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by

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Abstract

We use simple methods from harmonic maps to investigate singularities of period mappings at infinity. More precisely, we derive a harmonic map version of Schmid's nilpotent orbit theorem.

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1 Constructions, Results, and Consequences

Let $\Delta^*(\subset \Delta)$ be the punctured disk $\{z \in \mathbb{C} | 0 < |z| < 1\}$ with the Poincaré metric

$$\sqrt{-1}\frac{dz \wedge d\bar{z}}{|z|^2(\log|z|)^2},$$

G/K a symmetric space of noncompact type, here, G is a semisimple Lie group and K the corresponding maximal compact subgroup. Suppose that $\rho : \pi_1(\Delta^*) \to G$ is a unipotent representation, i.e. its images are unipotent elements in G, and $u : \mathcal{H} \to G/K$ a ρ-equivariant harmonic map with finite energy. Here, $\mathcal{H}$ is the upper half plane which is considered as the universal covering space of Δ^* under the map $w = e^{\sqrt{-1}z}$. The purpose of this note is, by using techniques from harmonic maps, to consider the singularities of u at the infinity of $\mathcal{H}$ (or equivalently, near the puncture of Δ^*).

A prototype of the above problem is the degeneration of the period mapping [12, 3]. Let G/H be a period domain (more generally a homogeneous complex manifold, where H is the centralizer of a circle subgroup of G; special examples are bounded symmetric domains, [4]); it can be considered as a fibre space over a symmetric space G/K of noncompact type with the compact fiber K/H, $p : G/H \to G/K$, where K is the maximal compact subgroup of G containing H. We remark that the G/K is not in general a complex manifold, unless it is a bounded symmetric domain. Then, locally, a *period mapping* is a ρ-equivariant map Φ from the universal covering space $\mathcal{H}$ of Δ^* into the

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homogeneous complex manifold G/H for a certain unipotent representation $\rho : \pi_1(\Delta^*) \to G$, where Φ is

1) holomorphic and

2) horizontal with respect to $p : G/H \to G/K$.

In general, the period mapping is singular at the puncture. This singularity was analyzed in depth in the fundamental paper of Wilfried Schmid [12]. The period mapping defined above arises naturally from a *Variation of Hodge Structure* defined over the rational number field $\mathbb{Q}$ up to a finite lifting.

An observation of Lu (cf. [10], Theorem 1.1) shows that p, restricted to a horizontal slice of G/H, is pluriharmonic. So, this, together with the horizontality of Φ, implies that *the composition* $p \circ \Phi$ *is harmonic.* On the other hand, Φ is of finite energy on Δ^* (or a fundamental domain), and so *the composition* $p \circ \Phi$ *is also of finite energy.* This can be seen by using two different arguments. One is to observe the asymptotic behavior (cf. e.g. [14]) of the Gauss-Manin connection (or the differential of Φ): $\frac{dz}{z}N$ (here N is a nilpotent element in the corresponding Lie algebra, see also the discussion below), which has finite L^2 norm under the Poincaré metric, and so Φ has finite energy; obviously, this argument is based on Schmid's theory [12]. The other[1], due to Lu and Sun, is that by computing the curvature of the Hodge metric one can get the finiteness of volume of holomorphic subvarieties under the Hodge metric, in particular Φ has finite energy (for details, cf. [11], Theorem 5.2 and its proof). This argument depends on the Schwarz lemma and is independent of Schmid's theory.

Return to the above equivariant harmonic maps situation. Without loss of generality, we can restrict ourself to the case of $G = SL(n, \mathbb{C})$ or one of its semisimple subgroups. Let γ be a generator of $\pi_1(\Delta^*)$. By the assumption, $\log \rho(\gamma)$ is nilpotent, denoted by N. Then, by the Jacobson-Morozov's theorem (cf. e.g.[12]), we can obtain a semisimple element Y so that $\{N, Y\}$ can be extended to an $\mathfrak{sl}_2$-triple $\{N, N^-, Y\}$. Moreover, such an $\mathfrak{sl}_2$-triple is unique up to conjugation in G [9].

Now, we can construct a canonical ρ-equivariant mapping from the universal covering $\mathcal{H}$ of (Δ^*) into the symmetric space $SL(n, \mathbb{C})/SU(n)$ as follows. Let $\mathcal{P}_n$ be the set of positive definite Hermitian symmetric matrices of order n with determinant 1. $SL(n, \mathbb{C})$ acts transitively on $\mathcal{P}_n$ by

$$g \circ H =: gH\bar{g}^t, \ H \in \mathcal{P}_n, g \in SL(n, \mathbb{C}).$$

Obviously, the action has the isotropy subgroup $SU(n)$ at the identity I_n. Thus, $\mathcal{P}_n$ can be identified with the symmetric space of noncompact type

[1]The second author thanks Zhiqin Lu for informing him of this observation.

$SL(n,\mathbb{C})/SU(n)$. In particular, under the invariant metric on $\mathcal{P}_n$, the geodesics through the identity I_n are of the form $\exp(tA)$, $t \in \mathbb{R}$, where A is a Hermitian symmetric matrix. Write $z = re^{\sqrt{-1}\theta}$; $(-\log r, \theta)$ can be considered as the coordinate of $\mathcal{H}$. Now, we can set

$$h_0(z) = \exp(\frac{1}{2\pi}\theta N) \circ \exp((\frac{1}{2}\log|\log r|)Y), \tag{1}$$

which is a ρ-equivariant map from $\mathcal{H}$ into the symmetric space $SL(n,\mathbb{C})/SU(n)$.

Geometrically, such a construction gives an equivariant geodesic embedding of the upper half plane into the symmetric space $SL(n,\mathbb{C})/SU(n)$; based on this, we can consider this map as a canonical one. So, as a map from $\mathcal{H}$ into $SL(n,\mathbb{C})/SU(n)$, h_0 is harmonic. In the sequel, for sake of convenience, we also consider a ρ-equivariant map from $\mathcal{H}$ as a ρ-equivariant map from the punctured disk. An easy computation also shows that it is of finite energy (on a fundamental domain w.r.t. the representation ρ).

In this note, we want to prove the following

Theorem. *Suppose $h : \Delta^* \to SL(n,\mathbb{C})/SU(n)$ is a ρ-equivariant harmonic map with finite energy. Then, h has the same asymptotic behavior as h_0 near the puncture of Δ^*; more precisely, under the invariant metric of $\mathcal{P}_n$, the distance function between h and h_0 is uniformly bounded on Δ^*.*

Remark. *The theorem can be considered as the version for harmonic maps of Schmid's nilpotent orbit theorem.*

The reprensentation ρ induces an n-dimensional flat complex vector bundle over Δ^* by considering ρ as a representation on $\mathbb{C}^n$, denoted by L_ρ. Then, the map h_0 (and h) can be considered as a metric on L_ρ—*harmonic metric* (cf. e.g.[13]), which is a natural generalization of the Hodge metric on a variation of Hodge strucutre. We shall now analyze the asymptotic behavior of h_0 (and h) as a metric on L_ρ. The main issue is the r-direction, since the θ-direction is compact (equivariant).

Without loss of generality, we may assume here that the representation is irreducible. This means that the Jordan normal form of N has only one Jordan block. Let $\{N, N^-, Y\}$ be the corresponding $\mathfrak{sl}_2$-triple. The semisimple element Y can actually be described as follows. Canonically, $\mathbb{C}^n$ has a filtration

$$0 \subset W_{-(n-1)} \subset W_{-(n-3)} \subset \cdots \subset W_{n-3} \subset W_{n-1} = \mathbb{C}^n, \tag{2}$$

with the properties that $N(W_i) \subset W_{i-2}$, Y preserves each W_i, and all the quotients W_i/W_{i-2} are 1-dimensional. Then the (induced) action of Y on

W_i/W_{i-2} is multiplication by i. Actually, one can also choose a basis

$$\{e_{-(n-1)}, e_{-(n-3)}, \cdots, e_{n-3}, e_{n-1}\}$$

of $\mathbb{C}^n$, which is compatible with the above filtration (i.e., $Ne_j = e_{j-2}$ and $\{e_j\}_{j\le i}$ generate W_i) and satisfy $Ye_i = ie_i$. $\exp((\frac{1}{2}\log|\log r|)Y)$, under the above basis, can then be written as

$$\begin{pmatrix} |\log r|^{\frac{-(n-1)}{2}} & 0 & \cdots & 0 & 0 \\ 0 & |\log r|^{\frac{-(n-3)}{2}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & |\log r|^{\frac{n-3}{2}} & 0 \\ 0 & 0 & \cdots & 0 & |\log r|^{\frac{n-1}{2}} \end{pmatrix}. \tag{3}$$

On the other hand, by using the invariant metric of $\mathcal{P}_n$, a simple computation also shows that the energy of h_0 is finite, namely,

$$E(h_0) = \int_{\Delta^*} |h_0^{-1}dh_0|^2 * 1 \le C\int_0^\alpha |\log r|^{-2}r^{-1}dr < \infty.$$

From the above argument, we can now see the asymptotic behavior of the norms near the puncture of flat sections of L_ρ under the metric h_0 (and hence h). For $p \in \Delta^*$, the fiber $(L_\rho)_p$ canonically has a weight filtration $\{W_l\}_{l=-k}^k$ (k is the weight of N) arising from N and satisfying $N(W_l) \subset W_{l-2}$; this filtration is moreover invariant w.r.t. the flat connection D of L_ρ and hence determines a filtration of L_ρ by some local subsystems, denoted by $\mathbf{W}_l$, $-k \le l \le k$. We remark that $\{W_l\}_{l=-k}^k$ can be decomposed into the direct sum of some subfiltrations, each of which corresponds to a unique Jordan block of N and is of the form as in (2), if the number of the Jordan blocks of N is > 1. By the construction of h_0 and (3), a flat section v of $\mathbf{W}_l$, if not lying in $\mathbf{W}_{l-1}$, has the following norm estimate[2] under h_0 (and hence h)

$$\|v\|^2 \sim |\log r|^l, \tag{4}$$

on any ray from the puncture of Δ^*. This is just Schmid's norm estimate for Hodge metrics [12].

The basic idea of this note was already utilized in our joint paper [7], where we discussed the cohomologies of harmonic bundles over noncompact curves; so, this note can also be considered as an appendix of [7].

[2]Here, we use the notation $\sim$ to mean "is within a bounded multiple of".

2 Proof of Theorem

The proof of the theorem utilizes some simple and well-known facts about harmonic maps, see e.g. [5] as a reference. From the point of view of geometric analysis, the key point is that the target space G/K as a symmetric space of noncompact type has non-positive sectional curvature. Let $\Delta_i^*(\subset \Delta^*)$ be the punctured disk with the radius $\frac{1}{i}$. By minimizing the energy among all ρ-equivariant maps with prescribed values on the boundary of $\Delta^* \setminus \Delta_i^*$ given by the values of h on that boundary, we obtain a ρ-equivariant harmonic map h_i on $\Delta^* \setminus \Delta_i^*$ with

$$\begin{aligned} h_i|_{\partial\Delta} &= h_0|_{\partial\Delta}, \\ h_i|_{\partial\Delta_i} &= h|_{\partial\Delta_i}, \\ E(h_i, \Delta^* \setminus \Delta_i^*) &\leq E(h, \Delta^* \setminus \Delta_i^*) + C \quad \leq E(h, \Delta^*) + C < +\infty, \end{aligned} \tag{5}$$

where C is some nonnegative constant independent of i. (That C need not be 0 stems from the fact that on $\partial\Delta$, we impose the boundary values of h_0 and not those of h.) On the other hand, by a standard computation, the function

$$(\mathrm{dist}_{\mathcal{P}_n}(h_i, h))^2$$

is subharmonic on $\Delta^* \setminus \Delta_i^*$. So, by the maximum principle, we have

$$\mathrm{dist}_{\mathcal{P}_n}(h_i, h)|_{\Delta^* \setminus \Delta_i^*} \leq \max_{\partial\Delta} \mathrm{dist}_{\mathcal{P}_n}(h, h_0). \tag{6}$$

Thus, (5) and (6) together with standard estimates imply that h_i (possibly after selecting a subsequence) converges uniformly on (any compact subset of) Δ^* to some harmonic limit $\tilde{h}$ that satisfies

$$\tilde{h}|_{\partial\Delta} = h_0|_{\partial\Delta}, \tag{7}$$

$$E(\tilde{h}, \Delta^*) < +\infty. \tag{8}$$

Altogether, we get a ρ-equivariant harmonic map $\tilde{h} : \Delta^* \to SL(n, \mathbb{C})/SU(n)$ with

$$1)\ E(\tilde{h}, \Delta^*) < \infty, \tag{9}$$

$$2)\ \mathrm{dist}_{\mathcal{P}_n}(h, \tilde{h}) \leq \max_{\partial\Delta} \mathrm{dist}_{\mathcal{P}_n}(h, h_0), \text{on } \Delta^* \tag{10}$$

$$3)\ \tilde{h}|_{\partial\Delta} = h_0|_{\partial\Delta}. \tag{11}$$

By 2), h and $\tilde{h}$ have the same asymptotic behavior near the puncture. We now want to show that actually $\tilde{h} \equiv h_0$ on Δ^*, and hence the theorem is obtained.

Consider the square of the distance function between $\tilde{h}$ and h_0

$$(\mathrm{dist}_{\mathcal{P}_n}(\tilde{h}, h_0))^2,$$

which is defined on Δ^* and subharmonic because of the nonpositive sectional curvature of G/K. Furthermore, $(\mathrm{dist}_{\mathcal{P}_n}(\tilde{h}, h_0))^2$ is of finite energy. Actually, the energy finiteness of $\tilde{h}$ and h_0, by Cheng's gradient estimate for harmonic maps, implies that they have bounded energy density under the Poincaré metric

$$\sqrt{-1}\frac{dz \wedge d\bar{z}}{|z|^2(\log|z|)^2}.$$

Consequently, $\mathrm{dist}_{\mathcal{P}_n}(\tilde{h}, h_0) \leq C \log|\log r|$, for some constant C. This, together with the energy finiteness of $\tilde{h}$ and h_0, implies that $(\mathrm{dist}_{\mathcal{P}_n}(\tilde{h}, h_0))^2$ is of finite energy. The theorem now follows from the following elementary lemma; for completeness, we reduce it to a standard result.

Lemma 1. *Let Δ^* be the punctured disk with the Poincaré metric. Then any finite energy non-negative subharmonic function w that vanishes on the exterior boundary vanishes identically in Δ^*.*

Proof. Because of the conformal invariance of the Dirichlet integral in the 2-dimensional case, neither the finiteness of the energy nor the subharmonicity depend on the metrics on the punctured disk (as long as it is conformally equivalent to the hyperbolic one). So, instead of the complete hyperbolic metric, we may as well use the incomplete Euclidean one on Δ^*. Consider the following sequence of harmonic functions

$$\begin{cases} u_i & \text{harmonic function on } \Delta^* \setminus \Delta_i^*, \\ u_i|_{\partial\Delta} & = w|_{\partial\Delta} = 0, \\ u_i|_{\partial\Delta_i} & = w|_{\partial\Delta_i}. \end{cases} \tag{12}$$

The maximum principle implies $u_i \geq w$ on $\Delta^* \setminus \Delta_i^*$ and

$$E(u_i; \Delta^* \setminus \Delta_i^*) \leq E(w; \Delta^* \setminus \Delta_i^*) \leq E(w).$$

The standard elliptic estimates then implies that u_i converges uniformly on (any compact subset of) Δ^* to a harmonic function u, which satisfies $u \geq w$, $u|_{\partial\Delta} = 0$, and $E(u) \leq E(w)$. Then, $u \equiv 0$, and hence $w \equiv 0$, follows from the standard removibility of isolated singularities of finite energy harmonic functions, see e.g. Corollary 11.2.2. in [6], and the maximum principle. □

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References

[1] E. Cattani, A. Kaplan, and W. Schmid, *Degeneration of Hodge structures.* Annales of Mathematics, **123** (1986), 457–535.

[2] M. Cornalba and P. Griffiths, *Analytic cycles and vector bundles on non-compact algebraic varieties.* Inventiones Math., **28** (1975), 1-106.

[3] P. Griffiths (ed.), *Topics in transcendental Algebraic Geometry.* Ann. of Math. Stud., vol. **106**, Princeton University Press, 1984.

[4] P. Griffiths and W. Schmid, *Locally homogeneous complex manifolds.* Acta Mathematica, **123** (1969), 253-302.

[5] J. Jost, *Riemannian geometry and geometric analysis*, 6th edition, Springer, 2011

[6] J. Jost, *Partial differential equations*, 3rd edition, Springer, 2013

[7] J. Jost, Y.-H. Yang, & K. Zuo, *Cohomologies on unipotent bundles over non-compact curves.* J. reine angew. Math. **609** (2007), 137-159.

[8] J. Jost, Y.-H. Yang, & K. Zuo, *Harmonic metrics on unipotent bundles over quasi-compact Kähler manifolds.* Preprint.

[9] B. Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math. **81** (1959), 973C1032.

[10] Z. Lu, *On the geometry of classifying spaces and horizontal slices.* American Journal of Mathematics, **121**(1999), 177-198.

[11] Z. Lu & Xiaofeng Sun, *Weil-Petersson geometry on moduli space of polarized Calabi-Yau manifolds*, J. Inst. Math. Jussieu, **3(2)** (2004), 185-229.

[12] W. Schmid, *Variation of Hodge structure: the sigularities of the period mapping.* Invent. Math., **22** (1973), 211-319.

[13] C. Simpson, *Harmonic bundles on noncompact curves.* J. Amer. Math. Soc., **3** (1990), 513-570.

[14] S. Zucker, *Hodge theory with degenerating coeffcients: L^2 cohomology in the Poincaré metric.* Ann. Math. **109** (1979), 415C476.

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