Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

Local Unitary Invariants for Multipartite Quantum Systems

by

Jing Wang, Ming Li, Shao-Ming Fei, and Xianqing Li-Jost

Preprint no.: 97 2014

Local Unitary Invariants for Multipartite Quantum Systems

Jing Wang[1,2], Ming Li[2,3], Shao-Ming Fei[1,3] and Xianqing Li-Jost[3]

[1] School of Mathematical Sciences, Capital Normal University, 100048 Beijing, China

[2] College of the Science, China University of Petroleum, 266580 Qingdao, China

[3] Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany

Abstract

We present an approach of constructing invariants under local unitary transformations for multipartite quantum systems. The invariants constructed in this way can be complement to that in [Science 340 (2013) 1205-1208]. Detailed examples are given to compute such invariant in detail. It is shown that these invariants can be used to detect the local unitary equivalence of degenerated quantum states.

PACS numbers: 03.67.-a, 02.20.Hj, 03.65.-w

As the characteristic trait of quantum theory, quantum entanglement has been extensively studied in recent years. One approach to study the multipartite quantum entanglement is to study the local unitary (LU) invariants of the system. Actually, there are many characters related to the quantum entanglement, such as the degree of entanglement [1, 2], the maximal violations of Bell inequalities [3–6] and the teleportation fidelity [7, 8]. The quantities related are invariant under local unitary transformations, while two quantum states with the same entanglement may be not equivalent under local unitary transformations. It is of great importance to investigate the invariants under LU transformations.

In [11] the authors presented a complete set of 18 polynomial invariants for the local unitary(LU) equivalence of two-qubit mixed states. For three qubits states, nice results have been obtained in [12, 13]. The authors have proposed invariants for some generic mixed states in [14–16], tripartite pure and mixed states in [17]. For bipartite mixed quantum systems, Zhou et.al [18] have presented a complete set of invariants such that two density matrices are locally equivalent if and only if all these invariants have equal values in these density matrices. In [19], the authors have investigated the LU equivalence problem in terms of matrix realignment and partial transposition. A necessary and sufficient criterion for the local unitary equivalence of multipartite states, together with explicit forms of the local unitary operators have been presented. The criterion is shown to be operational for states having eigenvalues with multiplicity of no more than 2. Recently, we have given a complete classification under LU operation for multi-qubit quantum states[20], which also supplies a numerically computable protocol to detect the LU equivalence of any two multi-qubit mixed states. In [21] the case for multipartite system is studied and a complete set of invariants is presented for a class of mixed states. The authors in [22] have derived a set of invariants by using the partial transpose and realignment. However, generally a complete set of LU invariants is still missing. In this paper, we construct invariants under local unitary transformation for multipartite quantum systems. We further show by examples that these invariants are computable and can be used as criterion for detecting local unitary equivalence of degenerating quantum states.

Consider a pure quantum state $\rho = |\psi\rangle\langle\psi|$ in multipartite quantum system $H_1 \otimes H_2 \otimes \cdots \otimes H_N$ systems with $dim\, H_i = d_i,\ i = 1, 2, \cdots, N$. We first recall some results in [23]. Let $\rho_i = Tr_{\hat{i}}\rho,\ i = 1, 2, \cdots, N$, be the one-body reduced matrices of ρ, where $Tr_{\hat{i}}$ denotes the trace over all the subsystems except the ith. Denote $S = \{\lambda_1^1, \lambda_2^1, \cdots, \lambda_{d_1}^1, \lambda_1^2, \cdots, \lambda_{d_2}^2, \cdots, \lambda_1^N, \cdots, \lambda_{d_N}^N\}$ the set of all the eigenvalues of ρ_i. Walter et. al showed that S is in fact a convex polytope which represents an entanglement class. If the collection of eigenvalues S of the one-body reduced density matrices of a pure quantum state does not lie in an entanglement polytope, then the state does not belong to the corresponding entanglement class. The elements in the set S are just the invariants of $\rho = |\psi\rangle\langle\psi|$ under local unitary transformation. In the following we present a set of invariants under local unitary transformation that is complement to S, by using the idea in constructing invariants of bipartite systems [21]. The invariants obtained in this way are independent of the detailed spectral expressions of a density matrix.

Let ρ_{ij}, $1 \leq i \neq j \leq N$, denote the reduced density matrices with rank r_{ij} of a pure state ρ, acting on $H_i \otimes H_j$. The spectral decomposition of ρ_{ij} is represented as

$$\rho_{ij} = \sum_k^{r_{ij}} \Lambda_k^{ij} |X_k^{ij}\rangle\langle X_k^{ij}|, \tag{1}$$

where $\Lambda_k^{ij},\ k = 1, 2, ..., r_{ij}$, are the eigenvalues of ρ_{ij}, with the corresponding eigenvectors $|X_k^{ij}\rangle$. Set $|\tilde{X}_k^{ij}\rangle = \sqrt{\Lambda_k^{ij}}|X_k^{ij}\rangle$. One has that $\rho_{ij} = \sum_k^{r_{ij}} |\tilde{X}_k^{ij}\rangle\langle\tilde{X}_k^{ij}|$. Let A_k^{ij} be the matrix with entries given by the coefficients of the bipartite state vector $|\tilde{X}_k^{ij}\rangle$ in computational basis. Define the matrix Ω^{ij} with entries $(\Omega^{ij})_{lk} = Tr(A_l^{ij}(A_k^{ij})^T)$, where T stands for the transposition of a matrix. The character polynomial of Ω^{ij} is given by

$$\det\{\lambda I - \Omega^{ij}\} = \lambda^{r_{ij}^2} + C_1^{ij}\lambda^{r_{ij}^2 - 1} + \cdots + C_{r_{ij}}^{ij}. \tag{2}$$

Theorem 1: The coefficients C_α^{ij} in (2) must be the invariants of ρ under local unitary transformation, i.e. $\{C_\alpha^{ij},\ \alpha = 1, ..., r_{ij},\ 1 \leq i \neq j \leq N\}$ form a set of invariants for $\rho = |\psi\rangle\langle\psi|$ under local unitary transformations.

Proof: Assume $|\psi'\rangle = U_1 \otimes U_2 \otimes \cdots \otimes U_N|\psi\rangle$. One has that

$$\begin{aligned}\rho'_{ij} &= Tr_{\hat{i}\hat{j}}\rho' = Tr_{\hat{i}\hat{j}}U_1 \otimes U_2 \otimes \cdots \otimes U_N|\psi\rangle\langle\psi|U_1^\dagger \otimes U_2^\dagger \otimes \cdots \otimes U_N^\dagger \\ &= Tr_{\hat{i}\hat{j}}\rho = U_i \otimes U_j \rho_{ij} U_i^\dagger \otimes U_j^\dagger.\end{aligned} \tag{3}$$

Hence we have

$$\rho'_{ij} = \sum_k^{r'_{ij}} |\tilde{X}'^{ij}_k\rangle\langle\tilde{X}'^{ij}_k| = \sum_k^{r_{ij}} U_i \otimes U_j |\tilde{X}^{ij}_k\rangle\langle\tilde{X}^{ij}_k| U_i^\dagger \otimes U_j^\dagger, \tag{4}$$

where r'_{ij} is the rank of ρ'_{ij}.

Therefore, we have $|\tilde{X}'^{ij}_k\rangle = U_i \otimes U_j |\tilde{X}^{ij}_k\rangle$. Correspondingly we derive that $A'^{ij}_l = U_i A^{ij}_l U_j^T$. Hence $\Omega'^{ij} = \Omega^{ij}$, which leads to that the coefficients of the characteristic polynomials of Ω'^{ij} and Ω^{ij} are the same, $C'^{ij}_\alpha = C^{ij}_\alpha$. They are the invariants under local unitary transformations. ■

It is obvious that if two multipartite pure states have different values for one or more invariants, they can not be LU equivalent. In the following we give two examples to compute these invariants.

Example 1: Consider the generalized GHZ state $|GHZ\rangle = (\cos\theta, 0, 0, 0, 0, 0, 0, \sin\theta)^T$. The two-body reduced matrices are all of the same form:

$$\rho_{12} = \rho_{13} = \rho_{23} = \begin{pmatrix} \cos\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin\theta \end{pmatrix}. \tag{5}$$

Correspondingly,

$$\Omega^{12} = \Omega^{13} = \Omega^{23} = \begin{pmatrix} \sin^2\theta & 0 & 0 & 0 \\ 0 & \cos^2\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

From (2) we get the LU invariants: $C_1^{ij} = 1$, $C_2^{ij} = -1$, $C_3^{ij} = \cos^2\theta\sin^2\theta$, $C_4^{ij} = 0$, $ij \in \{12, 13, 23\}$.

Example 2: The generalized W state can be written as $|W\rangle = (0, \alpha, \beta, 0, \gamma, 0, 0, 0)^T$, with $\alpha^2 + \beta^2 + \gamma^2 = 1$. The two-body reduced matrices are of the forms:

$$\rho_{12} = \begin{pmatrix} \alpha^2 & 0 & 0 & 0 \\ 0 & \beta^2 & \beta\gamma & 0 \\ 0 & \beta\gamma & \gamma^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \rho_{13} = \begin{pmatrix} \beta^2 & 0 & 0 & 0 \\ 0 & \alpha^2 & \alpha\gamma & 0 \\ 0 & \alpha\gamma & \gamma^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \rho_{23} = \begin{pmatrix} \gamma^2 & 0 & 0 & 0 \\ 0 & \alpha^2 & \alpha\beta & 0 \\ 0 & \alpha\beta & \beta^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

The corresponding Ω^{ij} are given by

$$\Omega^{12}=\begin{pmatrix}\frac{(1-\alpha^2)^2}{\gamma^2} & 0 & 0 & 0\\ 0 & \alpha^2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix};\quad \Omega^{13}=\begin{pmatrix}\frac{(1-\beta^2)^2}{\gamma^2} & 0 & 0 & 0\\ 0 & \beta^2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix};\quad \Omega^{23}=\begin{pmatrix}\frac{(1-\gamma^2)^2}{\beta^2} & 0 & 0 & 0\\ 0 & \gamma^2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}. \quad (8)$$

Therefore we get the LU invariants of $|W\rangle$:

$$\{C^{12}_\alpha|\alpha=1,2,3,4\}=\{1,-1-\frac{\beta^2(\beta^2+\gamma^2)}{\gamma^2},\frac{\alpha^2(1-\alpha^2)}{\gamma^2}\},$$
$$\{C^{13}_\alpha|\alpha=1,2,3,4\}=\{1,-1-\frac{\alpha^2(\alpha^2+\gamma^2)}{\gamma^2},\frac{\beta^2(1-\beta^2)}{\gamma^2}\},$$
$$\{C^{23}_\alpha|\alpha=1,2,3,4\}=\{1,-1-\frac{\gamma^2(\beta^2+\gamma^2)}{\beta^2},\frac{\alpha^2(1-\alpha^2)}{\beta^2}\}.$$

Example 3: Consider a three-qutrit pure state:

$$|\Psi\rangle=\frac{1}{3}(1,0,0,0,1,0,0,0,1,0,x,0,0,0,x,x,0,0,0,0,x^2,x^2,0,0,0,x^2,0)^T,$$

where $x=e^{-\frac{2i\pi}{3}}$. It is one of the maximally entangled states, since the concurrence of $|\Psi\rangle$ is the same as that of $|GHZ\rangle$. The two-body reduced matrices of $\rho=|\Psi\rangle\langle\Psi|$ are all the same. And the corresponding Ω^{ij}s are given by $\Omega^{12}=\Omega^{13}=\Omega^{23}=Diag\{1/9,1/9,1/9,0,0,0,0,0,0\}$. Therefore we obtain the invariants for all states that are LU equivalent to $|\Psi\rangle$: $C^{ij}_1=1$, $C^{ij}_2=-1$, $C^{ij}_3=0.333333$, $C^{ij}_4=-0.037037$, and $C^{ij}_k=0$ for $ij\in\{12,13,23\}$, $k=5,6,7,8,9$.

Generally, we can construct local unitary invariants also from n-body ($2\leq n\leq N$) reduced density matrices of ρ. We use the notation in [24] to define the matrix unfolding of pure states $|\Psi\rangle\in H_{i_1}\otimes H_{i_2}\otimes\cdots\otimes H_{i_n}$ with the kth index as

$$A_k\in H_{i_k}\otimes(H_{i_{k+1}}\otimes\cdots\otimes H_{i_n}\otimes H_{i_1}\otimes\cdots\otimes H_{i_{k-1}}). \quad (9)$$

Here A_k Let A^{ij}_k be the matrix with entries given by the coefficients of the bipartite state vector $|\tilde{X}^{ij}_k\rangle$ in computational basis. here is a $d_{i_k}\times(d_{i_{k+1}}\times\cdots\times d_{i_n}\times d_{i_1}\times\cdots\times d_{i_{k-1}})$ matrix.

For $\rho=|\psi\rangle\langle\psi|\in H_1\otimes H_2\otimes\cdots\otimes H_N$, the k-body reduced matrices $\rho_{i_1i_2\cdots i_k}$ are given by by tracing over all the subsystems except $H_{i_1}H_{i_2}\cdots H_{i_k}$, $\rho_{i_1i_2\cdots i_k}=Tr_{\hat{i_1}\cdots\hat{i_k}}\rho$. Denote r_I the rank of $\rho_{i_1i_2\cdots i_k}$ and $I=\{i_1i_2\cdots i_k\}$. Let $\rho_I=\sum_m^{r_I}\Lambda^I_m|X^I_m\rangle\langle X^I_m|$ be the spectral decomposition of ρ_I. Set $|\tilde{X}^I_m\rangle=\sqrt{\Lambda^I_m}|X^I_m\rangle$. Thus

$$\rho_I=\sum_m^r|\tilde{X}^I_m\rangle\langle\tilde{X}^I_m|. \quad (10)$$

By using the matrix unfolding of multi-tensor one can represent $|\tilde{X}_m^I\rangle$ in the matrix form $(A_x^I)_m$ with $x \in I$. There are totally k matrix unfolding forms of $|\tilde{X}_m^I\rangle$. Let Ω_x^I denote the matrix with entries given by $(\Omega_x^I)_{mn} = Tr((A_x^I)_m (A_x^I)_n^T)$.

Theorem 2: The coefficients $(C_\alpha^I)_x$ of the character polynomial of Ω_x^I,

$$\det\{\lambda I - \Omega_x^I\} = \lambda^{r_I^2} + (C_x^I)_1 \lambda^{r_I^2-1} + \cdots + (C_x^I)_{r_I^2}, \tag{11}$$

$1 \leq x \leq r_I^2$, $1 \leq \alpha \leq k$, are the invariants of $\rho = |\psi\rangle\langle\psi|$ under local unitary transformations.

Proof: Let $|\psi'\rangle = U_1 \otimes U_2 \otimes \cdots \otimes U_N |\psi\rangle$. Similar to (3) one has that

$$\begin{aligned} \rho'_I &= Tr_{\hat{I}}\rho' = Tr_{\hat{I}} U_1 \otimes U_2 \otimes \cdots \otimes U_N |\psi\rangle\langle\psi| U_1^\dagger \otimes U_2^\dagger \otimes \cdots \otimes U_N^\dagger \\ &= Tr_{\hat{I}}\rho = U_{i_1} \otimes U_{i_2} \otimes \cdots \otimes U_{i_k} \rho_I U_{i_1}^\dagger \otimes U_{i_2}^\dagger \otimes \cdots \otimes U_{i_k}^\dagger. \end{aligned} \tag{12}$$

From the spectral decomposition $\rho_I = \sum_x |\tilde{X}_x^I\rangle\langle\tilde{X}_x^I|$ we derive that

$$\rho'_I = \sum_x |\tilde{X'}_x^I\rangle\langle\tilde{X'}_x^I| = \sum_x U_{i_1} \otimes U_{i_2} \otimes \cdots \otimes U_{i_k} |\tilde{X}_x^I\rangle\langle\tilde{X'}_x^I| U_{i_1}^\dagger \otimes U_{i_2}^\dagger \otimes \cdots \otimes U_{i_k}^\dagger. \tag{13}$$

Namely, $|\tilde{X'}_x^I\rangle = U_{i_1} \otimes U_{i_2} \otimes \cdots \otimes U_{i_k} |\tilde{X}_x^I\rangle$. As $(\Omega_\alpha)_{lm}^I = Tr[(A_\alpha)_l^I ((A_\alpha)_m^I)^T]$ and $(\Omega'_\alpha)_{lm}^I = Tr(A'^I_{\alpha l}(A'^I_{\alpha m})^T)$ with $(A_\alpha)_l^I$ and $A'^I_{\alpha l}$ the αth matrix representation of $|\tilde{X}_l^I\rangle$ and $|\tilde{X'}_l^I\rangle$, respectively. One gets that $A'^I_{\alpha l} = U_{i_\alpha} A_{\alpha l}^I U_{\hat{i_\alpha}}^T$ with $U_{\hat{i_\alpha}} = U_{i_1} \otimes \cdots \otimes U_{i_{\alpha-1}} \otimes U_{i_{\alpha+1}} \otimes \cdots \otimes U_{i_k}$. Then we get $\Omega'^I_\alpha = \Omega_\alpha^I$ which illustrates the equivalence of the coefficients of characteristic polynomials, i.e. $((C')_\alpha^I)_x = (C_\alpha^I)_x$. ∎

When two multipartite density matrices have degenerated eigenvalues, it becomes a challenging problem to judge their LU equivalence. The invariants derived in our Theorem 1 and 2 can be used to detect the LU equivalent problem for multipartite degenerated quantum states.

Example 4: Let us consider two three-qutrit mixed quantum states:

$$\rho = \frac{1}{2}|\psi_+\rangle\langle\psi_+| + \frac{1}{54}\sum_{i,j=0}^{2}|0ij\rangle\langle 0ij| + \frac{1}{81}\sum_{i,j=0}^{2}|1ij\rangle\langle 1ij| + \frac{2}{81}\sum_{i,j=0}^{2}|2ij\rangle\langle 2ij|;$$

and

$$\sigma = \frac{1}{2}|\phi_+\rangle\langle\phi_+| + \frac{1}{54}\sum_{i,j=0}^{2}|0ij\rangle\langle 0ij| + \frac{1}{81}\sum_{i,j=0}^{2}|1ij\rangle\langle 1ij| + \frac{2}{81}\sum_{i,j=0}^{2}|2ij\rangle\langle 2ij|;$$

where $|\psi_+\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)$ and $|\phi_+\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |111\rangle + |222\rangle)$.

σ and ρ have the same eigenvalues: three non-degenerated ones 0.51857, 0.02206, 0.01493, and three eigenvalues 0.02469, 0.01852, 0.01235 with multiplicity 8 each. The criteria in [25, 26] becomes less operational for such degenerated states. From our Theorem we have that the coefficients C_α^{12} corresponding to ρ_{12} are

$\{0, 0, 0, 0.00006, -0.00107, 0.01269, -0.09385, 0.41564, -1, 1\}$, which is different from that from σ_{12}: $\{0, 0, 0, 0.00001, -0.00185, 0.02246, -0.16676, 0.7079, -1.46571, 1\}$. Therefore by using our LU invariants we can easily conclude that ρ and σ are not equivalent under LU transformations.

To classify quantum states under local unitary transformations is a fundamental problem in the theory of quantum entanglement and correlations. We have introduced sets of invariants under LU transformations derived from the reduced matrices. Invariants from hyperdeterminants have been also constructed in [21]. However, hyperdeterminants become quite difficult to compute for systems except for three-qubit ones. The invariants proposed in this manuscript is easy to compute. Moreover, our invariants can be used to detect the LU equivalence of multipartite degenerated mixed states. Since for two multipartite mixed states ρ and ρ', if they are are LU equivalent, the corresponding reduced density matrices must be also LU equivalent. Therefore our LU invariants give rise to necessary conditions of LU equivalence for multipartite mixed states too.

Acknowledgments This work is supported by the NSFC 11105226, 11275131; the Fundamental Research Funds for the Central Universities No. 12CX04079A, No. 24720122013; Research Award Fund for outstanding young scientists of Shandong Province No. BS2012DX045.

[1] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A 54, 3824 (1996); C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, ibid. 53, 2046 (1996); V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. 78, 2275 (1997).

[2] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).

[3] R. Horodecki, P. Horodecki, and M. Horodecki, Phys. Lett. A 200, 340 (1995).

[4] R. F. Werner and M. M. Wolf, Phys. Rev. A 64, 032112 (2001).

[5] M. Zukowski and C. Brukner, Phys. Rev. Lett. 88, 210401 (2002).

[6] M. Li and S.M. Fei, Phys. Rev. A 86, 052119 (2012).

[7] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A, 60, 1888(1999).

[8] S. Albeverio, S. M. Fei, and W. L. Yang, Phys. Rev. A, 66, 012301(2002).

[9] K. Chen, S. Albeverio and S.M. Fei, Phys. Rev. Lett. 95, 040504 (2005); K. Chen, S. Albeverio and S.M. Fei, Phys. Rev. Lett. 95, 210501 (2005).

[10] K. Chen and L.A. Wu, Quant. Inf. Comput.3,193(2003). K. Chen and L.A. Wu, Phys. Lett.

A 306, 14(2002). O. Rudolph, Phys. Rev. A 67,032312(2003). K. Chen, S. Albeverio and S.M. Fei, Phys. Rev. A 68, 062313(2003).

[11] Y. Makhlin, Quant. Info. Proc. 1, 243 (2002).

[12] N. Linden, S. Popescu and A. Sudbery, Phys. Rev. Lett. 83, 243 (1999).

[13] N. Linden and S. Popescu, Fortsch. Phys. 46, 567 (1998).

[14] S. Albeverio, S.M. Fei, P.Parashar,W.L.Yang, Phys. Rev. A 68, 010303 (2003).

[15] S. Albeverio, S.M. Fei, and D.Goswami, Phys. Lett. A 340, 37 (2005).

[16] B.Z. Sun, S.M. Fei, X.Q. Li-Jost and Z.X.Wang, J. Phys. A 39, L43-L47(2006).

[17] S. Albeverio, L. Cattaneo, S.M. Fei and X.H. Wang, Int. J. Quant. Inform. 3, 603 (2005).

[18] C. Zhou, T.G. Zhang, S.M. Fei, N. Jing, and X. Li-Jost, Phys. Rev. A 86(R), 010303 (2012).

[19] T.G. Zhang, M.J. Zhao, M. Li, S.M. Fei, and X. Li-Jost, Phys. Rev. A, 88, 042304 (2013).

[20] M. Li, T.G. Zhang, S.M. Fei, X. Li-Jost, and N. Jing, Phys. Rev. A, 89, 062325 (2014).

[21] T.G. Zhang, M.J. Zhao, X. Li-Jost, S.M. Fei, Int. J. Theor. Phys. 52, 3020 (2013).

[22] U.T. Bhosale, K.V. Shuddhodan, and A. Lakshminarayan, Phys. Rev. A 87, 052311(2013).

[23] M. Walter, B. Doran, D. Gross, and M. Christandl, Science, vol. 340, no. 6137, pp. 1205-1208 (2013).

[24] L. D. Lathauwer, B.D. Moor, and J. Vandewalle, SIAM J. Matrix Anal. Appl.21, 1253(2000); Tamara G. Kolda and Brett W. Bader, SIAM Rev.51, 455(2009).

[25] B. Liu, J.L. Li, X. Li, C.F. Qiao, Phys. Rev. Lett.108, 050501 (2012).

[26] J.L. Li and C.F. Qiao, J. Phys. A: Math. Theor. 46, 075301 (2013).