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CORRELATION STRUCTURE OF THE CORRECTOR IN STOCHASTIC HOMOGENIZATION

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ABSTRACT. Recently, the quantification of errors in the stochastic homogenization of divergence-form operators has witnessed important progress. Our aim now is to go beyond error bounds, and give precise descriptions of the effect of the randomness, in the large-scale limit. This paper is a first step in this direction. Our main result is to identify the correlation structure of the corrector, in dimension 3 and higher. This correlation structure is similar to, but different from that of a Gaussian free field.

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1. INTRODUCTION

Consider the solution $u_\varepsilon : \mathbb{R}^d \to \mathbb{R}$ of the equation

$$\left(1 - \nabla \cdot A\left(\frac{\cdot}{\varepsilon}\right) \nabla\right) u_\varepsilon = f,$$

where f is a bounded smooth function, A is a random field of symmetric matrices on $\mathbb{R}^d$, and $\varepsilon > 0$. If A is uniformly elliptic and has a stationary ergodic law, then u_ε is known to converge as $\varepsilon \to 0$ to u_{h}, the solution of

$$(1 - \nabla \cdot A_{\mathsf{h}} \nabla)\, u_{\mathsf{h}} = f,$$

where A_{h} is the (constant in space, deterministic) *homogenized matrix*. This asymptotic result becomes more interesting if we can

(1) devise (provably) efficient techniques to compute the homogenized matrix;
(2) estimate the error in the convergence of u_ε to u_{h}.

Doing so requires to introduce some additional assumption on the type of correlations displayed by the random coefficients; we assume from now on that they have a finite range of dependence. These problems were discussed in several works [Yu86, CI03, BP04, Bo09, Mo11, CS14, CF15] (see also [CS10, AS14] for non-divergence form operators), but optimal error bounds were worked out only recently in [GO11, GO12, GM12, GNO15] for (1), and in [Mo14, GNO14] for (2) (in the discrete-space setting).

While controlling the size of the errors in homogenization is useful, it would be better (and it is our aim) to describe precisely what the errors look like when ε is small. As an analogy, if the convergence of u_ε to u_{h} is a law of large numbers, then we are looking for a central limit theorem.

The present paper is a first step towards this goal. In a discrete-space setting, it was proved in [GO11] that stationary correctors exist for $d \geqslant 3$ (recall that we assume that the random coefficients have a finite range of dependence). In this case, let us write ϕ_ξ for the (stationary) corrector in the direction ξ [see (2.2)]. Under a minor smoothness assumption on the random coefficients, we show that for large x, the correlation $\langle \phi_\xi(0)\ \phi_\xi(x)\rangle$ becomes very close to

$$\mathscr{K}_\xi(x) := \int \nabla \mathcal{G}_{\mathsf{h}}(y) \cdot \mathsf{Q}^{(\xi)}\, \nabla \mathcal{G}_{\mathsf{h}}(x-y)\ \mathrm{d}y, \tag{1.1}$$

where $\mathcal{G}_{\mathsf{h}}$ is the Green function of the homogenized operator $-\nabla \cdot A_{\mathsf{h}} \nabla$, and $\mathsf{Q}^{(\xi)}$ is a $d \times d$ symmetric matrix that can be expressed in terms of correctors, see (2.4).

This result paves the way for the understanding of the full scaling limit of $\varepsilon^{-\frac{d-2}{2}} \phi(\cdot/\varepsilon)$, seen as a random distribution. Indeed, the main missing ingredient is now to show that for any bounded, smooth test function f, the properly rescaled random variable $\sum_{x \in \mathbb{Z}^d} \phi(x) f(\varepsilon x)$ converges in law to a Gaussian. This will be done in [MN15].

This result on the corrector suggests (via a formal two-scale expansion) a scaling limit for $\varepsilon^{-d/2}\,(u_\varepsilon(\cdot) - \langle u_\varepsilon(\cdot)\rangle)$ as well. This will be addressed in [MG15].

Related works. We now give a brief overview of related works. These can be divided into three groups.

First, the questions that we consider here in dimension $d \geqslant 3$ have been investigated in dimension 1. One can benefit from this setting to gain a better understanding of the effect of long-range correlations of the coefficients [BGMP08, GB12].

Second, similar questions have been explored for the homogenization of operators other than those considered here. Typically, one considers a deterministic operator perturbed by the addition of a rapidly oscillating random potential [FOP82, Ba08, Ba10, Ba11, BJ11, BGGJ12, GB15]. We refer to [BG13] for a review.

Third, there is a deep connection between the corrector studied in the present paper and so-called $\nabla\varphi$ interface models [Fu05]. At a heuristic level, one can think of the corrector as the zero-temperature limit of such an interface model (with a bond-dependent potential). The scaling limit of the interface model with convex, homogeneous potential was shown to be the Gaussian free field [NS97, GOS01, Mi11]. In view of this, one may expect (as was suggested in [BB07, Conjecture 5]) the correlations of the corrector to be described by a Gaussian free field as well. However, our results show that such is not the case in general. One way to see this is to observe that the Fourier transform of $\mathscr{K}_\xi$ is

$$\frac{p \cdot \mathsf{Q}^{(\xi)} p}{(p \cdot A_{\mathsf{h}} p)^2} \qquad (p \in \mathbb{R}^d), \tag{1.2}$$

while it should be of the form

$$\frac{1}{p \cdot Bp} \qquad (p \in \mathbb{R}^d) \tag{1.3}$$

for some symmetric, positive definite matrix B, if the correlations were those of a Gaussian free field. By considering coefficients with small ellipticity ratio, one can produce examples where (1.2) cannot be reduced to (1.3).

The proof given in [NS97] that the interface model rescales to the Gaussian free field (and the proof of the dynamical version of this in [GOS01]) uses a Helffer-Sjöstrand representation of the correlations. We will also use this representation here, but with an important difference. In the case of the interface model, the Helffer-Sjöstrand representation readily enables to express the correlations of the interface as the averaged Green function of some operator, and the crux is then to show that this operator can be homogenized. In our case, the representation has a less clear interpretation. But it has to be so, since otherwise this would lead to Gaussian-free-field correlations.

Recently, a very interesting and direct connection was put forward in [BS11] between certain interface models with homogeneous but possibly non-convex potentials and the corrector considered here. The authors obtained the scaling limit of interface models with such potentials and zero tilt. They point out that the understanding of models with non-zero tilt could be obtained from the understanding of the scaling limit of the corrector. We refer to [BS11, Section 6] for more on this.

Organization of the paper. The precise setting and results of this paper are laid down in the next section. The Helffer-Sjöstrand representation of correlations is introduced in Section 3. Section 4 recalls several crucial estimates on the corrector and the Green function. The goal of Section 5 is to justify, in a weak sense, the two-scale expansion of the gradient of the Green function. The proof of the main result is then completed in Section 6.

2. Precise setting and results

We consider the (non-oriented) graph $(\mathbb{Z}^d, \mathbb{B})$ with $d \geqslant 3$, where $\mathbb{B}$ is the set of nearest-neighbor edges. Let $(\mathbf{e}_1, \ldots, \mathbf{e}_d)$ be the canonical basis of $\mathbb{Z}^d$. For every edge $e \in \mathbb{B}$, there exists a unique pair $(\underline{e}, i) \in \mathbb{Z}^d \times \{1, \ldots d\}$ such that e links $\underline{e}$ to $\underline{e} + \mathbf{e}_i$. Given such a pair, we write $\overline{e} = \underline{e} + \mathbf{e}_i$. We call $\underline{e}$ the *base point* of the edge e. For $f : \mathbb{Z}^d \to \mathbb{R}$, we let $\nabla f : \mathbb{B} \to \mathbb{R}$ be the gradient of f, defined by

$$\nabla f(e) = f(\overline{e}) - f(\underline{e}).$$

We write ∇^* for the formal adjoint of ∇, that is, for $F : \mathbb{B} \to \mathbb{R}$, $\nabla^* F : \mathbb{Z}^d \to \mathbb{R}$ is defined via

$$(\nabla^* F)(x) = \sum_{i=1}^{d} F((x - \mathbf{e}_i, x)) - F((x, x + \mathbf{e}_i)).$$

For such F, we define $AF(e) = a_e F(e)$, where (a_e) are real numbers taking values in a compact subset of $(0, +\infty)$. The operator of interest is $\nabla^* A \nabla$.

While a standard assumption for our purpose would be that (a_e) are independent and identically distributed, the technicalities of the proof will be reduced by assuming that they are also smooth in the following sense. We give ourselves a family $(\zeta_e)_{e \in \mathbb{B}}$ of independent standard Gaussian random variables (we write $\mathbb{P}$ for the law of this family on $\Omega = \mathbb{R}^{\mathbb{B}}$, and $\langle \cdot \rangle$ for the associated expectation). The coefficients $(a_e)_{e \in \mathbb{B}}$ are then defined by $a_e = a(\zeta_e)$, where $a : \mathbb{R} \to \mathbb{R}$ is a fixed twice differentiable function with bounded first and second derivatives (and taking values in a compact subset of $(0, +\infty)$).

Under these conditions, it is well-known that there exists a constant matrix A_{h} such that $\nabla^* A \nabla$ homogenizes over large scales to the continuous operator $-\nabla \cdot A_{\mathsf{h}} \nabla$.

Let ξ be a fixed vector of $\mathbb{R}^d$. For $\mu > 0$, let $\phi_{\xi,\mu}$ be the unique stationary solution of

$$\mu \phi_{\xi,\mu} + \nabla^* A(\xi + \nabla \phi_{\xi,\mu}) = 0. \tag{2.1}$$

It is proved in [GO11] that (recall that we assume $d \geqslant 3$) $\phi_{\xi,\mu}$ converges in $L^2(\Omega)$ to the unique stationary solution ϕ_ξ of

$$\nabla^* A(\xi + \nabla \phi_\xi) = 0. \tag{2.2}$$

The function ϕ_ξ is called the (stationary) *corrector* in the direction ξ. We use ϕ_i as shorthand for $\phi_{\mathbf{e}_i}$. In equations such as (2.2), ξ is to be understood as the function from $\mathbb{B}$ to $\mathbb{R}$ such that $\xi(e) = \xi \cdot (\overline{e} - \underline{e})$.

Let ∂_e denote the weak derivative with respect to the random variable ζ_e, which we may call a *vertical derivative*. The formal adjoint of ∂_e is

$$\partial_e^* = -\partial_e + \zeta_e.$$

We write $\partial f = (\partial_e f)_{e \in \mathbb{B}}$. For $F = (F_e)_{e \in \mathbb{B}}$, we write $\partial^* F = \sum_e \partial_e^* F_e$, and we let

$$\mathscr{L} = \partial^* \partial. \tag{2.3}$$

We write $|x|$ for the L^2-norm of $x \in \mathbb{Z}^d$. In order to keep light notation, we let $|x|_* = |x| + 2$ (so that for instance $\log |x|_*$ is bounded away from 0).

Here is our main result.

Theorem 2.1 (structure of correlations)**.** *Recall that we assume $d \geqslant 3$. Let $\mathcal{E}_0$ be the set of edges with base-point $0 \in \mathbb{Z}^d$, let $\mathcal{G}_{\mathsf{h}} : \mathbb{R}^d \to \mathbb{R}$ be the Green function of the (continuous-space) homogenized operator $-\nabla \cdot A_{\mathsf{h}} \nabla$, let $\mathsf{Q}^{(\xi)} = (\mathsf{Q}^{(\xi)}_{jk})_{1 \leqslant j,k \leqslant d}$ be the matrix defined by*

$$\mathsf{Q}^{(\xi)}_{jk} = \sum_{e \in \mathcal{E}_0} \left\langle (\mathbf{e}_j + \nabla \phi_j)(e)(\xi + \nabla \phi_\xi)(e)\ \partial_e a_e (\mathscr{L} + 1)^{-1} \partial_e a_e\ (\mathbf{e}_k + \nabla \phi_k)(e)(\xi + \nabla \phi_\xi)(e) \right\rangle, \tag{2.4}$$

and let $\mathscr{K}_\xi(x)$ be defined by (1.1). There exists a constant $C < \infty$ such that for every $x \in \mathbb{Z}^d \setminus \{0\}$,

$$|\langle \phi_\xi(0)\ \phi_\xi(x)\rangle - \mathscr{K}_\xi(x)| \leqslant C\ \frac{\log^2 |x|_*}{|x|^{d-1}}. \tag{2.5}$$

Remark 2.2. When $\mathbf{e}_j$ is interpreted as a function over $\mathbb{B}$ as in (2.4), it is to be understood as ξ is in (2.2), that is, $\mathbf{e}_j(e)$ is 1 if the edge e is parallel to the basis vector $\mathbf{e}_j$, and is 0 otherwise.

Remark 2.3. The operator $\mathscr{L}$ is the infinitesimal generator of the Ornstein-Uhlenbeck semigroup on $\mathbb{R}^{\mathbb{B}}$, and $\mathbb{P}$ is a reversible measure for the associated dynamics. For more general distributions of coefficients, one may replace $\mathscr{L}$ by the infinitesimal generator of the Glauber dynamics, that is, to keep the definition (2.3), but with ∂_e changed for

$$\partial_e f = \mathbb{E}[f \mid (a_{e'})_{e' \neq e}] - f$$

(in which case $(\mathscr{L} + 1)^{-1}$ must be replaced by $\mathscr{L}^{-1}$ in (2.4)). The setting we have chosen reduces the amount of technicality mostly by allowing us to use the chain rule for derivation.

Remark 2.4. We learn from Proposition 3.2 and Theorem 4.1 that the tensor $\mathsf{Q}^{(\xi)}$ is well-defined. From the identity

$$\xi' \cdot \mathsf{Q}^{(\xi)} \xi' = \sum_{e \in \mathcal{E}_0} \left\langle (\xi' + \nabla \phi_{\xi'})(e)(\xi + \nabla \phi_\xi)(e)\ \partial_e a_e (\mathscr{L} + 1)^{-1} \partial_e a_e\ (\xi' + \nabla \phi_{\xi'})(e)(\xi + \nabla \phi_\xi)(e) \right\rangle, \tag{2.6}$$

which follows from the linearity of ϕ_ξ in ξ, we learn that $\mathsf{Q}^{(\xi)}$ is positive semi-definite. In particular, the Fourier transform of $\mathscr{K}_\xi$ is non-negative.

Moreover, $\mathsf{Q}^{(\xi)}$ is non-degenerate as soon as the derivative of the function $a : \mathbb{R} \to \mathbb{R}$ is everywhere positive. Indeed, if the expression (2.6) vanishes for $\xi' = \xi$, the strict positivity of the operator implies that for any $e \in \mathcal{E}_0$, $\partial_e a_e (\xi + \nabla \phi_\xi)^2(e)$ and thus $(\xi + \nabla \phi_\xi)^2(e)$ vanishes almost surely. This in turn implies that

$$\sum_{e \in \mathcal{E}_0} \langle (\xi + \nabla \phi_\xi)(e) a_e (\xi + \nabla \phi_\xi)(e) \rangle = \xi \cdot A_h \xi$$

vanishes. By the non-degeneracy of the homogenized tensor A_h, this yields as desired $\xi = 0$. The same argument also implies that the null space of $\mathsf{Q}^{(\xi)}$ is contained in the hyperplane orthogonal to $A_h \xi$.

Remark 2.5. There is no simple relation between the quartic form defined by $\mathsf{Q}^{(\xi)}_{jk}$ and the quadratic form A_h, besides that $\xi' \mathsf{Q}^{(\xi)} \xi'$ is bounded from below by $(\xi' \cdot A_h \xi)^2$ up to a multiplicative constant. As was noted in the introduction, $\mathscr{K}_\xi$ is not the

Green function of a second-order operator in general. While its Fourier transform has the right sign and homogeneity, it is not the inverse of a quadratic form.

Remark 2.6. By polarization of the quartic form $\mathsf{Q}^{(\xi)}_{jk}$ in the ξ-variables, one also obtains a result for covariances $\langle \phi_\xi(0)\ \phi_{\xi'}(x)\rangle$ with $\xi' \neq \xi$.

Remark 2.7. We expect that at least if the environment is sufficiently mixing (as for instance when its correlations are of finite range), then there exists a matrix $\mathsf{Q}^{(\xi)}$ [whose explicit expression may differ from that given in (2.4)] such that the large-scale correlations of the correctors are described by (1.1) and (2.5).

3. Hellfer-Sjöstrand representation

Proposition 3.1 (Helffer-Sjöstrand representation of correlations, [HS94, Sj96, NS97])**.** *Let $f, g : \Omega \to \mathbb{R}$ be centered square-integrable functions such that for every $e \in \mathbb{B}$, $\partial_e f$ and $\partial_e g$ are in $L^2(\Omega)$. We have*

$$\langle f\ g\rangle = \sum_{e\in\mathbb{B}} \left\langle \partial_e f\ (\mathscr{L}+1)^{-1}\ \partial_e g\right\rangle .$$

Proof. The claim is similar to (and simpler than) that obtained in [NS97, Section 2.1]. We recall the proof briefly for the reader's convenience. By density, we can restrict our attention to functions f, g that depend only on a finite number of $(\zeta_e)_{e\in\mathbb{B}}$, and also by density, we may assume f and g to be smooth functions. Note that the commutator $[\partial_e, \partial^*_{e'}]$ satisfies

$$[\partial_e, \partial^*_{e'}] = \mathbf{1}_{e=e'}. \tag{3.1}$$

Let us first assume that there exists a function $u \in L^2(\Omega)$ such that $g = \mathscr{L}u$. Writing $G = \partial u$, we observe that

$$\begin{aligned} \partial_e g &= \partial_e \partial^* G \\ &= \sum_{e'} \partial_e \partial^*_{e'} G_{e'} \\ &= \sum_{e'} \left([\partial_e, \partial^*_{e'}] + \partial^*_{e'}\partial_e\right) G_{e'} \\ &= G_e + \sum_{e'} \partial^*_{e'} \partial_{e'} G_e, \end{aligned}$$

where we used (3.1) and the fact that $\partial_{e'} G_e = \partial_{e'}\partial_e u = \partial_e G_{e'}$ in the last step. Recalling the definition of $\mathscr{L}$ in (2.3), we arrive at

$$\partial_e g = (\mathscr{L}+1)G_e = (\mathscr{L}+1)\partial_e u.$$

In particular, $\partial_e u \in L^2(\Omega)$ and

$$\langle f\ g\rangle = \langle f\ \mathscr{L}u\rangle = \sum_e \langle \partial_e f\ \partial_e u\rangle = \sum_e \left\langle \partial_e f\ (\mathscr{L}+1)^{-1}\partial_e g\right\rangle .$$

In order to conclude, it suffices to check that the range of the operator $\mathscr{L}$ is dense in the set of centered square-integrable functions. If $g \in L^2(\Omega)$ is smooth, depends on a finite number of $(\zeta_e)_{e\in\mathbb{B}}$ and is in the orthogonal complement of $\mathrm{Ran}(\mathscr{L})$, then

$$\langle g\ \mathscr{L}g\rangle = 0 = \sum_e \left\langle |\partial_e g|^2\right\rangle ,$$

so g is constant. It follows that the orthogonal complement of $\mathrm{Ran}(\mathscr{L})$ is the set of constant functions, and this completes the proof. □

The following additional information on $(\mathscr{L}+1)^{-1}$ will turn out to be useful.

Proposition 3.2 (contraction of L^p)**.** *For every $p \geqslant 2$, the operator $(\mathscr{L}+1)^{-1}$ is a contraction from $L^p(\Omega)$ to itself.*

Proof. Let Λ be a finite subset of $\mathbb{B}$, and let $\mathcal{F}_\Lambda$ the set of real functions of $(\zeta_e)_{e\in\Lambda}$. We define H^1_Λ as the completion of the set of smooth functions in $\mathcal{F}_\Lambda$ for the scalar product

$$(u,v)_{H^1_\Lambda} = \langle u\ v\rangle + \sum_{e\in\Lambda} \langle \partial_e u\ \partial_e v\rangle \,.$$

For every $f \in H^1_\Lambda$, there exists a unique $u \in H^1_\Lambda$ such that

$$\forall v \in H^1_\Lambda,\ (u,v)_{H^1_\Lambda} = \langle f\ v\rangle \,, \tag{3.2}$$

and this is nothing but the weak formulation of the equation $(\mathscr{L}+1)u = f$. For every $\varepsilon > 0$, let $\psi_\varepsilon(x) = \varepsilon^{-1}\arctan(\varepsilon x)$ be a "nice" (in particular, bounded) approximation of the function $x \mapsto x$. One can check that if $v \in H^1_\Lambda$, then $\psi_\varepsilon(v\ |v|^{p-2}) \in H^1_\Lambda$. Hence, for $u \in H^1_\Lambda$ satisfying (3.2),

$$(u, \psi_\varepsilon(u\ |u|^{p-2}))_{H^1_\Lambda} = \left\langle f\ \psi_\varepsilon(u\ |u|^{p-2})\right\rangle ,$$

and we recall that

$$(u, \psi_\varepsilon(u\ |u|^{p-2}))_{H^1_\Lambda} = \left\langle u\ \psi_\varepsilon(u\ |u|^{p-2})\right\rangle + \sum_{e\in\Lambda} \left\langle \partial_e u\ \partial_e \psi_\varepsilon(u\ |u|^{p-2})\right\rangle .$$

Since $u \mapsto \psi_\varepsilon(u\ |u|^{p-2})$ is an increasing function, it follows that for every e,

$$\left\langle \partial_e u\ \partial_e \psi_\varepsilon(u\ |u|^{p-2})\right\rangle \geqslant 0,$$

and thus

$$\left\langle u\ \psi_\varepsilon(u\ |u|^{p-2})\right\rangle \leqslant \left\langle f\ \psi_\varepsilon(u\ |u|^{p-2})\right\rangle .$$

By the monotone convergence theorem, the left-hand side converges to $\langle |u|^p\rangle = \|u\|_p^p$ as ε tends to 0. The right-hand side is bounded by

$$\|f\|_p\ \left\langle |\psi_\varepsilon(u\ |u|^{p-2})|^{p/p-1}\right\rangle^{1-1/p} \leqslant \|f\|_p \|u\|_p^{p-1},$$

where we have used $|\psi_\varepsilon(x)| \leqslant |x|$. We have thus shown

$$\|u\|_p^p \leqslant \|f\|_p\ \|u\|_p^{p-1},$$

that is, $\|u\|_p \leqslant \|f\|_p$, and this implies the theorem. □

Using the fact that $(\mathscr{L}+1)^{-1}$ is a contraction on $L^2(\Omega)$, we deduce the following covariance estimate, which parallels those appearing in [NS97, NS98] (Brascamp-Lieb inequality), [GNO15, Definition 1] and [GO12, Lemma 3].

Corollary 3.3 (covariance estimate)**.** *For f and g as in Proposition 3.1,*

$$|\langle f\ g\rangle| \leqslant \sum_{e\in\mathbb{B}} \left\langle (\partial_e f)^2\right\rangle^{1/2} \left\langle (\partial_e g)^2\right\rangle^{1/2} .$$

4. Estimates on the corrector and the Green function

The aim of this section is to gather several known estimates on the Green function and on the corrector.

Theorem 4.1 (existence and integrability of the corrector [GO11])**.** *Recall that we assume $d \geqslant 3$. For every $\mu > 0$, there exists a unique stationary solution $\phi_{\xi,\mu}$ to equation (2.1). Moreover, for every $p \geqslant 1$, $\langle |\phi_{\xi,\mu}(0)|^p\rangle$ and $\langle |\nabla \phi_{\xi,\mu}(e)|^p\rangle$ $(e \in \mathbb{B})$ are uniformly bounded in $\mu > 0$. The limit*

$$\phi_\xi = \lim_{\mu\to 0} \phi_{\xi,\mu}$$

is well-defined in $L^p(\Omega)$ and is the unique stationary solution to (2.2).

A direct consequence of this result is:

Corollary 4.2 (almost-sure control of the corrector)**.** *Let $B_n = \{-n, \ldots, n\}^d$ and let $\mathbb{B}_n$ be the set of edges whose base-point is in B_n. For every $\beta > 0$, almost surely,*

$$\lim_{n \to +\infty} \; n^{-\beta} \max_{x \in B_n} |\phi_\xi(x)| = 0$$

and

$$\lim_{n \to +\infty} \; n^{-\beta} \max_{e \in \mathbb{B}_n} |\nabla \phi_\xi(e)| = 0.$$

Proof. Let $p \geqslant 1$. By Chebyshev's inequality,

$$\mathbb{P}[|\phi_\xi(0)| \geqslant x] \leqslant \frac{\mathbb{E}[|\phi_\xi(0)|^p]}{x^p} \qquad (x > 0),$$

so for any $\varepsilon > 0$, by a union bound,

$$\mathbb{P}\left[n^{-\beta} \max_{x \in B_n} |\phi_\xi(x)| \geqslant \varepsilon\right] \leqslant |B_n| \frac{\mathbb{E}[|\phi_\xi(0)|^p]}{(\varepsilon n^\beta)^p}.$$

The first part of the corollary follows by taking p large enough and applying the Borel-Cantelli lemma. The second part is obtained in the same way. □

We write $G(x, y)$ for the Green function between points x and y in $\mathbb{Z}^d$, i.e. $G(x, y) = (\nabla^* A \nabla)^{-1}(x, y)$ (the dependence on $(a_e)_{e \in \mathbb{B}}$ is kept implicit in the notation). For $\mu > 0$, we also let $G_\mu(x, y) = (\mu + \nabla^* A \nabla)^{-1}(x, y)$.

Regularity theory ensures the following decay properties of the Green function (see for instance [Mo15, Proposition 3.6] for a proof adapted to our context).

Theorem 4.3 (pointwise estimates on the Green function)**.** *There exist $C < \infty$, $c > 0$ and $\alpha > 0$ such that for every $\mu \in [0, 1/2]$ and $\zeta \in \Omega$,*

$$G_\mu(0, x) \leqslant \frac{C}{|x|_*^{d-2}} \, e^{-c\sqrt{\mu}|x|} \qquad (x \in \mathbb{Z}^d), \tag{4.1}$$

$$|\nabla G_\mu(0, e)| \leqslant \frac{C}{|\underline{e}|_*^{d-2+\alpha}} \, e^{-c\sqrt{\mu}|\underline{e}|} \qquad (e \in \mathbb{B}). \tag{4.2}$$

It was recently shown in [MO13] that, after averaging over the environment, the rates of decay of the gradient and mixed second gradient of the Green function behave as in the homogeneous case (see also [Mo15, Remark 11.2] for the fact that the estimates hold uniformly over μ).

Theorem 4.4 (annealed estimates on the gradients of the Green function [MO13])**.** *For every $1 \leqslant p < \infty$, there exists $C < \infty$ such that for every $\mu \in [0, 1/2]$ and every $e, e' \in \mathbb{B}$,*

$$\langle |\nabla G_\mu(0, e)|^p \rangle^{1/p} \leqslant \frac{C}{|\underline{e}|_*^{d-1}},$$

$$\left\langle |\nabla \nabla G_\mu(e, e')|^p \right\rangle^{1/p} \leqslant \frac{C}{|\underline{e}' - \underline{e}|_*^{d}}.$$

Remark 4.5. Notice that $\nabla G(x, e)$ (for $x \in \mathbb{Z}^d$ and $e \in \mathbb{B}$) denotes the gradient of $G(x, \cdot)$ evaluated at the edge e. Similarly, $\nabla \nabla G(e, e')$ denotes the gradient of $\nabla G(\cdot, e')$ evaluated at the edge e.

We conclude this section by recalling useful computations of vertical derivatives. The following two propositions are borrowed from [GO11, Lemmas 2.4 and 2.5].

Proposition 4.6 (derivatives of the corrector [GO11])**.** *For every $\mu > 0$, $x \in \mathbb{Z}^d$ and $e \in \mathbb{B}$, the approximate corrector $\phi_{\xi,\mu}(x)$ is differentiable with respect to ζ_e and*

$$\partial_e \phi_{\xi,\mu}(x) = -\partial_e a_e \nabla G_\mu(x,e)(\xi + \nabla\phi_{\xi,\mu})(e).$$

Remark 4.7. Recalling that we assume a_e to be of the form $a(\zeta_e)$ with a differentiable, we can rewrite $\partial_e a_e$ as $a'(\zeta_e)$.

Remark 4.8. Contrary to $\phi_{\xi,\mu}$, the corrector ϕ_ξ is not well-defined for every value of $(\zeta_e) \in \Omega$, but only on a set of full probability measure. In order to prove a statement similar to Proposition 4.6 for ϕ_ξ instead of $\phi_{\xi,\mu}$, it is thus necessary to show first that ϕ_ξ is defined on a subset of Ω large enough that speaking of $\partial_e \phi_\xi$ be meaningful. We will however not show this here, since for our purpose, it is always possible to bypass this problem by approximating ϕ_ξ by $\phi_{\xi,\mu}$, computing the derivatives, and then passing to the limit $\mu \to 0$.

Proposition 4.9 (derivatives of the Green function [GO11])**.** *For every $\mu \geqslant 0$, $x, y \in \mathbb{Z}^d$ and $e \in \mathbb{B}$, the Green function $G_\mu(x,y)$ is differentiable with respect to ζ_e and*

$$\partial_e G_\mu(x,y) = -\partial_e a_e \nabla G_\mu(x,e) \nabla G_\mu(y,e).$$

These two propositions can be proved by differentiating the defining equation of, respectively, the corrector and the Green function, namely

$$\mu\phi_{\xi,\mu} + \nabla^* A(\xi + \nabla\phi_{\xi,\mu}) = 0,$$

$$(\mu + \nabla^* A \nabla) G_\mu(x,\cdot) = \mathbf{1}_x.$$

We refer to [GO11] for details.

5. Two-scale expansion of the Green function

Note that since we assume the coefficients to be independent and identically distributed, the law of the coefficients is invariant under the rotations that preserve the lattice, and A_{h} is thus a multiple of the identity, say $A_{\mathsf{h}} = a_{\mathsf{h}}\mathrm{Id}$. We define the discrete homogenized Green function G_{h} as the unique bounded solution of the equation

$$\nabla^* A_{\mathsf{h}} \nabla G_{\mathsf{h}} = \mathbf{1}_0,$$

where A_{h} in the formula above acts as the multiplication by a_{h} on every edge. For $f : \mathbb{Z}^d \to \mathbb{R}$ and $x \in \mathbb{Z}^d$, we write $\nabla_j f(x)$ to denote $f(x + \mathbf{e}_j) - f(x)$. If instead we take $e \in \mathbb{B}$, we understand $\nabla_j f(e)$ to mean $\nabla_j f(\underline{e})$, that is, the gradient of f along the edge parallel to the vector $\mathbf{e}_j$ having the same base-point as e.

The goal of this section is to prove the following quantitative two-scale expansion of the gradient of the Green function.

Theorem 5.1 (quantitative two-scale expansion of the Green function)**.** *For every $p > 2$, there exists $C < \infty$ such that the following holds. If $g : \Omega \to \mathbb{R}$ is in $L^p(\Omega)$ and is differentiable with respect to ζ_b with $\partial_b g \in L^p(\Omega)$ for every $b \in \mathbb{B}$, then for every $e \in \mathbb{B}$,*

$$\left| \langle g\ \nabla G(0,e) \rangle - \sum_{j=1}^d \nabla_j G_{\mathsf{h}}(e) \, \langle g\ (\mathbf{e}_j + \nabla\phi_j)(e) \rangle \right| \leqslant C \left(\|g\|_p \frac{\log |\underline{e}|_*}{|\underline{e}|_*^d} + \sum_{\substack{y \in \mathbb{Z}^d \\ b \in \mathbb{B}}} \|\partial_b g\|_p \frac{1}{|\underline{e} - y|_*^{d-1} \ |\underline{b} - y|_*^d \ |y|_*^d} \right). \tag{5.1}$$

Remark 5.2. Applying Theorem 5.1 with $g = 1$, we obtain that

$$|\langle \nabla G(0,e)\rangle - \nabla G_{\mathsf{h}}(e)| \leqslant C\ \frac{\log |\underline{e}|_*}{|\underline{e}|_*^d}.$$

Remark 5.3. By translation, under the assumptions of Theorem 5.1, we also have

$$\left| \langle g\ \nabla G(x,e)\rangle - \sum_{j=1}^d \nabla_j G_{\mathsf{h}}(e-x)\, \langle g\ (\mathbf{e}_j + \nabla\phi_j)(e)\rangle \right| \\ \leqslant C \left(\|g\|_p \frac{\log |\underline{e}-x|_*}{|\underline{e}-x|_*^d} + \sum_{\substack{y\in\mathbb{Z}^d\\ b\in\mathbb{B}}} \|\partial_b g\|_p \, \frac{1}{|\underline{e}-y|_*^{d-1}\ |\underline{b}-y|_*^d\ |y-x|_*^d} \right),$$

where $e - x$ denotes the translation of the edge e by the vector $-x$.

We define $z : \mathbb{Z}^d \to \mathbb{R}$ by

$$z(x) = G(0,x) - G_{\mathsf{h}}(x) - \sum_{j=1}^d \phi_j(x)\ \nabla_j G_{\mathsf{h}}(x). \tag{5.2}$$

Proposition 5.4 (equation for z [PV81, GNO14])**.** *Let $A_i(x)$ stand for $a_{x,x+\mathbf{e}_i}$. Write $\nabla^2 G_{\mathsf{h}}$ for the matrix with entries $\nabla_i^*\nabla_j G_{\mathsf{h}}$ ($1 \leqslant i,j \leqslant d$). Let R be the matrix with entries (R_{ij}) satisfying*

$$(R - A_{\mathsf{h}})_{ij} = -\left[A_i(\mathbf{1}_j^i + \nabla_i\phi_j)\right](\cdot - \mathbf{e}_i) \qquad (1 \leqslant i,j \leqslant d),$$

where $\mathbf{1}_j^i = \mathbf{1}_{i=j}$. For $e \in \mathbb{B}$ in the direction of $\mathbf{e}_i$, let

$$h(e) = -\left(A \sum_{j=1}^d \phi_j(\cdot + \mathbf{e}_i)\nabla\nabla_j G_{\mathsf{h}} \right)(e),$$

and denote the $\mathbb{R}^{d\times d}$-scalar product of two matrices M and N by $M : N$ (that is, the sum of all terms after entry-wise product). We have

$$\nabla^* A \nabla z = R : \nabla^2 G_{\mathsf{h}} + \nabla^* h. \tag{5.3}$$

Remark 5.5. The crucial feature of the right-hand side of (5.3) is that it involves only the second derivatives of G_{h} (this is precisely what one aims for when defining z). Another aspect that will turn out to be important for our purpose is that $\langle R(x)\rangle = 0$. This follows from the fact (see e.g. [Kü83, (3.17)]) that the (i,j)-th entry of the homogenized matrix A_{h} is equal to

$$\left\langle A_i(\mathbf{1}_j^i + \nabla_i\phi_j)\right\rangle = \langle \mathbf{e}_i \cdot A(\mathbf{e}_j + \nabla\phi_j)\rangle\,.$$

Proof. We follow the line of argument given in the first step of the proof of [GNO14, Theorem 1] (itself inspired by the first proof of [PV81, Theorem 3]). For $f : \mathbb{Z}^d \to \mathbb{R}$, we write $\nabla_i^* f(x) = f(x - \mathbf{e}_i) - f(x)$. To begin with, we observe that the following discrete Leibniz rules hold, for $f, g : \mathbb{Z}^d \to \mathbb{R}$:

$$\nabla_i(fg) = (\nabla_i f)g + f(\cdot + \mathbf{e}_i)\nabla_i g,$$
$$\nabla_i^*(fg) = (\nabla_i^* f)g + f(\cdot - \mathbf{e}_i)\nabla_i^* g.$$

Recall that by definition,

$$\nabla^* A \nabla G(0,\cdot) = \mathbf{1}_0 = \nabla^* A_{\mathsf{h}} \nabla G_{\mathsf{h}},$$

and thus,

$$\nabla^* A \nabla (G(0,\cdot) - G_{\mathsf{h}}) = \nabla^*(A_{\mathsf{h}} - A)\nabla G_{\mathsf{h}}.$$

Writing $A_{\mathsf{h},i}$ for the i-th diagonal coefficient of the (diagonal) matrix A_{h}, we can express the right-hand side above as

$$\sum_{i=1}^{d} \nabla_i^*(A_{\mathsf{h},i} - A_i)\nabla_i G_{\mathsf{h}}.$$

We now need to compute

$$\nabla^* A \nabla \left(\phi_j \ \nabla_j G_{\mathsf{h}}\right). \tag{5.4}$$

By the Leibniz rule,

$$\nabla_i \left(\phi_j \ \nabla_j G_{\mathsf{h}}\right) = (\nabla_i \phi_j)\nabla_j G_{\mathsf{h}} + \phi_j(\cdot + \mathbf{e}_i)\nabla_i \nabla_j G_{\mathsf{h}}.$$

Hence, the term in (5.4) is equal to

$$\sum_{i=1}^{d} \nabla_i^* \left[A_i \left(\nabla_i \phi_j \ \nabla_j G_{\mathsf{h}} + \phi_j(\cdot + \mathbf{e}_i)\nabla_i \nabla_j G_{\mathsf{h}}\right)\right].$$

We can thus rewrite $\nabla^* A \nabla z$ as

$$\begin{aligned}
&\sum_{i=1}^{d} \left\{ \nabla_i^*(A_{\mathsf{h},i} - A_i)\nabla_i G_{\mathsf{h}} - \sum_{j=1}^{d} \nabla_i^* \left[A_i \left(\nabla_i \phi_j \nabla_j G_{\mathsf{h}} + \phi_j(\cdot + \mathbf{e}_i)\nabla_i \nabla_j G_{\mathsf{h}}\right)\right] \right\} \\
&= \sum_{i=1}^{d} \left\{ A_{\mathsf{h},i} \nabla_i^* \nabla_i G_{\mathsf{h}} - \sum_{j=1}^{d} \nabla_i^* \left[A_i \left((\mathbf{1}_j^i + \nabla_i \phi_j)\nabla_j G_{\mathsf{h}} + \phi_j(\cdot + \mathbf{e}_i)\nabla_i \nabla_j G_{\mathsf{h}}\right)\right] \right\},
\end{aligned}$$

where we used the fact that A_{h} is constant. By the definition of the corrector, we have

$$\sum_{i=1}^{d} \nabla_i^* A_i(\mathbf{1}_j^i + \nabla_i \phi_j) = \nabla^* A(\mathbf{e}_j + \nabla \phi_j) = 0,$$

so by the Leibniz rule,

$$\sum_{i,j=1}^{d} \nabla_i^* \left[A_i(\mathbf{1}_j^i + \nabla_i \phi_j)\nabla_j G_{\mathsf{h}}\right] = \sum_{i,j=1}^{d} \left[A_i(\mathbf{1}_j^i + \nabla_i \phi_j)\right](\cdot - \mathbf{e}_i) \ \nabla_i^* \nabla_j G_{\mathsf{h}},$$

and the conclusion follows. □

As a consequence, we get the following representation for z.

Proposition 5.6 (representation for z)**.** *For every $x \in \mathbb{Z}^d$,*

$$z(x) = \sum_{y \in \mathbb{Z}^d} G(x,y)\left(R : \nabla^2 G_{\mathsf{h}}\right)(y) + \sum_{b \in \mathbb{B}} \nabla G(x,b)h(b). \tag{5.5}$$

Proof. Let $\tilde{z}(x)$ denote the right-hand side of (5.5), which is well-defined by Corollary 4.2. Letting $\overline{z} = z - \tilde{z}$, one can check thanks to Proposition 5.4 that $\nabla^* A \nabla \overline{z} = 0$. In particular,

$$\sum_{x \in B_n} \overline{z}(x) \nabla^* A \nabla \overline{z}(x) = 0.$$

This sum differs from

$$\sum_{e \in \mathbb{B}_n} \nabla \overline{z}(e) \cdot A \nabla \overline{z}(e)$$

by no more than a constant times

$$\sum_{e \in \mathbb{B}_{n+1} \setminus \mathbb{B}_n} \left(|\overline{z}(\underline{e})| + |\overline{z}(\overline{e})|\right) |\nabla \overline{z}(e)|. \tag{5.6}$$

This sum tends to 0 as n tends to infinity. To see this, we come back to the definitions of z and $\tilde{z}$, given respectively in (5.2) and in the right-hand side of (5.5).

Using Corollary 4.2, Theorem 4.3 and Proposition A.1 of the appendix, we obtain that for every $\beta > 0$, almost surely,

$$|z(x)| = o\left(\frac{1}{|x|^{d-2-\beta}}\right) \qquad (|x| \to \infty),$$

$$|\nabla z(e)| = o\left(\frac{1}{|\underline{e}|^{d-2+\alpha-\beta}}\right) \qquad (|\underline{e}| \to \infty)$$

(where α comes from Theorem 4.3), and the same relations hold for z replaced by $\tilde{z}$, and thus also for z replaced by $\overline{z}$. Since $d \geqslant 3$, we can take $\beta > 0$ sufficiently small to ensure that $2(d-2)+\alpha-2\beta > d-1$, and we obtain that the sum in (5.6) tends to 0 as n tends to infinity.

To sum up, we obtained that

$$\lim_{n\to+\infty} \sum_{e\in\mathbb{B}_n} \nabla\overline{z}(e)\cdot A\nabla\overline{z}(e) = 0.$$

Since A is positive definite, we conclude that $\overline{z}$ is a constant. Now, both z and $\tilde{z}$ tend to 0 at infinity, so in fact $\overline{z} = 0$, and this concludes the proof. ☐

Proof of Theorem 5.1. Let us first see that it suffices to show that

$$|\langle g\ \nabla z(e)\rangle| \leqslant C\left(\|g\|_p \frac{\log|\underline{e}|_*}{|\underline{e}|_*^d} + \sum_{\substack{y\in\mathbb{Z}^d\\ b\in\mathbb{B}}} \|\partial_b g\|_p \frac{1}{|\underline{e}-y|_*^{d-1}\ |y-\underline{b}|_*^d\ |y|_*^d}\right). \tag{5.7}$$

Note that, by the Leibniz rule,

$$\nabla_i z(x) = \nabla_i G(0,x) - \nabla_i G_{\mathsf{h}}(x) - \sum_{j=1}^d \left[\nabla_i\phi_j(x)\ \nabla_j G_{\mathsf{h}}(x) + \phi_j(x+\mathbf{e}_i)\ \nabla_i\nabla_j G_{\mathsf{h}}(x)\right].$$

In order to prove that (5.7) implies (5.1), it is thus sufficient to show that

$$|\langle g\ \phi_j(x+\mathbf{e}_i)\ \nabla_i\nabla_j G_{\mathsf{h}}(x)\rangle| \leqslant C\|g\|_p \frac{\log|x|_*}{|x|_*^d}. \tag{5.8}$$

This is true since $|\nabla_i\nabla_j G_{\mathsf{h}}(x)| \lesssim |x|_*^{-d}$,

$$|\langle g\ \phi_j(x+\mathbf{e}_i)\rangle| \leqslant \|g\|_2\ \|\phi_j\|_2,$$

$\|\phi_j\|_2$ is finite by Theorem 4.1, and we assume $p \geqslant 2$.

We now turn to the proof of (5.7). From Proposition 5.6, we learn that

$$\nabla z(e) = \sum_{y\in\mathbb{Z}^d} \nabla G(e,y)(R:\nabla^2 G_{\mathsf{h}})(y) + \sum_{b\in\mathbb{B}} \nabla\nabla G(e,b)h(b).$$

We now proceed to show that each of the two terms

$$\sum_{y\in\mathbb{Z}^d} \left|\left\langle g\ \nabla G(e,y)(R:\nabla^2 G_{\mathsf{h}})(y)\right\rangle\right|, \tag{5.9}$$

$$\sum_{b\in\mathbb{B}} |\langle g\ \nabla\nabla G(e,b)h(b)\rangle| \tag{5.10}$$

is bounded by the right-hand side of (5.7).

Step I.1. We begin with (5.9), which is the more delicate. As noted in Remark 5.5, the random variable R is centered, so the expectation appearing within the absolute value in (5.9) is in fact a correlation. We thus wish to apply Corollary 3.3 and write

$$\left|\left\langle g\ \nabla G(e,y)(R:\nabla^2 G_{\mathsf{h}})(y)\right\rangle\right| \leqslant \sum_{b\in\mathbb{B}} \left\langle [\partial_b (g\ \nabla G(e,y))]^2\right\rangle^{1/2} \left\langle [\partial_b (R:\nabla^2 G_{\mathsf{h}})(y)]^2\right\rangle^{1/2}. \tag{5.11}$$

However, recalling that

$$(R-A_{\mathsf{h}})_{ij}(y) = -\left[A_i(\mathbf{1}^i_j + \nabla_i \phi_j)\right](y-\mathbf{e}_i),$$

we see that a slight difficulty appears because we have not given a meaning to $\partial_b \phi_j$. As was anticipated in Remark 4.8, this need not bother us. If we formally extend Proposition 4.6 to the case $\mu = 0$, we arrive at the formal expression

$$\partial_b(R_{ij}(y)) = \partial_b a_b\big(-\mathbf{1}_{b=(y-\mathbf{e}_i,y)}(\mathbf{1}^i_j+\nabla_i\phi_j)(y-\mathbf{e}_i) + A_i(y-\mathbf{e}_i)\nabla\nabla G(y-\mathbf{e}_i,b)(\xi+\nabla\phi_j)(b)\big). \tag{5.12}$$

The point now is that although we do not wish to discuss the sense of (5.12) as a derivative, we can take it as a *definition* of the random variable $\partial_b(R_{ij}(y))$, and observe that (5.11) holds. To see this, we approximate the left-hand side of (5.11) by introducing a small mass $\mu > 0$. We introduce

$$(A^\mu_{\mathsf{h}})_{ij} = \left\langle A_i(\mathbf{1}^i_j + \nabla_i\phi_{j,\mu})\right\rangle$$

and R^μ by setting

$$(R^\mu - A^\mu_{\mathsf{h}})_{ij}(y) = -\left[A_i(\mathbf{1}^i_j+\nabla_i\phi_{j,\mu})\right](y-\mathbf{e}_i)$$

(where of course $\phi_{j,\mu} = \phi_{\mathbf{e}_j,\mu}$). We can now write the left-hand side of (5.11) as the limit as μ tends to 0 of

$$\left|\left\langle g\ \nabla G(e,y)(R^\mu:\nabla^2 G_{\mathsf{h}})(y)\right\rangle\right|.$$

Applying Proposition 3.1 on this term is now legitimate, and by Proposition 4.6,

$$\partial_b(R^\mu_{ij}(y)) = \partial_b a_b\big(-\mathbf{1}_{b=(y-\mathbf{e}_i,y)}(\mathbf{1}^i_j+\nabla_i\phi_{j,\mu})(y-\mathbf{e}_i) + A_i(y-\mathbf{e}_i)\nabla\nabla G_\mu(y-\mathbf{e}_i,b)(\xi+\nabla\phi_{j,\mu})(b)\big). \tag{5.13}$$

By taking the limit $\mu \to 0$, it follows that (5.11) holds with $\partial_b R$ defined by (5.12).

Step I.2. By Hölder's inequality, it follows from Theorems 4.1 and 4.4 that

$$\left\langle(\partial_b R:\nabla^2 G_{\mathsf{h}})^2(y)\right\rangle^{1/2} \lesssim \frac{1}{|\underline{b}-y|^d_*\ |y|^d_*},$$

where $\lesssim$ stands for $\leqslant$ up to a multiplicative constant that only depends on d and the Lipschitz constant of a. On the other hand, using Proposition 4.9, we see that

$$\begin{aligned}\partial_b(g\ \nabla G(e,y)) &= (\partial_b g)\nabla G(e,y) + g\ \partial_b\nabla G(e,y)\\ &= (\partial_b g)\nabla G(e,y) - g\ (\partial_b a_b)\nabla\nabla G(e,b)\nabla G(y,b).\end{aligned}$$

Using Hölder's inequality (in conjunction with the strict inequality $p > 2$) and Theorem 4.4, we are led to

$$\left\langle[\partial_b(g\ \nabla G(e,y))]^2\right\rangle^{1/2} \lesssim \frac{\|\partial_b g\|_p}{|\underline{e}-y|^{d-1}_*} + \frac{\|g\|_p}{|\underline{b}-\underline{e}|^d_*\ |\underline{b}-y|^{d-1}_*}.$$

So we obtain from (5.11) the inequality

$$\left|\left\langle g\ \nabla G(e,y)(R:\nabla^2 G_{\mathsf{h}})(y)\right\rangle\right| \lesssim \sum_{b\in\mathbb{B}}\left(\frac{\|\partial_b g\|_p}{|\underline{e}-y|^{d-1}_*} + \frac{\|g\|_p}{|\underline{b}-\underline{e}|^d_*\ |\underline{b}-y|^{d-1}_*}\right)\frac{1}{|\underline{b}-y|^d_*\ |y|^d_*}, \tag{5.14}$$

and the term appearing in (5.9) is bounded (up to a constant) by

$$\sum_{\substack{y\in\mathbb{Z}^d\\ b\in\mathbb{B}}} \left(\frac{\|\partial_b g\|_p}{|\underline{e}-y|_*^{d-1}} + \frac{\|g\|_p}{|\underline{b}-\underline{e}|_*^d \, |\underline{b}-y|_*^{d-1}} \right) \frac{1}{|\underline{b}-y|_*^d \, |y|_*^d}.$$

To see that this is bounded by the right-hand side of (5.7), it suffices to observe that

$$\sum_{y\in\mathbb{Z}^d} \frac{1}{|\underline{b}-y|_*^{2d-1} \, |y|_*^d} \lesssim \frac{1}{|\underline{b}|_*^d}$$

and

$$\sum_{b\in\mathbb{B}} \frac{1}{|\underline{b}-\underline{e}|_*^d \, |\underline{b}|_*^d} \lesssim \frac{\log |\underline{e}|_*}{|\underline{e}|_*^d}. \tag{5.15}$$

These two facts are proved in Proposition A.1 of the appendix.

Step II. We now turn to the analysis of (5.10). We note that

$$\sum_{b\in\mathbb{B}} |\langle g \; \nabla\nabla G(e,b) h(b)\rangle| \leqslant \sum_{b\in\mathbb{B}} \|g\|_2 \left\langle (\nabla\nabla G(e,b)h(b))^2 \right\rangle^{1/2}.$$

Using the explicit form of h given by Proposition 5.4 together with Theorems 4.1 and 4.4, we arrive at

$$\left\langle (\nabla\nabla G(e,b)h(b))^2 \right\rangle^{1/2} \lesssim \frac{1}{|\underline{b}-\underline{e}|_*^d \, |\underline{b}|_*^d}.$$

In view of (5.15), we have shown that the term in (5.10) is bounded by a constant times

$$\|g\|_2 \, \frac{\log |\underline{e}|_*}{|\underline{e}|_*^d},$$

which is a better bound than needed. □

6. Proof of Theorem 2.1

Proof of Theorem 2.1. Our starting point is the identity

$$\langle \phi_\xi(0)\phi_\xi(x)\rangle = \sum_{e\in\mathbb{B}} \left\langle \partial_e\phi_\xi(0) \; (\mathscr{L}+1)^{-1} \; \partial_e\phi_\xi(x)\right\rangle, \tag{6.1}$$

with

$$\partial_e\phi_\xi(y) = -\partial_e a_e \nabla G(y,e)(\xi+\nabla\phi_\xi)(e) \qquad (y\in\mathbb{Z}^d). \tag{6.2}$$

As in Step I.1 of the proof of Theorem 5.1, we do not mean to discuss the meaning of $\partial_e\phi_\xi(y)$ as a derivative of $\phi_\xi(y)$. Rather, it suffices for our purpose to observe that the identity in (6.1) holds with $\partial_e\phi_\xi(0)$ and $\partial_e\phi_\xi(x)$ *defined* by (6.2). This follows easily by approximating ϕ_ξ by $\phi_{\xi,\mu}$, applying Propositions 3.1 and 4.6, and letting μ tend to 0.

Replacing $\partial_e\phi_\xi(0)$ and $\partial_e\phi_\xi(x)$ by their definitions, the summand in the right-hand side of (6.1) becomes

$$\left\langle \partial_e a_e \nabla G(0,e)(\xi+\nabla\phi_\xi)(e) \; (\mathscr{L}+1)^{-1} \; \partial_e a_e \nabla G(x,e)(\xi+\nabla\phi_\xi)(e)\right\rangle. \tag{6.3}$$

We see that two ∇G terms appear in this expectation. We will "pull out of the expectation" each of these ∇G terms using Theorem 5.1. These form the two first steps of the proof. The last step discusses how to replace ∇G_{h} by its continuous-space counterpart $\nabla\mathcal{G}_{\mathsf{h}}$.

Step 1. Defining

$$g_e(x) = \partial_e a_e(\xi+\nabla\phi_\xi)(e) \; (\mathscr{L}+1)^{-1} \; \partial_e a_e \nabla G(x,e)(\xi+\nabla\phi_\xi)(e),$$

we see that we can rewrite the term in (6.3) as

$$\langle g_e(x)\ \nabla G(0,e)\rangle\,,$$

and we wish to justify that

$$\sum_{e\in\mathbb{B}}\left|\langle g_e(x)\ \nabla G(0,e)\rangle - \sum_{j=1}^{d}\nabla_j G_{\mathsf{h}}(e)\,\langle g_e(x)\ (\mathbf{e}_j+\nabla\phi_j)(e)\rangle\right| \lesssim \frac{\log^2|x|_*}{|x|_*^{d-1}}. \tag{6.4}$$

In order to apply Theorem 5.1 for this purpose, we need to compute $\partial_b g_e(x)$ for every $b\in\mathbb{B}$. From the commutation relation in (3.1), it follows that

$$\partial_b\ \mathscr{L} = (\mathscr{L}+1)\ \partial_b,$$

and thus

$$\partial_b\ (\mathscr{L}+1)^{-1} = (\mathscr{L}+2)^{-1}\ \partial_b.$$

From this observation, we get that

$$\partial_b g_e(x) = g^{(1)}_{b,e}(x) + g^{(2)}_{b,e}(x) + g^{(3)}_{b,e}(x) + g^{(4)}_{b,e}(x) \tag{6.5}$$

with

$$g^{(1)}_{b,e}(x) = -\partial_e a_e \partial_b a_b \nabla\nabla G(e,b)(\xi+\nabla\phi_\xi)(b)\ (\mathscr{L}+1)^{-1}\ \partial_e a_e \nabla G(x,e)(\xi+\nabla\phi_\xi)(e),$$

$$\begin{aligned} g^{(2)}_{b,e}(x) = {} & -\partial_e a_e(\xi+\nabla\phi_\xi)(e)\ (\mathscr{L}+2)^{-1} \\ & \partial_e a_e \partial_b a_b \nabla\nabla G(e,b)\left[\nabla G(x,b)(\xi+\nabla\phi_\xi)(e) + \nabla G(x,e)(\xi+\nabla\phi_\xi)(b)\right], \end{aligned}$$

$$g^{(3)}_{b,e}(x) = \mathbf{1}_{e=b}\ \partial_e^2 a_e(\xi+\nabla\phi_\xi)(e)\ (\mathscr{L}+1)^{-1}\ \partial_e a_e \nabla G(x,e)(\xi+\nabla\phi_\xi)(e),$$

and

$$g^{(4)}_{b,e}(x) = \mathbf{1}_{e=b}\ \partial_e a_e(\xi+\nabla\phi_\xi)(e)\ (\mathscr{L}+2)^{-1}\ \partial_e^2 a_e \nabla G(x,e)(\xi+\nabla\phi_\xi)(e).$$

As before, we do not wish to discuss the meaning of (6.5) as a derivative, but rather use the fact that if $\partial_b g_e(x)$ is defined in this way, then by the usual approximation argument,

$$\begin{aligned} & \left|\langle g_e(x)\ \nabla G(0,e)\rangle - \sum_{j=1}^{d}\nabla_j G_{\mathsf{h}}(e)\,\langle g_e(x)\ (\mathbf{e}_j+\nabla\phi_j)(e)\rangle\right| \\ & \qquad \lesssim \|g_e(x)\|_p \frac{\log|\underline{e}|_*}{|\underline{e}|_*^d} + \sum_{\substack{y\in\mathbb{Z}^d\\ b\in\mathbb{B}}} \|\partial_b g_e(x)\|_p\, \frac{1}{|\underline{e}-y|_*^{d-1}\ |\underline{b}-y|_*^d\ |y|_*^d}. \end{aligned}$$

From Proposition 3.2 and Theorems 4.1 and 4.4, we learn that

$$\|g_e(x)\|_p \lesssim \frac{1}{|\underline{e}-x|_*^{d-1}}$$

and

$$\|\partial_b g_e(x)\|_p \lesssim \frac{1}{|\underline{b}-\underline{e}|_*^d}\left(\frac{1}{|\underline{e}-x|_*^{d-1}} + \frac{1}{|\underline{b}-x|_*^{d-1}}\right).$$

Hence, up to a multiplicative constant, the left-hand side of (6.4) is smaller than the sum of the following two terms:

$$\sum_{e\in\mathbb{B}} \frac{\log|\underline{e}|_*}{|\underline{e}-x|_*^{d-1}\ |\underline{e}|_*^d}, \tag{6.6}$$

$$\sum_{\substack{e,b\in\mathbb{B}\\ y\in\mathbb{Z}^d}} \frac{1}{|\underline{b}-\underline{e}|_*^d}\left(\frac{1}{|\underline{e}-x|_*^{d-1}} + \frac{1}{|\underline{b}-x|_*^{d-1}}\right)\frac{1}{|\underline{e}-y|_*^{d-1}\ |\underline{b}-y|_*^d\ |y|_*^d}. \tag{6.7}$$

By Remark A.2 of the appendix, the sum in (6.6) is dominated by a constant times the right-hand side of (6.4). As for the sum in (6.7), we can further split it into the sum of

$$\sum_{\substack{e,b\in\mathbb{B}\\ y\in\mathbb{Z}^d}} \frac{1}{|\underline{b}-\underline{e}|_*^d\ |\underline{e}-x|_*^{d-1}\ |\underline{e}-y|_*^{d-1}\ |\underline{b}-y|_*^d\ |y|_*^d} \tag{6.8}$$

and

$$\sum_{\substack{e,b\in\mathbb{B}\\ y\in\mathbb{Z}^d}} \frac{1}{|\underline{b}-\underline{e}|_*^d\ |\underline{b}-x|_*^{d-1}\ |\underline{e}-y|_*^{d-1}\ |\underline{b}-y|_*^d\ |y|_*^d}. \tag{6.9}$$

By repeatedly applying Proposition A.1 of the appendix, we can bound the sum in (6.8) by

$$\sum_{\substack{e\in\mathbb{B}\\ y\in\mathbb{Z}^d}} \frac{\log|\underline{e}-y|_*}{|\underline{e}-y|_*^{2d-1}\ |\underline{e}-x|_*^{d-1}\ |y|_*^d} \lesssim \sum_{e\in\mathbb{B}} \frac{1}{|\underline{e}-x|_*^{d-1}\ |\underline{e}|_*^d} \lesssim \frac{\log|x|_*}{|x|_*^{d-1}},$$

and similarly, bound the sum in (6.9) by

$$\sum_{\substack{b\in\mathbb{B}\\ y\in\mathbb{Z}^d}} \frac{\log|\underline{b}-y|_*}{|\underline{b}-y|_*^{2d-1}\ |\underline{b}-x|_*^{d-1}\ |y|_*^d} \lesssim \sum_{b\in\mathbb{B}} \frac{1}{|\underline{b}|_*^d\ |\underline{b}-x|_*^{d-1}} \lesssim \frac{\log|x|_*}{|x|_*^{d-1}},$$

and the proof of (6.4) is complete.

Step 2. Recall that we have written $\langle \phi_\xi(0)\ \phi_\xi(x)\rangle$ as

$$\sum_{e\in\mathbb{B}} \langle g_e(x)\nabla G(0,e)\rangle\,,$$

so we proved in Step 1 that

$$\left| \langle \phi_\xi(0)\ \phi_\xi(x)\rangle - \sum_{e\in\mathbb{B}}\sum_{j=1}^d \nabla_j G_{\mathsf{h}}(e)\,\langle g_e(x)\ (\mathbf{e}_j+\nabla\phi_j)(e)\rangle \right| \lesssim \frac{\log^2|x|_*}{|x|_*^{d-1}}. \tag{6.10}$$

We now aim to show that

$$\sum_{e\in\mathbb{B}}\sum_{j=1}^d \left| \nabla_j G_{\mathsf{h}}(e)\,\langle g_e(x)\ (\mathbf{e}_j+\nabla\phi_j)(e)\rangle - \sum_{k=1}^d \nabla_j G_{\mathsf{h}}(e)\mathsf{Q}_{jk}^{(\xi,e)}\nabla_k G_{\mathsf{h}}(e-x)\right| \lesssim \frac{\log^2|x|_*}{|x|_*^{d-1}}, \tag{6.11}$$

where $\mathsf{Q}_{jk}^{(\xi,e)}$ is defined by

$$\mathsf{Q}_{jk}^{(\xi,e)} = \Big\langle \partial_e a_e(\mathbf{e}_j+\nabla\phi_j)(e)(\xi+\nabla\phi_\xi)(e)\ (\mathscr{L}+1)^{-1}\ \partial_e a_e(\mathbf{e}_k+\nabla\phi_k)(e)(\xi+\nabla\phi_\xi)(e)\Big\rangle\,. \tag{6.12}$$

For $j\in\{1,\ldots,d\}$, we let

$$\tilde g_{e,j} = \partial_e a_e(\xi+\nabla\phi_\xi)(e)\ (\mathscr{L}+1)^{-1}\ \partial_e a_e(\mathbf{e}_j+\nabla\phi_j)(e)(\xi+\nabla\phi_\xi)(e),$$

and observe that since $(\mathscr{L}+1)^{-1}$ is symmetric,

$$\langle g_e(x)\ (\mathbf{e}_j+\nabla\phi_j)(e)\rangle = \langle \tilde g_{e,j}\ \nabla G(x,e)\rangle\,. \tag{6.13}$$

We let

$$\partial_b \tilde g_{e,j} = \tilde g^{(1)}_{b,e,j} + \tilde g^{(1)}_{b,e,j} + \tilde g^{(3)}_{b,e,j} + \tilde g^{(4)}_{b,e,j},$$

where

$$\tilde{g}^{(1)}_{b,e,j} = -\partial_e a_e \partial_b a_b \nabla\nabla G(e,b)(\xi+\nabla\phi_\xi)(b)\ (\mathscr{L}+1)^{-1}\ \partial_e a_e(\mathbf{e}_j+\nabla\phi_j)(e)(\xi+\nabla\phi_\xi)(e),$$

$$\begin{aligned}\tilde{g}^{(2)}_{b,e,j} = -\partial_e a_e \xi + \nabla\phi_\xi)(e)\ (\mathscr{L}+2)^{-1}\\ \partial_e a_e \partial_b a_b \nabla\nabla G(e,b)\left[(\mathbf{e}_j+\nabla\phi_j)(b)(\xi+\nabla\phi_\xi)(e)+(\mathbf{e}_j+\nabla\phi_j)(e)(\xi+\nabla\phi_\xi)(b)\right],\end{aligned}$$

$$\tilde{g}^{(3)}_{b,e,j} = \mathbf{1}_{e=b}\ \partial_e^2 a_e(\xi+\nabla\phi_\xi)(e)\ (\mathscr{L}+1)^{-1}\ \partial_e a_e(\mathbf{e}_j+\nabla\phi_j)(e)(\xi+\nabla\phi_\xi)(e),$$

and

$$\tilde{g}^{(4)}_{b,e,j} = \mathbf{1}_{e=b}\ \partial_e a_e(\xi+\nabla\phi_\xi)(e)\ (\mathscr{L}+2)^{-1}\ \partial_e^2 a_e(\mathbf{e}_j+\nabla\phi_j)(e)(\xi+\nabla\phi_\xi)(e).$$

As before (and because of Remark 5.3), this definition ensures that

$$\begin{aligned}&\left|\langle \tilde{g}_{e,j}\ \nabla G(x,e)\rangle - \sum_{k=1}^d \nabla_k G_{\mathsf{h}}(e-x)\,\langle \tilde{g}_{e,j}\ (\mathbf{e}_k+\nabla\phi_k)(e)\rangle\right| \\ &\qquad \lesssim \|\tilde{g}_{e,j}\|_p \frac{\log|\underline{e}-x|_*}{|\underline{e}-x|_*^d} + \sum_{\substack{y\in\mathbb{Z}^d\\ b\in\mathbb{B}}} \|\partial_b \tilde{g}_{e,j}\|_p \frac{1}{|\underline{e}-y|_*^{d-1}\ |\underline{b}-y|_*^d\ |y-x|_*^d}.\end{aligned}$$

Moreover, we infer from Proposition 3.2 and Theorems 4.1 and 4.4 that for any $1 \leqslant p < \infty$ (and thus in particular the $p > 2$ needed above)

$$\|\tilde{g}_{e,j}\|_p \lesssim 1$$

and

$$\|\partial_b \tilde{g}_{e,j}\|_p \lesssim \frac{1}{|\underline{b}-\underline{e}|_*^d}.$$

Since

$$\langle \tilde{g}_{e,j}\ (\mathbf{e}_k+\nabla\phi_k)(e)\rangle = \mathsf{Q}^{(\xi,e)}_{jk},$$

we obtain that

$$\begin{aligned}&\left|\langle \tilde{g}_{e,j}\ \nabla G(x,e)\rangle - \sum_{k=1}^d \mathsf{Q}^{(\xi,e)}_{jk} \nabla_k G_{\mathsf{h}}(e-x)\right| \\ &\qquad \lesssim \frac{\log|\underline{e}-x|_*}{|\underline{e}-x|_*^d} + \sum_{\substack{y\in\mathbb{Z}^d\\ b\in\mathbb{B}}} \frac{1}{|\underline{b}-\underline{e}|_*^d\ |\underline{e}-y|_*^{d-1}\ |\underline{b}-y|_*^d\ |y-x|_*^d},\end{aligned}$$

and thus by (6.13), up to a multiplicative constant, the left-hand side of (6.11) is smaller than

$$\sum_{e\in\mathbb{B}} \frac{1}{|e|_*^{d-1}} \left(\frac{\log|\underline{e}-x|_*}{|\underline{e}-x|_*^d} + \sum_{\substack{y\in\mathbb{Z}^d\\ b\in\mathbb{B}}} \frac{1}{|\underline{b}-\underline{e}|_*^d\ |\underline{e}-y|_*^{d-1}\ |\underline{b}-y|_*^d\ |y-x|_*^d} \right). \tag{6.14}$$

From Remark A.2 of the appendix, we have

$$\sum_{e\in\mathbb{B}} \frac{1}{|e|_*^{d-1}} \frac{\log|\underline{e}-x|_*}{|\underline{e}-x|_*^d} \lesssim \frac{\log^2|x|_*}{|x|_*^{d-1}}.$$

The remaining sum from (6.14) can be bounded, using Proposition A.1 repeatedly, by

$$\sum_{\substack{y\in\mathbb{Z}^d\\ e\in\mathbb{B}}} \frac{\log|\underline{e}-y|_*}{|\underline{e}|_*^{d-1}\ |\underline{e}-y|_*^{2d-1}\ |y-x|_*^d} \lesssim \sum_{y\in\mathbb{Z}^d} \frac{1}{|y|_*^{d-1}\ |y-x|_*^d} \lesssim \frac{\log|x|_*}{|x|_*^{d-1}},$$

and this finishes the proof of (6.11).

Step 3. Note that by the stationarity of the environment, the matrix $\mathsf{Q}^{(\xi,e)}$ depends on the edge e only through its orientation. On the other hand, the quantities $\nabla_j G_{\mathsf{h}}(e)$ and $\nabla_j G_{\mathsf{h}}(e-x)$ depend on the edge e only through its base point. We also observe that the matrix $\mathsf{Q}^{(\xi)}$ introduced in (2.4) is by definition $\sum_{e\in\mathcal{E}_0}\mathsf{Q}^{(\xi,e)}$. Hence, the previous steps of the proof have led us, see (6.10) and (6.11), to

$$\left|\langle\phi_\xi(0)\ \phi_\xi(x)\rangle - \sum_{y\in\mathbb{Z}^d}\sum_{j,k=1}^d \nabla_j G_{\mathsf{h}}(y)\mathsf{Q}^{(\xi)}_{jk}\nabla_k G_{\mathsf{h}}(y-x)\right| \lesssim \frac{\log^2|x|_*}{|x|_*^{d-1}}. \tag{6.15}$$

In order to complete the proof of Theorem 2.1, it thus suffices to show that

$$\left|\sum_{y\in\mathbb{Z}^d}\sum_{j,k=1}^d \nabla_j G_{\mathsf{h}}(y)\mathsf{Q}^{(\xi)}_{jk}\nabla_k G_{\mathsf{h}}(y-x) - \mathscr{K}_\xi(x)\right| \lesssim \frac{\log|x|_*}{|x|_*^{d-1}},$$

where $\mathscr{K}_\xi$ was introduced in (1.1). We learn from Proposition A.3 of the appendix that

$$\left|\nabla_j G_{\mathsf{h}}(y) - \frac{\partial\mathcal{G}_{\mathsf{h}}}{\partial y_j}(y)\right| \lesssim \frac{1}{|y|^d}.$$

As a consequence,

$$\sum_{\substack{y\in\mathbb{Z}^d\setminus\{0\}\\ 1\leqslant j,k\leqslant d}}\left|\nabla_j G_{\mathsf{h}}(y)\mathsf{Q}^{(\xi)}_{jk}\nabla_k G_{\mathsf{h}}(y-x) - \frac{\partial\mathcal{G}_{\mathsf{h}}}{\partial y_j}(y)\mathsf{Q}^{(\xi)}_{jk}\nabla_k G_{\mathsf{h}}(y-x)\right|$$
$$\lesssim \sum_{y\in\mathbb{Z}^d\setminus\{0\}}\frac{1}{|y|^d\ |y-x|_*^{d-1}} \lesssim \frac{\log|x|_*}{|x|_*^{d-1}},$$

where we used Proposition A.1 of the appendix in the last step. Similarly,

$$\sum_{\substack{y\in\mathbb{Z}^d\setminus\{0,x\}\\ 1\leqslant j,k\leqslant d}}\left|\frac{\partial\mathcal{G}_{\mathsf{h}}}{\partial y_j}(y)\mathsf{Q}^{(\xi)}_{jk}\nabla_k G_{\mathsf{h}}(y-x) - \frac{\partial\mathcal{G}_{\mathsf{h}}}{\partial y_j}(y)\mathsf{Q}^{(\xi)}_{jk}\frac{\partial\mathcal{G}_{\mathsf{h}}}{\partial y_k}(y-x)\right| \lesssim \frac{\log|x|_*}{|x|_*^{d-1}}.$$

Moreover, one can check that

$$\sum_{\substack{y\in\mathbb{Z}^d\setminus\{0,x\}\\ 1\leqslant j,k\leqslant d}}\left|\frac{\partial\mathcal{G}_{\mathsf{h}}}{\partial y_j}(y)\mathsf{Q}^{(\xi)}_{jk}\frac{\partial\mathcal{G}_{\mathsf{h}}}{\partial y_k}(y-x) - \int_{y+[0,1]^d}\frac{\partial\mathcal{G}_{\mathsf{h}}}{\partial y_j}(y')\mathsf{Q}^{(\xi)}_{jk}\frac{\partial\mathcal{G}_{\mathsf{h}}}{\partial y_k}(y'-x)\ \mathrm{d}y'\right| \lesssim \frac{\log|x|_*}{|x|_*^{d-1}}.$$

In these computations, we have been forced to drop some terms indexed by $y\in\{0,x\}$. But it is easy to check that these terms are negligible, for example

$$\sum_{\substack{y\in\{0,x\}\\ 1\leqslant j,k\leqslant d}}\left|\int_{y+[0,1]^d}\frac{\partial\mathcal{G}_{\mathsf{h}}}{\partial y_j}(u)\mathsf{Q}^{(\xi)}_{jk}\frac{\partial\mathcal{G}_{\mathsf{h}}}{\partial y_k}(u-x)\ \mathrm{d}u\right| \lesssim \frac{1}{|x|_*^{d-1}} \qquad (x\in\mathbb{Z}^d\setminus\{0\}),$$

so the proof is complete. □

Appendix A. Basic estimates on discrete convolutions and Green functions

Proposition A.1. *For every $\alpha > d$ and $\beta\in(0,\alpha]$,*

$$\sum_{y\in\mathbb{Z}^d}\frac{1}{|y|_*^\alpha\ |y-x|_*^\beta} \lesssim \frac{1}{|x|_*^\beta},$$

while for $\beta\in(0,d]$,

$$\sum_{y\in\mathbb{Z}^d}\frac{1}{|y|_*^d\ |y-x|_*^\beta} \lesssim \frac{\log|x|_*}{|x|_*^\beta}.$$

(In both statements, the sign $\lesssim$ hides a multiplicative constant that does not depend on $x \in \mathbb{Z}^d$.)

Proof. We give a unified proof of these two results, although it will be apparent that the proof of the first statement alone can be slightly simplified. We thus assume $\alpha \geqslant d$ and $\beta \in (0, \alpha]$. We decompose the sum over $y \in \mathbb{Z}^d$ according to whether $|y| \geqslant 2|x|$ or not. If $|y| \geqslant 2|x|$, then $|y-x| \geqslant |y|/2$, and thus

$$\sum_{|y| \geqslant 2|x|} \frac{1}{|y|_*^\alpha \ |y-x|_*^\beta} \lesssim \sum_{|y| \geqslant 2|x|} \frac{1}{|y|_*^{\alpha+\beta}} \lesssim \frac{1}{|x|_*^{\alpha+\beta-d}} \leqslant \frac{1}{|x|_*^\beta}$$

(here and below, we understand that y is the variable of summation). We split the rest of the sum into two parts along the condition $|y-x| \geqslant |x|/2$. This gives us two contributions, the first of which is

$$\sum_{\substack{|y| \leqslant 2|x| \\ |y-x| \geqslant |x|/2}} \frac{1}{|y|_*^\alpha \ |y-x|_*^\beta} \lesssim \frac{1}{|x|_*^\beta} \sum_{|y| \leqslant 2|x|} \frac{1}{|y|_*^\alpha},$$

This last sum is uniformly bounded if $\alpha > d$, while it is bounded by $\log |x|_*$ if $\alpha = d$. For the second contribution to be considered, note that $|y-x| \leqslant |x|/2$ implies that $|y| \geqslant |x|/2$, and thus

$$\sum_{\substack{|y| \leqslant 2|x| \\ |y-x| \leqslant |x|/2}} \frac{1}{|y|_*^\alpha \ |y-x|_*^\beta} \lesssim \frac{1}{|x|_*^\alpha} \sum_{|y-x| \leqslant |x|/2} \frac{1}{|y-x|_*^\beta}.$$

Up to a constant, this last sum is bounded by

$$\left| \begin{array}{ll} 1 & \text{if } \beta > d, \\ \log |x|_* & \text{if } \beta = d, \\ |x|_*^{d-\beta} & \text{if } \beta < d. \end{array} \right.$$

Thus, this second contribution is always at most of the order of the first, and this concludes the proof. □

Remark A.2. The proof of Proposition A.1 can be adapted to yield, for every $\beta \in (0, d]$,

$$\sum_{y \in \mathbb{Z}^d} \frac{\log |y|_*}{|y|_*^d \ |y-x|_*^\beta} \lesssim \frac{\log^2 |x|_*}{|x|_*^\beta}.$$

Proposition A.3. *For every $k \in \{1, \ldots, d\}$,*

$$\left| \nabla_k G_{\mathsf{h}}(x) - \frac{\partial}{\partial x_k} \mathcal{G}_{\mathsf{h}}(x) \right| \lesssim \frac{1}{|x|^d}.$$

Proof. Recall that A_{h} is a diagonal matrix, the diagonal entries of which we denote by $A_{\mathsf{h},1}, \ldots, A_{\mathsf{h},d}$. For $p \in [-\pi, \pi]^d$, let

$$s(p) = 2 \sum_{j=1}^d A_{\mathsf{h},j} (1 - \cos(p_j)).$$

Using Fourier transforms, one can represent the Green function G_{h} as

$$G_{\mathsf{h}}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}} \frac{e^{-ip\cdot x}}{s(p)} \ \mathrm{d}p,$$

where $\mathbb{T} = [-\pi, \pi]^d$. Similarly,

$$\nabla_j G_{\mathsf{h}}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}} \frac{(e^{-ip_j} - 1)}{s(p)} \ e^{-ip\cdot x} \ \mathrm{d}p.$$

Let $\eta(x) = (2\pi)^{-d/2} e^{-|x|^2/2}$. We note that

$$\left| \frac{\partial \mathcal{G}_{\mathsf{h}}}{\partial x_k}(x) - \left(\frac{\partial \mathcal{G}_{\mathsf{h}}}{\partial x_k} * \eta \right)(x) \right| \lesssim \frac{1}{|x|^d},$$

where $*$ denotes the convolution. This can be seen for instance using the explicit formula for the Green function,

$$\mathcal{G}_{\mathsf{h}}(x) = \frac{1}{(d-2)\gamma_d \, |\det(A_{\mathsf{h}})| \, (x \cdot A_{\mathsf{h}}^{-1} x)^{(d-2)/2}},$$

where γ_d denotes the area measure of the unit sphere. The regularization by convolution permits us to write down the Fourier representation

$$\left(\frac{\partial \mathcal{G}_{\mathsf{h}}}{\partial x_k} * \eta \right)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} -\frac{i p_j}{p \cdot A_{\mathsf{h}} p} \, e^{-|p|^2/2} \, e^{-ip\cdot x} \, \mathrm{d}p.$$

In order to prove the proposition, it thus suffices to show that

$$\left| \int_{\mathbb{T}} \frac{(e^{-ip_j} - 1)}{s(p)} \, e^{-ip\cdot x} \, \mathrm{d}p - \int_{\mathbb{R}^d} -\frac{i p_j}{p \cdot A_{\mathsf{h}} p} \, e^{-|p|^2/2} \, e^{-ip\cdot x} \, \mathrm{d}p \right| \lesssim \frac{1}{|x|^d}.$$

We select a smooth cut-off function $\chi(p)$ that is equal to one near $p = 0$ and is compactly supported in $\mathbb{T}$. We use it to split the left-hand side into

$$\int_{\mathbb{T}} (1-\chi)(p) \frac{(e^{-ip_j} - 1)}{s(p)} \, e^{-ip\cdot x} \, \mathrm{d}p \quad \text{and} \quad \int_{\mathbb{R}^d} f(p) \, e^{-ip\cdot x} \, \mathrm{d}p,$$

where

$$f(p) = \chi(p) \frac{(e^{-ip_j} - 1)}{s(p)} + \frac{i p_j}{p \cdot A_{\mathsf{h}} p} \, e^{-|p|^2/2}$$

can be considered to be defined on all $\mathbb{R}^d$. By the properties of χ, $(1-\chi)(p)\frac{(e^{-ip_j}-1)}{s(p)}$ is a smooth periodic function on $\mathbb{T}$, so that we obtain by integrations by parts that

$$\int_{\mathbb{T}} (1-\chi)(p) \frac{(e^{-ip_j} - 1)}{s(p)} \, e^{-ip\cdot x} \, \mathrm{d}p$$

decays faster than any negative power of $|x|$. Hence it suffices to show that

$$\left| \int_{\mathbb{R}^d} f(p) \, e^{-ip\cdot x} \, \mathrm{d}p \right| \lesssim \frac{1}{|x|^d}. \tag{A.1}$$

One can decompose f as

$$f(p) = -\frac{p_j^2}{2p \cdot A_{\mathsf{h}} p} + \tilde{f}(p),$$

so that $\tilde{f}$ is "more regular" than f close to the origin. One can then show by integration by parts that

$$\left| \int_{\mathbb{R}^d} -\frac{p_j^2}{2p \cdot A_{\mathsf{h}} p} \, e^{-ip\cdot x} \, \mathrm{d}p - \left(\frac{\partial^2 \mathcal{G}_{\mathsf{h}}}{\partial x_j^2} * \eta \right)(x) \right| \lesssim \frac{1}{|x|^d}$$

and

$$\left| \int_{\mathbb{R}^d} \tilde{f}(p) \, e^{-ip\cdot x} \, \mathrm{d}p \right| \lesssim \frac{1}{|x|^d}$$

for any $x \in \mathbb{R}^d$. Since $(\partial^2 \mathcal{G}_{\mathsf{h}}/\partial x_j^2 * \eta)(x) \lesssim |x|^{-d}$, the proof is complete. □

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