)-j)| \right) \\
&\quad - \sum_{n=0}^{\lfloor j \rfloor} \left( \frac{1}{2j+1} - \frac{j-n}{j(j+1)(2j+1)} |(F(2j+1)-j)| \right) \\
&\quad \cdot \log_2 \left( \frac{1}{2j+1} - \frac{j-n}{j(j+1)(2j+1)} |(F(2j+1)-j)| \right) \\
&= \begin{cases} D(\rho^{ab}), & j \in \text{half-integer} \\ D(\rho^{ab}) + \frac{1}{d}, & j \in \text{integer}. \end{cases} \tag{14}
\end{aligned}$$

where  $d = 2j + 1$ . Especially, when  $j = \frac{1}{2}$ , the state becomes the  $2 \otimes 2$  Werner state, and the one-way deficit is equal to the quantum discord, which is in consistent with the result obtained in [30].

In Fig. 1 we show that quantum correlations as a function of  $F$  for different spin  $j$ . For half-integer  $j$ , we observe that as  $j$  increases, the one-way deficit increases for small  $F$  and decreases for large  $F$ . But for integer  $j$ , the one-way deficit decreases as  $j$  increases, see Fig.1(a). For high dimensional system (large  $j$ ), the one-way deficit becomes symmetric around the point  $F = \frac{1}{2}$  where  $\Delta^{\rightarrow}(\rho^{ab})$  vanishes, see Fig.1(b). Maximum of the one-way deficit is attained at  $F = 1$  and at  $F = 0$  for all dimensional systems.

From Eq.(14), one can see an interesting fact: for half-integer  $j$ , the one-way deficit is equal to the quantum discord, see Fig.1(c). For integer  $j$ , the difference between the one-

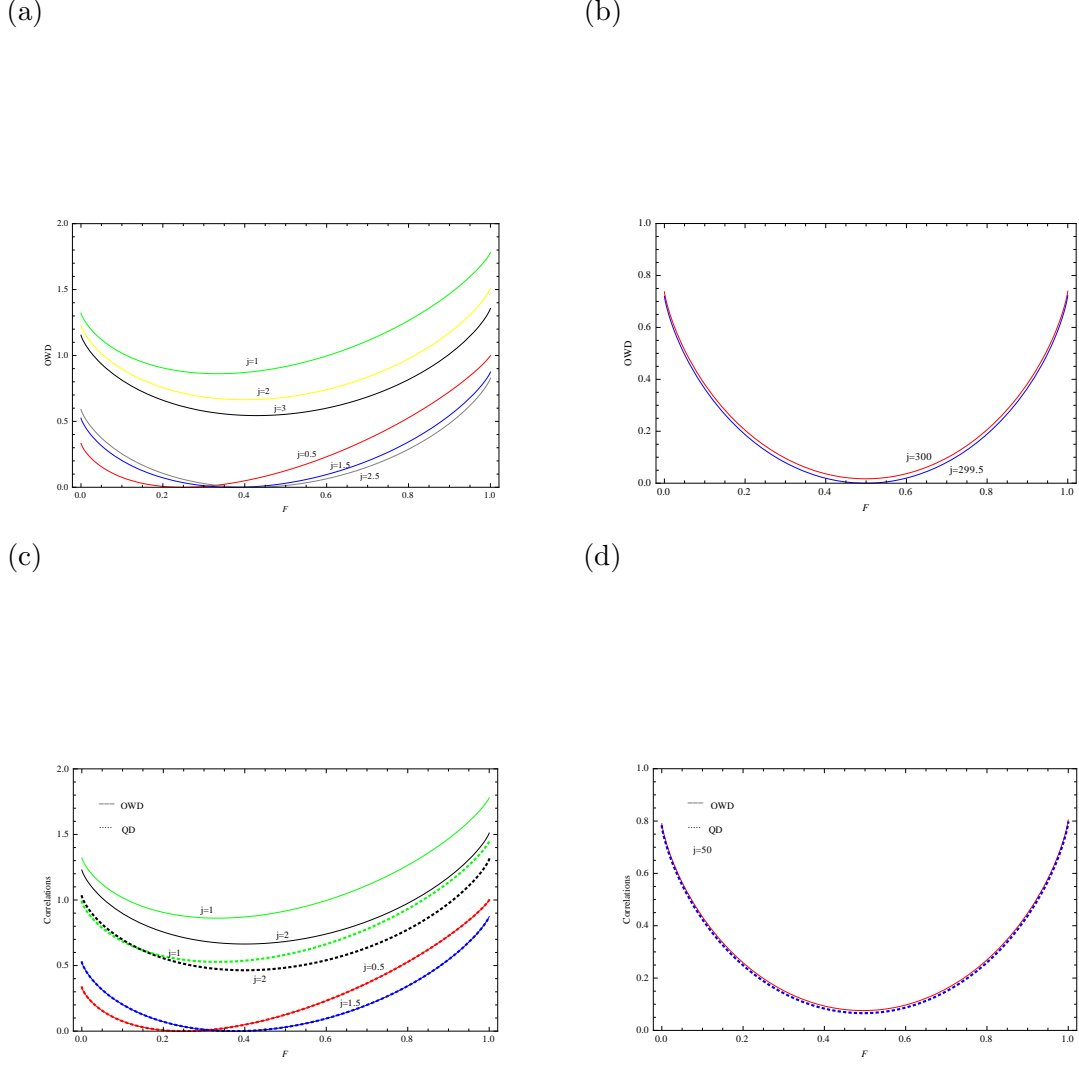


FIG. 1: (Color online) quantum correlations of the bipartite state composed of a spin- $j$  and a spin- $\frac{1}{2}$  vs  $F$ . (a) one-way deficit of  $j = 1, j = 2, j = 3$  and  $j = \frac{1}{2}, j = \frac{3}{2}, j = \frac{5}{2}$ , (b) one-way deficit of  $j = 300$  and  $j = \frac{599}{2}$ , (c) comparison of one-way deficit (solid) and quantum discord(dotted) of  $j = 1, j = 2$  and  $j = \frac{1}{2}, j = \frac{3}{2}$ , (d) comparison of one-way deficit(solid) and quantum discord (dotted) of  $j = 50$ .

way deficit and the quantum discord is  $\frac{1}{d}$ , see Fig.1(c), and the difference tends to be small for large  $j$ , see Fig.1(d).

We now compare the one-way deficit with the entanglement of formation (EoF). The

EoF for a spin- $\frac{1}{2}$  and a spin- $j$   $SU(2)$  invariant state  $\rho^{ab}$  is given by [35],

$$EoF = \begin{cases} 0, & F \in [0, \frac{2j}{2j+1}] \\ H\left(\frac{1}{2j+1} \left(\sqrt{F} - \sqrt{2j(1-F)}\right)^2\right), & F \in [\frac{2j}{2j+1}, 1], \end{cases} \quad (15)$$

where  $H(x) = -x\log x - (1-x)\log(1-x)$  is the binary entropy. It is shown that EoF becomes an upper bound for quantum discord in  $d \otimes d$  Werner states [41]. However, we can see that the one-way deficit always remains larger than EoF for half-integer  $j$ . The difference between EoF and  $\Delta^{\rightarrow}(\rho^{ab})$  increases as  $j \rightarrow \infty$  [39]]. But for integer  $j$ , the difference decreases as  $j \rightarrow \infty$ , see Fig.2 (a) and (b). It should be noted that as  $j \rightarrow \infty$ , the state  $\rho^{ab}$  becomes separable while its one-way deficit remains finite. In the region of zero EoF, the one-way deficit survives.

(a)

(b)

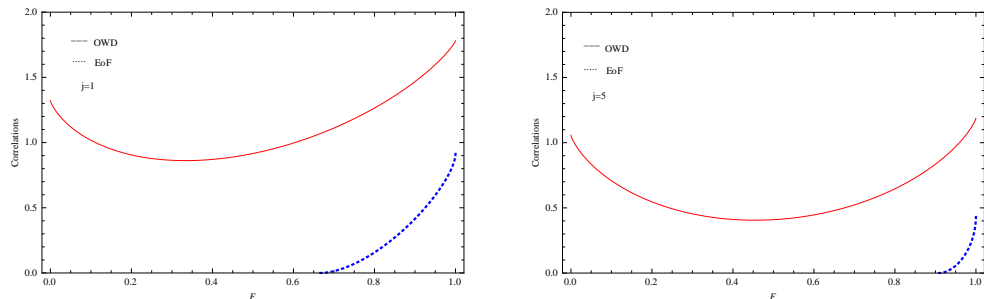


FIG. 2: (Color online) one-way deficit (solid line) and EoF (dotted line) vs.  $F$ : (a)  $j = 1$ , (b)  $j = 5$ .

### III. CONCLUSION

We have analytically calculated the one-way deficit of  $SU(2)$  invariant states consisting of a spin- $j$  and a spin- $\frac{1}{2}$  subsystems, with measurement on the spin- $\frac{1}{2}$  subsystem. By comparing the one-way deficit with the quantum discord of these states we have shown that the one-way

deficit is equal to the quantum discord for half-integer  $j$ , and the one-way deficit is larger than the quantum discord for integer  $j$ . We have also compared our results on one-way deficit with the quantum entanglement EoF. It is shown that in the large  $j$  limit, one-way deficit remains significantly larger than EoF. Moreover, the maximal value of one-way deficit decreases with the increasing system size. As there are abundance of  $SU(2)$  invariant states in real physical systems, our results can be used in quantum protocols that rely on one-way deficit.

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