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Systems

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# Entanglement Monogamy Relations of Qubit Systems

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We investigate the monogamy relations related to the concurrence and the entanglement of formation. General monogamy inequalities given by the  $\alpha$ th power of concurrence and entanglement of formation are presented for  $N$ -qubit states. The monogamy relation for entanglement of assistance is also established. Based on these general monogamy relations, the residual entanglement of concurrence and entanglement of formation are studied. Some relations among the residual entanglement, entanglement of assistance and three tangle are also presented.

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## I. INTRODUCTION

Quantum entanglement [1–6] is an essential feature of quantum mechanics, which distinguishes the quantum from classical world. As one of the fundamental differences between quantum entanglement and classical correlations, a key property of entanglement is that a quantum system entangled with one of other systems limits its entanglement with the remaining others. The monogamy relations give rise to the structures of entanglement in the multipartite setting. Monogamy is also an essential feature allowing for security in quantum key distribution [7].

For a tripartite system  $A$ ,  $B$  and  $C$ , the monogamy of an entanglement measure  $\varepsilon$  implies that [8], the entanglement between  $A$  and  $BC$  satisfies  $\varepsilon_{A|BC} \geq \varepsilon_{AB} + \varepsilon_{AC}$ . Such monogamy relations are not always satisfied by entanglement measures. Although the concurrence  $C$  and entanglement of formation  $E$  do not satisfy such monogamy inequality, it has been shown that the squared concurrence  $C^2$  [9, 10] and the squared entanglement of formation  $E^2$  [11] do satisfy the monogamy relations.

In this paper, we study the general monogamy inequalities satisfied by the  $\alpha$ th power of concurrence  $C^\alpha$  and the  $\alpha$ th power of entanglement of formation  $E^\alpha$ . We show that  $C^\alpha$  and  $E^\alpha$  satisfy the monogamy inequalities for  $\alpha \geq 2$  and  $\alpha \geq \sqrt{2}$ , respectively. The monogamy relations for the entanglement of assistance are also established. Correspondingly, the residual entanglement of concurrence and entanglement of formation are also investigated.

## II. MONOGAMY RELATION OF CONCURRENCE

For a bipartite pure state  $|\psi\rangle_{AB}$  in vector space  $H_A \otimes H_B$ , the concurrence is given by [12–14]

$$C(|\psi\rangle_{AB}) = \sqrt{2[1 - \text{Tr}(\rho_A^2)]}, \quad (1)$$

where  $\rho_A$  is reduced density matrix by tracing over the subsystem  $B$ ,  $\rho_A = \text{Tr}_B(|\psi\rangle_{AB}\langle\psi|)$ . The concurrence

for a tripartite mixed state  $\rho_{AB}$  is defined by the convex roof,

$$C(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle), \quad (2)$$

where the minimum (infimum) is taken over all possible decompositions of  $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ , with  $p_i \geq 0$  and  $\sum_i p_i = 1$  and  $|\psi_i\rangle \in H_A \otimes H_B$ .

For a tripartite state  $|\psi\rangle_{ABC}$ , the concurrence of assistance (CoA) is defined by [15, 16]

$$C_a(|\psi\rangle_{ABC}) \equiv C_a(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle), \quad (3)$$

where the maximum (supremum) is taken over all possible decompositions of  $\rho_{AB} = \text{Tr}_C(|\psi\rangle_{ABC}\langle\psi|) = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|$ . When  $\rho_{AB} = |\psi\rangle_{AB}\langle\psi|$  is a pure state, then one has  $C(|\psi\rangle_{AB}) = C_a(\rho_{AB})$ .

For an  $N$ -qubit state  $|\psi\rangle_{AB_1 \dots B_{N-1}} \in H_A \otimes H_{B_1} \otimes \dots \otimes H_{B_{N-1}}$ , the concurrence  $C(|\psi\rangle_{A|B_1 \dots B_{N-1}})$  of the state  $|\psi\rangle_{A|B_1 \dots B_{N-1}}$ , viewed as a bipartite with partitions  $A$  and  $B_1 B_2 \dots B_{N-1}$ , satisfies the Coffman-Kundu-Wootters (CKW) inequality [9, 10],

$$C_{A|B_1 B_2 \dots B_{N-1}}^2 \geq C_{AB_1}^2 + C_{AB_2}^2 + \dots + C_{AB_{N-1}}^2, \quad (4)$$

where  $C_{AB_i} = C(\rho_{AB_i})$  is the concurrence of  $\rho_{AB_i} = \text{Tr}_{B_1 \dots B_{i-1} B_{i+1} \dots B_{N-1}}(\rho)$ ,  $C_{A|B_1 B_2 \dots B_{N-1}} = C(|\psi\rangle_{A|B_1 \dots B_{N-1}})$ .

Dual to the CKW inequality, the generalized monogamy relation based on the concurrence of assistance was proved in Ref. [17],

$$C^2(|\psi\rangle_{A|B_1 \dots B_{N-1}}) \leq \sum_{i=1}^{N-1} C_a^2(\rho_{AB_i}). \quad (5)$$

The inequalities (4) and (5) are valid because, instead of the concurrence and CoA, the squared concurrence and CoA are used. In fact, besides the squared concurrence, one can get the following general monogamy inequalities:

**Theorem 1** For any  $2 \otimes 2 \dots \otimes 2 \otimes 2$  mixed state  $\rho \in H_A \otimes H_{B_1} \otimes \dots \otimes H_{B_{N-1}}$ , we have

$$C_{A|B_1 B_2 \dots B_{N-1}}^\alpha \geq C_{AB_1}^\alpha + \dots + C_{AB_{N-1}}^\alpha \quad (6)$$

for all  $\alpha \geq 2$ .

[Proof] For arbitrary  $2 \otimes 2 \otimes 2^{n-2}$  tripartite state  $\rho_{ABC}$ , one has [9, 18],

$$C_{A|BC}^2 \geq C_{AB}^2 + C_{AC}^2.$$

If  $\min\{C_{AB}, C_{AC}\} = 0$ , obviously we have  $C_{A|BC}^\alpha \geq C_{AB}^\alpha + C_{AC}^\alpha$ . If  $\min\{C_{AB}, C_{AC}\} > 0$ , assuming  $C_{AB} \geq C_{AC}$ , we have

$$\begin{aligned} C_{A|BC}^\alpha &\geq (C_{AB}^2 + C_{AC}^2)^{\frac{\alpha}{2}} \\ &= C_{AB}^\alpha \left(1 + \frac{C_{AC}^2}{C_{AB}^2}\right)^{\frac{\alpha}{2}} \\ &\geq C_{AB}^\alpha \left(1 + \left(\frac{C_{AC}^2}{C_{AB}^2}\right)^{\frac{\alpha}{2}}\right) \\ &= C_{AB}^\alpha + C_{AC}^\alpha, \end{aligned} \quad (7)$$

where the second inequality is due to the inequality  $(1+x)^t \geq 1+x^t$  for  $x \leq 1$  and  $t \geq 1$ .

By partitioning the last qudit system C into two subsystems: a qubit system  $C_1$  and a  $2^{n-3}$ -dimensional qudit system  $C_2$ , and using the above inequality repeatedly, one gets (6).  $\square$

**Theorem 2** For any  $2 \otimes 2 \otimes \dots \otimes 2$  mixed state  $\rho \in H_A \otimes H_{B_1} \otimes \dots \otimes H_{B_{N-1}}$  with  $C_{AB_i} \neq 0$ ,  $i = 1, \dots, N-1$ , we have

$$C_{A|B_1 \dots B_{N-1}}^\alpha < C_{AB_1}^\alpha + \dots + C_{AB_{N-1}}^\alpha \quad (8)$$

for  $\alpha \leq 0$ .

[Proof] Similar to the proof of Theorem 1, we only need to prove the inequality is true for arbitrary  $2 \otimes 2 \otimes 2^{n-2}$  tripartite states,

$$\begin{aligned} C_{A|BC}^\alpha &\leq (C_{AB}^2 + C_{AC}^2)^{\frac{\alpha}{2}} \\ &= C_{AB}^\alpha \left(1 + \frac{C_{AC}^2}{C_{AB}^2}\right)^{\frac{\alpha}{2}} \\ &< C_{AB}^\alpha \left(1 + \left(\frac{C_{AC}^2}{C_{AB}^2}\right)^{\frac{\alpha}{2}}\right) \\ &= C_{AB}^\alpha + C_{AC}^\alpha, \end{aligned} \quad (9)$$

where the first inequality is due to  $\alpha \leq 0$  and the second inequality is due to  $C_{AB}^\alpha > 0$  and the inequality  $(1+x)^t < 1+x^t$  for  $x > 0$  and  $t \leq 0$ .  $\square$

In (8) we have assumed that all  $C_{AB_i}$ ,  $i = 1, \dots, N-1$ , are nonzero. In fact, if one of them is zero, the inequality still holds if one removes this term from the inequality. Namely, if  $C_{AB_i} = 0$ , then one has  $C_{A|B_1 \dots B_{N-1}}^\alpha < C_{AB_1}^\alpha + \dots + C_{AB_{i-1}}^\alpha + C_{AB_{i+1}}^\alpha + \dots + C_{AB_{N-1}}^\alpha$  for  $\alpha \leq 0$ .

Theorem 1 shows that the  $\alpha$ th power of concurrence  $C^\alpha$  satisfies the monogamy inequality (6) for  $\alpha \geq 2$ . While Theorem 2 shows that for  $\alpha \leq 0$ , the inequality is reversed. However, for  $0 < \alpha < 2$ , the situation is not clear. Let us consider the three-qubit case. Any three-qubit state  $|\psi\rangle$  can be written in the generalized Schmidt decomposition [19, 20],

$$\begin{aligned} |\psi\rangle &= \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|100\rangle + \lambda_2|101\rangle \\ &\quad + \lambda_3|110\rangle + \lambda_4|111\rangle, \end{aligned} \quad (10)$$

where  $\lambda_i \geq 0$ ,  $i = 0, \dots, 4$ , and  $\sum_{i=0}^4 \lambda_i^2 = 1$ . From Eq.(1) and Eq.(2), we have  $C_{A|BC} = 2\lambda_0 \sqrt{\lambda_2^2 + \lambda_3^2 + \lambda_4^2}$ ,  $C_{AB} = 2\lambda_0 \lambda_2$ , and  $C_{AC} = 2\lambda_0 \lambda_3$ . Without loss of generality, we set  $\lambda_0 = \cos \theta_0$ ,  $\lambda_1 = \sin \theta_0 \cos \theta_1$ ,  $\lambda_2 = \sin \theta_0 \sin \theta_1 \cos \theta_2$ ,  $\lambda_3 = \sin \theta_0 \sin \theta_1 \sin \theta_2 \cos \theta_3$ , and  $\lambda_4 = \sin \theta_0 \sin \theta_1 \sin \theta_2 \sin \theta_3$ ,  $\theta_i \in [0, \frac{\pi}{2}]$ . Then we have

$$\begin{aligned} C_{A|BC}^\alpha - C_{AB}^\alpha - C_{AC}^\alpha &= (2\lambda_0)^\alpha \left[ (\lambda_2^2 + \lambda_3^2 + \lambda_4^2)^{\frac{\alpha}{2}} - \lambda_2^\alpha - \lambda_3^\alpha \right] \\ &= (2\lambda_0)^\alpha \sin^\alpha \theta_0 \sin^\alpha \theta_1 \left[ 1 - \cos^\alpha \theta_2 - \sin^\alpha \theta_2 \cos^\alpha \theta_3 \right]. \end{aligned} \quad (11)$$

From (11) we have  $C_{A|BC}^\alpha \geq C_{AB}^\alpha + C_{AC}^\alpha$  for  $\alpha \geq 2$ . While for  $\alpha \leq 0$  one has  $C_{A|BC}^\alpha < C_{AB}^\alpha + C_{AC}^\alpha$ . However, for  $0 < \alpha < 2$ , one can see that the sign of  $(C_{A|BC}^\alpha - C_{AB}^\alpha - C_{AC}^\alpha)$  is not certain.

### III. RESIDUAL ENTANGLEMENT OF CONCURRENCE

Similar to the three tangle of concurrence, for the three qubit state  $|\psi\rangle_{ABC} \in H_A \otimes H_B \otimes H_C$ , we can define the residual entanglement

$$\tau_\alpha^C(|\psi\rangle_{ABC}) = C_{A|BC}^\alpha - C_{AB}^\alpha - C_{AC}^\alpha, \quad (12)$$

where  $\alpha \geq 2$ .

**Theorem 3** For any three qubit pure state  $|\psi\rangle \in H_A \otimes H_B \otimes H_C$ ,

(1)  $|\psi\rangle$  is bipartite separable state if and only if for any  $\alpha \geq 2$ ,

$$\tau_\alpha^C(|\psi\rangle) = 0;$$

(2)  $|\psi\rangle$  is genuine entangled if and only if there is an  $\alpha \geq 2$  such that

$$\tau_\alpha^C(|\psi\rangle) > 0.$$

[Proof] (1) If  $|\psi\rangle$  is bipartite separable state, without loss of generality, we assume that  $|\psi\rangle$  is a  $B|AC$  bipartite separable state, then we have  $C_{B|AC} = C_{BA} = C_{BC} = 0$ . From (10) we have  $\lambda_0 = 0$  and  $|\lambda_1 \lambda_4 e^{i\varphi} - \lambda_2 \lambda_3| = 0$ , or  $\lambda_3 = \lambda_4 = 0$  and  $|\lambda_1 \lambda_4 e^{i\varphi} - \lambda_2 \lambda_3| = 0$ . For both the

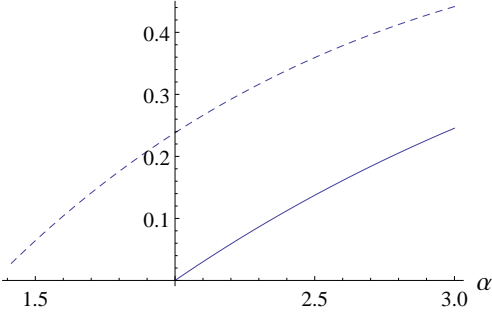


FIG. 1: Solid line:  $\tau_\alpha^C(|W\rangle)$  as a function of  $\alpha$  ( $\alpha \geq 2$ ); dashed line:  $\tau_\alpha^E(|W\rangle)$  as a function of  $\alpha$  ( $\alpha \geq \sqrt{2}$ ).

above two cases, we have  $C_{A|BC} = C_{AC}$  and  $C_{AB} = 0$ . Therefore  $\tau_\alpha^C(|\psi\rangle) = 0$  for all  $\alpha$ .

If  $\tau_\alpha^C(|\psi\rangle) = 0$  for any  $\alpha \geq 2$ , then we obtain  $C_{A|BC}^{2\alpha} - C_{AB}^{2\alpha} - C_{AC}^{2\alpha} = (C_{A|BC}^\alpha)^2 - C_{AB}^{2\alpha} - C_{AC}^{2\alpha} = (C_{AB}^\alpha + C_{AC}^\alpha)^2 - C_{AB}^{2\alpha} - C_{AC}^{2\alpha} = 2C_{AB}^\alpha C_{AC}^\alpha = 0$ , i.e. either  $C_{AB}^\alpha$  or  $C_{AC}^\alpha$  is zero. Without loss of generality, assuming  $C_{AB} = 0$ , we have  $\lambda_0 = 0$  or  $\lambda_3 = 0$ . If  $\lambda_0 = 0$ , then  $C_{A|BC} = 0$ . Hence  $|\psi\rangle$  is  $A|BC$  a bipartite separable state. If  $\lambda_0 \neq 0$  and  $\lambda_3 = 0$ , since  $\tau_2^C = 0$ , we have  $\lambda_4 = 0$ . Hence  $C_{B|AC} = 0$ , i.e.  $|\psi\rangle$  is  $B|AC$  bipartite separable.

(2) By using the above proof and the fact that  $\tau_\alpha^C(|\psi\rangle) \geq 0$  for all  $\alpha \geq 2$  from Theorem 1, one gets the result directly.  $\square$

*Example 1.* Let us consider the  $W$ -state

$$|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle). \quad (13)$$

We have  $\tau_\alpha^C(|W\rangle) = (\frac{2}{\sqrt{3}})^\alpha \left( \sqrt{(\frac{2}{\sqrt{3}})^\alpha} - 2(\frac{1}{\sqrt{3}})^\alpha \right)$ . For  $\alpha = 2$ ,  $\tau_\alpha^C$  is just the three tangle of concurrence. As  $\tau_2^C(|W\rangle) = 0$ , the three tangle of concurrence can not capture the genuine entanglement of the  $W$ -state. Nevertheless, for  $\alpha > 2$ , our residual entanglement of concurrence  $\tau_\alpha^C(|W\rangle) > 0$ , see Fig. (1).

#### IV. MONOGAMY INEQUALITY FOR EOF

The entanglement of formation (EoF) [21, 22] is a well-defined important measure of entanglement for bipartite systems. Let  $H_A$  and  $H_B$  be  $m$ - and  $n$ -dimensional ( $m \leq n$ ) vector spaces, respectively. The EoF of a pure state  $|\psi\rangle \in H_A \otimes H_B$  is defined by

$$E(|\psi\rangle) = S(\rho_A), \quad (14)$$

where  $\rho_A = \text{Tr}_B(|\psi\rangle\langle\psi|)$  and  $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$ . For a bipartite mixed state  $\rho_{AB} \in H_A \otimes H_B$ , the entanglement of formation is given by

$$E(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle), \quad (15)$$

with the minimum (infimum) taking over all possible decompositions of  $\rho_{AB}$  in a mixture of pure states  $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ , where  $p_i \geq 0$  and  $\sum_i p_i = 1$ . The corresponding entanglement of assistance (EoA) [23] is defined in terms of the entropy of entanglement [24] for a tripartite pure state  $|\psi\rangle_{ABC}$ ,

$$E_a(|\psi\rangle_{ABC}) \equiv E_a(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle), \quad (16)$$

where the maximum (supremum) is taken over all possible decompositions of  $\rho_{AB} = \text{Tr}_C(|\psi\rangle_{ABC}) = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ , with  $p_i \geq 0$  and  $\sum_i p_i = 1$ .

Denote  $f(x) = H\left(\frac{1+\sqrt{1-x}}{2}\right)$ , where  $H(x) = -x \log_2(x) - (1-x) \log_2(1-x)$ . From Eq. (14) and Eq. (15), one has  $E(|\varphi\rangle) = f(C^2(|\varphi\rangle))$  for  $2 \otimes m$  ( $m \geq 2$ ) pure state  $|\varphi\rangle$ , and  $E(\rho) = f(C^2(\rho))$  for two qubit mixed state  $\rho$  [25]. It is obviously that  $f(x)$  is a monotonically increasing function for  $0 \leq x \leq 1$ .  $f(x)$  satisfies the following relations:

$$f^{\sqrt{2}}(x^2 + y^2) \geq f^{\sqrt{2}}(x^2) + f^{\sqrt{2}}(y^2), \quad (17)$$

$$f(x^2 + y^2) \leq f(x^2) + f(y^2), \quad (18)$$

where  $f^{\sqrt{2}}(x^2 + y^2) = (f(x^2 + y^2))^{\sqrt{2}}$ .

It has been shown that the entanglement of formation does not satisfy the inequality  $E_{AB} + E_{AC} \leq E_{A|BC}$  [26]. In [27] the authors showed that EoF is a monotonic function  $E^2(C_{A|B_1 B_2 \dots B_{N-1}}^2) \geq E^2(\sum_{i=1}^{N-1} C_{AB_i}^2)$ . It is further proved that for  $N$ -qubit systems, one has [11],

$$E_{A|B_1 B_2 \dots B_{N-1}}^2 \geq E_{AB_1}^2 + E_{AB_2}^2 + \dots + E_{AB_{N-1}}^2.$$

In fact, generally we can prove the following results:

**Theorem 4** For any  $N$ -qubit mixed state  $\rho \in H_A \otimes H_{B_1} \otimes \dots \otimes H_{B_{N-1}}$ , the entanglement of formation  $E(\rho)$  satisfies

$$E_{A|B_1 B_2 \dots B_{N-1}}^\alpha \geq E_{AB_1}^\alpha + E_{AB_2}^\alpha + \dots + E_{AB_{N-1}}^\alpha, \quad (19)$$

where  $\alpha \geq \sqrt{2}$ ,  $E_{A|B_1 B_2 \dots B_{N-1}}$  is the entanglement of formation of  $\rho$  in bipartite partition  $A|B_1 B_2 \dots B_{N-1}$ , and  $E_{AB_i}$ ,  $i = 1, 2, \dots, N-1$ , is the entanglement of formation of the mixed states  $\rho_{AB_i} = \text{Tr}_{B_1 B_2 \dots B_{i-1} B_{i+1} \dots B_{N-1}}(\rho)$ .

[Proof] Denote  $t = \alpha/\sqrt{2}$ . For  $\alpha \geq \sqrt{2}$ , we have

$$\begin{aligned} f^\alpha(x^2 + y^2) &= (f^{\sqrt{2}}(x^2 + y^2))^t \\ &\geq (f^{\sqrt{2}}(x^2) + f^{\sqrt{2}}(y^2))^t \\ &\geq (f^{\sqrt{2}}(x^2))^t + (f^{\sqrt{2}}(y^2))^t \\ &= f^\alpha(x^2) + f^\alpha(y^2), \end{aligned} \quad (20)$$

where the first inequality is due to the inequality (17), and the second inequality is obtained from a similar consideration in the proof of the second inequality in (7).

Let  $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \in H_A \otimes H_{B_1} \otimes H_{B_2} \otimes \dots \otimes H_{B_{N-1}}$  be the optimal decomposition of  $E_{A|B_1 B_2 \dots B_{N-1}}(\rho)$  for the  $N$ -qubit mixed state  $\rho$ , we have

$$\begin{aligned}
& E_{A|B_1 B_2 \dots B_{N-1}}(\rho) \\
&= \sum_i p_i E_{A|B_1 B_2 \dots B_{N-1}}(|\psi_i\rangle) \\
&= \sum_i p_i f\left(C_{A|B_1 B_2 \dots B_{N-1}}^2(|\psi_i\rangle)\right) \\
&\geq f\left(\left[\sum_i p_i C_{A|B_1 B_2 \dots B_{N-1}}(|\psi_i\rangle)\right]^2\right) \\
&\geq f(C_{A|B_1 B_2 \dots B_{N-1}}^2(\rho)) \\
&\geq f(C_{AB_1}^2 + C_{AB_2}^2 + \dots + C_{AB_{N-1}}^2) \\
&\geq \sqrt[\alpha]{f^\alpha(C_{AB_1}^2) + f^\alpha(C_{AB_2}^2) + \dots + f^\alpha(C_{AB_{N-1}}^2)} \\
&= \sqrt[\alpha]{E_{AB_1}^\alpha + E_{AB_2}^\alpha + \dots + E_{AB_{N-1}}^\alpha},
\end{aligned}$$

where the first inequality is due to that  $f(x^2)$  is a convex function of  $x$ . Due to the definition of concurrence (2) and that  $f(x)$  is a monotonically increasing function, we obtain the second inequality. We have used the monogamy inequality (4) for  $N$ -qubit states  $\rho$  to obtain the third inequality. The last inequality is due to the inequality (20). Since for any  $2 \otimes 2$  quantum state  $\rho_{AB_i}$ ,  $E(\rho_{AB_i})$  satisfies  $E(\rho_{AB_i}) = H\left(\frac{1+\sqrt{1-C^2(\rho_{AB_i})}}{2}\right) = f(C^2(\rho_{AB_i}))$ , one gets the last equality.  $\square$

The inequality (19) in Theorem 4 shows that the  $\alpha$ th power of EoF satisfies the monogamy inequality for any  $\alpha \geq \sqrt{2}$ , which is a little different from the case of concurrence in which  $\alpha \geq 2$ . As for the entanglement of assistance, we have the following conclusion:

**Theorem 5** *For any  $N$ -qubit pure state  $|\psi\rangle \in H_A \otimes H_{B_1} \otimes \dots \otimes H_{B_{N-1}}$ , the entanglement of assistance satisfies*

$$E(|\psi\rangle_{A|B_1 B_2 \dots B_{N-1}}) \leq \sum_{i=1}^{N-1} E_a(\rho_{AB_i}), \quad (21)$$

where  $E(|\psi\rangle_{A|B_1 B_2 \dots B_{N-1}})$  is the entanglement of formation of  $|\psi\rangle$  in bipartite partition  $A|B_1 B_2 \dots B_{N-1}$ , and  $\rho_{AB_i} = \text{Tr}_{B_1 \dots B_{i-1} B_{i+1} \dots B_{N-1}}(|\psi\rangle\langle\psi|)$ .

[Proof] Let  $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$  be the optimal decomposition of  $C_a(\rho_{AB})$ . We have

$$\begin{aligned}
E_a(\rho_{AB}) &\geq \sum_i p_i E(|\psi_i\rangle) \\
&= \sum_i p_i f\left(C_a^2(|\psi_i\rangle)\right) \\
&\geq f\left(\left[\sum_i p_i C_a(|\psi_i\rangle)\right]^2\right) \\
&= f(C_a^2(\rho_{AB})),
\end{aligned} \quad (22)$$

where the first equality is due to that, for a pure state, one has  $\rho_{AB} = \rho_{AB}^2$  and  $C(\rho_{AB}) = C_a(\rho_{AB})$  [17]. The second inequality is due to that  $f(x^2)$  is a convex function of  $x$ .

Therefore for an  $N$ -qubit pure state  $|\psi\rangle_{A|B_1 \dots B_{N-1}}$ , we have

$$\begin{aligned}
E(|\psi\rangle_{A|B_1 \dots B_{N-1}}) &= f(C^2(|\psi\rangle_{A|B_1 \dots B_{N-1}})) \\
&\leq f\left(C_a^2(\rho_{AB_1}) + \dots + C_a^2(\rho_{AB_{N-1}})\right) \\
&\leq f\left(C_a^2(\rho_{AB_1})\right) + \dots + f\left(C_a^2(\rho_{AB_{N-1}})\right) \\
&\leq E_a(\rho_{AB_1}) + \dots + E_a(\rho_{AB_{N-1}}).
\end{aligned}$$

The first inequality is due to Eq. (5). We have used the inequality (18) to get the second inequality. The last inequality is due to (22).  $\square$

## V. RESIDUAL ENTANGLEMENT OF EOF

Similar to the residual entanglement (12) defined by the  $\alpha$ th ( $\alpha \geq 2$ ) power of concurrence, we can define the residual entanglement by the  $\alpha$ th ( $\alpha \geq \sqrt{2}$ ) power of EoF for a three qubit pure state  $|\psi\rangle_{ABC}$ :

$$\tau_\alpha^E(|\psi\rangle_{ABC}) = E_{A|BC}^\alpha - E_{AB}^\alpha - E_{AC}^\alpha \geq 0. \quad (23)$$

As an example, let us consider again the W-state (13). We have  $E_{AB} = E_{AC} = 0.550048$  and  $E_{A|BC} = 0.918296$ . Therefore  $\tau_\alpha^E = 0.918296^\alpha - 2(0.550048)^\alpha$ , see Fig. (1).

Here it should be noted that, different from the residual entanglement of concurrence, the residual entanglement of EoF depends on which qubit is chosen to be  $A$ .

In the following we give some relations among the residual entanglement of EoF, entanglement of assistance and three tangle.

**Theorem 6** *For a three qubit pure state  $|\psi\rangle_{ABC}$ , we have*

$$\tau_\alpha^E(|\psi\rangle_{ABC}) \geq f^2(\tau_2^C(|\psi\rangle_{ABC})), \quad (24)$$

and

$$E_a^\alpha(\rho_{AB}) \geq E^\alpha(\rho_{AB}) + f^\alpha(\tau_2^C(|\psi\rangle_{ABC})), \quad (25)$$

where  $\alpha \geq \sqrt{2}$ ,  $\rho_{AB} = \text{Tr}_C(|\psi\rangle_{ABC}\langle\psi|)$  and  $\tau_2^C(|\psi\rangle_{ABC})$  is the three tangle of concurrence.

[Proof] According to the definition of  $\tau_\alpha^E(|\psi\rangle_{ABC})$ , we have

$$\begin{aligned}
\tau_\alpha^E(|\psi\rangle_{ABC}) &= E_{A|BC}^\alpha - E_{AB}^\alpha - E_{AC}^\alpha \\
&= f^\alpha(C_{A|BC}^2) - f^\alpha(C_{AB}^2) - f^\alpha(C_{AC}^2) \\
&= f^\alpha(C_{AB}^2 + C_{AC}^2 + \tau_2^C(|\psi\rangle_{ABC})) \\
&\quad - f^\alpha(C_{AB}^2) - f^\alpha(C_{AC}^2) \\
&\geq f^\alpha(\tau_2^C(|\psi\rangle_{ABC})),
\end{aligned}$$

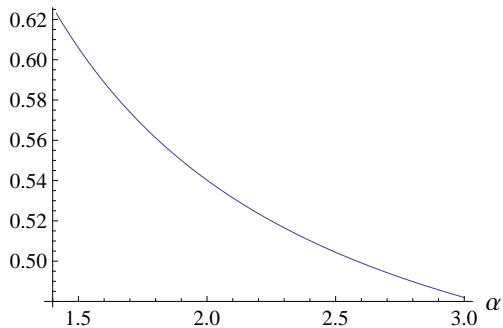


FIG. 2: The lower bound of  $E_a(\rho_{AB})$  for state (26) with  $\alpha \geq \sqrt{2}$ .

where the third equality is due to the definition of the three tangle  $\tau_2^C$ . We have used the Eq. (20) to get the last inequality.

Accounting to that for a  $2 \otimes 2 \otimes m$  quantum pure state  $|\psi\rangle_{ABC}$ ,  $C_a^2(\rho_{AB}) = C^2(\rho_{AB}) + \tau_2^C(|\psi\rangle_{ABC})$  [28], we have

$$\begin{aligned} E_a(\rho_{AB}) &\geq f(C_a^2(\rho_{AB})) \\ &= f(C^2(\rho_{AB}) + \tau_2^C(|\psi\rangle_{ABC})) \\ &\geq \sqrt[\alpha]{f^\alpha(C^2(\rho_{AB})) + f^\alpha(\tau_2^C(|\psi\rangle_{ABC}))} \\ &= \sqrt[\alpha]{E^\alpha(\rho_{AB}) + f^\alpha(\tau_2^C(|\psi\rangle_{ABC}))}, \end{aligned}$$

where we have used the inequality (22) to obtain the first inequality and the Eq. (17) to get the last inequality.  $\square$

The relations among entanglement of formation, entanglement of assistance and three tangle given in Theorem 6 can be used to obtain a lower bound of EoA. Let us consider the following example.

*Example 2.* Superpositions of the Greenberger-Horne-

Zeilinger (GHZ)-state and the W-state (13):

$$|\Psi\rangle_{ABC} = \sqrt{\frac{1}{2}}|GHZ\rangle - \sqrt{\frac{1}{2}}|W\rangle, \quad (26)$$

where  $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ . According to Theorem 6 we obtain the lower bound of  $E_a(\rho_{AB})$ ,  $E_a(\rho_{AB}) \geq (E^\alpha(\rho_{AB}) + f^\alpha(\tau_2^C(|\psi\rangle_{ABC})))^{\frac{1}{\alpha}}$ , where  $\rho_{AB} = T_{TC}(|\Psi\rangle_{ABC}\langle\Psi|)$ , see Fig. (2). From Fig. (2), one gets that the optimal lower bound of  $E_a(\rho_{AB})$  is 0.623 at  $\alpha = \sqrt{2}$ .

## VI. CONCLUSION

Entanglement monogamy is a fundamental property of multipartite entangled states. We have investigated the monogamy relations related to the concurrence and the entanglement of formation generally for  $N$ -qubit states. We also proved that the entanglement of assistance satisfies the monogamy inequality  $E(|\psi\rangle_{A|B_1B_2\dots B_{N-1}}) \leq \sum_{i=1}^{N-1} E_a(\rho_{AB_i})$ . To study the genuine tripartite entanglement, we investigated the residual entanglement of concurrence  $\tau_\alpha^C(|\psi\rangle_{ABC})$  and the residual entanglement of entanglement of formation  $\tau_\alpha^E(|\psi\rangle_{ABC})$ . By exploring the relations among the residual entanglement, entanglement of assistance and three tangle, we have presented a bound of  $E_a(\rho)$ . Our approach may be used to study further the monogamy properties related to other quantum entanglement measures such as negativity and to quantum correlations such as quantum discord.

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