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by

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Abstract

In this work we study the stability of a stationary solution to the thin-film equation with linear mobility and partial wetting boundary conditions. The method used is strongly based on the gradient-flow structure of the problem. We obtain natural relaxation rates of perturbations to the stationary solution by showing that the energy is in fact convex in a neighborhood around the stationary solution.

1 Introduction

The thin-film equation

$$h_t + (h^n h_{xxx})_x = 0, \tag{1.1}$$

describes the evolution of a fluid on a substrate, given by its height h. The dynamics are driven only by surface tension and viscosity. Equation (1.1) can be derived by a lubrication approximation in the case of thin viscous films. Various values of n correspond to different slip conditions at the solid, for an in detail discussion of the underlying physics, see [9]. We are restricting our considerations to the case $n = 1$. Mathematically speaking the thin-film equation is a fourth-order degenerate parabolic equation with a moving free boundary

$$h_t + (hh_{xxx})_x = 0 \text{ in } \{h > 0\},$$

which is complemented by three boundary conditions

$$h_x^2 = \alpha, h = 0 \text{ on } \partial\{h > 0\}, \tag{1.2}$$

$$\lim_{\{h>0\}\ni y\to\partial\{h>0\}} h_{xxx}(y) = V, \tag{1.3}$$

where V denotes the velocity of the moving boundary. Depending on the ratio of the surface tensions of different phases, different values for α arise. In this work here we are treating the so called partial wetting regime $\alpha = 1$, the complete wetting regime $\alpha = 0$ is fundamentally different.

Our equation thus reads

$$\begin{aligned} &h_t + (hh_{xxx})_x = 0 && \text{in } \{h > 0\}, \\ &h_x^2 = 1, h = 0 && \text{on } \partial\{h > 0\}, \\ &\lim\nolimits_{\{h>0\}\ni y\to\partial\{h>0\}} h_{xxx}(y) = V. && \end{aligned} \tag{1.4}$$

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The literature for the partial wetting case (1.4) is not so extensive, let us recall some results. A first important result is the existence of weak solutions to (1.4) first shown in [10]. The proof therein relies, as the present work, heavily on the gradient-flow structure of the problem. More recently in [8] it was shown that (1.4) arises rigorously as the lubrication approximation of a Hele-Shaw flow. Furthermore the authors provide a first existence and uniqueness result for classical solutions to (1.4), see [8, Theorem 3.5].

Let us explain in more detail the gradient flow structure of (1.4). It is known since the work [1] that (1.4) is the gradient flow with respect to the Wasserstein metric and the energy

$$F(h) := \frac{1}{2}\int h_x^2\,dx + \frac{1}{2}|\{h>0\}| = \frac{1}{2}\int_{\{h>0\}} h_x^2 + 1\,dx, \tag{1.5}$$

in the case of solutions $h \geq 0$ satisfying

$$\int_{\mathbb{R}} h\,dx = 1.$$

The situation we are interested in is the case where the free boundary at every time t is given by a single contact point $\chi(t,0)$,

$$\begin{aligned} &h_t + (hh_{xxx})_x = 0 && \text{in }]\chi(t,0),+\infty[, \\ &h_{xxx}(x) = \dot\chi(t,0),\ h_x(x) = 1,\ h(x) = 0 && \text{for } x = \chi(t,0). \end{aligned} \tag{1.6}$$

Note that (1.6) admits a family of stationary solutions h_{c_0,c_1} given by

$$h_{c_0,c_1}(x) = (x-c_0)_+ + c_1(x-c_0)_+^2, \qquad \text{for } x \in \{h_{c_0,c_1} > 0\} =]c_0,+\infty[.$$

Since equation (1.6) is mass-preserving, by prescribing the initial mass we are determining the constants c_0, c_1 of the possible limit profiles, and thus by a possible rescaling and shifting of the initial data, it suffices to restrict our attention to perturbations of the simplest stationary solution

$$h_0(t,x) = \begin{cases} x, & x>0, \\ 0, & \text{otherwise.} \end{cases} \tag{1.7}$$

Equation (1.6) can be seen to be a gradient flow with respect to the energy

$$E(h) = \frac{1}{2}\int_{\chi(0)}^{\infty} (h_x - 1)^2\,dx. \tag{1.8}$$

This energy at least formally arises as the energy gap with respect to the above energy F in the following sense. Since for h_0 we know $F(h_0) = +\infty$, we are defining the cut-off energies

$$F^R(h) := \frac{1}{2}\int_{\chi(0)}^{\chi(R)} \left(h_x^2 + 1\right)\,dx.$$

Now we can define the energy gap with respect to F^R by

$$\begin{aligned} E^R(h) &:= F^R(h) - F^R(h_0) = \frac{1}{2}\int_{\chi(0)}^{\chi(R)} \left(h_x^2+1\right)\,dx - R \\ &= \frac{1}{2}\int_{\chi(0)}^{\chi(R)} \left(h_x^2 - 1\right)\,dx - \chi(0) + (\chi(R) - R). \end{aligned}$$

Taking the limit $R \to \infty$ we obtain

$$\bar{E}(h) := \lim_{R\to\infty} E^R(h) = \lim_{R\to\infty} \frac{1}{2}\int_{\chi(0)}^{\chi(R)} \left(h_x^2 - 1\right)\, dx - \chi(0),$$

This can be reformulated as

$$\bar{E}(h) = E(h) = \frac{1}{2}\int_{\chi(0)}^{\infty} (h_x - 1)^2\, dx \geq 0.$$

Here and in the following we are assuming that $h(t,x)$ is a smooth solution of (1.6) so that $h(t,\cdot)$ is supposed to be admissible, meaning it satisfies

$$\begin{aligned} \exists\, \chi(0) :\ \{h > 0\} = &\quad]\chi(0), +\infty[, \\ h \in &\quad C^\infty([\chi(0), \infty[), \\ h_x(\chi(0)) = &\quad 1, \\ W_2^2(h, h_0) < &\quad \infty. \end{aligned} \tag{1.9}$$

Let us quickly comment on those admissibility criteria. The first one is requiring that there is no touchdown of the film to zero and thus no topological change of the set $\{h > 0\}$, the second one is that solutions are smooth in the set $\{h > 0\}$ up to the boundary. By the regularity result in [8] both can be achieved by assuming the initial data to be small in the appropriate norms used in [8, Theorem 3.5]. The third one is just saying that h satisfies the partial wetting boundary conditions. The fourth one can be seen as prescribing the initial mass and thus determining to which of the stationary solutions we actually converge.

Let us now define the crucial quantities we are interested in. They all arise naturally from the gradient-flow structure of the problem.

Definition 1.1. *Let $h_0(x) = x_+$ be the stationary solution and let h be a smooth solution to* (1.6) *such that $h(t,\cdot)$ satisfies* (1.9)*. We define the following three time-dependent quantities:*

- *the squared distance*

$$\mathcal{H}(t) := H(h(t,\cdot)) := W_2^2(h(t,\cdot), h_0(\cdot)) = \inf_{T: T_\# h(t,\cdot) = h_0(\cdot)} \int |T(x) - x|^2\, h(t,x)\, dx.$$

- *the energy gap*

$$\mathcal{E}(t) := E(h(t,\cdot)) = \frac{1}{2}\int_{\chi(0)}^{\infty} (h_x(t,x) - 1)^2\, dx.$$

- *the dissipation*

$$\mathcal{D}(t) := D(h(t,\cdot)) := \int_{\chi(t,0)}^{\infty} h(t,x)(h_{xxx}(t,x))^2\, dx = \text{“}\left|\nabla E\right|^2 (h(t,x))\text{”}.$$

Note that each of the above quantities measures how far away we are from the global minimum (corresponding to the stationary solution) in a way adapted to the energy landscape given by E and the Wasserstein metric. It should be in principle also possible to develop an existence theory based on our work here so that we would not have to rely on prior existence results. For this reason and to make the work more consistent, in the following we will only use that h is an admissible smooth solution such that $H(h), E(h), D(h) < \infty$ and we will not assume the finiteness of other norms, such as the norms used in [8, Theorem 3.5]. We will obtain relaxation rates for these quantities under the additional assumption that the initial data is close to the stationary solution measured in terms of a combination of the above intrinsic quantities. Namely we assume that

$$\mathcal{E}_0 \mathcal{D}_0^{\frac{1}{2}} \ll 1. \tag{1.10}$$

Observe that (1.10) is meaningful since $\mathcal{E}_0 \mathcal{D}_0^{\frac{1}{2}}$ is scaling invariant with respect to the scaling of equation (1.6), which is given by

$$h_\lambda(t,x) = \lambda^{-1} h(\lambda^3 t, \lambda x).$$

The main result of this work are the following relaxation rates.

Theorem 1.2. *Let h be a smooth solution to* (1.6) *satisfying* (1.9)*, such that*

$$\mathcal{E}_0 \mathcal{D}_0^{\frac{1}{2}} \ll 1.$$

Then

$$\begin{aligned} \mathcal{H}(t) &\leq \mathcal{H}_0, \\ \mathcal{E}(t) &\leq \frac{\mathcal{H}_0}{t}, \\ \mathcal{D}(t) &\leq \frac{4\mathcal{H}_0}{t^2}. \end{aligned}$$

One interesting consequence of these estimates is that they imply certain convergence rates of the contact point $\chi(t,0)$ to zero, the contact point of the stationary solution. Those are stated in the following Corollary.

Corollary 1.3. *Let h be a solution to* (1.6) *satisfying* (1.9)*, such that*

$$\mathcal{E}_0 \mathcal{D}_0^{\frac{1}{2}} \ll 1.$$

Then the contact point satisfies

$$|\chi(t,0)| \lesssim \left(\mathcal{E}(t)\mathcal{H}(t)^{\frac{1}{2}}\right)^{\frac{1}{3}} \lesssim \frac{1}{t^{\frac{1}{3}}}.$$

The strategy of proof for the main theorem relies on certain algebraic and differential relationships between $\mathcal{H}, \mathcal{E}$ and $\mathcal{D}$.

Lemma 1.4. *Let h be smooth solution to* (1.6) *satisfying* (1.9)*, such that*

$$\mathcal{E}_0 \mathcal{D}_0^{\frac{1}{2}} \ll 1.$$

Then the following relationships hold

$$\begin{aligned} \partial_t \mathcal{H} &\leq 0, && (1.11)\\ \partial_t \mathcal{E} &= -\mathcal{D}, && (1.12)\\ \partial_t \mathcal{D} &\leq 0, && (1.13)\\ \mathcal{E} &\leq \sqrt{\mathcal{H}\mathcal{D}}. && (1.14) \end{aligned}$$

Note that (1.12) and (1.13) ensure that if we start close to the stationary solution in the sense

$$\mathcal{E}_0 \mathcal{D}_0^{\frac{1}{2}} \ll 1,$$

then we stay close for all times $t \geq 0$ in the sense of

$$\mathcal{E}(t) \mathcal{D}(t)^{\frac{1}{2}} \ll 1.$$

This strategy is inspired by the recent work [11], where the authors establish effectively the same relationships as we do here. This work was itself inspired by an observation in [4] that these relationships hold in the case when the energy is convex, and imply by an ODE argument the rates of Theorem 1.2, as seen in the next lemma.

Lemma 1.5. *Suppose the quantities $\mathcal{H}, \mathcal{E}, \mathcal{D} \geq 0$ satisfy*

$$\begin{aligned} \partial_t \mathcal{H} &\leq 0,\\ \partial_t \mathcal{E} &= -\mathcal{D},\\ \partial_t \mathcal{D} &\leq 0,\\ \mathcal{E} &\leq \sqrt{\mathcal{H}\mathcal{D}}. \end{aligned}$$

Then it holds

$$\begin{aligned} \mathcal{H}(t) &\leq \mathcal{H}_0, && (1.15)\\ \mathcal{E}(t) &\leq \frac{\mathcal{H}_0}{t}, && (1.16)\\ \mathcal{D}(t) &\leq \frac{4\mathcal{H}_0}{t^2}. && (1.17) \end{aligned}$$

Our work here is in spirit close to the setting of [4], since we show (see Lemma 3.7) that the energy is indeed convex in a neighborhood of the stationary solution. In the complete wetting case (i.e. $\alpha = 0$ in (1.2)), there are several known stability results for specific solutions. In [3], calculating the spectrum of the linear stability problem, estimates on the rate of convergence to the self-similar solution are made. In [6] the authors show that in the case of finite mass or finite second moment, we have convergence in L^1 and L^∞ of all strong solutions to the unique self-similar solution with the same mass (see [6, Theorem 5.1.]). Under additional assumptions and in the framework of classical solutions it is shown in [5] that there is also convergence in H^1 to the self-similar solution(see [5, Theorem 1.1.]). In [7] it is shown that if the initial data

is close to the stationary solution, which in the case of complete wetting is given by $\frac{x_+^2}{2}$, the free boundary converges to zero (see [7, Theorem 1.4]), comparable to Corollary 1.3.
Let us give a quick overview over the structure of the paper. Instead of working in the original h-variables of equation (1.6), it turns out to be more convenient to think in terms of the variable χ_h defined by

$$\int_{-\infty}^{\chi_h(z)} h(y)\,dy = \frac{z^2}{2}, \tag{1.18}$$

which can be seen as a transformation into Lagrangian coordinates. Section 2 is devoted to reformulating the quantities from Definition 1.1 in terms of the corresponding quantities in χ-variables, denoted by $\hat{H}, \hat{E}$ and $\hat{D}$. The main merit of this transformation is that the initial intrinsic metric of the problem, the non-Euclidean Wasserstein metric is transformed into a Euclidean weighted L^2-metric. More precisely, recall that the Wasserstein metric tensor at h is given by

$$\langle \delta h, \delta h \rangle_h := \int v^2 h\,dx,$$

where

$$\delta h + (vh)_x = 0.$$

For an account of the optimal transport problem and the Wasserstein metric, see [2]. Therefore $\langle \cdot, \cdot \rangle_h$ obviously depends on h and thus is apparently non-Euclidean. In comparison to this, the transformed metric is given by (see the discussion in Lemma 2.1)

$$\langle \delta\chi, \delta\chi \rangle_\chi = \int \delta\chi^2 z\,dz,$$

independent of χ. This simplifies the proof of the geodesic convexity of the energy, since in a Euclidean space geodesics are just straight lines.

Exactly this fact is also the reason why we limit our discussion to the case $n = 1$ instead of more general mobilities: the case $n \neq 1$ leads to a non-Euclidean metric in the Lagrangian coordinates and thus the strategy applied here does not easily generalize to these cases. For a further discussion of the gradient flow structure in the case $n \neq 1$ see [12].
Section 3 is the main part of this work. The main statement is Lemma 3.7, which says that the energy $\hat{E}$ is convex in a neighborhood of the stationary solution, i.e. for χ such that

$$\hat{E}(\chi)\hat{D}(\chi)^{\frac{1}{2}} \ll 1,$$

which is, as noted above, scaling invariant and stable under the differential inequalities stated in Lemma 1.4. To prove this, we need that in this regime we have certain L^∞-bounds on the first and second derivative of χ, as stated in Lemma 3.3 and 3.6, given by

$$|\chi_z - 1|_\infty \lesssim \left(\hat{E}(\chi)\hat{D}(\chi)^{\frac{1}{2}}\right)^{\frac{1}{3}} \ll 1, \tag{1.19}$$

$$|z\chi_{zz}|_\infty \lesssim \left(\hat{E}(\chi)\hat{D}(\chi)^{\frac{1}{2}}\right)^{\frac{1}{3}} \ll 1. \tag{1.20}$$

The main technical problem lies in understanding the rather complicated non-linear quantity $\hat{D}(\chi)$, which is needed for proving (1.19) and (1.20). This is the content of Lemma 3.5, which states that close to the stationary solution, $\hat{D}(\chi)$ controls a certain semi norm of χ, namely

$$\hat{D}(\chi) \gtrsim \int z^3 \chi_{zzzz}^2\,dz.$$

This turns out (see Lemma 3.6) to be enough to conclude estimate (1.20). Lemma 3.8 then gives an estimate of the contact point $\chi(0)$ in terms of the quantities $\hat{H}$ and $\hat{E}$. This establishes Corollary 1.3. In Section 4 we establish inequalities (1.11), (1.12), (1.13) and (1.14), which are by then easy consequences of the convexity of the energy as stated in Lemma 3.7. This proves Lemma 1.4. For completeness a proof of Lemma 1.5 is provided there as well. In the Appendix we prove Lemmas concerning the boundary behavior of χ (Lemma 5.1, 5.3 and Lemma 5.4). We are also providing a self-contained proof of a Bernis-like estimate which was already proved in [10] in a slightly different setting, namely

$$|h'(x) - h'(y)| \lesssim D^{\frac{1}{3}} |x - y|^{\frac{2}{3}},$$

see Lemma 5.6.

Throughout the chapter we will write $a \lesssim b$ if $a \leq cb$ for some universal constant c. We will also write $a \approx b$ if $a \lesssim b \lesssim a$. We will furthermore write: if $L \ll 1$ then $a \lesssim b$ and mean that there exists universal $\delta > 0$ such that if $L \leq \delta$ then $a \lesssim b$.

2 The problem in different coordinates

The next lemma states precisely how our main quantities H, E and D transform under the reparametrization (1.18).

Lemma 2.1. *For h satisfying* (1.9) *define $\chi := \chi_h$ via*

$$\int_{-\infty}^{\chi_h(z)} h(y)\, dy = \frac{z^2}{2}. \tag{2.1}$$

Denote for simplicity

$$u := \chi_z(z),$$
$$p := \chi_{zz}(z).$$

Let $\chi_{h_0}(z) =: \chi_0(z) = z$ denote the transformation corresponding to the stationary solution $h_0(x) = x_+$. Then the functionals from Definition 1.1 in these coordinates read

$$H(h) = \hat{H}(\chi) = \int_0^\infty z(\chi(z) - \chi_0(z))^2\, dz,$$
$$E(h) = \hat{E}(\chi) = \frac{1}{2}\int_0^\infty \frac{z^2p^2}{u^5} + \frac{1}{3u^3} - \frac{4}{3} + u\, dz =: \int_0^\infty L(z,u,p)\, dz,$$
$$D(h) = \hat{D}(\chi) = \int_0^\infty \frac{1}{z}\left(-\Big((\partial_2 L)(z,u,p)\Big)_z + \Big((\partial_3 L)(z,u,p)\Big)_{zz}\right)^2 dz.$$

In particular this implies that the time-dependent quantities from Definition 1.1 can be expressed in terms of χ as

$$\mathcal{H}(t) = \hat{H}(\chi(t,\cdot)),$$
$$\mathcal{E}(t) = \hat{E}(\chi(t,\cdot)),$$
$$\mathcal{D}(t) = \hat{D}(\chi(t,\cdot)).$$

Proof. Let us start by rewriting E.
First observe that differentiating (2.1) yields

$$h(\chi(z))\chi_z(z) = z, \quad h_x(\chi(z)) = \frac{\chi_z(z) - z\chi_{zz}(z)}{(\chi_z(z))^3}.$$

Using this to rewrite the energy we obtain

$$\begin{aligned}
E(h) &= \frac{1}{2}\int_{\chi(0)}^\infty (h_x - 1)^2\, dx \\
&= \frac{1}{2}\int_0^\infty \left(h_x(\chi(z)) - 1\right)^2 \chi_z(z)\, dz \\
&= \frac{1}{2}\int_0^\infty \frac{\left(\chi_z(1-\chi_z^2) - z\chi_{zz}\right)^2}{\chi_z^5}\, dz \\
&= \frac{1}{2}\int_0^\infty \frac{z^2\chi_{zz}^2}{\chi_z^5} + \frac{1}{\chi_z^3} + \chi_z - 2\frac{z\chi_{zz}}{\chi_z^4} - \frac{2}{\chi_z} + \frac{2z\chi_{zz}}{\chi_z^2}\, dz \\
&= \frac{1}{2}\int_0^\infty \frac{z^2\chi_{zz}^2}{\chi_z^5} + \frac{1}{3\chi_z^3} + \chi_z - \frac{4}{3} + \frac{2}{3}\left(\frac{z(1-\chi_z^2)}{\chi_z^3} + 2\left(z - \frac{z}{\chi_z}\right)\right)_z dz.
\end{aligned}$$

Observe that

$$\int_0^\infty \left(\frac{z(1-\chi_z^2)}{\chi_z^3} + 2\left(z - \frac{z}{\chi_z}\right)\right)_z dz = 0,$$

which is due to

$$\frac{z(1-\chi_z^2)}{\chi_z^3} + 2\left(z - \frac{z}{\chi_z}\right) = \frac{1+2\chi_z(z)}{\chi_z(z)^3} z\,(1-\chi_z(z))^2 \to 0, \text{ for } z \to \infty,$$

which follows from Lemma 5.3. Plugging this in yields

$$\begin{aligned} E(h) = \hat{E}(\chi) &= \frac{1}{2}\int_0^\infty \frac{z^2\chi_{zz}^2}{\chi_z^5} + \frac{1}{3\chi_z^3} - \frac{4}{3} + \chi_z\, dz \\ &=: \int_0^\infty L(z, \chi_z(z), \chi_{zz}(z))\, dz. \end{aligned}$$

Now let us reformulate H.

To understand why H and D transform to $\hat{H}$ and $\hat{D}$ in the way they do, let us do some formal computations to motivate the resulting expressions.

For this let us derive the correct metric tensor in the new coordinates, corresponding to the Wasserstein metric tensor. First we will investigate how we identify perturbations of χ in terms of perturbations of h. Let δh denote a perturbation of h, i.e.

$$\int \delta h\, dx = 0.$$

This defines a perturbation $\delta\chi$ of χ by

$$\int_{-\infty}^{(\chi+s\delta\chi)(z)} (h + s\delta h)(x)\, dx = \frac{1}{2}z^2.$$

differentiating with respect to s at $s = 0$ yields

$$\delta\chi(z)h(\chi(z)) + \int_{-\infty}^{\chi(z)} \delta h(x)\, dx = 0.$$

Thus we can identify a perturbation δh of h with a perturbation $\delta\chi$ of χ by

$$\delta\chi(z) = -\frac{1}{h(\chi(z))}\int_{-\infty}^{\chi(z)} \delta h(x)\, dx. \tag{2.2}$$

Next we transform the metric tensor. For this let $\delta h_1, \delta h_2$ be perturbations of h. The Wasserstein metric tensor is then defined by

$$\langle \delta h_1, \delta h_2 \rangle_h = \int v_1(x)v_2(x)h(x)\, dx,$$

with

$$\delta h_i + (hv_i)_x = 0, \text{ for } i = 1, 2.$$

Observe that this and (2.2) yield that

$$\delta\chi_i(z) = v_i(\chi(z)), \text{ for } i = 1, 2.$$

Thus we can transform the metric tensor

$$\begin{aligned}\langle \delta h_1, \delta h_2 \rangle_h &= \int_{-\infty}^{\infty} v_1(x) v_2(x) h(x)\, dx = \int_0^{\infty} v_1(\chi(z)) v_2(\chi(z)) h(\chi(z)) \chi_z(z)\, dz \\ &= \int_0^{\infty} v_1(\chi(z)) v_2(\chi(z)) z\, dz = \int_0^{\infty} \delta\chi_1(z) \delta\chi_2(z) z\, dz =: \langle \delta\chi_1, \delta\chi_2 \rangle_\chi .\end{aligned}$$

As it is well known, by the Benamou-Brenier formula we can rewrite the Wasserstein distance by use of this tensor as

$$W_2^2(h, h_0) = \inf \int_0^1 \langle \partial_s g^s, \partial_s g^s \rangle_{g^s}\, ds,$$

the infimum being taken over all curves $s \mapsto g^s$ with

$$g^0 = h, g^1 = h_0.$$

By the above transformation of the metric this can be written in χ coordinates as

$$\inf \int_0^1 \langle \partial_s \chi_g^s, \partial_s \chi_g^s \rangle\, ds = \inf \int_0^1 \int_0^{\infty} (\partial_s \chi_g^s)^2(z) z\, dz\, ds,$$

the infimum being taken over all curves $s \mapsto \chi_g^s$ with

$$\chi_g^0 = \chi,\ \chi_g^1 = \chi_0.$$

This infimum is equal to

$$\hat{H}(\chi) = \int_0^{\infty} (\chi - \chi_0)^2 (z) z\, dz,$$

as conjectured. To prove this rigorously we use that as in the case of probability measures we know that in one dimension the optimal transport map T is given by the monotone map

$$T_{opt} = \bar{h_0}^{-1} \circ \bar{h},$$

where as before

$$\bar{h}(x) = \int_{-\infty}^{x} h(y)\, dy,$$

and in particular

$$\bar{h_0}^{-1}(x) = \sqrt{2x}.$$

Thus

$$H(h) = \int_{\mathbb{R}} \left(\sqrt{2\bar{h}(x)} - x \right)^2 h(x)\, dx. \tag{2.3}$$

Substituting $\chi(z)$ for x we obtain as desired

$$H(h) = \int_0^{\infty} (z - \chi(z))^2 h(\chi(z)) \chi_z(z)\, dz = \int_0^{\infty} (z - \chi(z))^2 z\, dz = \int_0^{\infty} (\chi_0(z) - \chi(z))^2 z\, dz.$$

To motivate the expression $\hat{D}$ we use the defining identity

$$\partial_t E(h(t, \cdot)) = -D(h(t, \cdot)).$$

For this we first identify the equation solved by χ by using the fact that it is a gradient flow with respect to $\hat{E}$ and the metric tensor calculated above. This means that for every perturbation $\delta\chi$ we have

$$\begin{aligned}
\int_0^\infty \chi_t(z)\delta\chi(z)z\,dz &= -\partial_s|_{s=0}\hat{E}(\chi+s\delta\chi) \\
&= -\int_0^\infty (\partial_2 L)(z,\chi_z,\chi_{zz})(\delta\chi)_z + (\partial_3 L)(z,\chi_z,\chi_{zz})(\delta\chi)_{zz}\,dz \\
&= -\int_0^\infty -\Big((\partial_2 L)(z,\chi_z,\chi_{zz})\Big)_z(\delta\chi) + \Big((\partial_3 L)(z,\chi_z,\chi_{zz})\Big)_{zz}(\delta\chi)\,dz.
\end{aligned}$$

Thus we get that χ satisfies the equation

$$\chi_t(z) = -\frac{1}{z}\left(-\Big((\partial_2 L)(z,\chi_z,\chi_{zz})\Big)_z + \Big((\partial_3 L)(z,\chi_z,\chi_{zz})\Big)_{zz}\right). \tag{2.4}$$

Since we know that D is defined by the identity

$$\partial_t E(h(t,\cdot)) = -D(h(t,\cdot)),$$

using (2.4) we obtain

$$D(h(t,\cdot)) = \int_0^\infty \frac{1}{z}\left(-\Big((\partial_2 L)(z,\chi_z,\chi_{zz})\Big)_z + \Big((\partial_3 L)(z,\chi_z,\chi_{zz})\Big)_{zz}\right)^2 dz =: \hat{D}(\chi(t,\cdot)).$$

as desired.

For a rigorous proof we need to use the defining identity of χ

$$\bar{h}(\chi(z)) = \frac{z^2}{2}, \tag{2.5}$$

and take the derivative four times to identify $h_{xxx}(\chi(z))$, which turns out to be given by

$$\begin{aligned}
h_{xxx}(\chi(z)) &= \frac{1}{u^5}\left(15\frac{p^2}{u} - 4p_z - 15\frac{zp^3}{u^2} + 10\frac{zpp_z}{u} - zp_{zz}\right) \\
&= \frac{1}{z}\big(\left(\partial_2 L\right)(z,u,p)_z - \left(\partial_3 L\right)(z,u,p)_{zz}\big).
\end{aligned}$$

Using this we can rewrite

$$\begin{aligned}
D(h) = \int h h_{xxx}^2\,dx &= \int_0^\infty h(\chi(z))h_{xxx}(\chi(z))^2\chi_z(z)\,dz \\
&= \int \frac{z}{u}\frac{1}{u^{10}}\left(15\frac{p^2}{u} - 4p_z - 15\frac{zp^3}{u^2} + 10\frac{zpp_z}{u} - zp_{zz}\right)^2 u\,dz \\
&= \int_0^\infty \frac{1}{z}\left(-\Big(\left(\partial_2 L\right)(z,u,p)\Big)_z + \Big(\left(\partial_3 L\right)(z,u,p)\Big)_{zz}\right)^2 dz.
\end{aligned}$$

Also taking first the x and then the t-derivative of (2.5) we obtain that if h is a solution to (1.6), then its corresponding χ satisfies

$$\chi_t(t,z) = -\frac{u(z)^2}{z}h_t(t,\chi(t,z)).$$

Using this we can obtain that indeed (2.4) holds.

□

3 Convexity of the energy by critical norm estimates

In this section we will ultimately prove the convexity of the energy E close to the stationary solution, see Lemma 3.7. For this we need L^∞-control on the distance of χ to the stationary solution χ_0 in the first and second derivative in terms of our scaling invariant quantity $ED^{\frac{1}{2}}$. We start out with two estimates which are suboptimal, Lemmas 3.1 and 3.2. The first of these estimates is the content of the next lemma, which gives control of the L^∞ distance of the first derivative of χ to the stationary solution. It is non optimal in the scaling with respect to $ED^{\frac{1}{2}}$, the optimal scaling is then achieved later in Lemma 3.6. In this section χ is supposed to be admissible, i.e. belonging to an h which satisfies (1.9).

Lemma 3.1. *Let χ be close to the stationary solution in the sense of*

$$\hat{E}(\chi)\hat{D}(\chi)^{\frac{1}{2}} \ll 1.$$

Then it holds

$$\sup_{z\geq 0} |(\chi(z) - \chi_0(z))_z| = |\chi_z - 1|_\infty \lesssim \left(\hat{E}(\chi)\hat{D}(\chi)^{\frac{1}{2}}\right)^{\frac{2}{9}} \ll 1.$$

Proof. Let us for convenience write in the following as always

$$u := \chi_z,\ p := \chi_{zz}.$$

Let first z be such that

$$z \geq \left(\frac{\hat{E}}{\hat{D}}\right)^{\frac{1}{3}}.$$

This in particular implies that

$$z \geq 2\hat{E},$$

since

$$\hat{E}\hat{D}^{\frac{1}{2}} \ll 1.$$

Observe that using Lemma 5.1

$$\frac{u^{\frac{3}{2}} - 1}{u^{\frac{3}{2}}}(z) = \int_z^\infty \left(1 - \frac{1}{u^{\frac{3}{2}}}\right)_z (y)\, dy \lesssim \left(\int \frac{p^2 y^2}{u^5}\, dy \int_z^\infty \frac{1}{y^2}\, dy\right)^{\frac{1}{2}} \lesssim \left(\frac{\hat{E}(\chi)}{z}\right)^{\frac{1}{2}}.$$

Here we used that

$$\hat{E}(\chi) = \int_0^\infty L(z, \chi_z(z), \chi_{zz}(z))\, dz = \frac{1}{2}\int_0^\infty \frac{z^2 p^2}{u^5} + f(u)\, dz,$$

with

$$f(u) \geq 0, \quad \text{for } u \geq 0.$$

From this we obtain

$$|u(z) - 1|^{\frac{3}{2}} \lesssim |u^{\frac{3}{2}}(z) - 1| \lesssim \hat{E}(\chi)^{\frac{1}{2}} \frac{u^{\frac{3}{2}}(z)}{z^{\frac{1}{2}}}.$$

Since $z \geq 2\hat{E}$ we obtain $u \lesssim 1$ and thus

$$|u(z) - 1| \lesssim \left(\frac{\hat{E}}{z}\right)^{\frac{1}{3}} \leq (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{2}{9}} \ll 1. \tag{3.1}$$

This is the desired estimate for z such that

$$z \geq \left(\frac{\hat{E}}{\hat{D}}\right)^{\frac{1}{3}}.$$

Now let z be such that

$$z \leq \left(\frac{\hat{E}}{\hat{D}}\right)^{\frac{1}{3}},$$

and let us work in the original h coordinates and use Lemma 5.6, which states that

$$[h_x]_{\frac{2}{3},[0,r]} \lesssim D(h)^{\frac{1}{3}}, \tag{3.2}$$

where

$$r = C^{-1}D^{-\frac{1}{2}},$$

for a constant C.

Let us now assume without loss of generality that

$$\chi(0) = 0,$$

i.e.

$$h(0) = 0.$$

This can be achieved by looking at the function

$$\tilde{h}(x) := h(x + \chi(0)).$$

Using (3.2) implies for $x \lesssim D^{-\frac{1}{2}}$ (using $h_x(0) = 1$)

$$1 - D^{\frac{1}{3}}x^{\frac{2}{3}} \leq h_x(x) \leq 1 + D^{\frac{1}{3}}x^{\frac{2}{3}}.$$

Integrating yields

$$x\left(1 - D^{\frac{1}{3}}x^{\frac{2}{3}}\right) \leq h(x) \leq x\left(1 + D^{\frac{1}{3}}x^{\frac{2}{3}}\right), \tag{3.3}$$

which holds for all

$$x \leq C^{-1}D^{-\frac{1}{2}}.$$

Since $\chi(0) = 0$ and χ is continuous we know that for $z \ll 1$ also $\chi(z) \ll 1$ and in particular

$$\chi(z) \leq \left(\frac{\hat{E}}{\hat{D}}\right)^{\frac{1}{3}} = (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{1}{3}}\hat{D}^{-\frac{1}{2}} \leq \frac{1}{C}\hat{D}^{-\frac{1}{2}},$$

since

$$\hat{E}\hat{D}^{\frac{1}{2}} \ll 1.$$

Thus integrating from 0 to $\chi(z)$ gives by the definition of χ

$$\frac{1}{2}z^2 = \int_0^{\chi(z)} h(x)\,dx, \tag{3.4}$$

and using (3.3)

$$\frac{1}{2}\chi(z)^2\left(1 - D^{\frac{1}{3}}\chi(z)^{\frac{2}{3}}\right) \le \frac{1}{2}z^2 \le \frac{1}{2}\chi(z)^2\left(1 + D^{\frac{1}{3}}\chi(z)^{\frac{2}{3}}\right). \tag{3.5}$$

Using the fact that for $\delta \ll 1$ we have

$$\sqrt{1+\delta} \approx 1 + \frac{\delta}{2},$$

or more precisely for $0 \le \delta \le \frac{1}{2}$ we have

$$1 - \delta \le \sqrt{1-\delta},$$

and

$$\sqrt{1+\delta} \le 1 + \delta,$$

so that we obtain that for $z \ll 1$

$$\chi(z)\left(1 - D^{\frac{1}{3}}\chi(z)^{\frac{2}{3}}\right) \le z \le \chi(z)\left(1 + D^{\frac{1}{3}}\chi(z)^{\frac{2}{3}}\right). \tag{3.6}$$

Since for $z \ll 1$ also $\chi(z) \ll 1$ this yields

$$\frac{1}{2}\chi(z) \le z \le 2\chi(z).$$

But using this in (3.6) then yields

$$\chi(z) - cD^{\frac{1}{3}}z^{\frac{5}{3}} \le z \le \chi(z) + cD^{\frac{1}{3}}z^{\frac{5}{3}},$$

which implies

$$z\left(1 - cD^{\frac{1}{3}}z^{\frac{2}{3}}\right) \le \chi(z) \le z\left(1 + cD^{\frac{1}{3}}z^{\frac{2}{3}}\right). \tag{3.7}$$

This was true for $z \ll 1$ since then we knew that $\chi(z) \ll 1$, but (3.7) implies that for all

$$z \le \left(\frac{E}{D}\right)^{\frac{1}{3}},$$

we know that

$$\chi(z) \le 2\left(\frac{E}{D}\right)^{\frac{1}{3}}.$$

This allows us to use (3.4) for all $z \le \left(\frac{E}{D}\right)^{\frac{1}{3}}$, and thus repeating above arguments, (3.7) holds for all those z. Thus we have, combining (3.7) and (3.3)

$$|z - h(\chi(z))| \le D^{\frac{1}{3}}z^{\frac{5}{3}},$$

and thus using again (3.3)

$$\left|\frac{z-h(\chi(z))}{h(\chi(z))}\right| \leq D^{\frac{1}{3}} z^{\frac{2}{3}}.$$

This means

$$|\chi_z(z)-1| \lesssim D^{\frac{1}{3}} z^{\frac{2}{3}} \leq \left(ED^{\frac{1}{2}}\right)^{\frac{2}{9}}. \tag{3.8}$$

Together we obtained for all $z \geq 0$

$$|\chi_z(z)-1| \lesssim \left(ED^{\frac{1}{2}}\right)^{\frac{2}{9}}.$$

□

The next aim is to prove Lemma 3.6, which states that we have control of the second derivative of χ in the form of

$$|z\chi_{zz}(z)|_\infty \lesssim \left(\hat{E}\hat{D}^{\frac{1}{2}}\right)^{\frac{1}{3}}. \tag{3.9}$$

For this the main step is to understand that in the regime where

$$\hat{E}\hat{D}^{\frac{1}{2}} \ll 1,$$

the dissipation $\hat{D}$ controls a certain norm of χ_{zz}. This is the content of Lemma 3.5, which states that

$$\hat{D}(\chi) \gtrsim \int z^3 \chi_{zzzz}^2 \, dz.$$

Once we have this, using implicitly a kind of linear estimate for large z

$$|z\chi_{zz}(z)|_\infty \lesssim \left(\int z^3 \chi_{zzzz}^2 \, dz\right)^{\frac{1}{6}} \left(\int z^2 \chi_{zz}^2 \, dz\right)^{\frac{1}{3}},$$

we can deduce Lemma 3.6. The following lemma is a sub-optimal estimate for $|z\chi_{zz}(z)|$, but a first important step towards proving (3.9) since for small z, meaning

$$z \leq \left(\frac{\hat{E}}{\hat{D}}\right)^{\frac{1}{3}},$$

it already implies (3.9).

Lemma 3.2. *Let χ be close to the stationary solution in the sense of*

$$\hat{E}(\chi)\hat{D}(\chi)^{\frac{1}{2}} \ll 1.$$

Then it holds

$$|z\chi_{zz}(z)| \lesssim \left(\hat{E}\hat{D}\, z\right)^{\frac{1}{4}} + |\chi_z - 1|(z). \tag{3.10}$$

Proof. Let us for convenience write in the following as always

$$u := \chi_z,\, p := \chi_{zz}.$$

Note first that since

$$\hat{E}(\chi)\hat{D}(\chi)^{\frac{1}{2}} \ll 1,$$

Lemma 3.1 yields

$$u \approx 1,$$

and thus

$$z^2p^2 \lesssim \frac{z^2p^2(z)}{2u^5(z)} \lesssim z^2p^2. \tag{3.11}$$

Note also that

$$\hat{D} = \int_0^\infty \frac{U_z^2}{z}\, dz,$$

with

$$U := -\partial_2 L + (\partial_3 L)_z = -\frac{5z^2p^2}{2u^6} + \frac{2zp}{u^5} + \frac{z^2 p_z}{u^5} + \frac{1}{2}\left(\frac{1}{u^4} - 1\right).$$

As a first estimate we obtain

$$U(z) = \int_0^z U_y(y)\, dy \le \left(\int_0^\infty \frac{U_z^2(y)}{y}\, dy \int_0^z y\, dy\right)^{\frac{1}{2}} \lesssim \hat{D}^{\frac{1}{2}} z. \tag{3.12}$$

Here we used that

$$\lim_{z \searrow 0} U(z) = 0,$$

which is due to the fact that

$$\begin{aligned} &\lim\nolimits_{z \searrow 0} (u-1)(z) = 0, \\ &\lim\nolimits_{z \searrow 0} zp(z) = 0, \\ &\lim\nolimits_{z \searrow 0} z^2 p_z(z) = 0. \end{aligned} \tag{3.13}$$

The fact that

$$\lim_{z \searrow 0} (u-1)(z) = 0,$$

is just due to the partial wetting and for example seen in (3.8). The other two limits are just due to the fact that by Lemma 5.4 we know that $\chi \in C^\infty([0,\infty[)$, which in particular implies that for all n

$$|\lim_{z \searrow 0} \partial_z^n \chi(z)| < \infty.$$

In the following we will write

$$L(z) = L(z, u(z), p(z)) = \frac{z^2p^2(z)}{2u^5(z)} + \frac{1}{2} f(u(z)).$$

with

$$f(u) = \frac{1}{3u^3} - \frac{4}{3} + u.$$

Start with the quantity

$$\begin{aligned}
&\int_0^z U(y)p(y)\,dy \\
&\quad = \int_0^z \left(-\partial_2 L(y) + \partial_y \partial_3 L(y)\right) p(y)\,dy = -\int_0^z \partial_2 L(y)p(y) + \partial_3 L(y)\partial_y p(y)\,dy + \partial_3 L(z)p(z) \\
&\quad = -\int_0^z \partial_y L(y) - \partial_1 L(y, u(y), p(y))\,dy + \partial_3 L(z)p(z) = -L(z) + \int_0^z \frac{yp^2}{u^5}\,dy + \frac{z^2p^2(z)}{u^5(z)} \\
&\quad = \frac{z^2p^2(z)}{2u^5(z)} - \frac{1}{2}f(u(z)) + \int_0^z \frac{yp^2}{u^5}\,dy.
\end{aligned}$$

Here we used that

$$\lim_{z\searrow 0} \partial_3 L(z)p(z) = 0,$$
$$\lim_{z\searrow 0} L(z) = 0,$$

which is due to (3.13). On the other hand we use estimate (3.12) and (3.11) to get

$$\int_0^z U(y)p(y)\,dy \le \int_0^z yp(y)\,dy\,\hat{D}^{\frac{1}{2}} \le \left(\int_0^\infty y^2p^2\,dy\,z\right)^{\frac{1}{2}} \hat{D}^{\frac{1}{2}} \lesssim \left(\hat{E}\hat{D}\,z\right)^{\frac{1}{2}}.$$

Thus we obtain

$$\frac{z^2p^2(z)}{2u^5(z)} \le \frac{z^2p^2(z)}{2u^5(z)} + \int_0^z \frac{yp^2}{u^5}\,dy \lesssim \left(\hat{E}\hat{D}\,z\right)^{\frac{1}{2}} + \frac{1}{2}f(u(z)).$$

This turns into the desired estimate by using again (3.11) and since it holds that

$$f(u(z)) \lesssim |u(z) - 1|^2.$$

□

With the sub-optimal Lemmas 3.1 and 3.2, we are in the position to prove the estimate of $|\chi_z - 1|_\infty$ which is optimal in powers of $\hat{E}\hat{D}^{\frac{1}{2}}$.

Lemma 3.3. *Let χ be close to the stationary solution in the sense of*

$$\hat{E}(\chi)\hat{D}(\chi)^{\frac{1}{2}} \ll 1.$$

Then it holds

$$\sup_{z\ge 0} |(\chi(z) - \chi_0(z))_z| = |\chi_z - 1|_\infty \lesssim \left(\hat{E}(\chi)\hat{D}(\chi)^{\frac{1}{2}}\right)^{\frac{1}{3}} \ll 1.$$

Proof. Write as always

$$u := \chi_z,\ p := \chi_{zz}.$$

Let first z be such that

$$z \geq \left(\frac{\hat{E}}{\hat{D}}\right)^{\frac{1}{3}}.$$

Then using Lemma 3.1, we know that $|u-1|_\infty \ll 1$, which implies

$$\int z^2 p^2\, dz \lesssim \hat{E}.$$

Using this we obtain immediately by Hölder

$$|(u-1)(z)| \leq \int_z^\infty |p(y)|\, dy \lesssim \left(\frac{\hat{E}}{z}\right)^{\frac{1}{2}} \leq \left(\hat{E}(\chi)\hat{D}(\chi)^{\frac{1}{2}}\right)^{\frac{1}{3}}.$$

For

$$z \leq z_* = \left(\frac{\hat{E}}{\hat{D}}\right)^{\frac{1}{3}},$$

we will use Lemma 3.2 which implies

$$|p(z)| \lesssim \left(\frac{\hat{E}\hat{D}}{z^3}\right)^{\frac{1}{4}} + \frac{1}{z}|(u-1)(z)|. \tag{3.14}$$

Estimate for some $\delta \ll 1$

$$\begin{aligned} |u(z)-1| &\leq \int_\delta^z |p(y)|\, dy + |u(\delta)-1| \lesssim |u(\delta)-1| + (\hat{E}\hat{D})^{\frac{1}{4}} \int_0^z y^{-\frac{3}{4}}\, dy + |u-1|_{\infty,[0,z_*]} \ln\left(\frac{z}{\delta}\right) \\ &\lesssim \left(\hat{E}(\chi)\hat{D}(\chi)^{\frac{1}{2}}\right)^{\frac{1}{3}} + |u(\delta)-1| + |u-1|_{\infty,[0,z_*]}\frac{z}{\delta}. \end{aligned}$$

Now since $|u(z)-1| \to 0$ for $z \to 0$, by using (3.8) we observe that for

$$\delta := \hat{E}(\chi)^{\frac{1}{2}}\hat{D}(\chi)^{-\frac{1}{4}},$$

it holds

$$|u(\delta)-1| \lesssim \left(\hat{E}(\chi)\hat{D}(\chi)^{\frac{1}{2}}\right)^{\frac{1}{3}}.$$

Also observe that

$$\frac{z}{\delta} \leq \left(\hat{E}(\chi)\hat{D}(\chi)^{\frac{1}{2}}\right)^{-\frac{1}{6}},$$

which yields, using Lemma 3.1

$$\frac{z}{\delta}|u-1|_{\infty,[0,z_*]} \lesssim \left(\hat{E}(\chi)\hat{D}(\chi)^{\frac{1}{2}}\right)^{\frac{1}{18}} \ll 1.$$

Thus we can obtain for all $z \leq z_*$

$$|u(z)-1| \lesssim \left(\hat{E}(\chi)\hat{D}(\chi)^{\frac{1}{2}}\right)^{\frac{1}{3}} + \left(\hat{E}(\chi)\hat{D}(\chi)^{\frac{1}{2}}\right)^{\frac{1}{18}} |u-1|_{\infty,[0,z_*]}.$$

Now taking the supremum over all $z \leq z_*$ and then absorbing the last term into the left-hand side, we obtain the desired result. □

The next lemma contains a linear estimate which is used in the proof of Lemma 3.5.

Lemma 3.4. *Let χ be such that*

$$\hat{E}(\chi)\hat{D}(\chi)^{\frac{1}{2}} \ll 1.$$

Then we have

$$\int |\chi_{zz}|^3\, dz \lesssim |\chi_z - 1|_\infty \int z\chi_{zzz}^2\, dz.$$

Proof. Let as always

$$u := \chi_z,\, p := \chi_{zz},$$

By proving

$$\int |g_z|^3\, dz \lesssim |g|_\infty \int z g_{zz}^2\, dz,$$

and considering

$$g := u - 1,$$

we obtain the above desired estimate.

Define

$$z_* := |g|_\infty \left(\int z g_{zz}^2\, dz\right)^{-\frac{1}{2}}.$$

First observe that Hölder with $(2, 3, 6)$ yields

$$g_z^2(z_*) = -2\int_{z_*}^{\infty} g_z g_{zz}\, dz \lesssim \left(\int z g_{zz}^2\, dz\right)^{\frac{1}{2}} \left(\int |g_z|^3\, dz\right)^{\frac{1}{3}} \frac{1}{z_*^{\frac{1}{3}}}. \tag{3.15}$$

Here we used that

$$g_z(z) = p(z) \to 0, \qquad \text{for } z \to \infty,$$

which follows e.g. from estimate (3.10).

Estimate via integration by parts using this

$$\int_{z_*}^{\infty} g_z^3\, dz = -g_z^2(z_*)g(z_*) - \int_{z_*}^{\infty} 2g_z g_{zz} g\, dz \lesssim |g|_\infty 2 \left(\int z g_{zz}^2\, dz\right)^{\frac{1}{2}} \left(\int |g_z|^3\, dz\right)^{\frac{1}{3}} \frac{1}{z_*^{\frac{1}{3}}}.$$

Here we used $(u-1)(z)p^2(z) \to 0$ for $z \to 0$ which follows e.g. from (3.1) and (3.10).

Then use Young's inequality with $(3, \frac{3}{2})$ to obtain

$$\int_{z_*}^{\infty} g_z^3\, dz \le \delta \int |g_z|^3\, dz + c(\delta)|g|_\infty^{\frac{3}{2}} \left(\int z g_{zz}^2\, dz\right)^{\frac{3}{4}} \frac{1}{z_*^{\frac{1}{2}}} = \delta \int |g_z|^3\, dz + c(\delta)|g|_\infty \int z g_{zz}^2\, dz,$$

where in the last step we used the definition of z_*. Writing

$$\int_{z_*}^{\infty} |g_z|^3\, dz = \int_{g_z > 0 \cap]z_*, \infty[} g_z^3\, dz - \int_{g_z < 0 \cap]z_*, \infty[} g_z^3\, dz,$$

we conclude

$$\int_{z_*}^{\infty} |g_z|^3\, dz \le \delta \int |g_z|^3\, dz + c(\delta)|g|_\infty \int z g_{zz}^2\, dz.$$

On the other hand first observe that

$$\int_0^{z_*} |g_z(z_*) - g_z(z)|^3 \, dz = \int_0^{z_*} \left| \int_z^{z_*} g_{zz}(y)\, dy \right|^3 dz$$

$$\leq \int_0^{z_*} \left| \left(\int y g_{zz}^2(y)\, dy \right)^{\frac{1}{2}} \left(\int_z^{z_*} \frac{1}{y^2}\, dy \right)^{\frac{1}{4}} (z_* - z)^{\frac{1}{4}} \right|^3 dz$$

$$\leq \left(\int z g_{zz}^2 \, dz \right)^{\frac{3}{2}} \int_0^{z_*} \frac{(z_* - z)^{\frac{3}{2}}}{(z z_*)^{\frac{3}{4}}} \, dz$$

$$\lesssim \left(\int z g_{zz}^2 \, dz \right)^{\frac{3}{2}} z_* = |g|_\infty \int z g_{zz}^2 \, dz.$$

Using (3.15) we obtain

$$z_* g_z^3(z_*) \lesssim \left(|g|_\infty \int z g_{zz}^2 \, dz \int |g_z|^3 \, dz \right)^{\frac{1}{2}}.$$

Young's inequality then yields

$$\int_0^{z_*} |g_z(z_*)|^3 \, dz \lesssim c(\delta) |g|_\infty \int z g_{zz}^2 \, dz + \delta \int |g_z|^3 \, dz.$$

Putting this together leads to

$$\int_0^{z_*} |g_z(z)|^3 \, dz \lesssim \int_0^{z_*} |g_z(z_*) - g_z(z)|^3 \, dz + \int_0^{z_*} |g_z(z_*)|^3 \, dz$$

$$\lesssim c(\delta) |g|_\infty \int z g_{zz}^2 \, dz + \delta \int |g_z|^3 \, dz.$$

So altogether

$$\int_0^{\infty} |g_z(z)|^3 \, dz \lesssim c(\delta) |g|_\infty \int z g_{zz}^2 \, dz + \delta \int |g_z|^3 \, dz.$$

Absorbing the last term on the left-hand side yields the desired estimate. □

Now with the help of Lemma 3.2 and 3.4 we are in the position to prove the following lemma, which states that the non-linear quantity $\hat{D}$ controls in our regime a semi-norm of χ, see (3.16).

Lemma 3.5. *If χ is such that*

$$\hat{E}(\chi) \hat{D}(\chi)^{\frac{1}{2}} \ll 1,$$

then we have the estimate

$$\hat{D}(\chi) \gtrsim \int z^3 \chi_{zzzz}^2 \, dz. \tag{3.16}$$

Proof. We will in the following prove

$$\hat{D}(\chi) \gtrsim \left(1 - c\left(\hat{E}(\chi)\hat{D}(\chi)^{\frac{1}{2}}\right)^{\frac{8}{3}}\right) \int z^3 \chi_{zzzz}^2 \, dz,$$

which then implies the desired estimate.

Denote as always

$$u := \chi_z, \; p := \chi_{zz}.$$

Recall that $\hat{D}$ is given by

$$\hat{D} = \int \frac{1}{z} \left(-\partial_2 L + (\partial_3 L)_z\right)_z^2 \, dz.$$

Note that

$$\begin{aligned}
\partial_2 L &= -\frac{5}{2u^6} z^2 p^2 + \frac{1}{2}\left(1 - \frac{1}{u^4}\right), \\
\partial_3 L &= \frac{z^2 p}{u^5}.
\end{aligned}$$

Thus

$$(\partial_3 L)_z = \frac{2zp}{u^5} - \frac{5z^2p^2}{u^6} + \frac{z^2 p_z}{u^5},$$

and so

$$-\partial_2 L + (\partial_3 L)_z = -\frac{5z^2p^2}{2u^6} + \frac{2zp}{u^5} + \frac{z^2 p_z}{u^5} + \frac{1}{2}\left(\frac{1}{u^4} - 1\right).$$

Thus we compute

$$\left(-\partial_2 L + (\partial_3 L)_z\right)_z = -10\frac{z^2 p p_z}{u^6} + 15\frac{z^2 p^3}{u^7} - 15\frac{zp^2}{u^6} + 4\frac{zp_z}{u^5} + \frac{z^2 p_{zz}}{u^5}.$$

Since by Lemma 3.3 it holds $|u-1|_\infty \ll 1$ we can rewrite this as

$$\begin{aligned}
\hat{D} &= \int \frac{1}{z}\left(-10\frac{z^2 p p_z}{u^6} + 15\frac{z^2 p^3}{u^7} - 15\frac{zp^2}{u^6} + 4\frac{zp_z}{u^5} + \frac{z^2 p_{zz}}{u^5}\right)^2 dz \\
&= \int \frac{z}{u^{10}}\left(-10\frac{zpp_z}{u} + 15\frac{zp^3}{u^2} - 15\frac{p^2}{u} + 4p_z + zp_{zz}\right)^2 dz \\
&\approx \int z\left(-10\frac{zpp_z}{u} + 15\frac{zp^3}{u^2} - 15\frac{p^2}{u} + 4p_z + zp_{zz}\right)^2 dz.
\end{aligned}$$

We divide this into the terms linear and nonlinear in p by defining

$$\begin{aligned}
A &:= 4p_z + zp_{zz}, \\
B &:= 5\left(-2\frac{zpp_z}{u} + 3\frac{zp^3}{u^2} - 3\frac{p^2}{u}\right).
\end{aligned}$$

Then this reads

$$\hat{D} \approx \int z\,(A+B)^2 \, dz = \int zA^2 \, dz + \int zB^2 \, dz + 2\int zAB \, dz.$$

A short calculation using $zp_z(z) \to 0$ for $z \to 0$ which follows from Lemma 5.4 shows

$$\begin{aligned}
\int zA^2\,dz &= 8\int z^2 p_z p_{zz}\,dz + 16\int zp_z^2\,dz + \int z^3 p_{zz}^2\,dz \\
&= 4\int z^2 (p_z)_z^2\,dz + 16\int zp_z^2\,dz + \int z^3 p_{zz}^2\,dz \\
&= -8\int zp_z^2\,dz + 4z^2 p_z^2|_0^\infty + 16\int zp_z^2\,dz + \int z^3 p_{zz}^2\,dz \\
&\geq 8\int zp_z^2\,dz + \int z^3 p_{zz}^2\,dz \geq \int z^3 p_{zz}^2\,dz.
\end{aligned} \tag{3.17}$$

Obviously

$$\int zB^2\,dz \geq 0.$$

Thus the remaining term is

$$\begin{aligned}
2\int zAB\,dz =& 10\int z\left(4p_z + zp_{zz}\right)\left(-2\frac{zpp_z}{u} + 3\frac{zp^3}{u^2} - 3\frac{p^2}{u}\right)\,dz \\
=& 10\left(-8\int \frac{z^2 pp_z^2}{u}\,dz + 12\int \frac{z^2 p^3 p_z}{u^2}\,dz - 12\int \frac{zp^2 p_z}{u}\,dz\right) \\
&+ 10\left(-2\int \frac{z^3 pp_z p_{zz}}{u}\,dz + 3\int \frac{z^3 p^3 p_{zz}}{u^2}\,dz - 3\int \frac{z^2 p^2 p_{zz}}{u}\,dz\right).
\end{aligned}$$

In the following we are estimating the above six terms. Observe that the above terms can be written as

$$c_0\int \sqrt{z}g\,\sqrt{z}p\left(p(zp-1) - c_1 zp_z\right)\,dz,$$

for

$$g = c_2 p_z,$$

or

$$g = zp_{zz},$$

with $c_0, c_1, c_2 > 0$. Since we are just interested in estimates and not in the constants, let us for convenience drop c_0. As a first step use Young's inequality to obtain

$$\left|\int \sqrt{z}g\,\sqrt{z}p\left(p(zp-1) - c_1 zp_z\right)\,dz\right| \leq \delta\int zg^2\,dz + \frac{1}{\delta}\int zp^2\left(p(zp-1) - c_1 zp_z\right)^2\,dz.$$

Observe that by Hardy's inequality (Lemma 5.8 with $k = 1$, $\psi = p_z$) we have

$$\int zp_z^2\,dz \lesssim \int z^3 p_{zz}^2\,dz. \tag{3.18}$$

To use Lemma 5.8 we need $p_z(z_n) \to 0$ for a subsequence $z_n \to \infty$. This can be seen e.g. using the estimate

$$\left|\frac{z^2 p_z}{u^5} - \frac{5z^2 p^2}{2u^6} + \frac{2zp}{u^5} + \frac{1}{2}\left(\frac{1}{u^4} - 1\right)\right| = \left|-\partial_2 L + (\partial_3 L)_z\right|(z) \lesssim \hat{D}^{\frac{1}{2}} z.$$

Dividing by z^2 and using that $p \lesssim \frac{1}{z^{\frac{3}{4}}} \to 0$, which itself follows from (3.10), we obtain as desired

$$p_z(z) \to 0, \quad \text{for } z \to \infty. \tag{3.19}$$

Thus for small δ we can absorb this term into the term we get from (3.17). For the following we choose a cut-off function η_1 such that

$$\begin{aligned} 0 \le \eta_1 \le 1&, \\ \eta_1(z) = 0 \quad &\text{for } z \le \frac{z_*}{2}, \\ \eta_1(z) = 1 \quad &\text{for } z \ge z_*, \end{aligned}$$

as well as

$$|(\eta_1)_z| \lesssim \frac{1}{z_*}\chi_{]z_*/2, z_*[}, \tag{3.20}$$

where we choose

$$z_* := \left(\frac{\hat{E}}{\hat{D}}\right)^{\frac{1}{3}}.$$

Also define $\eta_0 = 1 - \eta_1$, such that

$$\eta_0 + \eta_1 = 1.$$

We will heavily use Lemma 3.2 which using Lemma 3.3 reads

$$|zp(z)| \lesssim (z\,\hat{E}\hat{D})^{\frac{1}{4}} + (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{1}{3}}. \tag{3.21}$$

This in particular implies that for $z \le z_*$ we have

$$|zp(z)| \lesssim (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{1}{3}}, \tag{3.22}$$

which is the reason we chose the cut-off at this threshold.
Now first estimate

$$\int \eta_0 z p^2 \left(p(zp-1) - c_1 z p_z\right)^2 \, dz \lesssim \int \eta_0 z p^4 \, dz + \int \eta_0 z^3 p^6 \, dz + \int \eta_0 z^3 p^2 p_z^2 \, dz.$$

Then using (3.22)

$$\begin{aligned} &\int \eta_0 z p^2 \left(p(zp-1) - c_1 z p_z\right)^2 \, dz \\ &\lesssim \int \eta_0 z p^4 \, dz + \int \eta_0 z^3 p^6 \, dz + \int \eta_0 z^3 p^2 p_z^2 \, dz \\ &\lesssim \left(|zp|_{\infty,[0,z_*]} + |zp|^3_{\infty,[0,z_*]}\right) \int |p|^3 \, dz + |zp|^2_{\infty,[0,z_*]} \int z p_z^2 \, dz \\ &\lesssim \left((\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{1}{3}} + \hat{E}\hat{D}^{\frac{1}{2}}\right) \int |p|^3 \, dz + (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{2}{3}} \int z p_z^2 \, dz. \end{aligned}$$

Now with the help of Lemma 3.4 we obtain

$$\int |p|^3 \, dz \lesssim (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{1}{3}} \int z p_z^2 \, dz.$$

Therefore we get

$$\int \eta_0 z p^2 \left(p(zp-1) - c_1 z p_z\right)^2 \, dz \lesssim \left((\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{2}{3}} + (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{4}{3}}\right) \int z p_z^2 \, dz. \tag{3.23}$$

It remains to estimate

$$\begin{aligned}
&\int \eta_1 z p^2 \left(p(zp-1) - c_1 z p_z\right)^2 \, dz \\
&\lesssim \int \eta_1 z p^4 \, dz + \int \eta_1 z^3 p^6 \, dz + \int \eta_1 z^3 p^2 p_z^2 \, dz.
\end{aligned}$$

Let us treat those three terms separately. For this we write

$$\int \eta_1 \left(z p^4 + z^3 p^6\right) \, dz = \int \left(\frac{z^2}{2}\right)_z p^4 \eta_1 + \left(\frac{z^4}{4}\right)_z p^6 \eta_1 \, dz,$$

and integrate by parts to obtain

$$\begin{aligned}
&\int \eta_1 \left(z p^4 + z^3 p^6\right) \, dz = -\int \tfrac{z^2}{2} \left(\eta_1 p^4\right)_z + \tfrac{z^4}{4} \left(\eta_1 p^6\right)_z \, dz \\
&\lesssim \int |(\eta_1)_z| \left(z^2 p^4 + z^4 p^6\right) \, dz + \int \eta_1 z^2 |p|^3 |p_z| \left(1 + |pz|^2\right) \, dz.
\end{aligned} \tag{3.24}$$

Observe that using as before $p \lesssim z^{-\frac{3}{4}}$ which follows from (3.10) we know that

$$z^2 p^4(z) + z^4 p^6(z) \to 0, \qquad \text{for } z \to \infty,$$

and thus there are no boundary terms appearing. Using (3.20) and (3.22) we can estimate

$$\begin{aligned}
&\int |(\eta_1)_z| \left(z^2 p^4 + z^4 p^6\right) \, dz \lesssim \tfrac{1}{z_*} \int_{z_*/2}^{z_*} z^2 p^4 + z^4 p^6 \, dz \\
&\lesssim \int z^2 p^2 \, dz \tfrac{1}{z_*^3} \left(|zp|^2_{\infty,[0,z_*]} + |zp|^4_{\infty,[0,z_*]}\right) \lesssim \hat{D} \left((\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{2}{3}} + (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{4}{3}}\right).
\end{aligned} \tag{3.25}$$

Estimate the second term of (3.24) using (3.21) and choosing in the following $g = p_z$

$$\begin{aligned}
&\int \eta_1 z^2 |p|^3 |g| \left(1 + |pz|^2\right) \, dz \\
&\lesssim \int \eta_1 |pg| \left((\hat{E}\hat{D})^{\frac{1}{2}} z^{\frac{1}{2}} + \hat{E}\hat{D} z\right) \, dz + \left((\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{2}{3}} + (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{4}{3}}\right) \int \eta_1 |pg| \, dz.
\end{aligned} \tag{3.26}$$

Estimate right-hand side's terms individually by using that $\eta_1(z) = 0$ for $z \lesssim z_*$ and Hölder

$$\begin{aligned}
&\int \eta_1 |pg| \left((\hat{E}\hat{D})^{\frac{1}{2}} z^{\frac{1}{2}} + \hat{E}\hat{D} z\right) \, dz \lesssim \left((\hat{E}\hat{D})^{\frac{1}{2}} \frac{1}{z_*} + (\hat{E}\hat{D}) \frac{1}{z_*^{\frac{1}{2}}}\right) \int z^{\frac{3}{2}} |pg| \, dz \\
&\lesssim \left((\hat{E}\hat{D})^{\frac{1}{2}} \frac{1}{z_*} + (\hat{E}\hat{D}) \frac{1}{z_*^{\frac{1}{2}}}\right) \left(\int z^2 p^2 \, dz \int z g^2 \, dz\right)^{\frac{1}{2}} \\
&\lesssim \hat{D}^{\frac{1}{2}} \left((\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{2}{3}} + (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{4}{3}}\right) \left(\int z g^2 \, dz\right)^{\frac{1}{2}} \\
&\lesssim \hat{D} + \left((\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{4}{3}} + (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{8}{3}}\right) \int z g^2 \, dz.
\end{aligned}$$

Calling

$$\varepsilon := (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{2}{3}} + (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{4}{3}},$$

the other term in (3.26) can be estimated in a similar fashion, since

$$\begin{aligned}\varepsilon \int \eta_1 |pg| \, dz &\lesssim \frac{\varepsilon}{z_*^{\frac{3}{2}}} \left(\int z^2 p^2 \, dz \int z g^2 \, dz \right)^{\frac{1}{2}} \\ &\lesssim \hat{D}^{\frac{1}{2}} \, \varepsilon \left(\int z g^2 \, dz \right)^{\frac{1}{2}} \lesssim \hat{D} + \varepsilon^2 \int z g^2 \, dz.\end{aligned}$$

Thus we obtain

$$\int \eta_1 z^2 |p|^3 |p_z| \left(1 + |pz|^2\right) \, dz \lesssim \hat{D} + \left((\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{4}{3}} + (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{8}{3}} \right) \int z p_z^2 \, dz,$$

and therefore

$$\int \eta_1 \left(z p^4 + z^3 p^6 \right) \, dz \lesssim \left(1 + (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{2}{3}} + (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{4}{3}} \right) \hat{D} + \left((\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{4}{3}} + (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{8}{3}} \right) \int z p_z^2 \, dz.$$

It remains to estimate

$$\begin{aligned}\int \eta_1 z^3 p^2 p_z^2 \, dz &= -\int \left(\eta_1 z^3 p^2 p_z \right)_z p \, dz = -\int (\eta_1)_z z^3 p^3 p_z + 3\eta_1 z^2 p^3 p_z + \eta_1 z^3 p^3 p_{zz} + 2\eta_1 z^3 p^2 p_z^2 \, dz \\ &\lesssim \int |(\eta_1)_z| z^3 |p|^3 |p_z| + \eta_1 z^2 |p|^3 |p_z| + \eta_1 z^3 |p|^3 |p_{zz}| \, dz.\end{aligned}$$

To argue that there are no boundary terms appearing due to the integration by parts, we use the fact that since $\hat{E} < \infty$, we know that for a subsequence $z_n \to \infty$

$$z_n p(z_n) \to 0,$$

as well as (3.19). The term

$$\int \eta_1 z^2 |p|^3 |p_z| \, dz$$

is already estimated starting from (3.26) for $g = p_z$. The term

$$\int \eta_1 z^3 |p|^3 |p_{zz}| \, dz$$

is estimated starting from (3.26) for $g = z p_{zz}$. We estimate the remaining term by Young's inequality for some small δ

$$\int |(\eta_1)_z| z^3 |p|^3 |p_z| \, dz \leq \frac{1}{\delta} \int |(\eta_1)_z| z^4 p^6 \, dz + \delta \int |(\eta_1)_z| z^2 p_z^2 \, dz.$$

The first term is already estimated in (3.25), the second one can be estimated using (3.20) by

$$\delta \int |(\eta_1)_z| z^2 p_z^2 \, dz \lesssim \delta \frac{1}{z_*} \int_{z_*/2}^{z_*} z^2 p_z^2 \, dz \leq \delta \int z p_z^2 \, dz.$$

Thus together we obtain

$$\int \eta_1 z p^2 \left(p(zp-1) - c_1 z p_z\right)^2 \, dz$$
$$\lesssim \left(1 + (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{2}{3}} + (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{4}{3}}\right) \hat{D} + \left((\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{4}{3}} + (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{8}{3}}\right) \int z p_z^2 \, dz + \delta \int z p_z^2 \, dz,$$

and using (3.23)

$$\int z p^2 \left(p(zp-1) - c_1 z p_z\right)^2 \, dz$$
$$\lesssim \left(1 + (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{2}{3}} + (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{4}{3}}\right) \hat{D} + \left((\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{2}{3}} + (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{4}{3}} + (\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{8}{3}}\right) \int z p_z^2 \, dz + \delta \int z p_z^2 \, dz.$$

Thus we estimate using Young

$$\int z|AB| \, dz \le c\hat{D} + c(\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{8}{3}} \int z^3 p_{zz}^2 \, dz + \delta \int z^3 p_{zz}^2 \, dz.$$

Choosing δ small enough we have thus proven

$$\hat{D} \ge \int z A^2 \, dz - 2\left|\int z AB \, dz\right| \gtrsim \left(1 - c(\hat{E}\hat{D}^{\frac{1}{2}})^{\frac{8}{3}}\right) \int z^3 p_{zz}^2 \, dz,$$

which proves the claim. □

Lemma 3.6. *If χ is close to the stationary solution in the sense of*

$$\hat{E}\hat{D}^{\frac{1}{2}} \ll 1,$$

then it holds

$$|z\chi_{zz}(z)|_\infty \lesssim \left(\hat{E}\hat{D}^{\frac{1}{2}}\right)^{\frac{1}{3}}. \tag{3.27}$$

Proof. As usual denote

$$u := \chi_z, \; p := \chi_{zz}.$$

Define

$$z_* := \left(\frac{\hat{E}}{\hat{D}}\right)^{\frac{1}{3}}.$$

For $z_0 \le z_*$ we know by Lemma 3.2 and 3.3 that (3.27) holds. Let

$$g := zp.$$

Now let $z_0 \ge z_*$ and use that in the first step we already showed the desired estimate for $g^2(z_*)$, so that by the triangle inequality it suffices to estimate

$$|g^2(z_0) - g^2(z_*)| \lesssim \int_{z_*}^{z_0} \eta_1 |g_z g| \, dz \lesssim \left(\hat{E} \int \eta_1 g_z^2 \, dz\right)^{\frac{1}{2}}.$$

Via integration by parts we observe

$$\int_0^\infty \eta_1 g_z^2\,dz = -\int \eta_1 g_{zz} g\,dz - \int g_z g (\eta_1)_z\,dz. \tag{3.28}$$

To ensure that there are no boundary terms while integrating by parts, we use that since $\hat{E} < \infty$ there exists a sequence $z_n \to \infty$ such that $z_n p(z_n) \to 0$. Also

$$\left| \frac{z_n^2 p_z}{u^5} - \frac{5 z_n^2 p^2}{2u^6} + \frac{2 z_n p}{u^5} + \frac{1}{2}\left(\frac{1}{u^4} - 1\right) \right| = \left| -\partial_2 L + (\partial_3 L)_z \right|(z_n) \lesssim \hat{D}^{\frac{1}{2}} z_n,$$

from where we then obtain, by dividing by z_n, that $z_n p_z(z_n)$ stays bounded. Thus

$$(g_z g)(z_n) = (p(z_n) + z_n p_z(z_n)) z_n p(z_n) \to 0.$$

Now the first term of (3.28) can be estimated by

$$-\int \eta_1 g_{zz} g\,dz \le \frac{1}{z_*^{\frac{1}{2}}} \left(\int \eta_1 z g_{zz}^2\,dz \int \eta_1 g^2\,dz \right)^{\frac{1}{2}} \le \hat{D}^{\frac{2}{3}} \hat{E}^{\frac{1}{3}},$$

where we used that by Hardy's inequality (see (3.18))

$$\int z g_{zz}^2\,dz \lesssim \int z^3 p_{zz}^2\,dz,$$

as well as by Lemma 3.5

$$\int z^3 p_{zz}^2\,dz \lesssim \hat{D}.$$

The second term can be estimated by

$$\int g_z g (\eta_1)_z\,dz \lesssim \int_{z_*/2}^{z_*} g g_z\,dz \frac{1}{z_*} = \frac{1}{z_*}\int z p^2\,dz + \frac{1}{z_*}\int z^2 p p_z\,dz$$

$$\lesssim \frac{1}{z_*^2}\hat{E} + \frac{1}{z_*^{\frac{1}{2}}} \hat{E}^{\frac{1}{2}} \left(\int z p_z^2\,dz \right)^{\frac{1}{2}} \lesssim \hat{D}^{\frac{2}{3}} \hat{E}^{\frac{1}{3}}.$$

Altogether we obtain

$$g^2(z_0) \lesssim |g^2(z_0) - g^2(z_*)| + g^2(z_*) \lesssim \left(\int_{z_*}^\infty g_z^2\,dz \hat{E} \right)^{\frac{1}{2}} + \hat{D}^{\frac{1}{3}} \hat{E}^{\frac{2}{3}} \lesssim \hat{D}^{\frac{1}{3}} \hat{E}^{\frac{2}{3}}.$$

This closes the proof. □

Lemma 3.7. *Let*

$$\mathcal{B} = \left\{ \chi \,\middle|\, \hat{E}(\chi)\hat{D}(\chi)^{\frac{1}{2}} \ll 1 \right\}.$$

Then for every $\chi_0, \chi_1 \in \mathcal{B}$ and $z \in \mathbb{R}_+$ the map

$$s \mapsto L(z, \chi_z^s(z), \chi_{zz}^s(z))$$

is convex, where

$$\chi^s := (1-s)\chi_0 + s\chi_1.$$

Furthermore the energy

$$\hat{E} : \chi \mapsto \int L(z, \chi_z(z), \chi_{zz}(z))\,dz$$

is geodesically convex on the convex hull of the set $\mathcal{B}$, seen as a subset of $L^2(z\,dz)$.

Proof. Let $c_1, c_2 \ll 1$ be such that if

$$|\chi_z(z) - 1|_\infty \le c_1, \quad |z\chi_{zz}(z)|_\infty \le c_2,$$

then

$$\frac{z^2\chi_{zz}(z)^2}{\chi_z(z)^2} \le \frac{1}{5}.$$

Let

$$\mathcal{A} = \left\{\chi \,\middle|\, |\chi_z(z) - 1|_\infty \le c_1, |z\chi_{zz}(z)|_\infty \le c_2\right\}.$$

By Lemma 3.3 and 3.6 we know that if

$$\hat{E}(\chi)\hat{D}(\chi)^{\frac{1}{2}} \ll 1,$$

it holds that

$$\mathcal{B} \subset \mathcal{A}.$$

Since $\mathcal{A}$ is convex this implies that also for the convex hull we have that

$$\operatorname{conv}(\mathcal{B}) \subset \mathcal{A}.$$

Thus we know that $\chi^s \in \mathcal{A}$. It therefore suffices to show that for all $\chi \in \mathcal{A}$

$$D^2_{2,3}L\left(z, \chi_z(z), \chi_{zz}(z)\right) \ge 0.$$

This is equivalent to showing that

$$\operatorname{tr}\left(D^2_{2,3}L\right)\left(z, \chi_z(z), \chi_{zz}(z)\right) \ge 0,$$

and

$$\det\left(D^2_{2,3}L\right)\left(z, \chi_z(z), \chi_{zz}(z)\right) \ge 0.$$

Denote as always

$$u := \chi_z, p := \chi_{zz}.$$

Remark that $D^2_{2,3}L(z,u,p)$ is given by

$$\begin{aligned}
\partial_2^2 L\left(z,u,p\right) &= 15\frac{z^2p^2}{u^7} + \frac{2}{u^5},\\
\partial_2\partial_3 L\left(z,u,p\right) &= -\frac{5z^2p}{u^6},\\
\partial_3^2 L\left(z,u,p\right) &= \frac{z^2}{u^5}.
\end{aligned}$$

Then

$$\operatorname{tr}\left(D^2_{2,3}L\right)\left(z,u,p\right) = \frac{15z^2p^2 + (2+z^2)u^2}{u^7} \ge 0,$$

since by definition

$$u = \frac{z}{h(\chi(z))} \ge 0.$$

Furthermore

$$\det\left(D^2_{2,3}L\right)(z,u,p) = \frac{2z^2}{u^{10}}\left(1 - \frac{5z^2p^2}{u^2}\right) \geq 0,$$

since $\mathcal{A}$ is chosen in such a way that

$$\frac{5z^2p^2}{u^2} \leq 1.$$

This proves the first part of the lemma. The second part easily follows by the fact that the constant speed geodesic on $L^2(z\,dz)$ connecting χ_0 and χ_1 is indeed given by χ^s as above and integration is a linear and monotone operation. □

The next lemma provides a proof of Corollary 1.3.

Lemma 3.8. *Let χ be close to the stationary solution, in the sense of*

$$\hat{E}(\chi)\hat{D}(\chi)^{\frac{1}{2}} \ll 1.$$

Then we have the estimate

$$|\chi(0)| \lesssim \hat{E}(\chi)^{\frac{1}{3}}\hat{H}(\chi)^{\frac{1}{6}}. \tag{3.29}$$

Proof. Start by observing that Lemma 3.3 yields $|\chi_z - 1|_\infty \ll 1$, and thus we have that

$$\int_0^\infty z^2\chi_{zz}^2\,dz \lesssim \hat{E}(\chi).$$

Also it holds

$$\int_0^\infty (\chi - z)_z^2\,dz \lesssim \hat{E}(\chi),$$

by Hardy's inequality (Lemma 5.8 with $k = 0, g = u - 1$, $(u-1)(z) \to 0$ for $z \to 0$ due to (3.1)). Thus

$$\int (\chi_z - 1)^2\,dz \lesssim \int z^2\chi_{zz}^2\,dz \lesssim \hat{E}(\chi). \tag{3.30}$$

To obtain (3.29) as before we treat separately the cases for small and large z. Let z_* be defined by

$$z_* = \left(\frac{\hat{H}}{\hat{E}}\right)^{\frac{1}{3}}.$$

Using that since $\hat{H}(\chi) < \infty$ we know that for a subsequence $z_n \to \infty$ we have

$$\chi(z_n) - z_n \to 0.$$

Using this we estimate first

$$(\chi(z_*) - z_*)^2 \lesssim \int_{z_*}^\infty |(\chi - z)_z(\chi - z)|\,dz \leq \left(\int_0^\infty (\chi - z)_z^2\,dz \int_{z_*}^\infty \frac{z}{z}(\chi - z)^2\,dz\right)^{\frac{1}{2}} \lesssim (\hat{E}\hat{H})^{\frac{1}{2}} z_*^{-\frac{1}{2}}.$$

This yields

$$|\chi(z_*) - z_*| \lesssim \left(\frac{\hat{E}\hat{H}}{z_*}\right)^{\frac{1}{4}} = \hat{E}^{\frac{1}{3}}\hat{H}^{\frac{1}{6}}.$$

Now observe

$$
\begin{aligned}
|\chi(z_*) - z_* - \chi(0)| &\le \int_0^{z_*} |(\chi - z)_z| \, dz \le \left(\int_0^{z_*} (\chi - z)_z^2 \, dz \, z_* \right)^{\frac{1}{2}} \\
&\lesssim (\hat{E} z_*)^{\frac{1}{2}} = \hat{E}^{\frac{1}{3}} \hat{H}^{\frac{1}{6}}.
\end{aligned}
$$

Thus we combine

$$
|\chi(0)| \le |\chi(0) - (\chi(z_*) - z_*)| + |\chi(z_*) - z_*| \lesssim \hat{E}^{\frac{1}{3}} \hat{H}^{\frac{1}{6}},
$$

as desired. □

4 Establishing the differential and algebraic relationships

In this section we are first proving Lemma 1.4, in which the main differential and algebraic relationships are established. These relationships are in fact an easy consequence of the convexity as stated in Lemma 3.7. For completeness we are recalling the proof nevertheless. Note that as noted in Lemma 3.7

$$\Phi : s \mapsto L(z, \chi_z^s(z), \chi_{zz}^s(z)) \tag{4.1}$$

is convex, where

$$\chi^s := (1-s)\chi_0 + s\chi.$$

Proof of Lemma 1.4. Let us first prove the relation (1.12). For this we work in h coordinates. Let without loss of generality $\chi(0) = 0$. Choose a cut-off function $\eta = \eta^R \in C^\infty$ such that

$$\eta^R(x) = \begin{cases} 1, & \text{for } x \le R, \\ 0, & \text{for } x \ge 2R, \end{cases}$$

as well as

$$|(\eta^R)'|_\infty \lesssim R^{-1},$$
$$|(\eta^R)''|_\infty \lesssim R^{-2}.$$

Now calculate

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\int \eta(h_x-1)^2\,dx &= \int (h_x-1)h_{xt}\eta\,dx = -\int (h_x-1)(hh_{xxx})_{xx}\eta\,dx \\ &= \int h_{xx}(hh_{xxx})_x\eta + (h_x-1)(hh_{xxx})_x\eta_x\,dx \\ &= -\int hh_{xxx}^2\eta\,dx - \int h_{xx}hh_{xxx}\eta_x\,dx + \int (h_x-1)(hh_{xxx})_x\eta_x\,dx. \end{aligned}$$

Now for $R \to \infty$ this yields

$$\frac{d}{dt}E(h) = -D(h) + \limsup_{R\to\infty}\left(-\int h_{xx}hh_{xxx}\eta_x\,dx + \int (h_x-1)(hh_{xxx})_x\eta_x\,dx\right).$$

It thus remains to show that

$$\limsup_{R\to\infty}\left(-\int h_{xx}hh_{xxx}\eta_x\,dx + \int (h_x-1)(hh_{xxx})_x\eta_x\,dx\right) = 0. \tag{4.2}$$

For this observe that

$$\begin{aligned} &-\int h_{xx}hh_{xxx}\eta_x\,dx + \int (h_x-1)(hh_{xxx})_x\eta_x\,dx = \int \left((h_x-1)^{-1}hh_{xxx}\right)_x (h_x-1)^2\eta_x\,dx \\ &= -\int (h_x-1)^{-1}hh_{xxx}\left((h_x-1)^2\eta_x\right)_x\,dx = -\int hh_{xxx}(h_x-1)\eta_{xx}\,dx - 2\int hh_{xxx}h_{xx}\eta_x\,dx. \end{aligned}$$

Note that

$$|h(x) - x| \lesssim E^{\frac{1}{2}}x^{\frac{1}{2}}.$$

This yields that for $x \gg 1$ we know that

$$\frac{h(x)}{x} \approx 1. \tag{4.3}$$

we can therefore estimate

$$\begin{aligned}\int h h_{xxx}(h_x - 1)\eta_{xx}\, dx &\lesssim \frac{1}{R^{\frac{3}{2}}} \int_R^{2R} \left| h^{\frac{1}{2}} h_{xxx}(h_x - 1) \right| \left(\frac{h}{R}\right)^{\frac{1}{2}} dx \\ &\lesssim (ED)^{\frac{1}{2}} \frac{1}{R^{\frac{3}{2}}}.\end{aligned}$$

For the other term start by estimating

$$\int h h_{xxx} h_{xx} \eta_x\, dx \lesssim \left(\int h h_{xxx}^2\, dx \int h h_{xx}^2 \eta_x^2\, dx \right)^{\frac{1}{2}}.$$

Using (4.3) we know that

$$\int h h_{xx}^2 \eta_x^2\, dx \approx \int x h_{xx}^2 \eta_x^2\, dx.$$

We therefore estimate

$$\begin{aligned}&\int x h_{xx}^2 \eta_x^2\, dx = -\int (x h_{xx} \eta_x^2)_x (h_x - 1)\, dx \\ &= -\int h_{xx} \eta_x^2 (h_x - 1)\, dx - \int x h_{xxx} \eta_x^2 (h_x - 1)\, dx - 2\int x h_{xx} \eta_x \eta_{xx} (h_x - 1)\, dx.\end{aligned}$$

Estimating term by term we have

$$\begin{aligned}&\int h_{xx} \eta_x^2 (h_x - 1)\, dx \lesssim \frac{1}{R} \int \eta_x h_{xx} (h_x - 1)\, dx \lesssim \frac{1}{R^{\frac{3}{2}}} \int \eta_x x^{\frac{1}{2}} h_{xx} (h_x - 1)\, dx \\ &\leq \delta \int x h_{xx}^2 \eta_x^2\, dx + \frac{c}{\delta} E \frac{1}{R^3},\end{aligned}$$

as well as

$$\int x h_{xxx} \eta_x^2 (h_x - 1)\, dx \lesssim \left(\int x h_{xxx}^2\, dx \int (h_x - 1)^2\, dx \right)^{\frac{1}{2}} \frac{1}{R^{\frac{3}{2}}} \lesssim (ED)^{\frac{1}{2}} \frac{1}{R^{\frac{3}{2}}},$$

and

$$\int x h_{xx} \eta_x \eta_{xx} (h_x - 1)\, dx \leq \delta \int x h_{xx}^2 \eta_x^2\, dx + \frac{c}{\delta} E \frac{1}{R^3}.$$

Choosing $\delta \ll 1$ and absorbing, we obtain that for $R \to \infty$

$$\int h h_{xx}^2 \eta_x^2\, dx \approx \int x h_{xx}^2 \eta_x^2\, dx \to 0.$$

This proves (4.2) and thus (1.12) is proved.

It remains to show that the three other relations (1.11), (1.14) and (1.13) hold under the hypothesis that

$$\mathcal{E}_0 \mathcal{D}_0^{\frac{1}{2}} \ll 1.$$

Observe that it is enough to show (1.11),(1.14) and (1.13) for all t_0 under the assumption

$$\mathcal{E}(t_0)\mathcal{D}(t_0)^{\frac{1}{2}} \ll 1.$$

since then (1.12) and (1.13) guarantee that this is satisfied for all $t_0 \geq 0$ if it is true for $t_0 = 0$. Thus we can assume that (4.1) is convex. Let us in the following drop the boundary terms when integrating by parts, since we can argue in the same manner as seen above in the proof of (1.12) that they indeed vanish. Start by proving (1.11).
Using the thin-film equation in χ coordinates (2.4) we obtain

$$\begin{aligned}
\partial_t\left(\frac{\mathcal{H}(t)}{2}\right) &= \partial_t\frac{1}{2}\int_0^\infty z(\chi(t,z)-\chi_0(z))^2\,dz = \int_0^\infty z\chi_t(t,z)(\chi(t,z)-\chi_0(z))\,dz \\
&= -\int_0^\infty (\partial_2 L)(z,u,p)(\chi-\chi_0)_z + (\partial_3 L)(z,u,p)(\chi-\chi_0)_{zz}\,dz \\
&= -\int_0^\infty \langle(\nabla_{2,3}L)(z,u,p) - (\nabla_{2,3}L)(z,u_0,p_0),(u,p)-(u_0,p_0)\rangle\;dz \leq 0.
\end{aligned}$$

The last step is true since the integrand is positive due to the fact that the map (4.1) is convex.

Next we show (1.14), i.e.

$$\mathcal{E} \leq \sqrt{\mathcal{H}\mathcal{D}}.$$

For this note that due to the convexity of (4.1), we have

$$\Phi(1) \leq \Phi(0) + \Phi'(1),$$

which translates into

$$L(z,\chi_z,\chi_{zz}) \leq (\partial_2 L)(z,\chi_z,\chi_{zz})(\chi-\chi_0)_z + (\partial_3 L)(z,\chi_z,\chi_{zz})(\chi-\chi_0)_{zz}.$$

Integrating this yields

$$\begin{aligned}
\mathcal{E}(t) = \hat{E}(\chi(t)) &= \int_0^\infty L(z,\chi_z,\chi_{zz})\,dz \\
&\leq \int_0^\infty (\partial_2 L)(z,\chi_z,\chi_{zz})(\chi-\chi_0)_z + (\partial_3 L)(z,\chi_z,\chi_{zz})(\chi-\chi_0)_{zz}\,dz \\
&= \int_0^\infty \sqrt{\frac{z}{z}}(\chi-\chi_0)\left(-\Big((\partial_2 L)(z,u,p)\Big)_z + \Big((\partial_3 L)(z,u,p)\Big)_{zz}\right)\,dz \\
&\leq \sqrt{\mathcal{H}(t)\mathcal{D}(t)},
\end{aligned}$$

by Hölder's inequality. Thus as claimed it holds

$$\mathcal{E}(t) \leq \sqrt{\mathcal{H}(t)\mathcal{D}(t)}.$$

Next we prove

$$\partial_t\mathcal{D}(t) \leq 0.$$

Start by computing

$$\begin{aligned}
\partial_t\left(\frac{\mathcal{D}(t)}{2}\right) &= \int_0^\infty \frac{1}{z}\left(-(\partial_2 L)_z + (\partial_3 L)_{zz}\right)\left(-\left(\partial_2^2 L\,\chi_{tz} + \partial_2\partial_3 L\,\chi_{tzz}\right)_z\right) dz \\
&\quad + \int_0^\infty \frac{1}{z}\left(-(\partial_2 L)_z + (\partial_3 L)_{zz}\right)\left(\partial_2\partial_3 L\,\chi_{tz} + \partial_3^2 L\,\chi_{tzz}\right)_{zz} dz \\
&= -\int_0^\infty (\chi_{tz})\left(\partial_2^2 L\,\chi_{tz} + \partial_2\partial_3 L\,\chi_{tzz}\right) dz \\
&\quad - \int_0^\infty (\chi_{tzz})\left(\partial_2\partial_3 L\,\chi_{tz} + \partial_3^2 L\,\chi_{tzz}\right) dz \\
&= -\int_0^\infty V(z)\cdot (D_{2,3}^2 L)(z,u,p)V(z)\,dz,
\end{aligned}$$

where we used (2.4) and defined

$$V(z) = (\chi_{tz}, \chi_{tzz})(z).$$

Again using the convexity of (4.1) in the form of

$$D_{2,3}^2 L(z,u,p) \geq 0,$$

we conclude. This closes the proof of Lemma 1.4. □

Now for completeness we are giving the ODE argument which is the content of Lemma 1.5.

Proof of Lemma 1.5. Estimate (1.15) follows directly by (1.11).
To show (1.16) use (1.12), (1.14) and (1.15) to obtain

$$\partial_t \mathcal{E}(t) = -\mathcal{D}(t) \leq -\frac{\mathcal{E}^2(t)}{\mathcal{H}(t)} \leq -\frac{\mathcal{E}^2(t)}{\mathcal{H}_0}.$$

Thus

$$\partial_t \mathcal{E}^{-1}(t) \geq \frac{1}{\mathcal{H}_0},$$

and integrating

$$\mathcal{E}^{-1}(t) \geq \mathcal{E}^{-1}(t) - \mathcal{E}^{-1}(0) \geq \frac{t}{\mathcal{H}_0},$$

or

$$\mathcal{E}(t) \leq \frac{\mathcal{H}_0}{t}.$$

To prove (1.17), observe that (1.13) implies for all $s \leq 2T$

$$\mathcal{D}(s) \geq \mathcal{D}(2T).$$

Using this, as well as (1.12) and (1.16) we obtain

$$\frac{\mathcal{H}_0}{T} \geq \mathcal{E}(T) = -\int_T^\infty \partial_t \mathcal{E}(s)ds = \int_T^\infty \mathcal{D}(s)ds \geq \int_T^{2T} \mathcal{D}(s)ds \geq \mathcal{D}(2T)T,$$

which yields as desired (1.17)

$$\mathcal{D}(T) \leq \frac{4\mathcal{H}_0}{T^2}.$$

□

5 Appendix

In the following we are proving several statements about the boundary behavior of admissible functions in the sense of (1.9) with finite E, H, D.

Lemma 5.1. *Let h be admissible in the sense of* (1.9) *and*

$$E(h) < \infty.$$

Then

$$\lim_{z \to +\infty} \chi_z(z) = 1.$$

Proof. We will use the identity

$$\chi_z(z) = \frac{z}{h(\chi(z))},$$

which follows by taking the derivative of the defining equation of χ.

Let us first show that

$$\lim_{z \to +\infty} \left| \frac{\chi(z)}{z} - 1 \right| = 0. \tag{5.1}$$

Observe that for $x \geq \chi(0)$

$$\begin{aligned} (h(x) - x) + \chi(0) = \int_{\chi(0)}^{x} (h_x(y) - 1)\, dy &\leq \left(\int (h_x - 1)^2\, dy \right)^{\frac{1}{2}} (x - \chi(0))^{\frac{1}{2}} \\ &\leq (E(h))^{\frac{1}{2}} (x - \chi(0))^{\frac{1}{2}}. \end{aligned}$$

From this we obtain

$$h(x) - h_0(x) \leq C(x - \chi(0))^{\frac{1}{2}} - \chi(0). \tag{5.2}$$

Integrating this and using the definition of $\chi(z)$ we obtain

$$\frac{1}{2}(z - \chi(z))(z + \chi(z)) = \frac{z^2 - \chi(z)^2}{2} = \int_{-\infty}^{\chi(z)} (h - h_0)\, dx \tag{5.3}$$

$$\lesssim (\chi(z) - \chi(0))^{\frac{3}{2}} - \chi(0)(\chi(z) - \chi(0)). \tag{5.4}$$

For $z \gg 1$ this yields

$$|z - \chi(z)| \lesssim \chi(z)^{\frac{1}{2}}.$$

Dividing by z this yields

$$\left| \frac{\chi(z)}{z} - 1 \right| \lesssim \frac{1}{z} \left(|\chi(z) - z|^{\frac{1}{2}} + z^{\frac{1}{2}} \right).$$

Thus by Young for $z \gg 1$

$$\left| \frac{\chi(z)}{z} - 1 \right| \lesssim \frac{1}{z^{\frac{1}{2}}} \to 0.$$

Now from (5.2) we can deduce

$$\left|\frac{h(x)}{x} - 1\right| \lesssim \frac{1}{x^{\frac{1}{2}}} \to 0.$$

Thus we can conclude

$$\lim_{z\to+\infty} \chi_z(z) = \lim_{z\to+\infty} \frac{z}{h(\chi(z))} = \lim_{z\to+\infty} \frac{\chi(z)}{h(\chi(z))}\frac{z}{\chi(z)} = 1.$$

□

Lemma 5.2. *Let h be admissible in the sense of* (1.9) *and*

$$H(h) < \infty, E(h) < \infty.$$

Then for $x_0 \gg 1$

$$\int_{x_0}^{\infty} (\overline{h - h_0})^2 \frac{1}{x}\, dx \lesssim H(h).$$

Proof. As noted before in (2.3), we know that

$$H(h) = \int_{\chi(0)}^{\infty} \left(\sqrt{2\bar{h}(x)} - x\right)^2 h(x)\, dx,$$

with

$$\bar{h}(x) = \int_{-\infty}^{x} h(y)\, dy.$$

This can be rewritten as

$$H(h) = \int_{\chi(0)}^{\infty} \left(2\bar{h}(x) - x^2\right)^2 \frac{h(x)}{\left(\sqrt{2\bar{h}(x)} + x\right)^2}\, dx \gtrsim \int_{\chi(0)}^{\infty} \left(2\bar{h}(x) - x^2\right)^2 \frac{h(x)}{\left(2\bar{h}(x) + x^2\right)}\, dx.$$

Now observe that for $x \geq \chi(0)$

$$\begin{aligned}(h(x) - x) + \chi(0) = \int_{\chi(0)}^{x} (h_x(y) - 1)\, dy &\geq -\left(\int (h_x - 1)^2\, dy\right)^{\frac{1}{2}} (x - \chi(0))^{\frac{1}{2}} \\ &\geq -(E(h))^{\frac{1}{2}} (x - \chi(0))^{\frac{1}{2}}.\end{aligned}$$

From this we obtain

$$h(x) \geq (x - \chi(0)) - C(x - \chi(0))^{\frac{1}{2}}.$$

Similarly we obtain

$$2\bar{h}(x) \leq (x - \chi(0))^2 + C(x - \chi(0))^{\frac{3}{2}}.$$

Thus

$$H(h) \gtrsim \int_{\chi(0)}^{\infty} \left(2\bar{h}(x) - x^2\right)^2 \left(\frac{(x - \chi(0)) - C(x - \chi(0))^{\frac{1}{2}}}{x^2 + (x - \chi(0))^2 + C(x - \chi(0))^{\frac{3}{2}}}\right)_+ dx =: \int_{\chi(0)}^{\infty} (\overline{h - h_0})^2 g(x)\, dx.$$

Since $g(x) \geq 0$ and

$$g(x) \approx \frac{1}{x}, \quad x \gg 1,$$

we obtain

$$\int_{x_0}^{\infty} (\overline{h-h_0})^2 \frac{1}{x}\, dx \lesssim H(h),$$

as desired. □

Lemma 5.3. *Let h be admissible in the sense of* (1.9) *and*

$$H(h) < \infty, E(h) < \infty.$$

Then

$$\lim_{z\to\infty} z\,(1-\chi_z(z))^2 = 0.$$

Proof. Let $x_1 \geq x_0 \gg 1$, then

$$\begin{aligned}
\int_{x_0}^{x_1} (h-h_0)^2\, dx &= \int_{x_0}^{x_1} (\overline{h-h_0})_x (h-h_0)\, dx = -\int_{x_0}^{x_1} (\overline{h-h_0})(h-h_0)_x dx + (\overline{h-h_0})(h-h_0)|_{x_0}^{x_1} \\
&\leq x_1^{\frac{1}{2}} \left(\int_{x_0}^{x_1} (\overline{h-h_0})^2 \frac{1}{x}\, dx\, E(h) \right)^{\frac{1}{2}} + |\overline{h-h_0}||h-h_0|(x_0) + |\overline{h-h_0}||h-h_0|(x_1) \\
&\lesssim x_1^{\frac{1}{2}} + |\overline{h-h_0}||h-h_0|(x_0) + |\overline{h-h_0}||h-h_0|(x_1).
\end{aligned}$$

Using this we obtain

$$\begin{aligned}
(h-h_0)^2(x_1) - (h-h_0)^2(x_0) = 2\int_{x_0}^{x_1} (h-h_0)(h-h_0)_x dx &\lesssim \left(\int_{x_0}^{x_1} (h-h_0)^2\, dx\, E(h) \right)^{\frac{1}{2}} \\
&\lesssim x_1^{\frac{1}{4}} + \left(|\overline{h-h_0}||h-h_0|\right)^{\frac{1}{2}}(x_0) + \left(|\overline{h-h_0}||h-h_0|\right)^{\frac{1}{2}}(x_1).
\end{aligned}$$

Now using Young with with $p = 4, q = \frac{4}{3}$ we obtain

$$(h-h_0)^2(x_1) \lesssim x_1^{\frac{1}{4}} + \varepsilon(h-h_0)^2(x_1) + (h-h_0)^2(x_0) + |\overline{h-h_0}|^{\frac{2}{3}}(x_0) + |\overline{h-h_0}|^{\frac{2}{3}}(x_1),$$

and absorbing into the right hand side we get

$$(h-h_0)^2(x_1) \lesssim x_1^{\frac{1}{4}} + (h-h_0)^2(x_0) + |\overline{h-h_0}|^{\frac{2}{3}}(x_0) + |\overline{h-h_0}|^{\frac{2}{3}}(x_1).$$

We can also estimate

$$\begin{aligned}
(\overline{h-h_0})^2(x_1) - (\overline{h-h_0})^2(x_0) = 2\int_{x_0}^{x_1} (\overline{h-h_0})(\overline{h-h_0})_x dx &\lesssim \left(\int_{x_0}^{x_1} (\overline{h-h_0})^2\, dx \int_{x_0}^{x_1} (h-h_0)^2\, dx \right)^{\frac{1}{2}} \\
&\lesssim \left(x_1 H(h) \int_{x_0}^{x_1} (h-h_0)^2\, dx \right)^{\frac{1}{2}} \\
&\lesssim x_1^{\frac{1}{2}} \left(x_1^{\frac{1}{4}} + \left(|\overline{h-h_0}||h-h_0|\right)^{\frac{1}{2}}(x_0) + \left(|\overline{h-h_0}||h-h_0|\right)^{\frac{1}{2}}(x_1) \right).
\end{aligned}$$

This yields (for fixed x_0) by Young

$$\left|\overline{h-h_0}\right|^{\frac{2}{3}}(x_1) \lesssim 1 + x_1^{\frac{1}{4}} + \varepsilon\left(|\overline{h-h_0}||h-h_0|\right)^{\frac{1}{2}}(x_1).$$

Using this and absorbing in the left hand side implies

$$(h-h_0)^2(x_1) + \left|\overline{h-h_0}\right|^{\frac{2}{3}}(x_1) \lesssim 1 + x_1^{\frac{1}{4}}.$$

This yields

$$\frac{(h-h_0)^2(x_1)}{x_1} \lesssim \frac{1}{x_1^{\frac{3}{4}}} \to 0, \quad \text{for } x_1 \to \infty. \tag{5.5}$$

and

$$\frac{\left|\overline{h-h_0}\right|(x_1)}{x_1} \lesssim \frac{1}{x_1^{\frac{5}{8}}} \to 0, \quad \text{for } x_1 \to \infty,$$

which using the identity

$$\frac{1}{2}\left|(z-\chi(z))(z+\chi(z))\right| = \left|\overline{h-h_0}\right|(\chi(z)),$$

implies

$$|z-\chi(z)| \lesssim \frac{1}{\chi(z)^{\frac{5}{8}}}. \tag{5.6}$$

Putting things together start by rewriting

$$z\left(1-\chi_z(z)\right)^2 = z\left(\frac{z-h(\chi(z))}{h(\chi(z))}\right)^2 = z\left(\frac{\chi(z)}{h(\chi(z))}\right)^2\left(\frac{z-h(\chi(z))}{\chi(z)}\right)^2.$$

Since by Lemma 5.1

$$\left(\frac{\chi(z)}{h(\chi(z))}\right)^2 \to 1,$$

it remains to estimate

$$z\left(\frac{z-h(\chi(z))}{\chi(z)}\right)^2 \lesssim \left(\frac{z}{\chi(z)}\right)\left(\frac{(z-\chi(z))^2}{\chi(z)} + \frac{(\chi(z)-h(\chi(z)))^2}{\chi(z)}\right).$$

Since also by Lemma 5.1

$$\frac{z}{\chi(z)} \to 1,$$

using (5.5) and (5.6) for the last two terms then yields as desired

$$z\left(1-\chi_z(z)\right)^2 \to 0, \quad \text{for } z \to \infty.$$

□

Lemma 5.4. *Let χ be such that it belongs to an h satisfying* (1.9)*, as well as $E(h) < \infty$ and $D(h) < \infty$. Then we know that*

$$\chi \in C^\infty([0, \infty[).$$

Proof. Let without loss of generality be $\chi(0) = 0$. The defining identity for χ is

$$\bar{h}(\chi(z)) = \frac{z^2}{2}.$$

Consider the k-th Taylor approximation of h given by

$$\begin{aligned} h(x) &= h(0) + h'(0)x + \sum_{j=2}^{k} h^{(j)}(0)\frac{x^j}{j!} + \int_0^x \frac{(x-z)^k}{k!} h^{(k+1)}(z)\, dz \\ &= x + \sum_{j=2}^{k} h^{(j)}(0)\frac{x^j}{j!} + \int_0^x \frac{(x-z)^k}{k!} h^{(k+1)}(z)\, dz. \end{aligned}$$

Then we obtain that

$$\begin{aligned} \bar{h}(x) = \int_0^x h(y)\, dy &= \frac{x^2}{2} + \sum_{j=2}^{k} h^{(j)}(0)\frac{x^{j+1}}{(j+1)!} + \int_0^x \int_0^y \frac{(y-z)^k}{k!} h^{(k+1)}(z)\, dz\, dy \\ &= \frac{x^2}{2}\left(1 + \sum_{j=2}^{k} h^{(j)}(0)\frac{2x^{j-1}}{(j+1)!} + \frac{1}{x^2}\int_0^x \int_0^y \frac{2(y-z)^k}{k!} h^{(k+1)}(z)\, dz\, dy\right) =: \frac{x^2}{2}(1 + R_k(x)). \end{aligned}$$

Thus taking the square root we obtain

$$\sqrt{2\bar{h}}(\chi(z)) = z,$$

with

$$G(x) := \sqrt{2\bar{h}}(x) = x\,(1 + R_k(x))^{\frac{1}{2}} =: xL(x).$$

We thus know that

$$\chi'(z) = \frac{1}{G'(\chi(z))},$$

or more generally for some polynomial P_n and some $N \leq n$

$$\chi^{(n)}(z) = \frac{P_n(G', \ldots, G^{(n)})(\chi(z))}{(G')^N(\chi(z))}, \tag{5.7}$$

where $f^{(n)}$ denotes the n-th derivative. Since $G \in C^\infty(]0, \infty[)$ we thus know that $\chi \in C^\infty(]0, \infty[)$. What remains to show is that

$$|\lim_{z \searrow 0} \chi^{(n)}(z)| < \infty. \tag{5.8}$$

By formula (5.7) we thus just have to show that

$$\lim_{x \searrow 0} G'(x) > 0,$$

and

$$\left|\lim_{x\searrow 0} G^{(n)}(x)\right| < \infty. \tag{5.9}$$

For this first observe that for $l < k$

$$R_k^{(l)}(x) = \partial_x^l \left(\sum_{j=2}^{k} h^{(j)}(0) \frac{2x^{j-1}}{(j+1)!} \right) + \partial_x^l \left(\frac{1}{x^2} \int_0^x \int_0^y \frac{2(y-z)^k}{k!} h^{(k+1)}(z)\, dz\, dy \right).$$

Obviously the first term is always bounded, the second one can be estimated by

$$\partial_x^l \left(\frac{1}{x^2} \int_0^x \int_0^y \frac{2(y-z)^k}{k!} h^{(k+1)}(z)\, dz\, dy \right) \lesssim |h^{(k+1)}|_{\infty,[0,x]} x^{k-l}.$$

Thus for $l < k$

$$\lim_{x\searrow 0} |R_k^{(l)}(x)| < \infty. \tag{5.10}$$

Also it holds

$$\lim_{x\searrow 0} R_k(x) = 0,$$

and thus

$$\lim_{x\searrow 0} L(x) = 1.$$

Observe that

$$L'(x) = \frac{R'(x)}{2L(x)}$$

and thus for some polynomial $\bar{P}_l$ and some $M \leq l$

$$L^{(l)}(x) = \frac{\bar{P}_l(R', \dots, R^{(l)})(x)}{(L(x))^M},$$

which yields by (5.10) that

$$\lim_{x\searrow 0} |L^{(l)}(x)| < \infty.$$

Now taking the derivative of G yields

$$G'(x) = L(x) + xL'(x) \to 1 > 0, \qquad \text{for } x \to 0.$$

Since

$$G^{(l)}(x) = lL^{(l-1)}(x) + xL^{(l)}(x),$$

we thus obtain (5.9) as desired and thus (5.8) for all $n = l < k$. Since k was arbitrary, the claim is proven. □

Apart from minor changes the next two lemmas are basically already contained in Proposition 2.2 and Proposition 2.3 of[10]. For convenience we are nevertheless recalling their proof including minor modifications.

Lemma 5.5. *Let h be admissible as in* (1.9)*, and $E(h) < \infty$. Let also w.l.o.g. $\chi(0) = 0$. Then there exists a C such that*

$$\frac{1}{2} \leq h'(x) \leq \frac{3}{2}, \quad \text{for } x \in [0, r],$$

with

$$r \geq \frac{1}{C} D^{-\frac{1}{2}}(h).$$

Proof. First observe that $E(h) < \infty$ implies the existence of a sequence $x_n \to +\infty$ such that

$$\lim_{n \to +\infty} h'(x_n) = 1.$$

Assume that r is maximal with the property that

$$\frac{1}{2} \leq h'(x) \leq \frac{3}{2}, \quad \text{for } x \in [0, r].$$

Then it holds that

$$h'(r) \in \left\{ \frac{1}{2}, \frac{3}{2} \right\}.$$

After possible rescaling

$$\tilde{h}(x) := r^{-1} h(rx),$$

we know that

$$\tilde{h}(0) = 0,\, \tilde{h}'(0) = 1,\, \exists x_n \to \infty : \tilde{h}'(x_n) \to 1,\, \tilde{h}'(1) \in \left\{ \frac{1}{2}, \frac{3}{2} \right\}, \tag{5.11}$$

$$\frac{1}{2} \leq \tilde{h}'(x) \leq \frac{3}{2}, \quad \text{for } x \in [0, 1]. \tag{5.12}$$

Since

$$D(\tilde{h}) = r^2 D(h),$$

we have to show that

$$D(\tilde{h}) \geq \frac{1}{C}.$$

Let us from now on for convenience write h instead of $\tilde{h}$. The first case is

$$h'(1) = \frac{3}{2}.$$

Define $\bar{h}$ to be the second order polynomial

$$\bar{h}(x) = \frac{1}{4}(x + 2)^2 - 1.$$

This is made in such a way that

$$\left(\bar{h}(0), \bar{h}'(0), \bar{h}'(1) \right) = \left(0, 1, \frac{3}{2} \right) = \left(h(0), h'(0), h'(1) \right).$$

The claim is now that

$$\forall \varepsilon > 0\, \exists\, \delta > 0 \text{ such that: } D(h) < \delta \Rightarrow |h'' - \bar{h}''|_{\infty, [1, +\infty[} < \varepsilon. \tag{5.13}$$

Because of (5.11) we know that for every $x \in [0,1]$

$$\frac{x}{2} \leq \int_0^x h'(y)\, dy \leq \frac{3x}{2},$$

which yields

$$\frac{x}{2} \leq h(x) \leq \frac{3x}{2}. \tag{5.14}$$

Since

$$\int_0^1 yh'''(y)\, dy = h''(1) - \int_0^1 h''(y)\, dy, \tag{5.15}$$

and

$$\int_0^1 h''(y)\, dy = \frac{1}{2},$$

we obtain using (5.14)

$$\left| h''(1) - \frac{1}{2} \right| \leq \left(\int_0^1 y(h''')^2 \right)^{\frac{1}{2}} \lesssim D^{\frac{1}{2}}.$$

Also

$$\int_0^1 hh'''\, dy = h(1)h''(1) - \int_0^1 h'h''\, dy = h(1)h''(1) - \frac{1}{2}(h')^2(1) + \frac{1}{2}, \tag{5.16}$$

yields using (5.14)

$$\left| h(1)h''(1) - \frac{5}{8} \right| \leq \int_0^1 h|h'''|\, dy \leq D^{\frac{1}{2}} \left(\int h\, dy \right)^{\frac{1}{2}} \lesssim D^{\frac{1}{2}}.$$

Thus if $D(h) \ll 1$ we know that

$$h''(1) \approx \frac{1}{2} = \bar{h}''(1),\ h'(1) = \frac{3}{2} = \bar{h}'(1),\ h(1) \approx \frac{5}{4} = \bar{h}(1),$$

and in particular

$$h(1) \geq 1,\ h'(1) = \frac{3}{2},\ h''(1) > \frac{1}{4}. \tag{5.17}$$

Let x_1 be maximal such that

$$h''(x) \geq \frac{1}{4}, \text{ for } x \in [1, x_1].$$

This implies that for $x \in [1, x_1]$ we have for a $\xi \in [1, x]$

$$h(x) = h(1) + h'(1)(x-1) + h''(\xi)\frac{(x-1)^2}{2} \geq 1 + \frac{1}{8}(x-1)^2.$$

Using this we can estimate for $x \in [1, x_1]$

$$|h''(x) - h''(1)| \leq \int_1^x |h'''|\, dy \leq \left(\int_0^\infty \frac{1}{\frac{1}{8}y^2 + 1}\, dy D \right)^{\frac{1}{2}} \lesssim D^{\frac{1}{2}}.$$

Thus for $x \in [1, x_1]$

$$|h''(x) - \bar{h}''(x)| \leq |h''(x) - h''(1)| + |h''(1) - \tfrac{1}{2}| \lesssim D^{\frac{1}{2}}.$$

Thus for all $\varepsilon > 0 \, \exists \, \delta > 0$ such that

$$D(h) < \delta \Rightarrow |h'' - \bar{h}''|_{\infty,[1,x_1]} < \varepsilon.$$

But then for $\varepsilon < \frac{1}{4}$ this implies

$$h''(x_1) > \bar{h}''(x_1) - \varepsilon = \frac{1}{4}.$$

This is a contradiction to the maximality of x_1 if $x_1 < \infty$, thus $x_1 = +\infty$ and we proved (5.13). But now since there exists $x_n \to \infty$ such that $h'(x_n) \to 1$, choose $R \geq 1$ such that

$$h'(R) \leq \frac{3}{2}.$$

Then

$$\frac{1}{R-1} \int_1^R h'' \, dy = \frac{h'(R) - h'(1)}{R-1} \leq 0.$$

But if $D \ll 1$ due to (5.13) the right-hand side would be close to $\frac{1}{2} > 0$, which is a contradiction. Thus there exists C such that

$$D \geq \frac{1}{C},$$

as desired.

It remains to prove

$$D \geq \frac{1}{C},$$

in the case that

$$h'(1) = \frac{1}{2}.$$

Similar to the other case we will compare to a second order polynomial, which is given by

$$\bar{h}(x) = -\frac{1}{4}(x-2)^2 + 1.$$

This is made in such a way that

$$\left(\bar{h}(0), \bar{h}'(0), \bar{h}'(1)\right) = \left(0, 1, \frac{1}{2}\right) = \left(h(0), h'(0), h'(1)\right).$$

Use in a similar fashion as above (5.15) and (5.16) to obtain

$$\left| h''(1) + \frac{1}{2} \right| \lesssim \left(\int_0^1 h (h''')^2 \, dy \right)^{\frac{1}{2}},$$

and

$$\left| h(1)h''(1) + \frac{3}{8} \right| \lesssim \left(\int_0^1 h (h''')^2 \, dy \right)^{\frac{1}{2}}.$$

Thus if $\int_0^1 h(h''')^2\,dy \ll 1$ we know that

$$h''(1) \approx -\frac{1}{2} = \bar{h}''(1),\ h'(1) = \frac{1}{2} = \bar{h}'(1),\ h(1) \approx \frac{3}{4} = \bar{h}(1). \tag{5.18}$$

Let $x_0 \in]1,4[$ arbitrary. Choose $\int_0^1 h(h''')^2\,dy \ll 1$ such that

$$h(1) > \frac{5}{8},$$

then

$$h(1) > \frac{1}{2} = \frac{1}{2}\max_{y\in]1,4[} \bar{h}(y) \geq \frac{1}{2}\bar{h}(x_0) > 0.$$

Now let $x_1 \in]1,x_0]$ be maximal such that

$$h(x) \geq \frac{1}{2}\bar{h}(x_0) \quad \forall x \in [1,x_1].$$

Then for $x \in [1,x_1]$ we obtain

$$|h''(x) - h''(1)| \leq \int_1^{x_1} |h'''|\,dy \lesssim \left(\frac{x_0-1}{\bar{h}(x_0)}\int_1^{x_1} h(h''')^2\,dy\right)^{\frac{1}{2}},$$

and thus together with (5.18) we get for all $\varepsilon > 0$, $\exists\,\delta > 0$ such that

$$\int_0^{x_1} h(h''')^2\,dy < \delta \Rightarrow |h'' - \bar{h}''|_{\infty,[1,x_1]} < \varepsilon.$$

Since $x_0 < 4$ this implies together with (5.18)

$$\int_0^{x_1} h(h''')^2\,dy < \delta \Rightarrow |h - \bar{h}|_{C^2([1,x_1])} < \varepsilon.$$

But then for $\varepsilon \leq \frac{1}{4}\bar{h}(x_0)$ we obtain by using the explicit form of $\bar{h}$ that

$$h(x_1) > \bar{h}(x_1) - \frac{1}{4}\bar{h}(x_0) \geq \frac{1}{2}\bar{h}(x_0).$$

This yields $x_1 = x_0$. Thus for all $x_0 \in]1,4[$ we know that for all $\varepsilon > 0$, $\exists\,\delta > 0$

$$\int_0^{x_0} h(h''')^2\,dy < \delta \Rightarrow |h - \bar{h}|_{C^2([1,x_0])} < \varepsilon. \tag{5.19}$$

Now observe that

$$\left(\bar{h}(4), \bar{h}'(4), \bar{h}''(4)\right) = \left(0, -1, -\frac{1}{2}\right).$$

Let us prove that for all $\delta > 0$ there exists $x_0 \in [1,4[$ and $\varepsilon > 0$ such that

$$|h - \bar{h}|_{C^2([1,x_0])} < \varepsilon \Rightarrow \int_{x_0}^{\infty} h(h''')^2\,dy > \frac{1}{16} - \delta. \tag{5.20}$$

Let δ be given. For this let x_0 be close to 4 and ε small enough such that

$$|h - \bar{h}|_{C^2([1,x_0])} < \varepsilon$$

implies

$$h(x_0) + 4h'(x_0) < 0,\, h''(x_0) < 0.$$

Now let $x_1 \in]x_0, 8]$ such that

$$h'(x) < h'(x_0) \text{ for } x \in]x_0, x_1[\text{ and } h'(x_1) = h'(x_0). \tag{5.21}$$

Such an x_1 does indeed exist since on the one hand $h''(x_0) < 0$ implies that there exists an $\tilde{x} > x_0$ such that

$$h'(x) < h'(x_0) \text{ for } x \in]x_0, \tilde{x}[.$$

on the other hand assume that this holds for $\tilde{x} = 8$. Then using that $h'(x_0) \le 0$ and $x_0 \le 4$ leads to

$$0 \le h(8) = h(x_0) + \int_{x_0}^{8} h'(y)\, dy \le h(x_0) + h'(x_0)(8 - x_0) \le h(x_0) + 4h'(x_0) < 0,$$

a contradiction. Thus an x_1 as in (5.21) exists. Let $x \in [x_0, x_1]$. Since $h'(x) \le h'(x_0) < 0$, we know that h is monotone decreasing on $[x_0, x_1]$. Using this we compute

$$\begin{aligned} h''(x) = h''(x_0) + \int_{x_0}^{x} h'''\, dy &\le h''(x_0) + \left(\int_{x_0}^{\infty} h(h''')^2\, dy \right)^{\frac{1}{2}} \left(\int_{x_0}^{x} \frac{1}{h(y)}\, dy \right)^{\frac{1}{2}} \\ &\lesssim h''(x_0) + \left(\int_{x_0}^{\infty} h(h''')^2\, dy \right)^{\frac{1}{2}} \left(\frac{x_1 - x_0}{h(x)} \right)^{\frac{1}{2}}. \end{aligned}$$

Multiplying this by $h'(x) < 0$ yields

$$\left(\frac{1}{2}(h'(x))^2 \right)' \ge h''(x_0)h'(x) + \left(\int_{x_0}^{\infty} h(h''')^2\, dy \right)^{\frac{1}{2}} (x_1 - x_0)^{\frac{1}{2}}\, 2(h^{\frac{1}{2}})'(x).$$

Now integrating from x_0 to x_1 and using (5.21) yields

$$\begin{aligned} 0 &= \frac{1}{2}(h'(x_1))^2 - \frac{1}{2}(h'(x_0))^2 \\ &\ge h''(x_0)(h(x_1) - h(x_0)) + \left(\int_{x_0}^{\infty} h(h''')^2\, dy \right)^{\frac{1}{2}} (x_1 - x_0)^{\frac{1}{2}}\, 2 \left(h^{\frac{1}{2}}(x_1) - h^{\frac{1}{2}}(x_0) \right). \end{aligned}$$

Observe that

$$h(x_0) = \int_{x_0}^{x_1} (-h'(y))\, dy + h(x_1) \ge (x_1 - x_0)(-h'(x_0)).$$

Using this as well as $h(x_1) \le h(x_0)$ we obtain

$$\begin{aligned} \int_{x_0}^{\infty} h(h''')^2\, dy &\ge \left(\frac{h''(x_0)(h(x_1) - h(x_0))}{(x_1 - x_0)^{\frac{1}{2}}\, 2 \left(h^{\frac{1}{2}}(x_1) - h^{\frac{1}{2}}(x_0) \right)} \right)^2 = h''(x_0)^2 \frac{(h^{\frac{1}{2}}(x_0) + h^{\frac{1}{2}}(x_1))^2}{4(x_1 - x_0)} \\ &\ge h''(x_0)^2 \frac{h(x_0)}{4(x_1 - x_0)} \ge h''(x_0)^2 \frac{-h'(x_0)}{4}. \end{aligned}$$

Now since x_0 is close to 4 and

$$|h - \bar{h}|_{C^2([1,x_0])} < \varepsilon,$$

we know that

$$h''(x_0) \approx -\frac{1}{2},\; h'(x_0) \approx -1.$$

Using this we derive

$$\int_{x_0}^{\infty} h(h''')^2\, dy > \frac{1}{16} - \delta,$$

as desired. Thus we know that (5.19) and (5.20) hold which imply the existence of a C such that

$$D(h) \geq \frac{1}{C}.$$

This closes the proof of the lemma. □

Lemma 5.6. *Let h be admissible and $E(h) < \infty$. Let*

$$r = \frac{1}{C} D^{-\frac{1}{2}},$$

where C is the constant from Lemma 5.5. Then

$$[h']_{\frac{2}{3},[0,r]} \lesssim D^{\frac{1}{3}}.$$

Proof. Using Lemma 5.5 we know that for

$$r = \frac{1}{C} D^{-\frac{1}{2}}$$

and for $x \in [0, r]$ we have

$$h(x) \geq \frac{x}{2} \tag{5.22}$$

and

$$\left| \int_0^r h''\, dy \right| \leq \frac{1}{2}. \tag{5.23}$$

Using the formula

$$\begin{aligned} h''(x) - \frac{1}{r}\int_0^r h''\, dy &= \int_0^x \frac{y}{r} h'''(y)\, dy + \int_x^r \left(\frac{y}{r} - 1\right) h'''(y)\, dy \\ &= \int_0^r y \left(\chi_{[0,x[}\frac{1}{r} + \chi_{[x,r]}\left(\frac{1}{r} - \frac{1}{y}\right)\right) h'''(y)\, dy, \end{aligned}$$

we obtain

$$|h''(x)| \leq \left|\frac{1}{r}\int_0^r h''\, dy\right| + \left(\int_0^r y(h'''(y))^2\, dy \int y \left(\chi_{[0,x[}\frac{1}{r} + \chi_{[x,r]}\left(\frac{1}{r} - \frac{1}{y}\right)\right)^2 dy\right)^{\frac{1}{2}}.$$

Compute

$$\int y \left(\chi_{[0,x[}\frac{1}{r} + \chi_{[x,r]}\left(\frac{1}{r} - \frac{1}{y}\right)\right)^2 dy = \frac{1}{r^2}\int_0^r y\,dy - \int_x^r \frac{2}{r}\,dy + \int_x^r \frac{1}{y}\,dy$$
$$= -\frac{3}{2} + \frac{2x}{r} + \ln\left(\frac{r}{x}\right) \leq \frac{1}{2} + \ln\left(\frac{r}{x}\right).$$

Using this as well as (5.22) and (5.23) we obtain

$$|h''(x)| \lesssim \left(1 + \ln\frac{r}{x}\right)\left(\frac{1}{2r} + D^{\frac{1}{2}}\right).$$

Since for $0 \leq y_0 \leq y_1 \leq 1$ it holds

$$\int_{y_0}^{y_1} 1 + \ln\frac{1}{y}\,dy = y_1 \ln\frac{1}{y_1} + 2y_1 - y_0 \ln\frac{1}{y_0} - 2y_0 \lesssim |y_1 - y_0|^{\frac{2}{3}},$$

we thus estimate for $0 \leq x_0 \leq x_1 \leq r$

$$|h'(x_1) - h'(x_0)| \leq \int_{x_0}^{x_1} |h''(y)|\,dy \lesssim |x_1 - x_0|^{\frac{2}{3}} r^{\frac{1}{3}} \left(\frac{1}{2r} + D^{\frac{1}{2}}\right) \lesssim D^{\frac{1}{3}} |x_1 - x_0|^{\frac{2}{3}},$$

since

$$r = \frac{1}{C} D^{-\frac{1}{2}}.$$

□

Remark 5.7. Observe that

$$E^{\frac{1}{3}} D^{-\frac{1}{3}} = (ED^{\frac{1}{2}})^{\frac{1}{3}}\, D^{-\frac{1}{2}}.$$

Thus for

$$x \leq E^{\frac{1}{3}} D^{-\frac{1}{3}},$$

we know that for

$$ED^{\frac{1}{2}} \ll 1,$$

it holds that

$$x \leq \frac{1}{C} D^{-\frac{1}{2}} = r,$$

and the above estimate holds.

Lemma 5.8 (Hardy's Inequality). *Let $k \neq 1$. Assume*

$$\exists\, z_n \searrow 0: \quad \psi(z_n) \to 0 \quad \text{if } k < -1,$$
$$\exists\, z_n \nearrow \infty: \quad \psi(z_n) \to 0 \quad \text{if } k > -1.$$

Then

$$\int_0^\infty z^k \psi^2\,dz \leq \frac{4}{(k+1)^2} \int_0^\infty z^{k+2} \psi_z^2\,dz.$$

Proof. See for example [7, Lemma A.1]. □

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