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Convergence of the thresholding scheme for multi-phase mean-curvature flow

by

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Abstract

We consider the thresholding scheme, a time discretization for mean curvature flow introduced by Merriman, Bence and Osher. We prove a convergence result in the multi-phase case. The result establishes convergence towards a weak formulation of mean curvature flow in the BV-framework of sets of finite perimeter. The proof is based on the interpretation of the thresholding scheme as a minimizing movement scheme by Esedoglu et. al.. This interpretation means that the thresholding scheme preserves the structure of (multi-phase) mean curvature flow as a gradient flow w. r. t. the total interfacial energy. More precisely, the thresholding scheme is a minimizing movement scheme for an energy functional that Γ-converges to the total interfacial energy. In this sense, our proof is similar to the convergence results of Almgren, Taylor and Wang and Luckhaus and Sturzenhecker, which establish convergence of a more academic minimizing movement scheme. Like the one of Luckhaus and Sturzenhecker, ours is a *conditional* convergence result, which means that we have to assume that the time-integrated energy of the approximation converges to the time-integrated energy of the limit. This is a natural assumption, which however is not ensured by the compactness coming from the basic estimates.

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1 Introduction

1.1 Context

The thresholding scheme, a time discretization for mean curvature flow introduced by Merriman, Bence and Osher [24], has because of its conceptual and practical simplicity become a very popular scheme, see Algorithm 1 for its definition in a more general context. It has a natural extension from the two-phase case to the multi-phase case with triple junctions in local equilibrium, well-known in case for equal surface tensions since some time [25]. Multi-phase mean-curvature flow models the slow relaxation of grain boundaries in poly-crystals (called grain growth), where each grain corresponds to a phase. Elsey, Esedoglu and Smereka have shown that (a modification of) the thresholding scheme is practical in handling a numbers of grains over time intervals sufficiently large to extract significant statistics of the coarsening (also called aging) of the grain configuration [10, 11, 12]. In grain growth, the surface tension (and the mobility) of a grain boundary is both dependent on the misorientation between the crystal lattice of the two adjacent grains and on the orientation of its normal. In other words, the surface tension σ_{ij} of an interface is indexed by the pair (i, j) of phases it separates, and is anisotropic. Esedoglu and the second author have shown in [13] the thresholding scheme can be extended to handle the first extension in a very general way, including in particular the most popular Ansatz for a misorientation-dependent grain boundary energy [31]. How to handle general anisotropies in the framework of the thresholding scheme, even in case of two phases, seems not yet to be completely settled, see however [5] and [18]. — Hence in this work, we will focus on the isotropic case, ignore mobilities, but make the attempt to be as general as [13] when it comes to the dependence of σ_{ij} on the pair (i, j). However, in this first version of the paper, the main result, Theorem 1.3, will be limited by the result in Section 4, where we restrict ourselves to just three phases, and thus three types of interfaces, of equal interfacial energy.

In the two-phase case, the convergence of the thresholding scheme is well-understood: Two-phase mean curvature flow satisfies a geometric *comparison principle*, and it is easy to see that the thresholding scheme preserves this structure. Partial differential equations and geometric motions that allow for a comparison principle can typically be even characterized by comparison with very simple solutions, which opens the way for a definition of a very robust notion of weak solutions, namely what bears the somewhat misleading name of viscosity solutions. If one allows for what the experts know as fattening, two-phase mean-curvature flow is well-posed in this framework [15]. Barles and Georgelin [4] and Evans [14] proved independently that the thresholding scheme converges to mean-curvature flow in this sense. — Hence the main novelty of this work is a (conditional) convergence result in the *multi-phase* case; where clearly a geometric comparison principle is absent. However, the result has also some interest in the two-phase case, since it establishes convergence even in situations where the viscosity solution features fattening. Together with Drew Swartz [20], the first author uses similar arguments to treat another version of mean curvature flow that does not allow for a comparison principle, namely volume-preserving (two-phase) mean curvature flow. They prove (conditioned) convergence of a scheme introduced by Ruuth and Wetton in [33]. We also draw the reader's attention to the recent work of Mugnai, Seis and Spadaro [30], where they prove a (conditional) convergence result as in [22] of a modification of the scheme in [2, 22] to volume-preserving mean curvature flow. Note that due to the only conditional convergence, our result does not provide a long-time existence result for (weak solutions of) multi-phase mean curvature flow. Short-time existence results of smooth solutions go back to the work of Bronsard and Reitich [7]. Mantegazza et. al. [23] and Schnürer et. al.[36] were able to construct long-time solutions close to a self-similar singularity.

For the present work, the structural substitute for the comparison principle is the *gradient flow* structure. Folklore says that mean curvature flow, also in its multi-phase version, is the gradient flow of the total interfacial energy. It is by now well-appreciated that the gradient flow structure also requires fixing a Riemannian structure, that is, an inner product on the tangent space, which here is given by the space of normal velocities. Mean curvature flow is then the gradient flow with respect to the L^2-inner product, in case of grain growth weighted by grain-boundary-dependent and anisotropic mobilities. Loosely speaking, Brakke's existence proof in the framework of varifolds [6] is based on this structure in the sense that the solution monitors weighted versions of the interfacial energy. And so does Ilmanen's convergence proof of the Allen-Cahn equation, a diffuse interface approximation of computational relevance in the world of phase-field models, to mean curvature flow [17]. It was only discovered recently that the thresholding algorithm preserves also this gradient flow structure [13], which in that paper was taken as a guiding principle to extend the scheme to surface tensions σ_{ij} and mobilities that depend on the phase pair (i,j). — In this paper, we take the gradient flow structure, which we make more precise in the following paragraphs, as a guiding principle for the convergence proof.

On the abstract level, every gradient flow has a natural discretization in time, which comes in form of a sequence of variational problems: The configuration Σ^k at time step k is obtained by minimizing $\frac{1}{2}\mathrm{dist}^2(\Sigma,\Sigma^{k-1}) + hE(\Sigma)$, where Σ^{k-1} is the configuration at the preceding time step, h is the time-step size and dist denotes the induced distance on the configuration space endowed with the Riemannian structure. In the Euclidean case, the Euler-Lagrange equation (i. e. the first variation) of this variational problem yields the implicit (or backwards) Euler scheme. This variational scheme has been named "minimizing movement scheme" by De Giorgi [9], and has recently gained popularity because it allows to interpret diffusion equations as gradient flows of an entropy functional w. r. t. the Wasserstein metric ([19], see [3] for the abstract framework) – an otherwise unrelated problem. However, the formal Riemannian structure in case of mean curvature flow is completely degenerate: $\mathrm{dist}^2(\Sigma,\tilde\Sigma)$ as defined as the infimal energy of curves in configuration space that connect Σ to $\tilde\Sigma$ vanishes identically, cf. [26].

Hence when formulating a minimizing movement scheme in case of mean curvature flow, one has to come up with a proxy for $\mathrm{dist}^2(\Sigma,\tilde\Sigma)$. This has been independently achieved by Almgren, Taylor and Wang [2] on the one side and Luckhaus and Sturzenhecker [22] on the other side of the Atlantic. $\Sigma = \partial\Omega$ and $\tilde{\Sigma} = \partial\tilde{\Omega}$, $2\int_{\Omega\triangle\tilde{\Omega}} d(x,\Sigma)dx$ is one possible substitute for $\mathrm{dist}^2(\Sigma,\tilde{\Sigma})$ in the minimizing movement scheme, where $d(x,\Sigma)$ denotes the unsigned distance of the point x to the surface Σ — it is easy to see that to leading order in the proximity of $\tilde{\Omega}$ to Ω, this expression behaves as the metric tensor $\int_\Sigma V^2 dx$, where V is the normal velocity leading from Ω to $\tilde{\Omega}$ in one unit time. Their work makes this point by proving that this minimizing movement scheme (which is completely academic from a computational point of view) converges to mean curvature flow. To be more precise, they consider a time-discrete solution $\{\Omega^k\}_k$ of the minimizing movement scheme, interpolated as a piecewise constant function Ω^h in time and assume that for a subsequence $h \downarrow 0$, the time-dependent sets Ω^h converge to Ω in a stronger sense than the given compactness provides. Almgren, Taylor and Wang assume that $\Sigma^h(t)$ converges to $\Sigma(t)$ in the Hausdorff distance and show that Σ solves the mean curvature flow equation in the above mentioned viscosity sense. The argument was later substantially simplified by Chambolle and Novaga in [8]. Luckhaus and Sturzenhecker start from a weaker convergence assumption than the one in [2]: They assume that for the finite time horizon T under consideration, $\int_0^T |\Sigma^h(t)|dt$ converges to $\int_0^T |\Sigma(t)|dt$. Then they show that Ω evolves according to a weak formulation of mean curvature flow, using a distributional formulation of mean curvature that is available for sets of finite perimeter, see Definition 1.1 for the case multi-phase case of this formulation. Those are both only *conditional* convergence results: While the natural estimates coming from the minimizing movement scheme, namely the uniform boundedness of $\sup_k |\Sigma^k|$ and $\sum_k 2\int_{\Omega^k\triangle\Omega^{k+1}} d(x,\Sigma^k)dx$, are enough to ensure $\int_0^T |\Omega^h(t)\triangle\Omega(t)|dt$ and $\int_0^T |\Sigma(t)|dt \leq \liminf \int_0^T |\Sigma^h(t)|dt$, they are not sufficient to yield $\limsup \int_0^T |\Sigma^h(t)|dt \leq \int_0^T |\Sigma(t)|dt$ or even the convergence of $\Sigma^h(t)$ to $\Sigma(t)$ in the Hausdorff distance. — Our result will be a conditional convergence result very much in the same sense as the one in [22] but it turns out that our convergence result for the thresholding scheme requires no regularity theory for (almost) minimal surfaces, in contrast to the one of [22].

Following [13], we now explain in which sense the thresholding scheme may be considered as a minimizing movement scheme for mean curvature flow. Each step in Algorithm 1 is equivalent to minimizing a functional of the form $E_h(\chi) - E_h(\chi - \chi^{n-1})$, where the functional E_h, defined below in (3) is an approximation to the total interfacial energy. It is a little more subtle to see that the second term, $-E_h(\chi^n - \chi^{n-1})$, is comparable to the metric tensor $\int_\Sigma V^2 dx$. The Γ-convergence of functionals of the type (3) to the area functional has a long history: For the two-phase case, cf. Alberti and Bellettini [1] and Miranda et. al. [27]. The multi-phase case, also for arbitrary surface tensions was investigated by Esedoglu and the second author in [13]. Incidentally, it is easy to see that Γ-convergence of the energy functionals is not sufficient for the convergence of the corresponding gradient flows; Sandier and Serfaty [34] have identified sufficient conditions on both the functional and the metric tensor for this to be true.

Identically, the approach of Saye and Sethian [35] for multi-phase evolutions can also be seen as coming from the gradient flow structure when applied to N-phase mean curvature flow. More precisely, it can be understood as a time splitting of an L^2-gradient flow with an additional phase whose volume is strongly penalized: The first step is $(N+1)$-phase gradient flow w. r. t. the total interfacial energy and the second step is $(N+1)$-gradient flow w. r. t. the volume penalization (so geometrical optics leading to the Voronoi construction.

1.2 Informal summary of the proof

We now give a summary of the main steps and ideas of the convergence proof. In Section 2, we draw consequences from the basic estimate (9) in a minimizing movement scheme, like compactness, Proposition 2.4, coming from a uniform (integrated) modulus of continuity in space, Lemma 2.5, and in time, Lemma 2.6, which we prove for arbitrary surface tensions and any number of phases. We also draw the first consequence from the strengthened convergence (8) in the case of equal surface tensions in Lemma 2.10. We strongly advise the reader to familiarize him- or herself with the argument for the modulus of continuity in time, Lemma 2.6, since it is there that the mesoscopic time scale $\sqrt{h}$ appears for the first time in a simple context before being used in Section 4 in a more complex context. In the same vein, the fudge factor α in the mesoscopic time scale $\alpha\sqrt{h}$, which will be crucial in Section 4, will first be introduced and used in the simple context when estimating the normal velocity V of the limit in Lemma 2.10.

Starting from Section 3, we also use the Euler-Lagrange equation (31) of the minimizing movement scheme. By Euler-Lagrange equation we understand the first variation w. r. t. the independent variables, as generated by a test vector field ξ. In Section 3, we pass to the limit in the energetic part of the first variation, recovering the mean curvature H via the term $\int_\Sigma H\,\xi\cdot\nu = \int_\Sigma \nabla\cdot\xi - \nu\cdot\nabla\xi\,\nu$. This amounts to show that under our assumption of strengthened convergence (8), the Γ-convergence of the *functionals* can be upgraded to a distributional convergence of their *first variations*, cf. Proposition 3.1. Although Proposition 3.1 is formulated for equal surface tensions, the arguments in this section are given for arbitrary surface tensions. It is a classical result credited to Reshetnyak [32] that under the strengthened convergence of sets of finite perimeter, the measure-theoretic normals and thus the distributional expression for mean curvature also converge. The fact that this convergence of the first variation may also hold when combined with a diffuse interface approximation is known for instance in case of the Ginzburg-Landau approximation of the area functional (also known by the names of Modica and Mortola, who established this Γ-convergence [28, 29]), see [21]. In our case the convergence of the first variations relies on a localization of the ingredients for the Γ-convergence worked out in [13], like monotonicity of the functional in h.

Section 4 constitutes the central and, as we believe, most innovative piece of the paper; we pass to the limit in the dissipation/metric part of the first variation, recovering the normal velocity V via the term $\int_\Sigma V\,\xi\cdot\nu$. Here, we restrict ourselves to the case of three phases and equal surface tensions. In fact, we think of the test-field ξ as localizing this expression in time and space, and recover the desired limiting expression only up to an error that measures how well the limiting configuration can be approximated by a configuration with only two phases and a flat interface in the space-time patch under consideration; this is measured both in terms of area (leading to a multi-phase excess in the language of the regularity theory of minimal surfaces) and volume, see Proposition 4.1. The main difficulty of recovering the metric term $\int_\Sigma V\,\xi\cdot\nu$ in comparison to recovering the distributional form $\int_\Sigma \nabla\cdot\xi - \nu\cdot\nabla\xi\,\nu$ of the energetic term is that one has to recover both the normal velocity V, which is distributionally characterized by $\partial_t\chi - V|\nabla\chi| = 0$ on the level of the characteristic function χ, and the (spatial) normal ν. In short: one has to pass to the limit in a *product*. More precisely, the main difficulty is that there is no good bound on the discrete normal velocity V at hand on the level of the *microscopic* time scale h; only on the level of the above-mentioned mesoscopic time scale $\sqrt{h}$, such an estimate is available. This comes from the fact that the basic estimate yields control of the time derivative of the characteristic function χ only when mollified on the spatial scale $\sqrt{h}$ in $u = G_h * \chi$. The main technical ingredient to overcome this lack of control in Proposition 4.1: If one of the two (spatial) functions $u, \tilde{u}$ is not too far from being strictly monotone in a given direction (a consequence of the control of the tilt excess, see Lemma 4.4), then the spatial L^1-difference between the level sets $\chi = \{u > \frac{1}{2}\}$ and $\tilde{\chi} = \{\tilde{u} > \frac{1}{2}\}$ is controlled by the squared L^2-difference between u and $\tilde{u}$.

In Section 5, we combine the results of the previous two sections yielding the weak formulation of $V = H$ on some space-time patch up to an error expressed in terms of the above mention (multi-phase) tilt excess of the limit on that patch. Complete localization in time and partition of unity in space allows us to assemble this to obtain $V = H$ globally, up to an error expressed by the time integral of the sum of the tilt excess over the spatial patches of finite overlap. De Giorgi's structure theorem for sets of finite perimeter (cf. Theorem 4.4 in [16]), adapted to a multi-phase situation but just used for a fixed time slice, implies that the error expression can be made arbitrarily small by sending the length scale of the spatial patches to zero.

1.3 Notation

We denote by

$$G_h(z) := \frac{1}{(2\pi h)^{d/2}} \exp\left(-\frac{|z|^2}{2h}\right)$$

the Gaussian kernel of variance h. Note that $G_{2t}(z)$ is the fundamental solution to the heat equation and thus

$$\partial_h G - \frac{1}{2}\Delta G = 0 \quad \text{in } (0,\infty) \times \mathbb{R}^d,$$
$$G = \delta_0 \quad \text{for } h = 0.$$

We recall some basic properties, such as the normalization, non-negativity and boundedness and the factorization property:

$$\int_{\mathbb{R}^d} G_h \, dz = 1, \quad 0 \leq G_h \leq C h^{-d/2}, \quad \nabla G_h(z) = -\frac{z}{h} G_h(z), \quad G(z) = G^1(z_1) G^{d-1}(z'),$$

where G^1 denotes the 1-dimensional and G^{d-1} the $(d-1)$-dimensional Gaussian kernel; let us also mention the semi-group property

$$G_{s+t} = G_t * G_t.$$

Throughout the paper, we will work on the flat torus $[0, \Lambda)^d$. The thresholding scheme for multiple phases, introduced in [13], for arbitrary surface tensions σ_{ij} and mobilities $\mu_{ij} = 1/\sigma_{ij}$

is the following.

Algorithm 1. *Given the partition $\Omega_1^{n-1},\dots,\Omega_P^{n-1}$ of $[0,\Lambda)^d$ at time $t=(n-1)h$, obtain the new partition $\Omega_1^n,\dots,\Omega_P^n$ at time $t=nh$ by:*

1. *Convolution step:*

$$\phi_i := G_h * \left(\sum_{j=1}^{P}\sigma_{ij}\mathbf{1}_{\Omega_j^{n-1}}\right). \tag{1}$$

2. *Thresholding step:*

$$\Omega_i^n := \left\{x\in[0,\Lambda)^d\colon \phi_i(x)<\phi_j(x) \text{ for all } j\neq i\right\}. \tag{2}$$

We will denote the characteristic functions of the phases Ω_i^n at the nth time step by χ_i^n and interpolate these functions piecewise constantly in time, i.e.

$$\chi_i^h(t) := \chi_i^n = \mathbf{1}_{\Omega_i^n} \quad \text{for } t\in[nh,(n+1)h).$$

As in [13], we define the *approximate energies*

$$E_h(\chi) := \frac{1}{\sqrt{h}}\sum_{i,j}\sigma_{ij}\int \chi_i\, G_h*\chi_j\,dx \tag{3}$$

for *admissible* measurable functions:

$$\chi=(\chi_1,\dots,\chi_P)\colon [0,\Lambda)^d\to\{0,1\}^P \quad \text{s.t.} \quad \sum_{i=1}^{P}\chi_i = 1 \quad \text{a.e..} \tag{4}$$

Here and in the sequel $\int dx$ stands short for $\int_{[0,\Lambda)^d}dx$, whereas $\int dz$ stands short for $\int_{\mathbb{R}^d}dz$. The minimal assumption on the matrix of surface tensions $\{\sigma_{ij}\}$, next to the obvious

$$\sigma_{ij}=\sigma_{ji}\geq\sigma_{\min}>0 \quad \text{if } i\neq j, \quad \sigma_{ii}=0$$

is the following triangle inequality

$$\sigma_{ij}\leq\sigma_{ik}+\sigma_{kj}.$$

It is known that (e.g. [13]), under the conditions above, these energies Γ-converge w.r.t. the L^1-topology to the *optimal partition energy* given by

$$E(\chi) := c_0\sum_{i,j}\sigma_{ij}\frac{1}{2}\Big(\int|\nabla\chi_i| + \int|\nabla\chi_j| - \int|\nabla(\chi_i+\chi_j)|\Big)$$

for *admissible* χ:

$$\chi = (\chi_1, \ldots, \chi_P)\colon [0,\Lambda)^d \to \{0,1\}^P \in BV \quad \text{s.t.} \quad \sum_{i=1}^P \chi_i = 1 \quad \text{a.e.}$$

The constant c_0 is given by

$$c_0 := \omega_{d-1} \int_0^\infty G(r) r^d dr = \frac{1}{\sqrt{2\pi}}.$$

We recall the minimizing movement interpretation from [13] which is easy to check. The combination of convolution and thesholding step in Algorithm 1 is equivalent to solving the following minimization problem

$$\chi^n = \arg\min_\chi \left\{ E_h(\chi) - E_h(\chi - \chi^{n-1}) \right\}, \tag{5}$$

where χ runs over (4). The proof will mostly be based on the interpretation (5) and only once uses the original form (1) and (2) in Lemma 4.2. Following [13], we will additionally assume that σ is *conditionally negative-definite*, i.e.

$$\sigma \leq -\underline{\sigma} \quad \text{on } (1, \ldots, 1)^\perp,$$

where $\underline{\sigma} > 0$ is a constant. That means, that σ is negative as a bilinear form on $(1, \ldots, 1)^\perp$. This ensures that $-E_h(\chi - \chi^{n-1})$ in (5) is non-negative and penalizes the distance to the previous step.
In the following we write $A \lesssim B$ to express that $A \leq CB$ for some generic constant $C < \infty$ that only depends on the dimension d and later on also on the total number of phases P and on the constants $\sigma_{\min}$ and $\underline{\sigma}$.

1.4 Main result

For the definition of our weak notion of mean curvature flow, we assume that all surface tensions are equal since our main result is also stated in this context.

Definition 1.1 (Motion by mean curvature)**.** Fix some finite time horizon $T < \infty$ and initial data $\chi^0\colon [0,\Lambda)^d \to \{0,1\}^P$ with $E_0 := E(\chi^0) < \infty$. We say that the network

$$\chi = (\chi_1, \cdots, \chi_P) : (0,T) \times [0,\Lambda)^d \to \{0,1\}^P$$

with $\sum_i \chi_i = 1$ a.e. and $\chi(t) \in BV([0,\Lambda)^d, \{0,1\}^P)$ for a.e. t *moves by mean curvature* if there exist functions $V_i\colon (0,T) \times [0,\Lambda)^d \to \mathbb{R}$ which satisfy

$$\sum_{i=1}^P \int_0^T \int \left(\nabla \cdot \xi - \nu_i \cdot \nabla \xi \, \nu_i + 2\, \xi \cdot \nu_i \, V_i \right) |\nabla \chi_i| \, dt = 0 \tag{6}$$

for all $\xi \in C_0^\infty((0,T) \times [0,\Lambda)^d, \mathbb{R}^d)$ and which are normal velocities in the sense that for all $\zeta \in C^\infty([0,T] \times [0,\Lambda)^d)$ with $\zeta(T) = 0$ and all $i \in \{1,\dots,P\}$

$$\int_0^T \int \partial_t \zeta \, \chi_i \, dx \, dt + \int \zeta(0) \chi_i^0 \, dx = - \int_0^T \int \zeta V_i \, |\nabla \chi_i| \, dt. \tag{7}$$

Note that (7) also encodes the inital condition.

Remark 1.2. To prove the convergence of the scheme, we will need the following convergence assumption:

$$\int_0^T E_h(\chi^h) \, dt \to \int_0^T E(\chi) \, dt. \tag{8}$$

This assumption makes sure that there is no loss of area in the limit $h \to 0$.

Theorem 1.3. *Let $P = 3$ and $\sigma_{ij} = 1 - \delta_{ij}$, $T < \infty$ and χ^0 be given with $E(\chi^0) < \infty$. Then the approximate solutions χ^h obtained by Algorithm 1 converge to some $\chi\colon (0,T) \times [0,\Lambda)^d \to \{0,1\}^P$. Given (8), χ moves by mean curvature in the sense of Definition 1.1 with inital data χ^0.*

Remark 1.4. Our proof uses the following three different time scales:

1. The *macroscopic time scale*, $T < \infty$, given by the finite time horizon,
2. the *mesoscopic time scale*, $\tau = \alpha\sqrt{h} \sim \sqrt{h} > 0$ and
3. the *microscopic time scale*, $h > 0$, coming from the time discretization.

The mesoscopic time scale arises naturally from the scheme: Due to the parabolic scaling, the microscopic time scale h corresponds to the length scale $\sqrt{h}$ as can be seen from the kernel G_h. Since for a smooth evolution, the normal velocity V is of order 1, this prompts the mesoscopic time scale $\sqrt{h}$.
The parameter α will be kept fixed most of the time until the very end, where we send $\alpha \to 0$. Therefore, it is natural to think of $\alpha \sim 1$, but small.
These three time scales go hand in hand with the following numbers, which we will for simplicity assume to be natural numbers throughout the proof:

1. N - the total number of microscopic time steps in a macroscopic time interval $(0,T)$,
2. K - the number of microscopic time steps in a mesoscopic time interval $(0,\tau)$ and
3. L - the number of mesoscopic time intervals in a macroscopic time interval.

The following simple identities linking these different parameters will be used frequently:

$$T = Nh = L\tau, \quad \tau = Kh, \quad L = \frac{N}{K} = \frac{T}{\tau}.$$

2 Compactness

Lemma 2.1 (Energy-dissipation estimate)**.** *The approximate solutions satisfy*

$$E_h(\chi^N) - \sum_{n=1}^N E_h(\chi^n - \chi^{n-1}) \leq E_0. \tag{9}$$

$\sqrt{-E_h}$ *defines a norm on the* process space $\{\omega\colon [0,\Lambda)^d \to \mathbb{R}^P | \sum_i \omega_i = 0\}$. *In particular, the algorithm dissipates energy.*

Proof. By the minimality condition (5), we have in particular

$$E_h(\chi^n) - E_h(\chi^n - \chi^{n-1}) \leq E_h(\chi^{n-1})$$

for each $n = 1, \dots, N$. Iterating this estimate yields (9) with $E_h(\chi^0)$ instead of $E_0 = E(\chi^0)$. (9) follows from the short argument after this proof.
We claim that the pairing $-\frac{1}{\sqrt{h}} \int \omega \cdot \sigma \left(G_h * \tilde{\omega}\right) dx$ defines a scalar product on the process space. It is bilinear and symmetric thanks to the symmetry of σ and G_h. Since σ is conditionally negative-definite,

$$-\frac{1}{\sqrt{h}} \int \omega \cdot \sigma \left(G_h * \omega\right) dx = -\frac{1}{\sqrt{h}} \int \left(G_{h/2} * \omega\right) \cdot \sigma \left(G_{h/2} * \omega\right) dx \geq \frac{\underline{\sigma}}{\sqrt{h}} \|G_{h/2} * \omega\|_{L^2}^2 \geq 0.$$

Furthermore, we have equality only if $\omega \equiv 0$. Thus, $\sqrt{-E_h}$ is the induced norm on the process space. □

Let us mention two results The following monotonicity statement is a key tool for the Γ-convergence in [13] and will be used frequently in our proof. We will also refine this statement in Section 3.

Lemma 2.2 (Approximate monotonicity)**.** *For all* $0 < h \leq h_0$ *and any admissible* χ*, we have*

$$E_h(\chi) \geq \left(\frac{\sqrt{h_0}}{\sqrt{h_0} + \sqrt{h}}\right)^{d+1} E_{h_0}(\chi). \tag{10}$$

Another important tool for the Γ-convergence in [13] is the following consistency, or pointwise convergence of the functionals E_h to E, which we will also refine in Section 3.

Lemma 2.3 (Consistency)**.** *For any admissible* $\chi \in BV$*, we have*

$$\lim_{h \to 0} E_h(\chi) = E(\chi). \tag{11}$$

Taking the limit $h \to 0$ in (10) with $\chi = \chi^0$ and using (11), we see that that the interfacial energy E_0 of the initial data $\chi(0) \equiv \chi^0$ bounds the approximate energy of the initial data:

$$E_0 := E(\chi(0)) \geq E_h(\chi^0).$$

Proposition 2.4 (Compactness)**.** *There exists a sequence $h \searrow 0$ and an admissible $\chi \in L^1((0,T) \times [0,\Lambda)^d, \mathbb{R}^P)$ such that*

$$\chi^h \longrightarrow \chi \quad \text{in } L^1((0,T) \times [0,\Lambda)^d). \tag{12}$$

Moreover,

$$\chi^h \longrightarrow \chi \quad \text{a.e. in } (0,T) \times [0,\Lambda)^d \tag{13}$$

and $\chi(t) \in BV([0,\Lambda)^d, \{0,1\}^P)$ for a.e. $t \in (0,T)$.

Lemma 2.5 (Almost BV in space)**.** *The approximate solutions satisfy*

$$\int_0^T \int \left|\chi^h(x+\delta e, t) - \chi^h(x,t)\right| dx\, dt \lesssim (1+T) E_0 \left(\delta + \sqrt{h}\right) \tag{14}$$

for each $\delta > 0$ and $e \in S^{d-1}$.

Proof of Lemma 2.5. Step 1: We have

$$\int_0^T \int \left|\nabla G_h * \chi^h\right| dx\, dt \lesssim (1+T) E_0. \tag{15}$$

Indeed, for any characteristic function $\chi : [0,\Lambda)^d \to \{0,1\}$ we have

$$\nabla (G_h * \chi)(x) = -\int \nabla G_h(z) \left(\chi(x+z) - \chi(x)\right) dz.$$

Therefore, since $|\nabla G_h(z)| \lesssim \frac{1}{\sqrt{h}} |G_{2h}(z)|$,

$$\int |\nabla (G_h * \chi)|\, dx \lesssim \frac{1}{\sqrt{h}} \int G_{2h}(z) \int |\chi(x+z) - \chi(x)|\, dx\, dz.$$

By $\chi \in \{0,1\}$, we have $|\chi(x+z) - \chi(x)| = \chi(x) \left(1-\chi\right)(x+z) + \left(1-\chi\right)(x) \chi(x+z)$ and thus by translation invariance:

$$\int |\nabla G_h * \chi|\, dx \lesssim \frac{1}{\sqrt{h}} \int \left(1-\chi\right)\, G_{2h} * \chi\, dx.$$

Applying this on χ_i^h, summing over $i = 1, \ldots, P$, using $\chi_i^h = 1 - \sum_{j\neq i} \chi_j^h$ and $\sigma_{ij} \geq \sigma_{\min} > 0$ for $i \neq j$ we obtain

$$\int \left|\nabla G_h * \chi^h(t)\right| dx \lesssim E_{2h}(\chi^h) \lesssim E_h(\chi^h),$$

where we used the approximate monotonicity of E_h, cf. (10) in Lemma 2.2. Using the energy-dissipation estimate (9), we have

$$\int \left|\nabla G_h * \chi^h(t)\right| \, dx \lesssim E_0.$$

Integrating in time yields (15).

Step 2: By (15) and Hadamard's trick, we have on the one hand

$$\int_0^T \int \left|G_h * \chi^h(x+\delta e,t) - G_h * \chi^h(x,t)\right| \, dx \, dt \lesssim (1+T)E_0\delta.$$

Since $\chi \in \{0,1\}$, we have on the other hand

$$(\chi - G_h * \chi)_+ = \chi \, G_h * (1-\chi) \quad \text{and} \quad (\chi - G_h * \chi)_- = (1-\chi) \, G_h * \chi,$$

which yields

$$\int |\chi - G_h * \chi| \, dx = 2\int (1-\chi) \, G_h * \chi. \tag{16}$$

Using the translation invariance and (16) for the components of $\chi^h(x+\delta,t)$, we have

$$\begin{aligned}
\int_0^T \int \left|\chi^h(x+\delta e,t) - \chi^h(x,t)\right| \, dx \, dt \leq & 2\int_0^T \int \left|G_h * \chi^h(t) - \chi^h(t)\right| \, dx \, dt \\
& + \int_0^T \int \left|G_h * \chi^h(x+\delta e,t) - G_h * \chi^h(x,t)\right| \, dx \, dt \\
\lesssim & (1+T) \, E_0 \left(\sqrt{h} + \delta\right).
\end{aligned}$$

□

Lemma 2.6 (Almost BV in time)**.** *The approximate solutions satisfy*

$$\int_\tau^T \int \left|\chi^h(t) - \chi^h(t-\tau)\right| \, dx \, dt \lesssim (1+T)E_0 \left(\tau + \sqrt{h}\right) \tag{17}$$

for every $\tau > 0$.

Proof. In this proof, we make use of the mesoscopic time scale τ, see Remark 1.4 for the notation. First, we argue that it is enough to prove

$$\int_\tau^T \int \left|\chi^h(t) - \chi^h(t-\tau)\right| \, dx \, dt \lesssim (1+T)E_0\tau \tag{18}$$

for $\alpha \in [1,2]$. If $\alpha \in (0,1)$, we can apply (18) twice, once for $\tau = \sqrt{h}$ and once for $\tau = (1+\alpha)\sqrt{h}$

and obtain (17). If $\alpha > 2$, we can iterate (18). Thus, we may assume that $\alpha \in [1,2]$. We have

$$\begin{aligned}\int_\tau^T \int \left|\chi^h(t) - \chi^h(t-\tau)\right| dx\, dt =& h \sum_{k=0}^{K-1} \sum_{l=1}^{L} \int \left|\chi^{Kl+k} - \chi^{K(l-1)+k}\right| dx \\ =& \frac{1}{K} \sum_{k=0}^{K-1} \tau \sum_{l=1}^{L} \int \left|\chi^{Kl+k} - \chi^{K(l-1)+k}\right| dx.\end{aligned}$$

Thus, it is enough to prove

$$\tau \sum_{l=1}^{L} \int \left|\chi^{Kl+k} - \chi^{K(l-1)+k}\right| dx \lesssim (1+T)E_0$$

for any $k = 0, \ldots, K-1$. By the energy-dissipation estimate (9), we have $E_h(\chi^k) \leq E_0$ for all these k's. Hence we may assume w.l.o.g. that $k = 0$ and prove only

$$\sum_{l=1}^{L} \int \left|\chi^{Kl} - \chi^{K(l-1)}\right| dx \lesssim (1+T)E_0. \tag{19}$$

Note that for any two characteristic functions $\chi, \tilde\chi$ we have

$$\begin{aligned}|\chi - \tilde\chi| =& (\chi - \tilde\chi)\, G_h * (\chi - \tilde\chi) + (\chi - \tilde\chi)(\chi - \tilde\chi - G_h * (\chi - \tilde\chi)) \\ \leq& (\chi - \tilde\chi)\, G_h * (\chi - \tilde\chi) + |\chi - G_h * \chi| + |\tilde\chi - G_h * \tilde\chi|.\end{aligned} \tag{20}$$

Now, we post-process the energy-dissipation estimate (9). Using the triangle inequality for the norm $\sqrt{-E_h}$ on the process space and Jensen's inequality, we have

$$-E_h\left(\chi^{Kl} - \chi^{K(l-1)}\right) \leq \left(\sum_{n=K(l-1)+1}^{Kl} \Big(- E_h\left(\chi^n - \chi^{n-1}\right)\Big)^{\frac{1}{2}} \right)^2 \leq K \sum_{n=K(l-1)+1}^{Kl} -E_h\left(\chi^n - \chi^{n-1}\right).$$

Using (20) for χ_i^{Kl} and $\chi_i^{K(l-1)}$, and (16) for the second term on the right-hand side, we obtain

$$\sum_{l=1}^{L} \int \left|\chi_i^{Kl} - \chi_i^{K(l-1)}\right| dx \lesssim \sqrt{h} K \sum_{n=1}^{N} -E_h\left(\chi^n - \chi^{n-1}\right) + L \max_n \int \left(1 - \chi_i^n\right) G_h * \chi_i^n \, dx.$$

Since $(1 - \chi_i^n) = \sum_{j\neq i} \chi_j^n$ a.e. and $\sigma_{ij} \geq \sigma_{\min} > 0$ for all $i \neq j$, the energy-dissipation estimate (9) yields

$$\sum_{l=1}^{L} \int \left|\chi^{Kl} - \chi^{K(l-1)}\right| dx \lesssim \alpha E_0 + \frac{1}{\alpha} T E_0 \lesssim (1+T)E_0,$$

which establishes (19) ant thus concludes the proof. □

Proof of Proposition (2.4). The proof is an adaptation of the proof of the Riesz-Kolmogorov

L^p-compactness theorem. By Lemma 2.5 and Lemma 2.6, we have

$$\int_0^T \int \left|\chi^h(x+\delta e, t+\tau) - \chi^h(t)\right| dx\, dt \lesssim (1+T)E_0 \left(\delta + \tau + \sqrt{h}\right) \tag{21}$$

for any $\delta, \tau > 0$ and $e \in S^{d-1}$. For $\delta > 0$ consider the mollifier φ_δ given by the scaling $\varphi_\delta(x) := \frac{1}{\delta^{d+1}}\varphi(\frac{x}{\delta}, \frac{t}{\delta})$ and $\varphi \in C_0^\infty((-1,0) \times B_1)$ such that $0 \leq \varphi \leq 1$ and $\int_{-1}^0 \int_{B_1} \varphi = 1$. We have the estimates

$$\left|\varphi_\delta * \chi^h\right| \leq 1 \quad \text{and} \quad \left|\nabla(\varphi_\delta * \chi^h)\right| \lesssim \frac{1}{\delta}.$$

Hence, on the one hand, the mollified functions are equicontinuous and by Arzelà-Ascoli precompact in $C^0([0,T] \times [0,\Lambda)^d)$: For given $\epsilon, \delta > 0$ there exist functions $u_i \in C^0([0,T] \times [0,\Lambda)^d)$, $i = 1, \ldots, n(\epsilon, \delta)$ such that

$$\left\{\varphi_\delta * \chi^h \colon h > 0\right\} \subset \bigcup_{i=1}^{n(\epsilon,\delta)} B_\epsilon(u_i),$$

where the balls $B_\epsilon(u_i)$ are given w.r.t. the C^0-norm. On the other hand, for any function χ we have

$$\begin{aligned}\int_0^T \int |\varphi_\delta * \chi - \chi|\, dx\, dt &\leq \int \varphi_\delta(z,s) \int |\chi(x-z, t-s) - \chi(x,t)|\, d(x,t)\, d(z,s)\\ &\leq \sup_{(z,s)\in \operatorname{supp}\varphi_\delta} \int_0^T \int |\chi(x-z,t-s) - \chi(x,t)|\, dx\, dt.\end{aligned}$$

Using this for χ^h and plugging in (21) yields

$$\int_0^T \int \left|\varphi_\delta * \chi^h - \chi^h\right| dx\, dt \lesssim (1+T)E_0\left(\delta + \sqrt{h}\right).$$

Given $\rho > 0$, fix $\delta, h_0 > 0$ such that

$$\int_0^T \int \left|\varphi_\delta * \chi^h - \chi^h\right| dx\, dt \leq \frac{\rho}{2} \quad \text{for all } h \in (0, h_0).$$

Then set $\epsilon := \frac{\rho}{TL^d}$ and find $u_1, \ldots, u_n$ from above. Note that only finitely many of the elements in the sequence $\{h\}$ are greater than h_0. Therefore,

$$\{\chi^h\}_h \subset \bigcup_{i=1}^n B_\rho(u_i) \cup \{\chi^h\}_{h>h_0} \subset \bigcup_{i=1}^n B_\rho(u_i) \cup \bigcup_{h>h_0} B_\rho(\chi^h)$$

is a finite covering of balls (w.r.t. L^1-norm) of given radius $\rho > 0$. Therefore, $\{\chi^h\}_h$ is precompact and hence relatively compact in L^1. Hence we can extract a converging subsequence. After passing to another subsequence, we can w.l.o.g. assume that we also have pointwise convergence almost everywhere in $(0,T) \times [0,\Lambda)^d$. $\square$

Lemma 2.7 ($C^{1/2}$-Bounds)**.** *We have uniform Hölder-type bounds for the approximate solutions. I.e. for each pair $s,t \in [0,T]$ with $|s-t| \geq h$ we have*

$$\int \left|\chi^h(s) - \chi^h(t)\right| \, dx \lesssim E_0 |s-t|^{1/2}. \tag{22}$$

In particular, $\chi \in C^{1/2}([0,T], L^1([0,\Lambda)^d))$: For almost every $s,t \in (0,T)$, we have

$$\int |\chi(s) - \chi(t)| \, dx \lesssim E_0 |s-t|^{1/2}. \tag{23}$$

Proof. First note that (23) follows directly from (22) since we also have $\chi^h(t) \to \chi(t)$ in L^1 for almost every t. The argument for (22) comes in two steps. Let $s > t$, $\tau := s - t$ and $t \in [nh, (n+1)h)$.

Step 1: Let τ be a multiple of h. We may assume w.l.o.g. that $\tau = m^2 h$ for some $m \in \mathbb{N}$. As in the proof of Lemma 2.6, we derive

$$\int |\chi^{n+m} - \chi^n| \, dx \lesssim m\sqrt{h} \sum_{k=1}^{m} -E_h(\chi^{n+k} - \chi^{n+k-1}) + \sqrt{h} \max_t E_h(\chi^h(t)).$$

As before, we sum these estimates:

$$\begin{aligned}
\int \left|\chi^{n+m^2} - \chi^n\right| \, dx \leq & \sum_{l=0}^{m-1} \int \left|\chi^{n+m(l+1)} - \chi^{n+ml}\right| \, dx \\
\lesssim & m\sqrt{h} \sum_{n'=n}^{n+m^2} -E_h(\chi^{n'} - \chi^{n'-1}) + m\sqrt{h} \max_t E_h(\chi^h(t)) \\
\lesssim & m\sqrt{h} E_0 = E_0 \sqrt{\tau}.
\end{aligned}$$

Step 2: Let $\tau \geq h$ be arbitrary. Take $m \in \mathbb{N}$ such that $s \in [(m+n)h, (m+n+1)h)$. From Step 2 we obtain the bound in terms of mh instead of τ. If $\tau \geq mh$, we are done. If $h \leq \tau < mh$, then $m \geq 2$ and thus $mh \leq \frac{m}{m-1}\tau \lesssim \tau$. $\square$

For the following estimates, it is useful to define certain measures which are induced by the metric term.

Definition 2.8 (Dissipation measure)**.** For $h > 0$, we define the *approximate dissipation measures* (associated to the approximate solution χ^h) μ_h on $[0,T] \times [0,\Lambda)^d$ by

$$\iint \zeta \, d\mu_h := \sum_{n=1}^{N} \frac{1}{\sqrt{h}} \int \overline{\zeta}_n \left(\left|G_{h/2} * (\chi^n - \chi^{n-1})\right|^2 + \left|G_h * (\chi^n - \chi^{n-1})\right|^2 \right) dx,$$

where $\zeta \in C_0^\infty((0,T)\times[0,\Lambda)^d)$ and $\overline{\zeta}_n$ is the time average of ζ on the interval $[nh,(n+1)h)$. By the monotonicity of $h\mapsto \|G_h * u\|_{L^2}$ and the energy-dissipation estimate (9), we have

$$\mu_h([0,T]\times[0,\Lambda)^d) \lesssim E_0$$

and $\mu_h \rightharpoonup \mu$ after passage to a further subsequence for some finite, non-negative measure μ on $[0,T]\times[0,\Lambda)^d$ with $\mu([0,T]\times[0,\Lambda)^d)\lesssim E_0$. We call μ the *dissipation measure*.

Remark 2.9 (Implications of convergence assumption)**.** The convergence assumption (8) ensures that for any $i\in\{1,\dots,P\}$ and $\zeta\in C^\infty([0,T]\times[0,\Lambda)^d)$,

$$\int_0^T \frac{1}{\sqrt{h}}\int \zeta\left[\left(1-\chi_i^h\right)G_h*\chi_i^h+\chi_i^h\,G_h*\left(1-\chi_i^h\right)\right]dx\,dt \to 2c_0\int_0^T\int \zeta\,|\nabla\chi_i|\,dt, \tag{24}$$

as $h\to 0$.

Proof. Step 1: It is enough to prove that $\chi^h\to\chi$ in $L^1([0,\Lambda)^d,\mathbb{R}^P)$ and $E_h(\chi^h)\to E(\chi)$ imply

$$\frac{1}{\sqrt{h}}\int \zeta\left[\left(1-\chi_i^h\right)G_h*\chi_i^h+\chi_i^h\,G_h*\left(1-\chi_i^h\right)\right]dx \to 2c_0\int\zeta\,|\nabla\chi_i|\,,\quad \text{as } h\to 0. \tag{25}$$

for any $i\in\{1,\dots,P\}$ and $\zeta\in C^\infty([0,\Lambda)^d)$. Indeed, this already implies the integrated version (24). For the argument, we first prove that indeed

$$E_h(\chi^h)\to E(\chi)\quad \text{for a.e. } t. \tag{26}$$

Writing $\left|E_h(\chi^h)-E(\chi)\right| = 2\big(E(\chi)-E_h(\chi^h)\big)_+ + E_h(\chi^h)-E(\chi)$, and using the lim inf-inequality of the Γ-convergence of E_h to E, we have

$$\lim_{h\to0}\Big(E(\chi)-E_h(\chi^h)\Big)_+ = 0\quad\text{for a.e. } t.$$

By Lebesgue's dominated convergence and the convergence assumption (8), we have

$$\lim_{h\to0}\int_0^T\Big|E_h(\chi^h)-E(\chi)\Big|\,dt = 0$$

and thus (26) after passage to a subsequence. Therefore, we can apply (25) for a.e. t and the claim follows by Lebesgue's dominated convergence theorem.

Step 2: Given $\chi^h\to\chi$ in $L^1([0,\Lambda)^d,\mathbb{R}^P)$ and $E_h(\chi^h)\to E(\chi)$, for any $i\in\{1,\dots,P\}$, we have

$$\frac{1}{\sqrt{h}}\int\left(1-\chi_i^h\right)G_h*\chi_i^h\,dx\to c_0\int|\nabla\chi_i|\,,\quad\text{as } h\to0. \tag{27}$$

Argument: We define the two-phase analogon of the approximate energies

$$\widetilde{E}_h(\tilde\chi) := \frac{1}{\sqrt{h}}\int(1-\tilde\chi)\,G_h*\tilde\chi\,dx,$$

for measurable functions $\tilde{\chi}\colon [0,\Lambda)^d \to \{0,1\}$. These functionals Γ-converge to

$$c_0 \int |\nabla \tilde{\chi}| \,.$$

Thus, using the convergence $E_h(\chi^h) \to E(\chi)$ and the $\liminf$-inequality of the Γ-convergence of the functionals $\widetilde{E}_h$, we obtain a $\limsup$-inequality:

$$c_0 \int |\nabla \chi_i| = E(\chi) - c_0 \sum_{j\neq i} \int |\nabla \chi_j| \geq \lim_{h\to 0} E_h(\chi^h) - \sum_{j\neq i} \liminf_{h\to 0} \widetilde{E}_h(\chi_j^h) = \limsup_{h\to 0} \widetilde{E}_h(\chi_i^h).$$

Step 3: Given $\chi^h \to \chi$ in $L^1([0,\Lambda)^d, \mathbb{R}^P)$ and $E_h(\chi^h) \to E(\chi)$, for any $i \in \{1,\ldots,P\}$ and any $\zeta \in C^\infty([0,\Lambda)^d)$, we have (25).
Argument: W.l.o.g. we may assume that $0 \leq \zeta \leq 1$ for this argument. Indeed, by linearity this is sufficient since every function $\zeta \in C^\infty([0,\Lambda)^d)$ can be written as

$$\zeta = \alpha\tilde{\zeta} + \beta, \quad \alpha,\, \beta \in \mathbb{R},\, \tilde{\zeta} \in C^\infty([0,\Lambda)^d),\, 0 \leq \tilde{\zeta} \leq 1.$$

The approximate monotonicity (Lemma 3.4) yields

$$\begin{aligned}
&\frac{1}{\sqrt{h}} \int \zeta \left[\left(1-\chi_i^h\right) G_h * \chi_i^h + \chi_i^h\, G_h * \left(1-\chi_i^h\right) \right] dx \\
&\geq \left(\frac{\sqrt{h_0}}{\sqrt{h}+\sqrt{h_0}} \right)^{d+1} \frac{1}{\sqrt{h_0}} \int \zeta \left[\left(1-\chi_i^h\right) G_{h_0} * \chi_i^h + \chi_i^h\, G_{h_0} * \left(1-\chi_i^h\right) \right] dx \\
&- C \|\nabla \zeta\|_\infty \frac{\sqrt{h_0}}{\sqrt{h}} \int \left(1-\chi_i^h\right) G_h * \chi_i^h dx
\end{aligned}$$

for any $0 < h \leq h_0$. Since $\chi^h \to \chi$ in L^2 and by Step 2, we can pass to the limit $h \to 0$ on the right-hand side:

$$\begin{aligned}
&\liminf_{h\to 0} \frac{1}{\sqrt{h}} \int \zeta \left[\left(1-\chi_i^h\right) G_h * \chi_i^h + \chi_i^h\, G_h * \left(1-\chi_i^h\right) \right] dx \\
&\geq \frac{1}{\sqrt{h_0}} \int \zeta \left[(1-\chi_i)\, G_{h_0} * \chi_i + \chi_i\, G_{h_0} * (1-\chi_i) \right] dx \\
&- C \|\nabla \zeta\|_\infty \sqrt{h_0} \int |\nabla \chi_i|
\end{aligned}$$

for all $h_0 > 0$. By Lemma 3.5, we can pass to the limit $h_0 \to 0$ and obtain

$$\liminf_{h\to 0} \frac{1}{\sqrt{h}} \int \zeta \left[\left(1-\chi_i^h\right) G_h * \chi_i^h + \chi_i^h\, G_h * \left(1-\chi_i^h\right) \right] dx \geq 2c_0 \int \zeta\, |\nabla \chi_i| \,.$$

Using the same argument for $1-\zeta$ instead of ζ, linearity of both sides and the convergence in Step 2, we obtain the inverse bound:

$$\limsup_{h\to 0} \frac{1}{\sqrt{h}} \int \zeta \left[\left(1-\chi_i^h\right) G_h * \chi_i^h + \chi_i^h\, G_h * \left(1-\chi_i^h\right) \right] dx \leq 2c_0 \int \zeta\, |\nabla \chi_i| \,.$$

This concludes the proof. $\square$

Lemma 2.10. *If the convergence assumption (8) holds, the limit $\chi = \lim_{h\to 0}\chi^h$ has the following properties.*

(i) $\partial_t\chi$ is a Radon measure with

$$\iint |\partial_t \chi_i| \lesssim (1+T)E_0$$

for each $i \in \{1, \cdots, P\}$.

(ii) For each $i \in \{1, \cdots, P\}$, $\partial_t\chi_i$ is absolutely continuous w.r.t. $|\nabla\chi_i|\,dt$. In particular, there exists a density $V_i \in L^1(|\nabla\chi_i|\,dt)$, such that

$$-\int_0^T\int \partial_t\zeta\,\chi_i\,dx\,dt = \int_0^T\int \zeta\,V_i\,|\nabla\chi_i|\,dt$$

for all $\zeta \in C_0^\infty((0,T)\times[0,\Lambda)^d)$.

(iii) We have a strong L^2-bound:

$$\int_0^T\int V_i^2\,|\nabla\chi_i|\,dt \lesssim E_0.$$

Proof. We make use of the mesoscopic time scale τ, see Remark 1.4 for the notation.

Argument for (i): Let $\zeta \in C_0^\infty((0,T)\times[0,\Lambda)^d)$. We have to show that

$$-\int_0^T\int \partial_t\zeta\,\chi_i\,dx\,dt \lesssim (1+T)E_0\,\|\zeta\|_\infty\,.$$

In this part, we will choose $\alpha = 1$. Using the notation $\partial^\tau\zeta$ for the discrete time derivative $\frac{1}{\tau}\left(\zeta(t+\tau)-\zeta(t)\right)$, by the smoothness of ζ,

$$\partial^\tau\zeta \to \partial_t\zeta \quad \text{in } L^\infty((0,T)\times[0,\Lambda)^d) \quad \text{as } h\to 0.$$

Since $\chi^h \to \chi$ in $L^1((0,T)\times[0,\Lambda)^d)$, the product converges:

$$\int_0^T\int \partial_t\zeta\,\chi_i\,dx\,dt = \lim_{h\to 0}\int_0^T\int \partial^\tau\zeta\,\chi_i^h\,dx\,dt.$$

Since $\operatorname{supp}\zeta$ is compact, we have for sufficiently small h

$$-\int_0^T\int \partial^\tau\zeta\,\chi_i^h\,dx\,dt = \int_0^T\int \zeta\,\partial^{-\tau}\chi_i^h\,dx\,dt \le \|\zeta\|_{L^\infty}\int_\tau^T\int \left|\partial^{-\tau}\chi_i^h\right|dx\,dt \lesssim (1+T)\,E_0\,\|\zeta\|_{L^\infty}$$

by Lemma 2.6.

Argument for (ii): First, we prove

$$-\int_0^T\int \partial_t\zeta\,\chi_i\,dx\,dt \lesssim \frac{1}{\alpha}\int_0^T\int |\zeta|\,|\nabla\chi_i|\,dt + \alpha\iint |\zeta|\,d\mu \tag{28}$$

for any $\alpha > 0$ and any $\zeta \in C_0^\infty((0,T)\times[0,\Lambda)^d)$. Let $\zeta \in C_0^\infty((0,T)\times[0,\Lambda)^d)$. By linearity, we may assume that $\zeta \geq 0$ if we prove the inequality with absolute values on the left-hand side. We use the identity from above

$$-\int_0^T\int \partial_t\zeta\,\chi_i\,dx\,dt = \lim_{h\to 0}\int_0^T\int \zeta\,\partial^{-\tau}\chi_i^h dx\,dt.$$

Setting

$$\zeta^n := \frac{1}{h}\int_{nh}^{(n+1)h} \zeta(t)\,dt,$$

we have

$$\left|\int_0^T\int \zeta\,\partial^{-\tau}\chi_i^h dx\,dt\right| \leq \frac{1}{K}\sum_{k=1}^K\sum_{l=1}^L\int \zeta^{Kl+k}\left|\chi_i^{Kl+k} - \chi_i^{K(l-1)+k}\right| dx.$$

Now fix $k \in \{1,\dots,K\}$. For simplicity, we will ignore k at first. Since for any $\chi \in \{0,1\}$,

$$\frac{1}{\sqrt{h}}\int \zeta\,|G_h*\chi - \chi|\,dx = \frac{1}{\sqrt{h}}\int \zeta\,[(1-\chi)\,G_h*\chi + \chi\,G_h*(1-\chi)]\,dx =: \widetilde{E}_h(\chi,\zeta)$$

and as in the proof of Lemma 2.6,

$$\left|\int \left(\zeta^{K(l+1)} - \zeta^{Kl}\right)(1-\chi)\,G_h*\chi dx\right| \leq \|\partial_t\zeta\|_\infty\alpha\sqrt{h}\int (1-\chi)\,G_h*\chi\,dx,$$

we obtain

$$\begin{aligned}\sum_{l=1}^L\int \zeta^{Kl}\left|\chi_i^{Kl} - \chi_i^{K(l-1)}\right| dx \lesssim & \frac{\tau}{\alpha}\sum_{l=1}^L \widetilde{E}_h(\chi_i^{Kl},\zeta^{Kl}) + \sqrt{h}\,\|\partial_t\zeta\|_\infty\,\tau\sum_{l=1}^L E_h(\chi^{Kl})\\ &+\alpha\iint \zeta\,d\mu_h + \|\nabla\zeta\|_\infty\sqrt{h}\sum_{l=1}^L\int\left|\chi^{Kl} - \chi^{K(l-1)}\right| dx,\end{aligned}$$

where μ_h denote the approximate dissipation measures in Definition 2.8. The last term comes from a manipulation of the metric term to obtain the dissipation measures: For any $\zeta \in$

$C^\infty([0,\Lambda)^d)$ and any $\chi,\, \tilde\chi \in \{0,1\}$ we have

$$\begin{aligned}
&\left| \int \zeta \left[G_{h/2} * (\chi - \tilde\chi)\right]^2 dx - \int \zeta\, (\chi - \tilde\chi)\, G_h * (\chi - \tilde\chi)\, dx \right| \\
&= \left| \int \left(\zeta \left[G_{h/2} * (\chi - \tilde\chi)\right] - G_{h/2} * \left[\zeta\, (\chi - \tilde\chi)\right]\right) G_{h/2} * (\chi - \tilde\chi)\, dx \right| \\
&\leq \int G_{h/2}(z) \int \left|\zeta(x+z) - \zeta(x)\right| \left|\chi - \tilde\chi\right| (x+z) \left|G_{h/2} * (\chi - \tilde\chi)\right| (x)\, dx\, dz \\
&\lesssim \left\|\nabla\zeta\right\|_\infty \sqrt{h} \int \frac{|z|}{\sqrt{h}} G_{h/2}(z)\, dz \int \left|\chi - \tilde\chi\right| dx \\
&\lesssim \left\|\nabla\zeta\right\|_\infty \sqrt{h} \int \left|\chi - \tilde\chi\right| dx.
\end{aligned}$$

Taking the mean over the k's and using the energy-dissipation estimate (9) and Lemma 2.6, we obtain

$$\left| \int_0^T \int \zeta\, \partial^{-\tau} \chi_i^h dx\, dt \right| \lesssim \frac{1}{\alpha} \int_0^T \widetilde{E}_h(\chi_i^h, \zeta)\, dt + \alpha \iint \zeta\, d\mu_h + \sqrt{h}\, \left\|\partial_t \zeta\right\|_\infty T E_0 + \sqrt{h}\, \left\|\nabla\zeta\right\|_\infty \frac{1}{\alpha}(1+T) E_0.$$

Passing to the limit $h \to 0$, (24), which is guaranteed by the convergence assumption (8), implies (28).
Now let $U \subset (0,T) \times [0,\Lambda)^d$ be open such that

$$\iint_U \left|\nabla\chi_i\right| dt = 0.$$

If we take $\zeta \in C_0^\infty(U)$, the first term on the right-hand side of (28) vanishes. Thus, we have

$$-\int_0^T \int \partial_t \zeta\, \chi_i\, dx\, dt \lesssim \alpha \iint |\zeta|\, d\mu.$$

Since the left-hand side does not depend on α, we have

$$-\int_0^T \int \partial_t \zeta\, \chi_i\, dx\, dt \leq 0.$$

Taking the supremum over all $\zeta \in C_0^\infty(U)$, we have

$$\iint_U \left|\partial_t \chi_i\right| = 0.$$

Thus, $\partial_t \chi_i$ is absolutely continuous w.r.t. $\left|\nabla\chi_i\right| dt$ and the Radon-Nikodym theorem completes the proof.

Argument for (iii): We refine the estimate in the argument for (ii). Instead of estimating the right-hand side of (28) and optimizing afterwards, we localize. Starting from (28), we notice that we can localize with the test function ζ. Thus, we can post-process the estimate and obtain

$$\left| \int_0^T \int V_i\, \zeta\, |\nabla \chi_i|\, dt \right| \le C \int_0^T \int \frac{1}{|\alpha|}\, |\zeta|\, |\nabla \chi_i|\, dt + C \iint |\alpha|\, |\zeta|\, d\mu$$

for any integrable $\zeta : (0,T) \times [0,\Lambda)^d \to \mathbb{R}$, any measurable $\alpha\colon (0,T) \times [0,\Lambda)^d \to \mathbb{R}\backslash\{0\}$ and some constant C which depends only on the dimension. Now choose

$$\zeta = V_i \quad \text{and} \quad \alpha = \frac{2C}{V_i},$$

where we set $\alpha := 1$ if $V_i = 0$, in which case all other integrands vanish. Then, the first term on the right-hand side can be absorbed in the left-hand side and we obtain

$$\int_0^T \int V_i^2\, |\nabla \chi_i|\, dt \lesssim \mu([0,T] \times [0,\Lambda)^d) \lesssim E_0.$$

□

3 Energy Functional and Curvature

It is a classical result by Reshetnyak [32] that the convergence $\chi^h \to \chi$ in L^1 and

$$\int |\nabla \chi^h| \to \int |\nabla \chi| =: E(\chi)$$

imply convergence of the first variation

$$\delta E(\chi, \xi) = \int \left(\nabla \cdot \xi - \nu \cdot \nabla \xi\, \nu \right) |\nabla \chi| \,.$$

A result by Luckhaus and Modica [21] shows that this may extend to a Γ-convergence situation, namely in case of the Ginzburg-Landau functional

$$E_h(u) := \int h\, |\nabla u|^2 + \frac{1}{h} \left(1 - u^2\right)^2 dx.$$

We show that this also extends to our Γ-converging functionals E_h. All proofs in this section are adaptations of the Γ-convergence proof in [13] with some generalizations and minor changes. Let us first address why the first variation of the approximate energies is of interest in view of our minimizing movement scheme. We recall (5): the approximate solution χ^n at time nh minimizes $E_h(\chi) - E_h(\chi - \chi^{n-1})$ among all χ. The natural variations of such a minimization problem are *inner variations*, i.e. variations of the independent variable. Given a vector field $\xi \in C^\infty([0,\Lambda)^d, \mathbb{R}^d)$ and an admissible χ, we define the deformation χ_s of χ along ξ by the distributional equation

$$\frac{\partial}{\partial s}\chi_{i,s} + \nabla\chi_{i,s}\cdot\xi = 0, \quad \chi_{i,s}\big|_{s=0} = \chi_i,$$

which means that the phases are deformed by the flow generated through ξ. The inner variation δE_h of the energy E_h at χ along the vector field ξ is then given by

$$\delta E_h(\chi,\xi) := \frac{d}{ds}E_h(\chi_s)\big|_{s=0} = \frac{1}{\sqrt{h}}\sum_{i,j}\sigma_{ij}\int (-\nabla\chi_i\cdot\xi)\; G_h * \chi_j + \chi_i\, G_h * (-\nabla\chi_j\cdot\xi)\, dx. \tag{29}$$

For a given admissible $\overline{\chi}$ we define the inner variation of the metric term $-E_h(\chi-\overline{\chi})$ as

$$-\delta E_h(\,\cdot\, - \overline{\chi})(\chi,\xi) := \frac{d}{ds} - E_h(\chi_s - \overline{\chi})\big|_{s=0}. \tag{30}$$

The (chosen and not necessarily unique) minimizer χ^n in Algorithm 1 therefore satisfies the Euler-Lagrange equation

$$\delta E_h(\chi^n,\xi) - \delta E_h(\,\cdot\, - \chi^{n-1})(\chi^n,\xi) = 0 \tag{31}$$

for any vector field $\xi\in C^\infty([0,\Lambda)^d,\mathbb{R}^d)$.
The goal of this section is to prove the following statement about the convergence of the first term in the Euler-Lagrange equation. Although the arguments in this section are given for general surface tensions σ, we state the main result in the easier case of equal surface tensions.

Proposition 3.1. *Let $\{\chi^h\}_h$ be a sequence of admissible functions such that*

$$\chi^h \longrightarrow \chi \quad \textit{a.e. in ,} \tag{32}$$

$$\chi^h \longrightarrow \chi \quad \textit{in } L^1, \tag{33}$$

$$\int_0^T E_h(\chi^h)\,dt \longrightarrow \int_0^T E(\chi)\,dt.. \tag{34}$$

Then, for any $\xi\in C_0^\infty((0,T)\times[0,\Lambda)^d,\mathbb{R}^d)$, we have

$$\lim_{h\to 0}\int_0^T \delta E_h(\chi^h,\xi)\,dt = c_0\sum_{i=1}^P\int_0^T\int (\nabla\cdot\xi - \nu_i\cdot\nabla\xi\,\nu_i)\,|\nabla\chi_i|\,dt.$$

Proof of Proposition 3.1. The proposition is an immediate consequence of Proposition 3.2. Indeed, according to Step 1 in the proof of Remark 2.9 we have $E_h(\chi^h)\to E(\chi)$ for a.e. t. Thus all conditions of Proposition 3.2 are fulfilled. Proposition 3.1 follows then from Lebesgue's dominated convergence theorem. $\square$

Proposition 3.2. *Let $\{\chi^h\}_h$ be a sequence of admissible functions and let $\chi\in BV$ be admissible* *such that*

$$\chi^h \longrightarrow \chi \quad a.e., \tag{35}$$

$$\chi^h \longrightarrow \chi \quad in\ L^1, \tag{36}$$

$$E_h(\chi^h) \longrightarrow E(\chi). \tag{37}$$

Then, for any $\xi \in C_0^\infty([0,\Lambda)^d, \mathbb{R}^d)$, *we have*

$$\lim_{h\to 0} \delta E_h(\chi^h, \xi) = c_0 \sum_{i,j} \sigma_{ij} \frac{1}{2} \int (\nabla\cdot\xi - \nu_i \cdot \nabla\xi\,\nu_i)\left(|\nabla\chi_i| + |\nabla\chi_j| - |\nabla(\chi_i+\chi_j)|\right).$$

The following first lemma brings the first variation δE_h of E_h into a more convenient form, up to an error vanishing as $h \to 0$ because of the smoothness of ξ.

Lemma 3.3. *Let* χ *be admissible and* $\xi \in C^\infty([0,\Lambda)^d, \mathbb{R}^d)$. *Setting* $K(z) := z \otimes z\, G(z)$, *we have*

$$\begin{aligned}\delta E_h(\chi,\xi) = &-\frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \chi_i\, \nabla\xi : K_h * \chi_j\, dx \\ &+ \frac{2}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \chi_i\,(\nabla\cdot\xi)\, G_h * \chi_j\, dx + O\left(\|\nabla^2\xi\|_\infty E_h(\chi)\sqrt{h}\right).\end{aligned} \tag{38}$$

Proof. Recall the definition of δE_h in (29). Since $-\nabla\chi\cdot\xi = -\nabla\cdot(\chi\xi) + \chi\nabla\cdot\xi$ for functions $\chi\colon [0,\Lambda)^d \to \mathbb{R}$, we can rewrite the integrals:

$$\begin{aligned}&\int -\nabla\chi_i\cdot\xi\, G_h*\chi_j + \chi_i\, G_h*(-\nabla\chi_j\cdot\xi)\, dx \\ =&\int -\nabla\cdot(\chi_i\xi)\, G_h*\chi_j + \chi_i\,\nabla\cdot\xi\, G_h*\chi_j - \chi_i\, G_h*(\nabla\cdot(\chi_j\xi)) + \chi_i\, G_h*(\chi_j\,\nabla\cdot\xi)\, dx \\ =&\int \chi_i\xi\cdot\nabla G_h*\chi_j - \chi_i\,\nabla G_h*(\chi_j\xi) + \chi_i\,\nabla\cdot\xi\, G_h*\chi_j + \chi_i\, G_h*(\chi_j\,\nabla\cdot\xi)\, dx.\end{aligned}$$

Hence, using $\nabla G_h(z) = \frac{1}{\sqrt{h}}(\nabla G)_h(z/\sqrt{h})$, we find

$$\begin{aligned}\delta E_h(\chi,\xi) =& \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int\int \chi_i(x)\left(\xi(x+\sqrt{h}z) - \xi(x)\right)\cdot\nabla G(z)\frac{1}{\sqrt{h}}\chi_j(x+\sqrt{h}z) \\ &+ \chi_i(x)\left((\nabla\cdot\xi)(x) + (\nabla\cdot\xi)(x+\sqrt{h}z)\right) G(z)\chi_j(x+\sqrt{h}z)\,dx\,dz.\end{aligned}$$

A Taylor expansion of ξ around x yields

$$\begin{aligned}\delta E_h(\chi,\xi) =& \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int\int \chi_i(x)\nabla G(z)\cdot\nabla\xi(x)z\,\chi_j(x+\sqrt{h}z)\,dx\,dz \\ &+ \frac{2}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int\int \chi_i(x)(\nabla\cdot\xi)(x)G(z)\chi_j(x+\sqrt{h}z)\,dx\,dz + O\left(\|\nabla^2\xi\|_\infty E_h(\chi)\sqrt{h}\right).\end{aligned}$$

Indeed, the error term is estimated by $\|\nabla^2\xi\|_\infty$ times

$$\begin{aligned}&\frac{1}{\sqrt{h}}\sum_{i,j}\sigma_{ij}\int\int\chi_i(x)\left(|\sqrt{h}z|^2\frac{1}{\sqrt{h}}|\nabla G(z)|+|\sqrt{h}z|\,G(z)\right)\chi_j(x+\sqrt{h}z)dx\,dz\\ =&\sum_{i,j}\sigma_{ij}\int\int\chi_i(x)\left(|z|^3+|z|\right)G(z)\chi_j(x+\sqrt{h}z)dx\,dz.\end{aligned}$$

Since $\left(|z|^3+|z|\right)G(z)\lesssim G(z/2)$, this can be estimated by

$$\begin{aligned}\sum_{i,j}\sigma_{ij}\int\int\chi_i(x)G(z/2)\chi_j(x+\sqrt{h}z)dx\,dz\sim&\sum_{i,j}\sigma_{ij}\int\int\chi_i(x)G(z)\chi_j(x+2\sqrt{h}z)dx\,dz\\ =&2\sqrt{h}E_{4h}(\chi)\lesssim\sqrt{h}E_h(\chi),\end{aligned}$$

by using the approximate monotonicity (10) of E_h. $\square$

The next lemma shows that the monotonicity (10) of $E_h(\chi)$ in h, which is crucial for the lower semi-continuity part of the Γ-convergence of E_h to E in [13], approximately survives if the energy functional is localized by a smooth $\zeta\geq 0$.

Lemma 3.4 (Perturbed approximate monotonicity)**.** *Let k be a non-negative, radially non-increasing kernel, i.e.*

$$\nabla k(z)\cdot z\leq 0. \tag{39}$$

Let $\zeta\in C^\infty([0,\Lambda)^d)$, $\zeta\geq 0$ and let χ be admissible. Set

$$f_h(\chi,\zeta):=\frac{1}{\sqrt{h}}\sum_{i,j}\sigma_{ij}\int\zeta\chi_i\,k_h*\chi_j\,dx,\quad g_h(\chi):=\frac{1}{\sqrt{h}}\sum_{i,j}\sigma_{ij}\int\chi_i\,\tilde{k}_h*\chi_j\,dx,$$

where $\tilde{k}(z):=|z|k(z)$. Then we have

$$f_h(\chi,\zeta)\geq\left(\frac{\sqrt{h_0}}{\sqrt{h}+\sqrt{h_0}}\right)^{d+1}f_{h_0}(\chi,\zeta)-\|\nabla\zeta\|_\infty g_h(\chi)\sqrt{h_0} \tag{40}$$

for all $0<h\leq h_0$.

The proof follows the idea of the proof of Lemma 3 in Section 7.1 of [13].

Proof. Step 1: Reduction of the statement. Write f_h instead of $f_h(\chi,\zeta)$ and g_h instead of $g_h(\chi)$. The statement can be reduced to the following two statements. On a discrete level, the real function f_h is approximately decreasing in h:

$$f_{N^2h}\leq f_h+\|\nabla\zeta\|_\infty g_h(N-1)\sqrt{h}\quad\text{for all } N\in\mathbb{N}, \tag{41}$$

and it is logarithmic increasing:

$$\sqrt{h_1}^{d+1}f_{h_1}\leq\sqrt{h_2}^{d+1}f_{h_2}\quad\text{for all } 0<h_1\leq h_2. \tag{42}$$

Indeed: Fix $0 < h < h_0$ and let $N \in \mathbb{N}$ be such that

$$(N-1)\sqrt{h} < \sqrt{h_0} \leq N\sqrt{h}.$$

In particular, we have $\sqrt{h} + \sqrt{h_0} \geq N\sqrt{h}$, thus

$$\frac{\sqrt{h_0}}{N\sqrt{h}} \geq \frac{\sqrt{h_0}}{\sqrt{h} + \sqrt{h_0}}. \tag{43}$$

Then,

$$\begin{aligned} f_h &\overset{(41)}{\geq} f_{N^2h} - \|\nabla\zeta\|_\infty g_h(\chi)(N-1)\sqrt{h} \\ &\overset{(42)}{\geq} (N^2h)^{-\frac{d+1}{2}} h_0^{\frac{d+1}{2}} f_{h_0} - \|\nabla\zeta\|_\infty g_h(\chi)\sqrt{h_0} \\ &\overset{(43)}{\geq} \left(\frac{\sqrt{h_0}}{\sqrt{h} + \sqrt{h_0}}\right)^{d+1} f_{h_0} - \|\nabla\zeta\|_\infty g_h(\chi)\sqrt{h_0}. \end{aligned}$$

Step 2: Proof of (42). The logarithmic monotonicity, the infinitesimal version of which is

$$\frac{d}{d\sqrt{h}}\left(\sqrt{h}^{d+1} f_h\right) \geq 0,$$

can be reformulated as

$$\frac{d}{d\sqrt{h}} f_h \geq -\frac{d+1}{\sqrt{h}} f_h. \tag{44}$$

In order to see (44), we write

$$f_h = \frac{1}{\sqrt{h}} \int k_h(z) F_\zeta(z)\, dz = \int \frac{1}{\sqrt{h}^{d+1}} k(\frac{z}{\sqrt{h}}) F_\zeta(z)\, dz,$$

where

$$F_\zeta(z) := \sum_{i,j} \sigma_{ij} \int \zeta(x)\chi_i(x)\chi_j(x+z)\, dx. \tag{45}$$

Straightforward differentiation now yields (44):

$$\begin{aligned} \frac{d}{d\sqrt{h}} f_h &= -\frac{1}{\sqrt{h}^{d+2}} \int \left((d+1)k(\frac{z}{\sqrt{h}}) + (\nabla k)(\frac{z}{\sqrt{h}}) \cdot \frac{z}{\sqrt{h}}\right) F_\zeta(z)\, dz \\ &\overset{(39)}{\geq} -\frac{1}{\sqrt{h}^{d+2}} \int (d+1)k(\frac{z}{\sqrt{h}}) F_\zeta(z)\, dz \\ &= -\frac{d+1}{\sqrt{h}} f_h. \end{aligned}$$

Step 3: Proof of (41). We start by claiming the following perturbed triangle inequality for F_ζ

introduced in (45):

$$F_\zeta(z+w) \le F_\zeta(z) + F_\zeta(w) + \|\nabla\zeta\|_\infty |z| F_1(w) \quad \text{for all } z,\, w \in \mathbb{R}^d. \tag{46}$$

Argument for (46): Using the admissibility of χ in the form of $\sum_k \chi_k = 1$, we obtain the following identity for any pair $1 \le i, j \le P$ of phases and any points $x, x', x'' \in [0,\Lambda)^d$:

$$\begin{aligned}
&\chi_i(x)\chi_j(x'') - \chi_i(x)\chi_j(x') - \chi_i(x')\chi_j(x'')\\
=&\chi_i(x)\sum_k \chi_k(x')\chi_j(x'') - \chi_i(x)\chi_j(x')\sum_k \chi_k(x'') - \sum_k \chi_k(x)\chi_i(x')\chi_j(x'')\\
=&\sum_k \left[\chi_i(x)\chi_k(x')\chi_j(x'') - \chi_i(x)\chi_j(x')\chi_k(x'') - \chi_k(x)\chi_i(x')\chi_j(x'')\right].
\end{aligned}$$

Note that the contribution of $k \in \{i,j\}$ to the sum has a sign:

$$\begin{aligned}
&\sum_{k\in\{i,j\}} \left[\chi_i(x)\chi_k(x')\chi_j(x'') - \chi_i(x)\chi_j(x')\chi_k(x'') - \chi_k(x)\chi_i(x')\chi_j(x'')\right]\\
=&\chi_i(x)\chi_i(x')\chi_j(x'') - \chi_i(x)\chi_j(x')\chi_i(x'') - \chi_i(x)\chi_i(x')\chi_j(x'')\\
&+ \chi_i(x)\chi_j(x')\chi_j(x'') - \chi_i(x)\chi_j(x')\chi_j(x'') - \chi_j(x)\chi_i(x')\chi_j(x'')\\
=&-\left[\chi_i(x)\chi_j(x')\chi_i(x'') + \chi_j(x)\chi_i(x')\chi_j(x'')\right] \le 0.
\end{aligned}$$

We now multiply with $\zeta \ge 0$ and obtain

$$\begin{aligned}
&\zeta(x)\chi_i(x)\chi_j(x'') - \zeta(x)\chi_i(x)\chi_j(x') - \zeta(x')\chi_i(x')\chi_j(x'')\\
\le&\zeta(x)\sum_{k\notin\{i,j\}} \left\{\chi_i(x)\chi_k(x')\chi_j(x'') - \chi_i(x)\chi_j(x')\chi_k(x'') - \chi_k(x)\chi_i(x')\chi_j(x'')\right\}\\
&+ \left(\zeta(x) - \zeta(x')\right)\chi_i(x')\chi_j(x'').
\end{aligned}$$

Since ζ is smooth, we have

$$\left(\zeta(x) - \zeta(x')\right)\chi_i(x')\chi_j(x'') \le \|\nabla\zeta\|_\infty |x - x'|\, \chi_i(x')\chi_j(x'').$$

We now fix $z,\, w \in \mathbb{R}^d$ and use the above inequality for $x' = x + z$, $x'' = x + z + w$ so that after multiplication with σ_{ij}, summation over $1 \le i, j \le P$ and integration over x, we obtain $F_\zeta(z+w) - F_\zeta(z) - F_\zeta(w)$ on the left-hand side. Indeed, using the translation invariance for the term appearing in $F_\zeta(w)$, we have

$$
\begin{aligned}
&F_\zeta(z+w)-F_\zeta(z)-F_\zeta(w)\\
&=\int\zeta(x)\sum_{i\neq j}\sigma_{ij}\left[\chi_i(x)\chi_j(x+z+w)-\chi_i(x)\chi_j(x+z)-\chi_i(x+z)\chi_j(x+z+w)\right]dx\\
\leq&\int\zeta(x)\sum_{i\neq j,k\neq i,j}\sigma_{ij}\big[\chi_i(x)\chi_k(x+z)\chi_j(x+z+w)-\chi_i(x)\chi_j(x+z)\chi_k(x+z+w)\\
&\qquad\qquad-\chi_k(x)\chi_i(x+z)\chi_j(x+z+w)\big]dx\\
&+\|\nabla\zeta\|_\infty|z|\sum_{i,j}\sigma_{ij}\int\chi_i(x)\chi_j(x+w)\,dx.
\end{aligned}
$$

Using the triangle inequality for the surface tensions, we see that the first right-hand side integral is non-positive:

$$
\begin{aligned}
&\sum_{i\neq j,k\neq i,j}\sigma_{ij}\left(\chi_i(x)\chi_k(x')\chi_j(x'')-\chi_i(x)\chi_j(x')\chi_k(x'')-\chi_k(x)\chi_i(x')\chi_j(x'')\right)\\
\leq&\sum_{i\neq j,k\neq i,j}\sigma_{ik}\chi_i(x)\chi_k(x')\chi_j(x'')+\sum_{i\neq j,k\neq i,j}\sigma_{kj}\chi_i(x)\chi_k(x')\chi_j(x'')\\
&-\sum_{i\neq j,k\neq i,j}\sigma_{ij}\chi_i(x)\chi_j(x')\chi_k(x'')-\sum_{i\neq j,k\neq i,j}\sigma_{ij}\chi_k(x)\chi_i(x')\chi_j(x'')=0.
\end{aligned}
$$

Indeed, the first and the third term, and the second and the last term cancel since the domain of indices in the sums is symmetric.
Applying (46) iteratively on z and $w=(N-1)z$, $(N-2)z,\ldots,$ z, one obtains

$$
F_\zeta(Nz)\leq N\,F_\zeta(z)+\|\nabla\zeta\|_\infty|z|\sum_{n=1}^{N-1}F_1(nz)\quad\text{for all } z\in\mathbb{R}^d. \tag{47}
$$

Now let $h>0$ and $h_0:=N^2h$. Then

$$
\begin{aligned}
\sqrt{h_0}f_{h_0}=&\int k_{h_0}(\hat z)F_\zeta(\hat z)\,d\hat z=\int k_h(z)F_\zeta(Nz)\,dz\\
\leq&N\int k_h(z)F_\zeta(z)\,dz+\|\nabla\zeta\|_\infty\sum_{n=1}^{N-1}\int|z|k_h(z)F_1(nz)\,dz.
\end{aligned}
$$

To handle the last term, use (47) for n instead of N and $\zeta\equiv1$. Then the correction term vanishes since $\|\nabla\zeta\|_\infty=0$ and

$$
\begin{aligned}
\sum_{n=1}^{N-1}\int|z|k_h(z)F_1(nz)\,dz\leq&\left(\sum_{n=1}^{N-1}n\right)\int|z|k_h(z)F_1(z)\,dz\\
=&h\frac{1}{2}(N-1)(N-2)\frac{1}{\sqrt h}\int\frac{|z|}{\sqrt h}k_h(z)F_1(z)\,dz\\
\leq&hN(N-1)g_h.
\end{aligned}
$$

Substituting this in the first estimate and dividing by $\sqrt{h_0} = N\sqrt{h}$, one obtains

$$f_{N^2h} \leq f_h + \|\nabla\zeta\|_\infty (N-1)\sqrt{h} g_h,$$

i.e. (41), which concludes the proof. $\square$

The following lemma yields in particular the construction part in the Γ-convergence result of E_h to E. Again, we need it in a localized form; the proof closely follows the proof of Lemma 4 in Section 7.2 of [13].

Lemma 3.5 (Consistency). *Let $\chi \in BV([0,\Lambda)^d, \{0,1\}^P)$ be admissible and $\zeta \in C^\infty([0,\Lambda)^d)$. Then*

$$\lim_{h\to 0} \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \zeta\, \chi_i\, G_h * \chi_j\, dx = c_0 \sum_{i,j} \sigma_{ij} \frac{1}{2} \int \zeta \left(|\nabla\chi_i| + |\nabla\chi_j| - |\nabla(\chi_i+\chi_j)|\right)$$

and for $k(z) := z_1^2 G(z)$ we have

$$\lim_{h\to 0} \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \zeta\, \chi_i\, k_h * \chi_j\, dx = c_0 \sum_{i,j} \sigma_{ij} \frac{1}{2} \int \zeta \left(1+\nu_1^2\right) \left(|\nabla\chi_i| + |\nabla\chi_j| - |\nabla(\chi_i+\chi_j)|\right).$$

Proof. We will only give the argument for k since argument for G is easier.
Note that it is enough to show that for $\zeta \in C^\infty([0,\Lambda)^d)$ and $\chi,\, \tilde\chi \in BV([0,\Lambda)^d, \{0,1\})$ such that

$$\chi\,\tilde\chi = 0 \quad \text{a.e.} \tag{48}$$

we have

$$\lim_{h\to 0} \frac{1}{\sqrt{h}} \int \zeta\tilde\chi\, k_h * \chi\, dx = c_0 \frac{1}{2} \int \zeta \left(1+\nu_1^2\right) \left(|\nabla\chi| + |\nabla\tilde\chi| - |\nabla(\chi+\tilde\chi)|\right). \tag{49}$$

We will prove this in five steps. Before starting, we introduce polar coordinates $z = r\xi$ on the left-hand side:

$$\begin{aligned} \frac{1}{\sqrt{h}} \int \zeta\tilde\chi\, k_h * \chi\, dx &= \frac{1}{\sqrt{h}} \int k(z) \int \zeta(x)\tilde\chi(x)\chi(x+\sqrt{h}z)\, dx\, dz \\ &= \int_0^\infty G(r) r^{d+2} \frac{1}{\sqrt{h}r} \int_{S^{d-1}} \xi_1^2 \int \zeta(x)\tilde\chi(x)\chi(x+\sqrt{h}r\xi)\, dx\, d\xi\, dr. \end{aligned} \tag{50}$$

In the first two steps of the proof, we simplify the problem by disintegrating in r (Step 1) and ξ (Step 2). Then we explicitly calculate an integral that arises in the second reduction and which translates the anisotropy of the kernel k into a geometric information about the normal (Step 3). We simplify further by disintegration in the vertical component (Step 4) and conclude by solving the one-dimensional problem (Step 5).

Step 1: Disintegration in r. We claim that it is sufficient to show that

$$\lim_{h\to 0}\frac{1}{\sqrt{h}}\int\limits_{S^{d-1}}\xi_1^2\int\zeta(x)\tilde\chi(x)\chi(x+\sqrt{h}\xi)\,dx\,d\xi$$
$$=\frac{|B^{d-1}|}{d+1}\frac{1}{2}\int\zeta\left(1+\nu_1^2\right)\left(|\nabla\chi|+|\nabla\tilde\chi|-|\nabla(\chi+\tilde\chi)|\right). \tag{51}$$

Argument: Note that since $G(z)=G(|z|)$ and $\frac{d}{dr}G(r)=-rG(r)$ we have, using integration by parts,

$$\int\limits_0^\infty G(r)r^{d+2}\,dr=-\int\limits_0^\infty\frac{d}{dr}(G(r))r^{d+1}\,dr=(d+1)\int\limits_0^\infty G(r)r^d\,dr.$$

Replacing $\sqrt{h}$ by $\sqrt{h}r$ on the left-hand side of (51), integrating w.r.t. the non-negative measure $G(r)r^{d+2}dr$ and using the equality from above shows that (51), in view of (50), formally implies (49). To make this step rigorous, we use Lebesgue's theorem. A dominating function can be obtained as follows:

$$\left|\frac{1}{\sqrt{hr}}\int\limits_{S^{d-1}}\xi_1^2\int\zeta(x)\tilde\chi(x)\chi(x+\sqrt{h}r\xi)\,dx\,d\xi\right|$$
$$\overset{(48)}{=}\left|\frac{1}{\sqrt{hr}}\int\limits_{S^{d-1}}\xi_1^2\int\zeta(x)\tilde\chi(x)\left(\chi(x+\sqrt{h}r\xi)-\chi(x)\right)\,dx\,d\xi\right|$$
$$\le\|\zeta\|_\infty\frac{1}{\sqrt{hr}}\int\limits_{S^{d-1}}\int\left|\chi(x+\sqrt{h}r\xi)-\chi(x)\right|\,dx\,d\xi$$
$$\le\|\zeta\|_\infty\,|S^{d-1}|\int|\nabla\chi|,$$

which is finite and independent of r. Hence, it is integrable w.r.t. the finite measure $G(r)r^{d+2}dr$.

Step 2: Disintegration in ξ. We claim that it is sufficient to show that for each $\xi\in S^{d-1}$,

$$\lim_{h\to 0}\frac{1}{\sqrt{h}}\int\zeta(x)\tilde\chi(x)\left(\chi(x+\sqrt{h}\xi)+\chi(x-\sqrt{h}\xi)\right)\,dx$$
$$=\frac{1}{2}\int\zeta\,|\xi\cdot\nu|\left(|\nabla\chi|+|\nabla\tilde\chi|-|\nabla(\chi+\tilde\chi)|\right). \tag{52}$$

Indeed, if we integrate w.r.t. $\frac{1}{2}\xi_1^2d\xi$ we obtain the left-hand side of (51) from the left-hand side of (52). At least formally, this is obvious because of the symmetry under $\xi\mapsto-\xi$. The dominating function to interchange limit and integration is obtained as in Step 1:

$$\left|\frac{1}{\sqrt{h}}\int\zeta(x)\tilde\chi(x)\left(\chi(x+\sqrt{h}\xi)+\chi(x-\sqrt{h}\xi)\right)\,dx\right|$$
$$\overset{(48)}{\le}\frac{1}{\sqrt{h}}\sup|\zeta|\int\left|\chi(x+\sqrt{h}\xi)-\chi(x)\right|+\left|\chi(x-\sqrt{h}\xi)-\chi(x)\right|\,dx\le 2\sup|\zeta|\int|\nabla\chi|.$$

For the passage from the right-hand side of (52) to the right-hand side of (51) we not that since

$$\int_{S^{d-1}} \xi_1^2 \int \zeta\,|\xi\cdot\nu|\,|\nabla\chi|\frac{1}{2}d\xi = \frac{1}{2}\int\int_{S^{d-1}} \xi_1^2\,|\xi\cdot\nu|\,d\xi\,\zeta\,|\nabla\chi|$$

and $|\nu| = 1$ $|\nabla\chi|$- a.e. it is enough to prove

$$\frac{1}{2}\int_{S^{d-1}} \xi_1^2|\xi\cdot\nu|\,d\xi = \frac{|B^{d-1}|}{d+1}\left(1+\nu_1^2\right) \quad \text{for all } \nu\in S^{d-1} \tag{53}$$

to obtain the equality for the right-hand side.

Step 3: Argument for (53): By symmetry of $\int_{S^{d-1}} d\xi$ under the reflection that maps e_1 into ν, we have

$$\int_{S^{d-1}} \xi_1^2|\xi\cdot\nu|\,d\xi = \int_{S^{d-1}} (\xi\cdot\nu)^2|\xi_1|\,d\xi.$$

Applying the divergence theorem to the vector field $|\xi_1|\,(\xi\cdot\nu)\,\nu$, we have

$$\int_{S^{d-1}} (\xi\cdot\nu)^2|\xi_1|\,d\xi = \int_B \nabla\cdot\left(|\xi_1|\,(\xi\cdot\nu)\,\nu\right)d\xi.$$

Since $\nabla\cdot(|\xi_1|\,(\xi\cdot\nu)\,\nu) = \operatorname{sign}\xi_1\,(\xi\cdot\nu)\,\nu_1 + |\xi_1|$, the right-hand side is equal to

$$\left(\int_B \operatorname{sign}\xi_1\,\xi\,d\xi\right)\cdot\nu\,\nu_1 + \int_B |\xi_1|\,d\xi.$$

By symmetry of $d\xi$ under rotations that leave e_1 invariant, we see that $\int_B \operatorname{sign}\xi_1\,\xi\,d\xi$ points in direction e_1, so that the above reduces to

$$(\nu_1^2+1)\int_B |\xi_1|\,d\xi.$$

We conclude by observing

$$\int_B |\xi_1|\,d\xi = \int_{-1}^{1} |\xi_1|\,|B^{d-1}|\left(1-\xi_1^2\right)^{\frac{d-1}{2}}d\xi_1 = 2|B^{d-1}|\int_0^1 \frac{d}{d\xi_1}\left[-\frac{1}{d+1}\left(1-\xi_1^2\right)^{\frac{d-1}{2}}\right]d\xi_1 = 2\frac{|B^{d-1}|}{d+1}.$$

Step 4: One-dimensional reduction. The problem reduces to the one-dimensional analogue, namely: For all χ, $\tilde\chi\in BV([0,\Lambda),\{0,1\})$ such that

$$\chi\,\tilde\chi = 0 \quad \text{a.e.} \tag{54}$$

and every $\zeta \in C^\infty([0,\Lambda))$ we have

$$\lim_{h\to 0}\frac{1}{\sqrt{h}}\int_0^L \zeta(s)\tilde\chi(s)\left(\chi(s+\sqrt{h})+\chi(s-\sqrt{h})\right)\,ds = \frac{1}{2}\int_0^L \zeta\left(|\frac{d\chi}{ds}|+|\frac{d\tilde\chi}{ds}|-|\frac{d(\chi+\tilde\chi)}{ds}|\right). \quad (55)$$

Indeed, by symmetry, it suffices to prove (52) for $\xi = e_d$. Using the decomposition $x = x' + se_d$ we see that (52) follows from (55) using the functions $\chi_{x'}(s) := \chi(x'+se_d)$, $\tilde\chi_{x'}$, $\zeta_{x'}$ in (55) and integrating w.r.t. dx'. For the left-hand side, this is formally clear. For the right-hand side, one uses BV-theory: If $\chi \in BV([0,\Lambda)^d)$, we have $\chi_{x'} \in BV([0,\Lambda))$ for a.e. $x' \in [0,\Lambda)^{d-1}$ and

$$\int_{[0,\Lambda)^{d-1}}\int_0^L \zeta_{x'}(s)\,|\frac{d\chi_{x'}}{ds}|\,dx' = \int_{[0,\Lambda)^d}\zeta\,|e_d\cdot\nu|\,|\nabla\chi|$$

for any $\zeta \in C^\infty([0,\Lambda)^d)$. To make the argument rigorous, we use again Lebesgue's dominated convergence. As before, using (54), we obtain

$$\begin{aligned}
&\left|\frac{1}{\sqrt{h}}\int_0^L \zeta_{x'}(s)\tilde\chi_{x'}(s)\left(\chi_{x'}(s+\sqrt{h})+\chi_{x'}(s-\sqrt{h})\right)\,ds\right|\\
\leq&\sup|\zeta|\frac{1}{\sqrt{h}}\int_0^L \left|\chi_{x'}(s+\sqrt{h})-\chi_{x'}(s)\right|+\left|\chi_{x'}(s-\sqrt{h})-\chi_{x'}(s)\right|\,ds\\
\leq&2\|\zeta\|_\infty\int_0^L|\frac{d\chi_{x'}}{ds}|.
\end{aligned}$$

Since

$$\int_{[0,\Lambda)^{d-1}}\int_0^L|\frac{d\chi_{x'}}{ds}|dx' = \int_{[0,\Lambda)^d}|e_d\cdot\nu|\,|\nabla\chi| \leq \int_{[0,\Lambda)^d}|\nabla\chi|,$$

this is indeed an integrable dominating function.

Step 5: Argument for (55). Since χ, $\tilde\chi$ are $\{0,1\}$-valued, every jump has height 1 and since χ, $\tilde\chi \in BV([0,\Lambda))$, the total number of jumps is finite. Let J, $\tilde J \subset [0,\Lambda)$ denote the jump sets of χ and $\tilde\chi$, respectively. Now, if $\sqrt{h}$ is smaller than the minimal distance between two different points in $J\cup\tilde J$, then in view of (54), the only contribution to the left-hand side of (55) comes from neighborhoods of points where both, χ and $\tilde{\chi}$, jump:

$$\frac{1}{\sqrt{h}}\int_0^L \zeta(s)\tilde{\chi}(s)\left(\chi(s+\sqrt{h})+\chi(s-\sqrt{h})\right)\,ds$$
$$=\sum_{s\in J\cap\tilde{J}}\frac{1}{\sqrt{h}}\int_{s-\sqrt{h}}^{s+\sqrt{h}}\zeta(\sigma)\tilde{\chi}(\sigma)\left(\chi(\sigma+\sqrt{h})+\chi(\sigma-\sqrt{h})\right)\,d\sigma.$$

Note that $\chi(\sigma+\sqrt{h})+\chi(\sigma-\sqrt{h})\equiv 1$ on each of these intervals and that

$$\tilde{\chi}=\mathbf{1}_{I_s^h}\quad\text{on } (s-\sqrt{h},s+\sqrt{h})$$

for intervals of the form

$$I_s^h=(s-\sqrt{h},s)\quad\text{or}\quad I_s^h=(s,s+\sqrt{h}).$$

Since $|I_s^h|=\sqrt{h}$, we have

$$\frac{1}{\sqrt{h}}\int_0^L \zeta(s)\tilde{\chi}(s)\left(\chi(s+\sqrt{h})+\chi(s-\sqrt{h})\right)\,ds=\sum_{s\in J\cap\tilde{J}}\frac{1}{\sqrt{h}}\int_{I_s^h}\zeta(\sigma)\,d\sigma\longrightarrow\sum_{s\in J\cap\tilde{J}}\zeta(s).$$

Note that by (54), $\chi+\tilde{\chi}$ jumps precisely where either χ or $\tilde{\chi}$ jumps. Thus

$$\frac{1}{2}\int_0^L\zeta\left(|\frac{d\chi}{ds}|+|\frac{d\tilde{\chi}}{ds}|-|\frac{d(\chi+\tilde{\chi})}{ds}|\right)=\frac{1}{2}\Big(\sum_{s\in J}\zeta(s)+\sum_{s\in\tilde{J}}\zeta(s)-\sum_{s\in J\Delta\tilde{J}}\zeta(s)\Big)=\sum_{s\in J\cap\tilde{J}}\zeta(s).$$

Therefore, (55) holds, which concludes the proof. $\square$

The next lemma shows that under our convergence assumption of χ^h to χ, the corresponding spatial covariance functions F_h and F are very close.

Lemma 3.6 (Error estimate)**.** *Let χ^h, χ satisfy the convergence assumptions (35)-(37) and let k be a non-negative kernel such that*

$$k(z)\le p(|z|)G(z)$$

for some polynomial p. Then

$$\lim_{h\to 0}\frac{1}{\sqrt{h}}\int k_h(z)|F_h(z)-F(z)|\,dz=0, \tag{56}$$

where

$$F_h(z) := \sum_{i,j} \sigma_{ij} \int \chi_i^h(x)\chi_j^h(x+z)\,dx, \quad \textit{and}$$
$$F(z) := \sum_{i,j} \sigma_{ij} \int \chi_i(x)\chi_j(x+z)\,dx$$

Proof. The proof is divided into two steps. First, we prove the claim for $k = G$, to generalize this result for arbitrary kernels k in the second step.

Step 1: $k = G$. By Lemma 3.5 and the convergence assumption (37), we already know

$$\lim_{h\to 0} \frac{1}{\sqrt{h}} \int G_h(z) \left(F_h(z) - F(z)\right) \, dz = 0.$$

Hence, it is sufficient to show that

$$\lim_{h\to 0} \frac{1}{\sqrt{h}} \int G_h(z) \left(F(z) - F_h(z)\right)_+ \, dz = 0.$$

Fix $h_0 > 0$ and $N \in \mathbb{N}$ and set $h := \frac{1}{N^2} h_0$. Recall that by iterating the exact triangle inequality for $F_{(h)} = F$, F_h we have

$$F_{(h)}(Nz) \le N F_{(h)}(z) \quad \text{for all } z \in \mathbb{R}^d.$$

Hence, by the definition of h,

$$\frac{1}{\sqrt{h_0}} F_{(h)}(\sqrt{h_0} z) \le \frac{1}{\sqrt{h}} F_{(h)}(\sqrt{h} z) \quad \text{for all } z \in \mathbb{R}^d. \tag{57}$$

Therefore, using (57) for F_h, the triangle inequality and finally (57) for F, we obtain

$$\begin{aligned}
&\left(\frac{1}{\sqrt{h}} F(\sqrt{h} z) - \frac{1}{\sqrt{h}} F_h(\sqrt{h} z)\right)_+ \\
\le &\left(\frac{1}{\sqrt{h}} F(\sqrt{h} z) - \frac{1}{\sqrt{h_0}} F_h(\sqrt{h_0} z)\right)_+ \\
\le &\left(\frac{1}{\sqrt{h}} F(\sqrt{h} z) - \frac{1}{\sqrt{h_0}} F(\sqrt{h_0} z)\right)_+ + \left(\frac{1}{\sqrt{h_0}} F(\sqrt{h_0} z) - \frac{1}{\sqrt{h_0}} F_h(\sqrt{h_0} z)\right)_+ \\
\le &\frac{1}{\sqrt{h}} F(\sqrt{h} z) - \frac{1}{\sqrt{h_0}} F(\sqrt{h_0} z) + \frac{1}{\sqrt{h_0}} \left| F(\sqrt{h_0} z) - F_h(\sqrt{h_0} z) \right|.
\end{aligned}$$

Integrating w.r.t. G yields

$$\begin{aligned}
\frac{1}{\sqrt{h}} \int G_h(z) \left(F(z) - F_h(z)\right)_+ \, dz \le &\frac{1}{\sqrt{h}} \int G(z) F(\sqrt{h} z) \, dz - \frac{1}{\sqrt{h_0}} \int G(z) F(\sqrt{h_0} z) \, dz \\
&+ \frac{1}{\sqrt{h_0}} \int G(z) \left| F(\sqrt{h_0} z) - F_h(\sqrt{h_0} z) \right| \, dz \\
= &E_h(\chi) - E_{h_0}(\chi) + \frac{1}{\sqrt{h_0}} \int G_{h_0}(z) \left| F(z) - F_h(z) \right| \, dz.
\end{aligned} \tag{58}$$

Given $\delta > 0$, by Lemma 3.5 we may first choose $h_0 > 0$ such that for all $0 < h < h_0$:

$$|E_h(\chi) - E_{h_0}(\chi)| < \frac{\delta}{2}.$$

We note that we may now choose $N \in \mathbb{N}$ so large that for all $0 < h < \frac{1}{N^2} h_0$:

$$\left| F(\sqrt{h_0} z) - F_h(\sqrt{h_0} z) \right| \leq \frac{\delta}{2} \sqrt{h_0} \quad \text{for all } z \in \mathbb{R}^d.$$

Indeed, using the triangle inequality and translation invariance we have

$$\begin{aligned}
&\left| F(\sqrt{h_0} z) - F_h(\sqrt{h_0} z) \right| \\
\leq &\sum_{i,j} \sigma_{ij} \int \left| \chi_i(x)\chi_j(x+z) - \chi_i(x)\chi_j^h(x+z) \right| + \left| \chi_i(x)\chi_j^h(x+z) - \chi_i^h(x)\chi_j^h(x+z) \right| \, dx \\
\leq &2 \left(\max_{1 \leq i,j \leq N} \sigma_{ij} \right) \sum_{i=1}^N \int \left| \chi_i(x) - \chi_i^h(x) \right| \, dx,
\end{aligned}$$

which tends to zero as $h \to 0$ because of the convergence assumption (35). Hence also the second term on the right-hand side of (58) is small:

$$\frac{1}{\sqrt{h_0}} \int G_{h_0}(z) \left(F(z) - F_h(z) \right)_+ \, dz \leq \frac{\delta}{2}.$$

Step 2: $k = pG$. Fix $\varepsilon > 0$. Since G is exponentially decaying, we can find a number $M = M(\varepsilon) < \infty$ such that

$$k(z) \leq \varepsilon \, G(\frac{z}{2}) \quad \text{for all } |z| > M.$$

Hence we can split the integral into two parts. On the one hand,

$$\frac{1}{\sqrt{h}} \int_{\{|z| \leq M\}} k(z) \, |F_h(\sqrt{h} z) - F(\sqrt{h} z)| \, dz \leq \left(\sup_{[0,M]} p \right) \frac{1}{\sqrt{h}} \int G(z) |F_h(\sqrt{h} z) - F(\sqrt{h} z)| \, dz$$

and on the other hand

$$\begin{aligned}
\frac{1}{\sqrt{h}} \int_{\{|z| > M\}} k(z) \, |F_h(\sqrt{h} z) - F(\sqrt{h} z)| \, dz \leq &\varepsilon \frac{1}{\sqrt{h}} \int G(z) \left(F_h(\sqrt{h} z) + F(\sqrt{h} z) \right) \, dz \\
\leq &\varepsilon \left(E_h(\chi^h) + E_h(\chi) \right).
\end{aligned}$$

Take the limit $h \to 0$ and use (56) for $k = G$ to obtain

$$\limsup_{h \to 0} \frac{1}{\sqrt{h}} \int k_h(z) |F_h(z) - F(z)| \, dz \leq 2 \varepsilon c_0 \sum_{i,j} \sigma_{ij} \frac{1}{2} \Big(\int |\nabla \chi_i| + \int |\nabla \chi_j| - \int |\nabla(\chi_i + \chi_j)| \Big).$$

Here, we used the consistency, cf. Lemma 3.5, and the convergence assumption (37). Since the left-hand side does not depend on $\varepsilon > 0$, this implies (56). $\square$

The following proposition is the main ingredient for Proposition 3.2. It establishes convergence of a functional that is an anisotropic version of the energy, localized by a tensor field A.

Proposition 3.7. *Let χ^h, χ satisfy the convergence assumptions (35)-(37), $K(z) := z \otimes z\, G(z)$ and let $A \in C^\infty([0,\Lambda)^d, \mathbb{R}^{d\times d})$. Then,*

$$\lim_{h\to 0} \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \chi_i^h\, A : K_h * \chi_j^h dx = c_0 \sum_{i,j} \sigma_{ij} \frac{1}{2} \int (\nu\cdot A\nu + \operatorname{tr} A)\left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|\right).$$

Proof. Step 1: Reduction of the statement. Since $K(z)$ is a symmetric matrix,

$$A : K(z) = A^{\mathrm{sym}} : K(z).$$

depends only the symmetric part A^{sym} of A; hence w.l.o.g. let A be a symmetric matrix field. But then there exist functions $\zeta_{ij} \in C^\infty([0,\Lambda)^d)$, such that

$$A(x) = \sum_{i,j} \frac{1}{2} \zeta_{ij}(x)\left(e_i \otimes e_j + e_j \otimes e_i\right).$$

We also note

$$e_i \otimes e_j + e_j \otimes e_i = (e_i + e_j) \otimes (e_i + e_j) - (e_i \otimes e_i + e_j \otimes e_j).$$

Hence by linearity, it is enough to prove the statement for A of the form

$$A(x) = \zeta(x)\, \xi \otimes \xi$$

for some $\xi \in S^{d-1}$. By rotational invariance, we may assume

$$A(x) = \zeta(x)\, e_1 \otimes e_1.$$

Hence, with $k(z) := z_1^2 G(z)$, the statement can be reduced to

$$\begin{aligned}\lim_{h\to 0} \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} &\int \zeta\, \chi_i^h\, k_h * \chi_j^h\, dx \\ &= c_0 \sum_{i,j} \sigma_{ij} \frac{1}{2} \int \zeta \left(1 + \nu_1^2\right) \zeta \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|\right)\end{aligned} \tag{59}$$

for any $\zeta \in C^\infty([0,\Lambda)^d)$.
As in the proof of (24), it is enough to show the convergence under the additional assumption $0 \le \zeta \le 1$. The proof of (59) will be given in the following way:

- modify k to obtain a radially non-increasing kernel $\tilde{k}$ and prove a $\liminf$-inequality for $\tilde{k}$ and arbitrary $0 \le \zeta \le 1$ using the monotonicity,

- show convergence for $\tilde{k}$ and $\zeta \equiv 1$ using consistency and estimating the error,
- deduce the convergence for $\tilde{k}$ and therefore for k.

Step 2: A $\liminf$*-inequality.* We first note that the kernel $\tilde{k}(z) := (z_1^2+2)G(z) = k(z) + 2G(z)$ is radially non-increasing, i.e.

$$\nabla \tilde{k}(z) \cdot z \leq 0.$$

Indeed,

$$\begin{aligned}
\nabla \tilde{k}(z) \cdot z &= \left[2z_1 e_1 - z(z_1^2+2)\right] \cdot z\, G(z) \\
&= \left[2z_1^2 - |z|^2(z_1^2+2)\right] G(z) \\
&\leq \left[2|z|^2 - |z|^2(z_1^2+2)\right] G(z) \leq 0.
\end{aligned}$$

Hence, we may apply Lemma 3.4 to $\tilde{k}$. Furthermore, since $|z|\tilde{k}(z) \lesssim G(z/2)$, the error functional g_h in Lemma 3.4 can be easily estimated as the error term in the proof of Lemma 3.3:

$$g_h(\chi) \lesssim E_h(\chi).$$

Let $0 < h < h_0$. By Lemma 3.4, we have

$$\begin{aligned}
&\frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \zeta \chi_i^h\, \tilde{k}_h * \chi_j^h\, dx \\
&\geq \left(\frac{\sqrt{h_0}}{\sqrt{h}+\sqrt{h_0}}\right)^{d+1} \frac{1}{\sqrt{h_0}} \sum_{i,j} \sigma_{ij} \int \zeta \chi_i^h\, \tilde{k}_{h_0} * \chi_j^h\, dx - C\sqrt{h_0} E_h(\chi^h).
\end{aligned}$$

On the right-hand side, we can pass to the limit since on the one hand $\chi_i^h \to \chi_i$ in $L^2([0,\Lambda)^d)$ by (36) and $|\chi_i| \leq 1$ and on the other hand we can use (37) for the last term. Hence

$$\begin{aligned}
&\liminf_{h\to 0} \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \zeta \chi_i^h\, \tilde{k}_h * \chi_j^h\, dx \\
&\geq \frac{1}{\sqrt{h_0}} \sum_{i,j} \sigma_{ij} \int \zeta \chi_i\, \tilde{k}_{h_0} * \chi_j\, dx - C\sqrt{h_0}\, c_0 \sum_{i,j} \sigma_{ij} \frac{1}{2} \int \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i+\chi_j)|\right).
\end{aligned}$$

By Lemma 3.5, as $h_0 \to 0$, the right-hand side converges to

$$c_0 \sum_{i,j} \sigma_{ij} \frac{1}{2} \int \zeta \left(3+\nu_1^2\right) \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i+\chi_j)|\right)$$

so that we obtain

$$\liminf_{h\to 0} \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \zeta\, \chi_i^h\, \tilde{k}_h * \chi_j^h dx$$
$$\geq c_0 \sum_{i,j} \sigma_{ij} \frac{1}{2} \int \zeta \left(3+\nu_1^2\right) \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i+\chi_j)|\right). \tag{60}$$

Step 3: Convergence for $\zeta \equiv 1$. By Lemma 3.5, we have

$$\lim_{h\to 0} \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \zeta\, \chi_i\, \tilde{k}_h * \chi_j\, dx$$
$$= c_0 \sum_{i,j} \sigma_{ij} \frac{1}{2} \int \zeta \left(3+\nu_1^2\right) \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i+\chi_j)|\right). \tag{61}$$

By Lemma 3.6, we can control the difference between the left-hand side of (60) and the left-hand side of (61) with $\zeta \equiv 1$:

$$\lim_{h\to 0} \frac{1}{\sqrt{h}} \int \tilde{k}_h(z) |F_h(\sqrt{h}z) - F(\sqrt{h}z)|\, dz = 0.$$

Hence we have proven (59) for $\zeta \equiv 1$.

Step 4: Conclusion of the proof. For this argument use the abbreviations

$$f_h(\chi,\zeta) := \frac{1}{\sqrt{h}} \sum_{i,j} \sigma_{ij} \int \zeta \chi_i^h\, \tilde{k}_h * \chi_j^h\, dx,$$
$$f(\chi,\zeta) := c_0 \sum_{i,j} \sigma_{ij} \frac{1}{2} \int \zeta \left(3+\nu_1^2\right) \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i+\chi_j)|\right).$$

In these terms, (59) reads

$$\lim_{h\to 0} f_h(\chi^h,\zeta) = f(\chi,\zeta) \quad \text{for all } \zeta \in C^\infty([0,\Lambda)^d),\ 0\leq \zeta \leq 1. \tag{62}$$

Fix the test function $0 \leq \zeta \leq 1$. Then we can apply Step 1 on both, ζ and $(1-\zeta)$: On the one hand we have

$$\liminf_{h\to 0} f_h(\chi^h,\zeta) \geq f(\chi,\zeta).$$

On the other hand, by linearity we have

$$\liminf_{h\to 0} \left\{ f_h(\chi^h,1) - f_h(\chi^h,\zeta) \right\} \geq f(\chi,1) - f_0(\chi,\zeta).$$

By Step 2 we have

$$\lim_{h\to 0} f_h(\chi^h,1) = f(\chi,1)$$

and hence

$$\limsup_{h\to 0} f_h(\chi^h,\zeta) \le f(\chi,\zeta),$$

which together with (60) implies (62). As in the proof of Remark 2.9, the respective terms for G converge due to the convergence assumption (37). Therefore, by linearity we also have the convergence for $k = \tilde{k} - 2G$. This concludes the proof. $\Box$

Proof of Proposition 3.2. We may apply Lemma 3.3 for χ^h and obtain by the energy-dissipation estimate (9) that

$$\begin{aligned}\delta E_h(\chi^h,\xi) = &-\frac{1}{\sqrt{h}}\sum_{i,j}\sigma_{ij}\int \chi_i^h\,\nabla\xi : K_h * \chi_j^h\,dx\\ &+\frac{2}{\sqrt{h}}\sum_{i,j}\sigma_{ij}\int \chi_i^h\,(\nabla\cdot\xi)\,G_h*\chi_j^h\,dx + O\left(\|\nabla^2\xi\|_\infty E_0\sqrt{h}\right).\end{aligned}$$

Applying Proposition 3.7 on the first summand with $A = \nabla\xi$ and Lemma 3.5 on the second summand with $\zeta = \nabla\cdot\xi$, we obtain in the limit $h\to 0$:

$$\delta E_h(\chi^h,\xi) \to c_0\sum_{i,j}\sigma_{ij}\frac{1}{2}\int\left[-(\nu\cdot\nabla\xi\,\nu + \operatorname{tr}\nabla\xi) + 2\,\nabla\cdot\xi\right]\left(|\nabla\chi_i| + |\nabla\chi_j| - |\nabla(\chi_i+\chi_j)|\right),$$

which concludes the proof. $\Box$

4 Dissipation Functional and Velocity

As for any minimizing movement scheme, the time derivative of the solution should arise from the metric term in the minimization scheme. For the minimizing movement scheme of our interfacial motion, the time derivative is the normal velocity. The goal of this section, which is the core of the paper, is to compare the first variation of the dissipation functional to the normal velocity. The main result of this section is the following proposition which will be used for small time intervals in Section 5 where we will control the limiting error terms which appear here with soft arguments from Geometric Measure Theory. In view of the definition of $\mathscr{E}_1^2$ below, the proposition assumes that χ_1 is the *minority phase* in the space-time cylinder $(0,T)\times B_r$; likewise it assumes that the normal between χ_2 and χ_3 is close to the first unit vector e_1. This can be assumed since we can relabel the phases in case we want to treat another phase as the minority phase. Furthermore, due to the rotational invariance, it is no restriction to assume that e_1 is the approximate normal.

Proposition 4.1. *For any $\alpha\in(0,1)$, $T>0$, $\xi\in C_0^\infty((0,T)\times B_r,\mathbb{R}^d)$ and any $\eta\in C_0^\infty(B_{2r})$ radially symmetric and radially non-increasing cut-off of B_r in B_{2r} with $|\nabla\eta|\lesssim\frac{1}{r}$ and $|\nabla^2\eta|\lesssim$*

$\frac{1}{r^2}$, *we have*

$$\limsup_{h\to 0}\left|\int_0^T -\delta E_h(\,\cdot\,-\chi^h(t-h))(\chi^h(t),\xi(t))\,dt - 2c_0\int_0^T\left(\int \xi_1\,V_2\,|\nabla\chi_2| - \int \xi_1\,V_3\,|\nabla\chi_3|\right)dt\right|$$
$$\lesssim \|\xi\|_\infty\left[\int_0^T \frac{1}{\alpha}\mathscr{E}_1^2(t) + \alpha^{1/3}r^{d-1}dt + \alpha^{1/3}\iint \eta\,d\mu\right]. \tag{63}$$

Here we use the notation

$$\mathscr{E}_1^2(t) := \int \eta\,|\nabla\chi_1(t)| + \inf_{\chi^*}\left\{\left|\int \eta\,(|\nabla\chi_2(t)| - |\nabla\chi^*|)\right| + \left|\int \eta\,(|\nabla\chi_3(t)| - |\nabla\chi^*|)\right|\right.$$
$$\left. + \frac{1}{r}\int_{B_{2r}} |\chi_2(t) - \chi^*|\,dx + \frac{1}{r}\int_{B_{2r}} |\chi_3(t) - (1-\chi^*)|\,dx\right\},$$

where the infimum is taken over all half spaces $\chi^* = \mathbf{1}_{\{x_1>\lambda\}}$ *in direction* e_1.

Let us comment on the structure of $\mathscr{E}_1^2$. The first term, describing the surface area of Phase 1 inside the ball B_{2r}, will be small in the application when χ_1 is indeed the minority phase. The second term, sometimes called the *excess energy* describes how far χ_2 and χ_3 are away from being half spaces in direction e_1 or $-e_1$, respectively. The terms comparing the surface energy inside B_{2r} do not see the orientation of the normal, whereas the bulk terms measuring the L^1-distance inside the ball B_{2r} see the orientation of the normal.

Proof. Step 1: The discrete analogue of (63). The statement follows easily from

$$\left|\int_0^T -\delta E_h(\,\cdot\,-\chi^h(t-h))(\chi^h(t),\xi(t))\,dt - 2c_0\int_0^T\left(\int \xi_1\,V_2\,|\nabla\chi_2| - \int \xi_1\,V_3\,|\nabla\chi_3|\right)dt\right|$$
$$\lesssim \|\xi\|_\infty\left[\frac{1}{\alpha}h\sum_{n=1}^N \varepsilon_1^2(n) + \alpha^{1/3}r^{d-1}T + \alpha^{1/3}\iint \eta\,d\mu_h\right] + o(1), \quad \text{as } h\to 0. \tag{64}$$

Here we use the notation $\varepsilon_1^2(n) := \varepsilon_1^2(\chi^n)$, where the functional ε_1^2 is defined via

$$\begin{aligned}
\varepsilon_1^2(\chi) :=& \frac{1}{\sqrt{h}}\int \eta\,[(1-\chi_1)\,G_h*\chi_1 + \chi_1\,G_h*(1-\chi_1)]\,dx\\
&+\inf_{\chi^*}\left\{\frac{1}{\sqrt{h}}\int \eta\,[(1-\chi_2)\,G_h*\chi_2 + \chi_2\,G_h*(1-\chi_2)]\,dx\right.\\
&+\frac{1}{\sqrt{h}}\int \eta\,[(1-\chi_3)\,G_h*\chi_3 + \chi_3\,G_h*(1-\chi_3)]\,dx\\
&-2\frac{1}{\sqrt{h}}\int \eta\,[(1-\chi^*)\,G_h*\chi^* + \chi^*\,G_h*(1-\chi^*)]\,dx\\
&\left.+\frac{1}{r}\int_{B_{2r}} |\chi_2-\chi^*|\,dx + \frac{1}{r}\int_{B_{2r}} |\chi_3-(1-\chi^*)|\,dx\right\}.
\end{aligned}$$

The infimum is taken over all half spaces $\chi^* = \mathbf{1}_{x_1>\lambda}$ in direction e_1. All terms appearing in ε_1^2 correspond to terms in $\mathscr{E}_1^2$. The first term is the localized approximate energy of χ_1, the second term describes the approximate excess energy of Phase 2 and 3. The convergence of these terms as $h \to 0$ for a fixed half space χ^* follows as in the proof of Remark 2.9. Taking the infimum over the half spaces yields (63).

Step 2: Choice of appropriately shifted mesoscopic time slices. In order to prove (64), we use the machinery that we develop later on in this section. There we work on the mesoscopic time scale $\tau = \alpha\sqrt{h}$ instead of the microscopic time scale h. To apply these results, we have to adjust the time shift of time slices of mesoscopic distance. At the end, we will choose a microscopic time shift $k_0 \in \{1, \dots, K\}$ such that the average over time slices of mesoscopic distance is controlled by the average over all time slices:

$$\tau \sum_{l=1}^{L} \left[\varepsilon_1^2(Kl + k_0) + \varepsilon_1^2(Kl + k_0 - 1)\right] \lesssim h \sum_{n=1}^{N} \varepsilon_1^2(n). \tag{65}$$

This follows from the simple fact that $\varepsilon_1^2(k_0) \leq \frac{1}{K}\sum_{k=1}^{K} \varepsilon_1^2(k)$ for some k_0. For notational simplicity, we shall assume that $k_0 = 0$ in (65).

Step 3: Argument for (64). Using Lemmas 4.6, 4.7 and 4.8 below, we obtain

$$\begin{aligned} &\int_0^T -\delta E_h(\,\cdot\,, \chi^h(t-h))(\chi^h(t), \xi(t))\, dt \\ &\approx 2c_0\tau \sum_{l=1}^{L} \left(\int \xi_1(l\tau) \frac{\chi_2^{Kl} - \chi_2^{K(l-1)}}{\tau}\, dx - \int \xi_1(l\tau) \frac{\chi_3^{Kl} - \chi_3^{K(l-1)}}{\tau}\, dx \right) \end{aligned} \tag{66}$$

up to an error

$$\|\xi\|_\infty \left(\frac{1}{\alpha} h \sum_{n=1}^{N} \varepsilon_1^2(n) + \alpha^{1/3} r^{d-1} T + \alpha^{1/3} \iint \eta\, d\mu_h \right) + o(1), \quad \text{as } h \to 0,$$

where we used the choice of time slices (65). Since ξ has compact support in $(0,T)$, a discrete integration by parts yields

$$\tau \sum_{l=1}^{L} \int \xi_1(l\tau) \frac{1}{\tau} \left(\chi_i^{Kl} - \chi_i^{K(l-1)} \right) dx = -\tau \sum_{l=0}^{L-1} \int \frac{1}{\tau} \left(\xi_1((l+1)\tau) - \xi_1(l\tau) \right) \chi_i^{Kl}\, dx.$$

By the Hölder-type bounds in Lemma 2.7 we can replace the mesoscopic scale on the right-hand side by the microscopic scale for χ:

$$\begin{aligned} &\left| \tau \sum_{l=0}^{L-1} \int \frac{1}{\tau} \left(\xi_1((l+1)\tau) - \xi_1(l\tau) \right) \chi_i^{Kl}\, dx - \tau \sum_{l=0}^{L-1} \frac{1}{K} \sum_{k=1}^{K} \int \frac{1}{\tau} \left(\xi_1((l+1)\tau) - \xi_1(l\tau) \right) \chi_i^{Kl+k}\, dx \right| \\ &\leq \|\partial_t \xi\|_\infty h \sum_{l=0}^{L-1} \sum_{k=1}^{K} \int \left| \chi^{Kl} - \chi^{Kl+k} \right| dx \lesssim \|\partial_t \xi\|_\infty E_0 T \sqrt{\tau}. \end{aligned}$$

By the smoothness of ξ, we can easily do the same for ξ to obtain by (iii) in Lemma 2.10 that for $h \to 0$

$$\tau \sum_{l=0}^{L-1} \int \frac{1}{\tau} \left(\xi_1((l+1)\tau) - \xi_1(l\tau)\right) \chi_i^{Kl} \, dx \to \int_0^T \int \partial_t \xi_1 \, \chi_i \, dx \, dt = - \int_0^T \int \xi_1 V_i \, |\nabla \chi_i| \, dt.$$

Using this for the right-hand side of (66) establishes (64) and thus concludes the proof. □

4.1 A digression to the two-phase case

The estimates in Chapter 2 are not sufficient to understand the link between the first variation of the metric term and the normal velocities. For this, we need refined estimates which we will first present for the two-phase case, where only one interface evolves. The main tool of the proof is the following one-dimensional lemma. For two functions u_0, u_1, it estimates the L^1-distance between the characteristic functions $\chi_i = \mathbf{1}_{\{u_i \geq 1/2\}}$ in terms of the L^2-distance between the u_i's - at the expense of a term that measures the strict monotonicity of one of the functions u_i. We will apply it in a rescaled version for x_1 being the normal direction.

Lemma 4.2. *Let $I \subset \mathbb{R}$ be an interval, u_0, $u_1 \in C^{0,1}(I)$ and $\chi_i := \mathbf{1}_{\{u_i \geq 1/2\}}$. Then*

$$\int_I |\chi_1 - \chi_0| \, dx_1 \lesssim \int_{|u_0 - 1/2| < s} (\partial_1 u_0 - 1)_-^2 \, dx_1 + s + \frac{1}{s^2} \int_I (u_1 - u_0)^2 \, dx_1 \tag{67}$$

for every $s > 0$.

Proof. Step 1: An easier inequality. For any function $u \in C^{0,1}(I)$, we have

$$|\{|u| \leq 1\}| \lesssim \int_{\{|u| \leq 1\}} (\partial_1 u - 1)_-^2 + 1. \tag{68}$$

Argument: We decompose the set that we want to measure on the left-hand side

$$\{|u| \leq 1\} = \bigcup_{J \in \mathscr{J}} J$$

into countably many pairwise disjoint intervals. We distinguish the following four different cases for an interval $J = [a, b] \in \mathscr{J}$:

(i) $J \in \mathscr{J}_{\nearrow}$: $u(a) = -1$ and $u(b) = 1$

(ii) $J \in \mathscr{J}_{\searrow}$: $u(a) = 1$ and $u(b) = -1$

(iii) $J \in \mathscr{J}_{\rightarrow}$: $u(a) = u(b)$,

(iv) $J \in \mathscr{J}_{\partial}$: J contains a boundary point of I.

By Jensen's inequality for the convex function $z \mapsto z_-^2$, we have

$$\frac{1}{|J|}\int\limits_J (\partial_1 u - 1)_-^2 \, dx_1 \geq \left(\frac{1}{|J|}\int\limits_J (\partial_1 u - 1)\, dx_1\right)_-^2 = \left(\frac{1}{|J|}(u(b) - u(a) - |J|)\right)_-^2$$

$$= \begin{cases} \left(1 - \frac{2}{|J|}\right)_+^2, & \text{if } J \in \mathscr{J}_\nearrow, \\ \left(1 + \frac{2}{|J|}\right)^2, & \text{if } J \in \mathscr{J}_\searrow, \\ 1, & \text{if } J \in \mathscr{J}_\rightarrow. \end{cases}$$

In case (iv), if $|J| \geq 4$, then $-1 \leq 2(u(b) - u(a))/|J| \leq 1$ and so

$$\frac{1}{|J|}\int\limits_J (\partial_1 u - 1)_-^2 \, dx_1 \geq \frac{1}{4}.$$

Thus,

$$|J| \lesssim 1 \vee \int\limits_J (\partial_1 u - 1)_-^2 \, dx_1.$$

Since $\#\mathscr{J}_\partial \leq 2$, this is enough for our purpose.
In case (iii), we have immediately

$$|J| \lesssim \int\limits_J (\partial_1 u - 1)_-^2 \, dx_1,$$

while in case (ii) we even have

$$\int\limits_J (\partial_1 u - 1)_-^2 \, dx_1 \gtrsim |J| \left(1 + \frac{2}{|J|}\right)^2 \gtrsim 1 \vee |J|$$

since $1 + z^2 \geq 1$ and $1 + z^2 \geq 2z$ for all $z \in \mathbb{R}$. Thus on the one hand we can estimate the measure of such an interval $J \in \mathscr{J}_\searrow$:

$$|J| \lesssim \int\limits_J (\partial_1 u - 1)_-^2 \, dx_1.$$

On the other hand, we can bound the total number of these intervals:

$$\#\mathscr{J}_\searrow \lesssim \sum_{J \in \mathscr{J}_\searrow} \int\limits_J (\partial_1 u - 1)_-^2 \, dx_1, \tag{69}$$

which we will use for case (i).

If in case (i) additionally $|J| \geq 4$,

$$|J| \lesssim \int_J (\partial_1 u - 1)_-^2 \, dx_1.$$

Using the estimate (69), can also bound the total number of these intervals:

$$\#\mathscr{J}_\nearrow \leq \#\mathscr{J}_\searrow + 1 \lesssim \sum_{J \in \mathscr{J}_\searrow} \int_J (\partial_1 u - 1)_-^2 \, dx_1 + 1 \leq \int_{\{|u| \leq 1\}} (\partial_1 u - 1)_-^2 \, dx_1 + 1.$$

Hence,

$$\begin{aligned}
\sum_{J \in \mathscr{J}_\nearrow} |J| &= \sum_{\substack{J \in \mathscr{J}_\nearrow \\ |J| \geq 4}} |J| + \sum_{\substack{J \in \mathscr{J}_\nearrow \\ |J| < 4}} |J| \\
&\lesssim \int_{\{|u| \leq 1\}} (\partial_1 u - 1)_-^2 \, dx_1 + \#\mathscr{J}_\nearrow \\
&\lesssim \int_{\{|u| \leq 1\}} (\partial_1 u - 1)_-^2 \, dx_1 + 1.
\end{aligned}$$

Using these estimates, we derive

$$|\{|u| \leq 1\}| = \sum_{J \in \mathscr{J}} |J| \lesssim \int_{\{|u| \leq 1\}} (\partial_1 u - 1)_-^2 \, dx_1 + 1.$$

Step 2: Rescaling (68). Let $s > 0$. We use Step 1 for $\hat{u}$ and set $u := s\hat{u}$, $x_1 = s\hat{x}_1$. Then $\partial_1 u = \hat{\partial}_1 \hat{u}$ and

$$|\{|u| \leq s\}| = s\,|\{\hat{u} \leq 1\}| \overset{(68)}{\lesssim} s \int_{\{|\hat{u}| \leq 1\}} \left(\hat{\partial}_1 \hat{u} - 1\right)_-^2 d\hat{x}_1 + s = \int_{\{|u| \leq s\}} (\partial_1 u - 1)_-^2 \, dx_1 + s.$$

Therefore, for $u_0 = u + 1/2$, we have

$$|\{|u_0 - 1/2| \leq s\}| \lesssim \int_{\{|u_0 - 1/2| \leq s\}} (\partial_1 u_0 - 1)_-^2 \, dx_1 + s.$$

Step 3: Introducing u_1. By Chebyshev's inequality, we have

$$|\{|u_1 - u_0| \geq s\}| \leq \frac{1}{s^2} \int_I (u_1 - u_0)^2 dx_1$$

for all $s > 0$. Set

$$E := \{|u_0 - 1/2| \leq s\} \cup \{|u_1 - u_0| \geq s\} \subset I.$$

Then, since e.g. $u_0 \geq 1/2 > u_1$ and $|u_0 - 1/2| > s$ imply $|u_1 - u_0| > s$,

$$\{\chi_0 \neq \chi_1\} = \{u_0 \geq 1/2\}\Delta\{u_1 \geq 1/2\} \subset E.$$

Hence,

$$\int_I |\chi_1 - \chi_0|\, dx_1 \leq |E| \lesssim \int_{|u_0 - 1/2| < s} (\partial_1 u_0 - 1)_-^2\, dx_1 + s + \frac{1}{s^2}\int_I (u_1 - u_0)^2\, dx_1,$$

which concludes the proof. $\square$

In the proof we will apply the following rescaled and modified version of Lemma 4.2.

Corollary 4.3. *Let $u_0, u_1 \in C^{0,1}(I)$, $\chi_i := \mathbf{1}_{\{u_i \geq 1/2\}}$ and $\eta \in C_0^\infty(\mathbb{R})$, $0 \leq \eta \leq 1$ radially non-increasing. Then*

$$\frac{1}{\sqrt{h}}\int \eta\, |\chi_1 - \chi_0|\, dx_1 \lesssim \frac{1}{\sqrt{h}}\int_{|u_0 - 1/2| < s} \eta\left(\sqrt{h}\,\partial_1 u_0 - 1\right)_-^2 dx_1 + s + \frac{1}{s^2}\frac{1}{\sqrt{h}}\int \eta\,(u_1 - u_0)^2\, dx_1$$

for any $s > 0$.

Proof. By rescaling $x_1 = \sqrt{h}\hat{x}_1$ and $\hat{u}_i(\hat{x}_1) = u_i(\sqrt{h}\hat{x}_1)$ and using Lemma 4.2 for the $\hat{u}_i$'s we obtain:

$$\frac{1}{\sqrt{h}}\int_I |\chi_1 - \chi_0|\, dx_1 \lesssim \frac{1}{\sqrt{h}}\int_{|u_0 - 1/2| < s} \left(\sqrt{h}\,\partial_1 u_0 - 1\right)_-^2 dx_1 + s + \frac{1}{s^2}\frac{1}{\sqrt{h}}\int_I (u_1 - u_0)^2\, dx_1. \quad (70)$$

Approximate η by simple functions: Let

$$\tilde{\eta} := \frac{[N\eta]}{N} = \frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{J_n}, \quad \text{where} \quad J_n := \left\{x \in I \colon \eta(x) > \frac{n}{N}\right\}.$$

Then $0 \leq \tilde{\eta} \leq \eta$, $|\eta - \tilde{\eta}| \leq \frac{1}{N}$ and since η is radially non-increasing, each J_n is an open interval. We can apply (70) with J_n playing the role of I. By linearity we have

$$\begin{aligned}
\frac{1}{\sqrt{h}}\int \tilde{\eta}\, |\chi_1 - \chi_0|\, dx_1 &\lesssim \frac{1}{\sqrt{h}}\int_{|u_0 - 1/2| < s} \tilde{\eta}\left(\sqrt{h}\,\partial_1 u_0 - 1\right)_-^2 dx_1 + s + \frac{1}{s^2}\frac{1}{\sqrt{h}}\int \tilde{\eta}\,(u_1 - u_0)^2\, dx_1 \\
&\leq \frac{1}{\sqrt{h}}\int_{|u_0 - 1/2| < s} \eta\left(\sqrt{h}\,\partial_1 u_0 - 1\right)_-^2 dx_1 + s + \frac{1}{s^2}\frac{1}{\sqrt{h}}\int \eta\,(u_1 - u_0)^2\, dx_1.
\end{aligned}$$

Passing to the limit $N \to \infty$, the left-hand side converges to $\frac{1}{\sqrt{h}}\int \eta\, |\chi_1 - \chi_0|\, dx_1$ and we obtain the claim. $\square$

In the previous corollary, it was crucial to control strict monotonicity of one of the two functions via the term

$$\frac{1}{\sqrt{h}}\int\limits_{|u-1/2|<s}\eta\left(\sqrt{h}\,\partial_1 u-1\right)_-^2\,dx_1.$$

In the following lemma, we consider the d-dimensional version

$$\frac{1}{\sqrt{h}}\int\limits_{|u-1/2|<s}\eta\left(\sqrt{h}\,\partial_1 u-1\right)_-^2\,dx$$

of this term in case of $u=G_h*\chi$. We show that this term can be controlled in terms of the local energy gap, measuring the energy difference to a half space χ^* in direction e_1.

Lemma 4.4. *Let $\chi\colon[0,\Lambda)^d\to\{0,1\}$, $\chi^*=\mathbf{1}_{\{x_1>\lambda\}}$ a half space in direction e_1 and $\eta\in C_0^\infty(B_{2r})$ be a cut-off of B_r in B_{2r} with $|\nabla\eta|\lesssim\frac{1}{r}$ and $\left|\nabla^2\eta\right|\lesssim\frac{1}{r^2}$. Then there exists a universal constant $\overline{c}>0$ such that*

$$\frac{1}{\sqrt{h}}\int\limits_{\{z_1\lessgtr 0\}}G_h(z)\int\eta(x)\left(\chi(x+z)-\chi(x)\right)_\pm\,dx\,dz\lesssim\varepsilon^2+\sqrt{h}\frac{1}{r^2},\tag{71}$$

$$\frac{1}{\sqrt{h}}\int\limits_{\left\{\frac13\le G_h*\chi\le\frac23\right\}}\eta\left(\sqrt{h}\,\partial_1(G_h*\chi)-\overline{c}\right)_-^2\,dx\lesssim\varepsilon^2+\sqrt{h}\frac{1}{r^2}+\sqrt{h}\frac{1}{r}E_h(\chi),\tag{72}$$

where ε^2 is defined via

$$\begin{aligned}\varepsilon^2:=&\frac{1}{\sqrt{h}}\int\eta\left[\chi\,G_h*(1-\chi)+(1-\chi)\,G_h*\chi\right]dx\\&-\frac{1}{\sqrt{h}}\int\eta\left[\chi^*\,G_h*(1-\chi^*)+(1-\chi^*)\,G_h*\chi^*\right]dx+\frac{1}{r}\int\limits_{B_{2r}}|\chi-\chi^*|\,dx.\end{aligned}$$

Proof. Argument for (71): Note that for any $\tilde\chi\in\{0,1\}$ we have

$$\begin{aligned}&\int\eta\left[(1-\tilde\chi)\,G_h*\tilde\chi+\tilde\chi\,G_h*(1-\tilde\chi)\right]dx\\&=\int G_h(z)\left(\int\eta(x)\,(1-\tilde\chi)\,(x)\tilde\chi(x+z)\,dx+\int\eta(x)\,(1-\tilde\chi)\,(x)\tilde\chi(x-z)\,dx\right)dz\\&=\int G_h(z)\int\eta(x)\left|\tilde\chi(x+z)-\tilde\chi(x)\right|\,dx\,dz.\end{aligned}\tag{73}$$

Using $|\chi^*(x+z)-\chi^*(x)|=\operatorname{sign}(z_1)\,(\chi^*(x+z)-\chi^*(x))$, and $2u_+=|u|+u$ on the set $\{z_1>0\}$

and $2u_- = |u| - u$ on $\{z_1 < 0\}$, we thus obtain

$$\begin{aligned}
&\frac{2}{\sqrt{h}} \int\limits_{\{z_1 \lessgtr 0\}} G_h(z) \int \eta(x)\,(\chi(x+z) - \chi(x))_\pm\,dx\,dz \\
=&\frac{1}{\sqrt{h}} \int G_h(z) \int \eta(x)\,(|\chi(x+z) - \chi(x)| - |\chi^*(x+z) - \chi^*(x)|)\,dx\,dz \\
&- \frac{1}{\sqrt{h}} \int \operatorname{sign}(z_1)\,G_h(z) \int \eta(x)\,((\chi^* - \chi)(x+z) - (\chi^* - \chi)(x))\,dx\,dz \\
\leq&\varepsilon^2 - \frac{1}{\sqrt{h}} \int \operatorname{sign}(z_1)\,G_h(z) \int (\eta(x) - \eta(x-z))\,(\chi^* - \chi)(x)\,dx\,dz.
\end{aligned}$$

Here, the integral on the left-hand side with the two cases is a short notation for the sum of the two integrals. We now can apply a Taylor expansion for η around x, i.e. write $\eta(x) - \eta(x-z) = \nabla\eta(x) \cdot z + O(|z|^2)$, where the constant in the $O(|z|^2)$-term depends linearly on $\left\|\nabla^2\eta\right\|_\infty$. By symmetry, the first-order term is

$$\frac{1}{\sqrt{h}} \int \operatorname{sign}(z_1)\,z\,G_h(z)\,dz \cdot \int \nabla\eta(x)(\chi^* - \chi)(x)\,dx = \int \frac{|z_1|}{\sqrt{h}} G_h(z)\,dz \int \partial_1\eta(x)(\chi^* - \chi)(x)\,dx.$$

Note that the right-hand side can be controlled by

$$\|\partial_1\eta\|_\infty \int\limits_{B_{2r}} |\chi - \chi^*|\,dx \lesssim \frac{1}{r} \int\limits_{B_{2r}} |\chi - \chi^*|\,dx \leq \varepsilon^2.$$

The second-order term is controlled by

$$\left\|\nabla^2\eta\right\|_\infty \frac{1}{\sqrt{h}} \int |z|^2 G_h(z)\,dz = \left\|\nabla^2\eta\right\|_\infty \sqrt{h} \int |z|^2 G(z)\,dz \lesssim \sqrt{h}\frac{1}{r^2}.$$

This completes the proof of (71).

Argument for (72): For the first arguments let w.l.o.g $h = 1$. The first ingredient is the identity

$$\partial_1(G * \chi)(x) = \int |z_1| G(z)\,|\chi(x+z) - \chi(x)|\,dz - 2 \int\limits_{\{z_1 \lessgtr 0\}} |z_1| G(z)\,(\chi(x+z) - \chi(x))_\pm\,dz, \quad (74)$$

where the last term is the sum of the two integrals. Indeed, since $\partial_1 G(z) = -z_1 G(z)$ is odd in z_1,

$$\partial_1(G * \chi)(x) = \int \partial_1 G(z)\chi(x-z)\,dz = \int z_1 G(z)\,(\chi(x+z) - \chi(x))\,dz$$

and splitting the integrand in the form $u = |u| - 2u_-$ on the set $\{z_1 > 0\}$ and $-u = |u| - 2u_+$

on $\{z_1 < 0\}$, respectively, we derive

$$\partial_1(G*\chi)(x) = \int_{\{z_1>0\}} |z_1|G(z)\,|\chi(x+z)-\chi(x)|\,dz + \int_{\{z_1<0\}} |z_1|G(z)\,|\chi(x+z)-\chi(x)|\,dz$$
$$-2\int_{\{z_1>0\}} |z_1|G(z)\,(\chi(x+z)-\chi(x))_-\,dz - 2\int_{\{z_1<0\}} |z_1|G(z)\,(\chi(x+z)-\chi(x))_+\,dz,$$

which is (74).
The second ingredient for (72) is

$$\int |z_1|G(z)|\chi(x+z)-\chi(x)|dz \gtrsim \left(\int G(z)|\chi(x+z)-\chi(x)|dz\right)^2. \tag{75}$$

To obtain (75), we estimate

$$\begin{aligned}\int |z_1|G(z)|\chi(x+z)-\chi(x)|dz \geq & \int_{\{|z_1|\geq\epsilon\}} |z_1|G(z)|\chi(x+z)-\chi(x)|dz \\ \geq & \epsilon\int_{\{|z_1|\geq\epsilon\}} G(z)|\chi(x+z)-\chi(x)|dz \\ = & \epsilon\int G(z)|\chi(x+z)-\chi(x)|dz \\ & -\epsilon\int_{\{|z_1|<\epsilon\}} G(z)|\chi(x+z)-\chi(x)|dz.\end{aligned}$$

The second integral can be estimated from above by $2G^1(0)\epsilon$ so that

$$\int |z_1|G(z)|\chi(x+z)-\chi(x)|dz \geq \epsilon\int G(z)|\chi(x+z)-\chi(x)|dz - 2G^1(0)\epsilon^2.$$

Optimizing in ϵ yields (75).
Using the fact that $\chi\in\{0,1\}$,

$$\int G(z)|\chi(x+z)-\chi(x)|dz = (1-\chi)(x)(G*\chi)(x) + \chi(x)(G*(1-\chi))(x)$$

implies the third ingredient:

$$\int G(z)\,|\chi(x+z)-\chi(x)|\,dz \geq \min\left\{(G*\chi)(x), (1-G*\chi)(x)\right\}. \tag{76}$$

Combining (74), (75) and (76), one finds a positive constant $\bar c$ such that

$$\partial_1(G*\chi)(x) \geq 18\,\bar c\,[(G*\chi)(x)\wedge(1-G*\chi)(x)]^2 - 2\int_{\{z_1\lessgtr 0\}} |z_1|G(z)\,(\chi(x+z)-\chi(x))_\pm\,dz,$$

where we recall that the last term is the sum of the two integrals. We consider the "bad" set

$$E := \left\{ x\colon \int\limits_{\{z_1 \lessgtr 0\}} |z_1| G(z) \left(\chi(x+z) - \chi(x)\right)_\pm dz \geq \frac{\overline{c}}{2} \right\}.$$

By construction of E we have a good estimate on E^c:

$$\partial_1 (G * \chi)(x) \geq 18\, \overline{c} \left[\min\left\{(G*\chi)(x),\, (1 - G*\chi)(x)\right\}\right]^2 - \overline{c} \quad \text{on } E^c,$$

and thus we obtain strict monotonicity of $G * \chi$ in e_1-direction outside E as long as the first term on the left-hand side dominates the second term:

$$\partial_1 (G*\chi) \geq \overline{c} \quad \text{on } E^c \cap \left\{ \frac{1}{3} \leq G * \chi \leq \frac{2}{3} \right\}.$$

Therefore

$$\int\limits_{\left\{\frac{1}{3} \leq G*\chi \leq \frac{2}{3}\right\}} \eta \left(\partial_1 (G*\chi) - \overline{c}\right)_-^2 dx = \int\limits_{E \cap \left\{\frac{1}{3} \leq G*\chi \leq \frac{2}{3}\right\}} \eta \left(\partial_1 (G*\chi) - \overline{c}\right)_-^2 dx \lesssim \int\limits_E \eta \, dx.$$

We introduce the parameter h again. By construction of E and since $|z| G_h(z) \lesssim \sqrt{h}\, G_h(z/2)$, we have

$$\begin{aligned}
\frac{1}{\sqrt{h}} \int\limits_E \eta \, dx \lesssim & \frac{1}{h} \int\limits_{\{z_1 \lessgtr 0\}} |z_1| G_h(z) \int \eta(x) \left(\chi(x+z) - \chi(x)\right)_\pm dx\, dz \qquad (77)\\
\lesssim & \frac{1}{\sqrt{h}} \int\limits_{\{z_1 \lessgtr 0\}} G_h(z/2) \int \eta(x) \left(\chi(x+z) - \chi(x)\right)_\pm dx\, dz \\
\lesssim & \frac{1}{\sqrt{h}} \int\limits_{\{z_1 \lessgtr 0\}} G_h(z) \int \eta(x) \left(\chi(x+z) - \chi(x)\right)_\pm dx\, dz \\
& + \frac{1}{\sqrt{h}} \int\limits_{\{z_1 \lessgtr 0\}} G_h(z) \int \eta(x) \left(\chi(x+2z) - \chi(x+z)\right)_\pm dx\, dz
\end{aligned}$$

by a change of coordinates $z \mapsto 2z$ and the subadditivity of the functions $u \mapsto u_\pm$. The last term can be handled using a Taylor expansion of η around x:

$$\begin{aligned}
& \frac{1}{\sqrt{h}} \int\limits_{\{z_1 \lessgtr 0\}} G_h(z) \int \eta(x) \left(\chi(x+2z) - \chi(x+z)\right)_\pm dx\, dz \\
= & \frac{1}{\sqrt{h}} \int\limits_{\{z_1 \lessgtr 0\}} G_h(z) \int \eta(x-z) \left(\chi(x+z) - \chi(x)\right)_\pm dx\, dz \\
= & \frac{1}{\sqrt{h}} \int\limits_{\{z_1 \lessgtr 0\}} G_h(z) \int \eta(x) \left(\chi(x+z) - \chi(x)\right)_\pm dx\, dz + O\left(\sqrt{h}\right),
\end{aligned}$$

where the constant in the $O(\sqrt{h})$-term depends linearly on $E_h(\chi)$ and $\|\nabla\eta\|_\infty$. Indeed, the error in the equation above is - up to a constant times $\|\nabla\eta\|_\infty$ - estimated by

$$\int \frac{|z|}{\sqrt{h}} G_h(z) \int |\chi(x+z)-\chi(x)|\,dx\,dz \lesssim \int G_h(\frac{z}{2}) \int |\chi(x+z)-\chi(x)|\,dx\,dz \lesssim \sqrt{h} E_h(\chi).$$

Using (71), we obtain

$$\frac{1}{\sqrt{h}} \int\limits_E \eta\,dx \lesssim \varepsilon^2 + \delta + \sqrt{h}\frac{1}{r^2} + \sqrt{h}\frac{1}{r} E_h(\chi)$$

and thus (72) holds. □

4.2 Tools for the multi-phase case

4.2.1 1d Lemma

In our application, we use the following corollary for three phases instead of Lemma 4.2 or Corollary 4.3. Nevertheless, we will make use of the estimates in Section 4.1 for the proof of this corollary. As in Proposition 4.1, we assume that χ_1 is the minority phase.

Corollary 4.5. *Let $\{\chi^n\}_n$ be obtained by Algorithm 1 and $\eta \in C_0^\infty(B_{2r})$ radially non-increasing cut-off of B_r in B_{2r} with $|\nabla\eta| \lesssim \frac{1}{r}$ and $\left|\nabla^2\eta\right| \lesssim \frac{1}{r^2}$. Then, for any m, n and for $s \ll 1$, we have for the majority phases χ_2 and χ_3,*

$$\begin{aligned}
&\frac{1}{\sqrt{h}} \int \eta\,|\chi_2^m - \chi_2^n|\,dx_1 + \frac{1}{\sqrt{h}} \int \eta\,|\chi_3^m - \chi_3^n|\,dx_1 \\
&\lesssim \varepsilon^2(x') + s + \frac{1}{s^2}\frac{1}{\sqrt{h}} \int \eta \left|G_h * \left(\chi^{m-1} - \chi^{n-1}\right)\right|^2 dx_1.
\end{aligned}$$

Here $s \ll 1$ means that there exists a universal constant $s_0 > 0$, such that the statement holds for all $0 < s \le s_0$. The function $\varepsilon^2 \colon B'_{2r} \subset \mathbb{R}^{d-1} \to [0,\infty)$ satisfies

$$\int \varepsilon^2(x')\,dx' \lesssim \varepsilon_1^2 + (1+E_0)\frac{1}{r^2}\sqrt{h},$$

where the number ε_1^2 is defined below. Furthermore, after integration in x', we have for the majority phases χ_2 and χ_3 and any $0 < s \le s_0$

$$\begin{aligned}
\frac{1}{\sqrt{h}} \int \eta\,|\chi_2^m - \chi_2^n|\,dx &+ \frac{1}{\sqrt{h}} \int \eta\,|\chi_3^m - \chi_3^n|\,dx \\
&\lesssim \varepsilon_1^2 + s r^{d-1} + \frac{1}{s^2}\frac{1}{\sqrt{h}} \int \eta \left|G_h * \left(\chi^{m-1} - \chi^{n-1}\right)\right|^2 dx \\
&+ \frac{1}{\sqrt{h}} \int \eta \left|G_{h/2} * (\chi^m - \chi^n)\right|^2 dx + (1+E_0)\,\frac{1}{r^2}\sqrt{h}.
\end{aligned}$$

For the minority phase χ_1, we have

$$\frac{1}{\sqrt{h}} \int \eta\,|\chi_1^m - \chi_1^n|\,dx \lesssim \varepsilon_1^2 + \frac{1}{\sqrt{h}} \int \eta \left|G_{h/2} * (\chi^m - \chi^n)\right|^2 dx + \frac{1}{r} \int |\chi^m - \chi^n|\,dx.$$

Here, $\varepsilon_1^2 := \varepsilon_1^2(\chi^m) + \varepsilon_1^2(\chi^n) + \varepsilon_1^2(\chi^{n-1})$ *and*

$$
\begin{aligned}
\varepsilon_1^2(\chi) :=& \frac{1}{\sqrt{h}} \int \eta \left[(1-\chi_1)\, G_h * \chi_1 + \chi_1\, G_h * (1-\chi_1) \right] dx \\
&+ \inf_{\chi^*} \Bigg\{ \frac{1}{\sqrt{h}} \int \eta \left[(1-\chi_2)\, G_h * \chi_2 + \chi_2\, G_h * (1-\chi_2) \right] dx \\
&\quad - \frac{1}{\sqrt{h}} \int \eta \left[(1-\chi^*)\, G_h * \chi^* + \chi^*\, G_h * (1-\chi^*) \right] dx + \frac{1}{r} \int_{B_{2r}} |\chi_2 - \chi^*|\, dx \\
&\quad + \frac{1}{\sqrt{h}} \int \eta \left[(1-\chi_3)\, G_h * \chi_3 + \chi_3\, G_h * (1-\chi_3) \right] dx \\
&\quad - \frac{1}{\sqrt{h}} \int \eta \left[(1-\chi^*)\, G_h * \chi^* + \chi^*\, G_h * (1-\chi^*) \right] dx + \frac{1}{r} \int_{B_{2r}} |\chi_3 - (1-\chi^*)|\, dx \Bigg\},
\end{aligned}
$$

where the infimum is taken over all half spaces $\chi^* = \mathbf{1}_{x_1 > \lambda}$ *in direction* e_1*.*

Proof. Step 1: Minority phase i=1. By (20) and (16), we have

$$
\begin{aligned}
\frac{1}{\sqrt{h}} \int \eta\, |\chi_1^m - \chi_1^n|\, dx \lesssim& \frac{1}{\sqrt{h}} \int \eta \left[(1-\chi_1^m)\, G_h * \chi_1^m + \chi_1^m\, G_h * (1-\chi_1^m) \right] dx \\
&+ \frac{1}{\sqrt{h}} \int \eta \left[(1-\chi_1^n)\, G_h * \chi_1^n + \chi_1^n\, G_h * (1-\chi_1^n) \right] dx \\
&+ \frac{1}{\sqrt{h}} \int \eta\, (\chi_1^m - \chi_1^n)\, G_h * (\chi_1^m - \chi_1^n)\, dx.
\end{aligned}
$$

As in the proof of Lemma 2.10 for ζ, we can deal with η to obtain

$$
\left| \frac{1}{\sqrt{h}} \int \eta\, (\chi_1^m - \chi_1^n)\, G_h * (\chi_1^m - \chi_1^n)\, dx_1 \right| \lesssim \frac{1}{\sqrt{h}} \int \eta \left| G_{h/2} * (\chi^m - \chi^n) \right|^2 dx + \frac{1}{r} \int |\chi^m - \chi^n|\, dx.
$$

Step 2: Minority phases $i = 2, 3$*.* Because of symmetry, we may restrict to $i = 2$. For the ease of notation, we define $\phi := G_h * \chi^{n-1}$, $\tilde{\phi} := G_h * \chi^{m-1}$ $\chi := \chi^n$, $\tilde{\chi} := \chi^m$. Setting (compare to Algorithm 1)

$$
u := \frac{1}{2} \left(\phi_2 - \phi_1 \vee \phi_3 + 1 \right) \in C^{0,1}(\mathbb{R}), \tag{78}
$$

and $\tilde{u}$ in the same way, we have $\chi_2 = \mathbf{1}_{u > 1/2}$, which allows us to apply Corollary 4.3 for u, χ_2 and $\tilde{u}$, $\tilde{\chi}_2$:

$$
\frac{1}{\sqrt{h}} \int \eta\, |\chi_2 - \tilde{\chi}_2|\, dx_1 \lesssim \frac{1}{\sqrt{h}} \int_{|u-1/2|<s} \eta \left(\sqrt{h}\, \partial_1 u - 2\overline{c} \right)_-^2 dx_1 + s + \frac{1}{s^2} \frac{1}{\sqrt{h}} \int \eta\, (u - \tilde{u})^2\, dx_1 \tag{79}
$$

for any $s > 0$. Since $x \mapsto x_+$ is 1-Lipschitz, we have

$$
|u - \tilde{u}| \le \frac{1}{2} \left| \phi_2 - \tilde{\phi}_2 \right| + \frac{1}{2} \left| \phi_1 \vee \phi_3 - \tilde{\phi}_1 \vee \tilde{\phi}_3 \right| \le \sum_{i=1}^{3} \left| \phi_i - \tilde{\phi}_i \right|.
$$

Therefore, the last term is estimated as desired. Now we turn to the first right-hand side term of (79). Now, we can estimate the first right-hand side term of (79). By

$$\{|u-1/2|<s\} = \{|\phi_1-\phi_2|<2s,\, \phi_1\geq\phi_3\} \cup \{|\phi_2-\phi_3|<2s,\, \phi_3\geq\phi_1\}$$

and since by the chain rule $\partial_1(\phi_1\vee\phi_3) = \partial_1\phi_1$ if $\phi_1\geq\phi_3$ and vice versa, we have

$$\begin{aligned}\frac{1}{\sqrt{h}}\int\limits_{|u-1/2|<s} \eta\left(\sqrt{h}\,\partial_1 u - 2\overline{c}\right)_-^2 dx_1 \leq & \frac{1}{\sqrt{h}}\int\limits_{|\phi_1-\phi_2|<2s,\,\phi_1\geq\phi_3} \eta\left(\sqrt{h}\,\partial_1(\phi_2-\phi_1) - 2\overline{c}\right)_-^2 dx_1\\ &+\frac{1}{\sqrt{h}}\int\limits_{|\phi_2-\phi_3|<2s,\,\phi_3\geq\phi_1} \eta\left(\sqrt{h}\,\partial_1(\phi_2-\phi_3) - 2\overline{c}\right)_-^2 dx_1.\end{aligned}$$

By the continuity of the map $(\phi_1,\phi_2,\phi_3)\mapsto u$ and $\phi_1+\phi_2+\phi_3=1$, we have for $s\ll 1$

$$\{|\phi_1-\phi_2|<2s,\,\phi_1\geq\phi_3\}\subset\{1/4<\phi_1<3/4\}\cup\{1/4<\phi_2<3/4\}.$$

Thus, by the subadditivity of $x\mapsto x_-$, we have for $s\ll 1$

$$\begin{aligned}\frac{1}{\sqrt{h}}\int\limits_{|\phi_1-\phi_2|<2s,\,\phi_1\geq\phi_3} \eta\left(\sqrt{h}\,\partial_1(\phi_2-\phi_1) - 2\overline{c}\right)_-^2 dx_1 \lesssim & \frac{1}{\sqrt{h}}\int\limits_{\frac14<\phi_2<\frac34} \eta\left(\sqrt{h}\,\partial_1\phi_2 - \overline{c}\right)_-^2 dx_1\\ &+\frac{1}{\sqrt{h}}\int\limits_{\frac14<\phi_1<\frac34} \eta\left(\sqrt{h}\,\partial_1\phi_1 + \overline{c}\right)_+^2 dx_1,\end{aligned}$$

and the same if we exchange the roles of ϕ_1 and ϕ_3. By Lemma 4.4, we can estimate the terms with ϕ_2 and ϕ_3. For the term with ϕ_1 - the minority phase - we note that since $|\sqrt{h}\partial_1\phi_1|\lesssim 1$, we can estimate

$$\begin{aligned}\frac{1}{\sqrt{h}}\int\limits_{\frac14<\phi_1<\frac34} \eta\left(\sqrt{h}\,\partial_1\phi_1 + \overline{c}\right)_+^2 dx_1 \lesssim & \frac{1}{\sqrt{h}}\int\limits_{G_h*(1-\chi_1^{n-1}),\,G_h*\chi_1^{n-1}>\frac14} \eta\left[\left(1-\chi_1^{n-1}\right)+\chi_1^{n-1}\right] dx_1\\ \lesssim & \frac{1}{\sqrt{h}}\int \eta\left[\left(1-\chi_1^{n-1}\right)G_h*\chi_1^{n-1}+\chi_1^{n-1}\,G_h*\left(1-\chi_1^{n-1}\right)\right] dx_1.\end{aligned}$$

□

4.2.2 Steps for Proposition 4.1

In the next two lemmas, we approximate the first variation of the metric term by an expression that makes the normal velocity appear. The main idea is to work, as for Lemma 2.6, on a mesoscopic time scale $\tau\sim\sqrt{h}$, introducing a fudge factor α, cf. Remark 1.4. The first lemma shows that we may coarsen the first variation from the microscopic time scale h to the mesoscopic time scale $\alpha\sqrt{h}$. It also shows that we may isolate the test vector field ξ.

Lemma 4.6 (First variation of the dissipation)**.** *Let $\xi \in C_0^\infty((0,T)\times B_r,\mathbb{R}^d)$. Then*

$$
\int_0^T -\delta E_h(\,\cdot\, - \chi^h(t-h))(\chi^h(t),\xi(t))\,dt
\approx \sum_{i=1}^{3}\tau\sum_{l=1}^{L}\int \frac{\chi_i^{Kl}-\chi_i^{K(l-1)}}{\tau}\,\xi(l\tau)\cdot\left(-\sqrt{h}\,\nabla G_h\right)*\left(\chi_i^{K(l-1)}+\chi_i^{Kl}\right)dx
$$

in the sense that the error is controlled by

$$
\|\xi\|_\infty\left(\frac{1}{\alpha}\varepsilon_1^2+\alpha^{1/3}\iint \eta\,d\mu_h+\alpha^{1/3}r^{d-1}T\right)+o(1),\quad \text{as } h\to 0,
$$

where $\eta\in C_0^\infty(B_{2r})$ is a radially symmetric, radially non-increasing cut-off for B_r in B_{2r} with $|\nabla\eta|\lesssim\frac{1}{r}$,

$$
\varepsilon_1^2 := h\sum_{n=1}^{N}\varepsilon_1^2(\chi^n)+\tau\sum_{l=1}^{L}\varepsilon_1^2(\chi^{Kl}),
$$

and the functional $\varepsilon_1^2(\chi)$ is defined in Corollary 4.5.

Proof. We recall the definition of the inner variation of $-E_h(\chi-\tilde\chi)$ in (30) and have for any pair of admissible functions $\chi,\tilde\chi$ and any test function $\xi\in C^\infty([0,\Lambda)^d,\mathbb{R}^d)$ for equal surface tensions:

$$
\begin{aligned}
-\delta E_h(\,\cdot\,-\tilde\chi)(\chi,\xi) &= \frac{d}{ds}\Big|_{s=0} -\frac{1}{\sqrt h}\sum_{i,j}\sigma_{ij}\int(\chi_{i,s}-\tilde\chi_i)\,G_h*(\chi_{j,s}-\tilde\chi_j)\,dx\\
&=\frac{2}{\sqrt h}\sum_{i,j}\sigma_{ij}\int(\chi_i-\tilde\chi_i)\,G_h*(\xi\cdot\nabla\chi_j)\,dx\\
&=\frac{2}{\sqrt h}\sum_{i}\int(\chi_i-\tilde\chi_i)\left[\nabla G_h*(\xi\chi_i)-G_h*((\nabla\cdot\xi)\,\chi_i)\right]dx.
\end{aligned}
$$

In our case, after integration in time, this yields

$$
\int_0^T -\delta E_h(\,\cdot\,-\chi^h(t-h))(\chi^h(t),\xi(t))\,dt
=\sum_{i=1}^{3}h\sum_{n=1}^{N}\frac{2}{\sqrt h}\int\left[-\nabla G_h*\left(\overline{\xi}^n\chi_i^n\right)+G_h*\left(\left(\nabla\cdot\overline{\xi}^n\right)\chi_i^n\right)\right]\left(\chi_i^n-\chi_i^{n-1}\right)dx,
$$

where

$$
\overline{\xi}^n := \frac{1}{h}\int_{nh}^{(n+1)h}\xi(t)\,dt
$$

denotes the time average of ξ over a microscopic time interval $[nh,(n+1)h)$.
Now we prove step by step that

1. the $(\nabla \cdot \xi)$-term is negligible as $h \to 0$;
2. we can freeze mesoscopic time for ξ, that is, substitute $\overline{\xi}^n$ by some nearby value $\xi(l_n\tau)$ at the expense of an $o(1)$-term;
3. we can smuggle in η at the expense of an $o(1)$-term;
4. we can freeze mesoscopic time for χ^h and substitute χ^n in the second factor by the mean $\frac{1}{2}\left(\chi^h((l_n-1)\tau)+\chi^h(l_n\tau)\right)$, which is the main step;
5. we can get rid of η again at the expense of an $o(1)$-term; and finally
6. we can pull ξ out of the convolution at the expense of an $o(1)$-term.

Note that Step 3 and Step 5 are just auxiliary steps for Step 4.

Step 1: The $(\nabla \cdot \xi)$-term vanishes as $h \to 0$. Using Jensen's inequality, we obtain

$$
\begin{aligned}
&\left| h\sum_{n=1}^{N} \frac{1}{\sqrt{h}} \int \left(\chi_i^n - \chi_i^{n-1}\right) G_h * \left(\left(\nabla \cdot \overline{\xi}^n\right) \chi_i^n\right) dx \right| \\
\leq& \|\nabla \xi\|_\infty T \frac{1}{\sqrt{h}} \frac{1}{N} \sum_{n=1}^{N} \int \left| G_h * \left(\chi_i^n - \chi_i^{n-1}\right)\right| dx \\
\lesssim& \|\nabla \xi\|_\infty T \frac{1}{\sqrt{h}} \left(\frac{1}{N} \sum_{n=1}^{N} \int \left| G_h * \left(\chi^n - \chi^{n-1}\right)\right|^2 dx \right)^{1/2}.
\end{aligned}
$$

Since the L^2-norm of $G_h * u$ is decreasing in h and by the energy-dissipation estimate (9), the error is controlled by

$$
\|\nabla \xi\|_\infty T \frac{1}{\sqrt{h}} \left(\frac{1}{N} \sqrt{h} E_0 \right)^{1/2} \leq \|\nabla \xi\|_\infty E_0^{1/2} T^{1/2} h^{1/4} = o(1).
$$

Step 2: Time freezing for ξ. We can approximate $\overline{\xi}^n$ by a nearby value $\xi(l_n\tau)$, where $l_n \in \{1, \dots L\}$ is chosen such that $K(l_n-1) < n \leq K l_n$. Note that $|\overline{\xi}^n - \xi^{l_n}| \leq \tau \|\partial_t \xi\|_\infty$. Therefore, by Jensen's inequality, we have

$$
\begin{aligned}
&\left| h\sum_{n=1}^{N} \frac{1}{\sqrt{h}} \int \left(\chi_i^n - \chi_i^{n-1}\right) \nabla G_h * \left(\left(\xi^{l_n} - \overline{\xi}^n\right) \chi_i^n\right) dx \right| \\
\leq& \alpha \|\partial_t \xi\|_\infty T \frac{1}{N} \sum_{n=1}^{N} \int \left| \nabla G_h * \left(\chi_i^n - \chi_i^{n-1}\right)\right| dx \\
\lesssim& \alpha \|\partial_t \xi\|_\infty T \left(\frac{1}{N} \sum_{n=1}^{N} \int \left| \nabla G_h * \left(\chi^n - \chi^{n-1}\right)\right|^2 dx \right)^{1/2}.
\end{aligned}
$$

But $\sqrt{h}\|\nabla G_h * u\|_{L^2} \lesssim \|G_{h/2} * u\|_{L^2}$ yields

$$
\int \left| \nabla G_h * \left(\chi^n - \chi^{n-1}\right)\right|^2 dx \lesssim \frac{1}{h} \int \left[G_{h/2} * \left(\chi^n - \chi^{n-1}\right)\right]^2 dx.
$$

Using the energy-dissipation estimate (9), the error is controlled by

$$\alpha\|\partial_t\xi\|_\infty T\left(\frac{1}{N}\frac{1}{\sqrt{h}}E_0\right)^{1/2}=\alpha\|\partial_t\xi\|_\infty E_0^{1/2}T^{1/2}h^{1/4}=o(1).$$

Step 3: Smuggling in η. We claim that

$$\begin{aligned}&h\sum_{n=1}^N\frac{1}{\sqrt{h}}\int\nabla G_h*(\xi(l_n\tau)\chi_i^n)\left(\chi_i^n-\chi_i^{n-1}\right)dx\\&=h\sum_{n=1}^N\frac{1}{\sqrt{h}}\int\eta\,\nabla G_{h/2}*(\xi(l_n\tau)\chi_i^n)\,G_{h/2}*\left(\chi_i^n-\chi_i^{n-1}\right)dx+o(1)\quad\text{as } h\to 0.\end{aligned}$$

Using $\nabla G_h=G_{h/2}*\nabla G_{h/2}$, the left-hand side is equal to

$$h\sum_{n=1}^N\frac{1}{\sqrt{h}}\int\nabla G_{h/2}*(\xi(l_n\tau)\chi_i^n)\,G_{h/2}*\left(\chi_i^n-\chi_i^{n-1}\right)dx.$$

Note that since $\eta\equiv 1$ on the support of ξ and $|z|\left|\nabla G_{1/2}(z)\right|\lesssim|z|^2G(z)$ has finite integral, we have for any $\chi\in\{0,1\}$,

$$\begin{aligned}\left|(1-\eta)\nabla G_{h/2}*(\xi\chi)\right|=&\left|\int\nabla G_{h/2}(z)(\eta(x+z)-\eta(x))\xi(x+z)\chi(x+z)\,dz\right|\\\lesssim&\|\nabla\eta\|_\infty\|\xi\|_\infty\int|z|\left|\nabla G_{h/2}(z)\right|dz\lesssim\|\nabla\eta\|_\infty\|\xi\|_\infty.\end{aligned}$$

Thus, using the Cauchy-Schwarz inequality and the energy-dissipation estimate (9), the error is controlled by

$$\begin{aligned}h^{1/4}\left(\sum_{n=1}^N\frac{1}{\sqrt{h}}\int\left|G_{h/2}*\left(\chi^n-\chi^{n-1}\right)\right|^2dx\right)^{1/2}&\left(h\sum_{n=1}^N\left(\|\nabla\eta\|_\infty\|\xi\|_\infty\right)^2\right)^{1/2}\\&\lesssim E_0^{1/2}T^{1/2}\|\nabla\eta\|_\infty\|\xi\|_\infty h^{1/4}=o(1).\end{aligned}$$

Step 4: Time freezing for χ^h. We claim that

$$\begin{aligned}&h\sum_{n=1}^N\frac{2}{\sqrt{h}}\int\eta\,\nabla G_{h/2}*(\xi(l_n\tau)\chi_i^n)\,G_{h/2}*\left(\chi_i^n-\chi_i^{n-1}\right)dx\\&\approx h\sum_{n=1}^N\frac{1}{\sqrt{h}}\int\eta\,\nabla G_{h/2}*\Big(\xi(l_n\tau)\left(\chi_i^h((l_n-1)\tau)+\chi_i^h(l_n\tau)\right)\Big)\,G_{h/2}*\left(\chi_i^n-\chi_i^{n-1}\right)dx,\end{aligned}$$

in the sense that the error is controlled by

$$\|\xi\|_\infty\left(\frac{1}{\alpha}\varepsilon_1^2+\alpha^{1/3}\iint\eta\,d\mu_h+\alpha^{1/3}r^{d-1}T\right)+o(1),\quad\text{as } h\to 0.$$

Here, we assumed that Phase 1 is the minority phase in the support of η. Indeed, we can control the error using the Cauchy-Schwarz inequality by

$$\left(\sum_{n=1}^{N}\frac{1}{\sqrt{h}}\int \eta^2\left|G_{h/2}*(\chi^n-\chi^{n-1})\right|^2dx\right)^{1/2}\times$$

$$\left(\tau\sum_{l=1}^{L}\frac{1}{K}\sum_{k=1}^{K}\frac{1}{\sqrt{h}}\int\left|\sqrt{h}\nabla G_{h/2}*\left[\xi(l\tau)\left(\chi_i^{K(l-1)+k}-\frac{1}{2}\left(\chi_i^{K(l-1)}+\chi_i^{Kl}\right)\right)\right]\right|^2dx\right)^{1/2}.$$

Since $0\le\eta\le 1$, the term in the first parenthesis is bounded by $\iint\eta\,d\mu_h$. For the term in the second parenthesis, we fix the mesoscopic block index l and the microscopic time step index k and sum at the end. Let $l=1$ and write ξ instead of $\xi(l\tau)$ for notational simplicity. We use the L^2-convolution estimate and introduce η in the second integral, which is equal to 1 on the support of ξ:

$$\begin{aligned}
&\frac{1}{\sqrt{h}}\int\left(\sqrt{h}\nabla G_{h/2}*\left[\xi\left(\chi_i^k-\frac{1}{2}\left(\chi_i^0+\chi_i^K\right)\right)\right]\right)^2dx\\
&\le\frac{1}{\sqrt{h}}\left(\int\left|\sqrt{h}\nabla G_{h/2}\right|dz\right)^2\int|\xi|^2\left[\chi_i^k-\frac{1}{2}\left(\chi_i^0+\chi_i^K\right)\right]^2dx\\
&\lesssim\|\xi\|_\infty^2\left(\frac{1}{\sqrt{h}}\int\eta\left|\chi^k-\chi^0\right|dx+\frac{1}{\sqrt{h}}\int\eta\left|\chi^K-\chi^k\right|dx\right).
\end{aligned}$$

With Corollary 4.5, we can estimate these terms and set for abbreviation

$$\alpha^2(k,k'):=\frac{1}{\sqrt{h}}\int\eta\left[G_h*\left(\chi^k-\chi^{k'}\right)\right]^2dx+\frac{1}{\sqrt{h}}\int\eta\left[G_{h/2}*\left(\chi^k-\chi^{k'}\right)\right]^2dx.$$

By Minkowski's triangle inequality w.r.t. the measure $\eta\,dx$, we see that α also satisfies a triangle inequality. Thus, thanks to Jensen's inequality,

$$\alpha^2(k-1,-1)\le\left(\sum_{n=0}^{k-1}\alpha(n,n-1)\right)^2\le k\sum_{n=0}^{k-1}\alpha^2(n,n-1)\le K\sum_{n=0}^{K-1}\alpha^2(n,n-1).$$

Therefore, using Corollary 4.5 and Lemma 2.7, we have

$$\begin{aligned}
\frac{1}{\sqrt{h}}\int\eta\left|\chi^k-\chi^0\right|dx\lesssim&\,\varepsilon_1^2(\chi^k)+\varepsilon_1^2(\chi^{k-1})+\varepsilon_1^2(\chi^0)+sr^{d-1}+\frac{1}{s^2}K\sum_{n=0}^{K}\alpha^2(n,n-1)\\
&+E_0\sqrt{\tau}+K\sum_{n=1}^{K}\alpha^2(n,n-1)+(1+E_0)\,\frac{1}{r^2}\sqrt{h}.
\end{aligned}$$

Note that since $s\le s_0\lesssim 1$, we can combine the two α^2-terms so that by $\sum_n\alpha^2(n,n-1)=\iint\eta\,d\mu_h$, after summation, we have

$$\tau\sum_{l=1}^{L}\frac{1}{K}\sum_{k=1}^{K}\frac{1}{\sqrt{h}}\int\eta\left|\chi^{Kl+k}-\chi^{Kl}\right|dx\lesssim\varepsilon_1^2+sr^{d-1}T+\frac{1}{s^2}\alpha^2\iint\eta\,d\mu_h+o(1),$$

as $h\to 0$. Now we choose the parameter s. If $(r^{d-1}T)^{-1}\iint\eta\,d\mu_h\le s_0^3$, we optimize in s.

Otherwise, we choose $s := s_0$. In the first case, we have

$$\min_{s\le s_0}\left\{sr^{d-1}T++\frac{1}{s^2}\alpha^2\iint\eta\,d\mu_h\right\}=\left(r^{d-1}T\right)^{2/3}\left(\alpha^2\iint\eta\,d\mu_h\right)^{1/3}.$$

Whereas in the second case, since $s_0>0$ is a universal constant and thus bounded away from 0, we obtain

$$s_0r^{d-1}T++\frac{1}{s_0^2}\alpha^2\iint\eta\,d\mu_h\lesssim\left(r^{d-1}T\right)^{2/3}\left(\alpha^2\iint\eta\,d\mu_h\right)^{1/3}+\alpha^2\iint\eta\,d\mu_h.$$

By Young's inequality, since $\alpha\le 1$, we can thus estimate in both cases

$$\min_{s\le s_0}\left\{sr^{d-1}T++\frac{1}{s^2}\alpha^2\iint\eta\,d\mu_h\right\}\lesssim\alpha^{2/3}r^{d-1}T+\alpha^{2/3}\iint\eta\,d\mu_h.$$

Using Young's inequality once more, the total error in this step is controlled by

$$\begin{aligned}&\|\xi\|_\infty\left(\iint\eta\,d\mu_h\right)^{1/2}\left(\varepsilon_1^2+\alpha^{2/3}r^{d-1}T+\alpha^{2/3}\iint\eta\,d\mu_h\right)^{1/2}+o(1)\\&\lesssim\|\xi\|_\infty\left(\frac{1}{\alpha^{1/3}}\varepsilon^2+\alpha^{1/3}r^{d-1}T+\alpha^{1/3}\iint\eta\,d\mu_h\right)+o(1).\end{aligned}$$

Step 5: Getting rid of η again. As in Step 3, we can estimate

$$\begin{aligned}&h\sum_{n=1}^N\frac{1}{\sqrt{h}}\int\eta\,G_{h/2}*\left(\chi_i^n-\chi_i^{n-1}\right)\nabla G_{h/2}*\left(\xi(l_n\tau)\left(\chi_i^h((l_n-1)\tau)+\chi_i^h(l_n\tau)\right)\right)dx\\&=h\sum_{n=1}^N\frac{1}{\sqrt{h}}\int\left(\chi_i^n-\chi_i^{n-1}\right)\nabla G_h*\left(\xi(l_n\tau)\left(\chi_i^h((l_n-1)\tau)+\chi_i^h(l_n\tau)\right)\right)dx+o(1),\end{aligned}$$

as $h\to 0$.

Step 6: Pulling out ξ. First, fix l and write $\xi=\xi(l\tau)$. For simplicity of the formula, we will ignore l and formally set $l=1$. Note that since ∇G is antisymmetric, we have for any $\chi,\,\tilde\chi\in\{0,1\}$,

$$\begin{aligned}&\int(\chi-\tilde\chi)\left[\xi\cdot\nabla G_h*\chi-\nabla G_h*(\xi\chi)\right]dx\\&\qquad=-\int\nabla G_h(z)\int(\chi-\tilde\chi)\,(x+z)\chi(x)\,(\xi(x+z)-\xi(x))\,dx\,dz.\end{aligned}$$

Set $K(z):=z\otimes z\;G(z)$, take a Taylor-expansion of ξ around x: $\xi(x+z)-\xi(x)=\nabla\xi(x)z+O(|z|^2)$, where the constant in the $O(|z|^2)$-term is depending linearly on $\|\nabla^2\xi\|_\infty$. Then the error on this single time interval splits into two terms.

The one coming from the first-order term in the expansion of ξ is

$$\begin{aligned}
&\left|\frac{1}{K}\sum_{k=1}^{K}\frac{1}{\sqrt{h}}\int \left(\chi_i^0+\chi_i^K\right)\nabla\xi \colon \left[K_h * \left(\chi_i^k-\chi_i^{k-1}\right)\right] dx\right| \\
&\lesssim \|\nabla\xi\|_\infty \frac{1}{\sqrt{h}}\left(\frac{1}{K}\sum_{k=1}^{K}\int \left|K_h * \left(\chi^k-\chi^{k-1}\right)\right|^2 dx\right)^{1/2},
\end{aligned}$$

where we used Jensen's inequality. Since $K_h = h\nabla^2 G_h + G_h\, Id$, $h\|\nabla^2 G_h * u\|_{L^2} \lesssim \|G_{h/2} * u\|_{L^2}$ for any u and since the L^2-norm of $G_h * u$ is non-increasing in h, we have for any characteristic functions $\chi, \tilde\chi$

$$\begin{aligned}
\int |K_h * (\chi-\tilde\chi)|^2\,dx \le& h^2\int \left|\nabla^2 G_h * (\chi-\tilde\chi)\right|^2 dx + \int \left[G_h * (\chi-\tilde\chi)\right]^2 dx \\
\lesssim& \int \left[G_{h/2} * (\chi-\tilde\chi)\right]^2 dx.
\end{aligned}$$

Plugging this into the inequality above, multiplying by τ, summing over the block index l and using Jensen's inequality, we can control the contribution to the error coming from the first-order term by

$$T\|\nabla\xi\|_\infty \frac{1}{\sqrt{h}}\left(\frac{1}{N}\sum_{n=1}^{N}\int \left|G_{h/2} * \left(\chi^n-\chi^{n-1}\right)\right|^2 dx\right)^{1/2} \le \|\nabla\xi\|_\infty E_0^{1/2} T^{1/2} h^{1/4} = o(1),$$

where we used the energy-dissipation estimate at the end.
By Lemma 2.6, the contribution coming from the second-order term in the expansion of ξ is controlled by

$$\begin{aligned}
&\|\nabla^2\xi\|_\infty h \sum_{n=1}^{N}\int \left(\frac{|z|}{\sqrt{h}}\right)^3 G_h(z)\int \left|\chi^n-\chi^{n-1}\right| dx\,dz \\
&\lesssim \|\nabla^2\xi\|_\infty \int_0^T\int \left|\chi^h(t)-\chi^h(t-h)\right| dx\,dt \lesssim \|\nabla^2\xi\|_\infty E_0(1+T)\sqrt{h} = o(1).
\end{aligned}$$

Finally, we note that by the time freezing in Step 4, we constructed a telescope sum: Rewriting the summation over the microscopic time step index $n = 1,\dots,N$ as the double sum over the microscopic time step index $k = 1,\dots,K$ in the respective mesoscopic time intervals and the mesoscopic block index $l = 1,\dots,L$, we have for each l,

$$\begin{aligned}
&\sum_{k=1}^{K}\left(\chi_i^{K(l-1)+k}-\chi_i^{K(l-1)+k-1}\right)\xi(l\tau)\cdot\nabla G_h * \left(\chi_i^{K(l-1)}+\chi_i^{Kl}\right) \\
&= \left(\chi_i^{Kl}-\chi_i^{K(l-1)}\right)\xi(l\tau)\cdot\nabla G_h * \left(\chi_i^{K(l-1)}+\chi_i^{Kl}\right).
\end{aligned}$$

Thus, we obtain

$$h\sum_{n=1}^{N}\frac{1}{\sqrt{h}}\int\left(\chi_i^n-\chi_i^{n-1}\right)\nabla G_h*\Big(\xi(l_n\tau)\left(\chi_i^h((l_n-1)\tau)+\chi_i^h(l_n\tau)\right)\Big)\,dx$$
$$=\frac{1}{\alpha}\tau\sum_{l=1}^{L}\int\left(\chi_i^{Kl}-\chi_i^{K(l-1)}\right)\xi(l\tau)\cdot\nabla G_h*\left(\chi_i^{K(l-1)}+\chi_i^{Kl}\right)dx+o(1),$$

which concludes the proof. □

While the first lemma made the mesoscopic time derivative $\frac{1}{\tau}\left(\chi_i^{Kl}-\chi_i^{K(l-1)}\right)$ appear, the upcoming second lemma makes the approximate normal, here e_1, appear.

Lemma 4.7. *Under the hypotheses of Lemma 4.6, we have*

$$\sum_{i=1}^{3}\tau\sum_{l=1}^{L}\int\xi(l\tau)\cdot\left(-\sqrt{h}\,\nabla G_h\right)*\left(\chi_i^{K(l-1)}+\chi_i^{Kl}\right)\frac{\chi_i^{Kl}-\chi_i^{K(l-1)}}{\tau}\,dx$$
$$\approx 2c_0\tau\sum_{l=1}^{L}\left(\int\xi_1(l\tau)\,\frac{\chi_2^{Kl}-\chi_2^{K(l-1)}}{\tau}dx-\int\xi_1(l\tau)\,\frac{\chi_3^{Kl}-\chi_3^{K(l-1)}}{\tau}dx\right),$$

in the sense that the error is controlled by

$$\|\xi\|_\infty\left[\frac{1}{\alpha}\tau\sum_{l=1}^{L}\varepsilon_1^2(\chi^{Kl})+\frac{1}{\alpha}\tau\sum_{l=1}^{L}\frac{1}{\sqrt{h}}\int\eta\left|\chi^{Kl}-\chi^{K(l-1)}\right|k_h*\left(\eta\left|\chi^{Kl}-\chi^{K(l-1)}\right|\right)dx\right]+o(1),$$

as $h\to 0$, where $0\le k(z)\le |z|G(z)$. Let us comment on the error term: The first part of the error term arises because e_1 is only the approximate normal. The second part arises in the passage from a diffuse to a sharp interface and formally is of quadratic nature.

Proof. Step 1: Substitution of ∇G for majority phases $i=2,3$. We want to replace the convolution with $-\nabla G$ on the left-hand side of the claim by a convolution with the anisotropic kernel

$$K(z):=\operatorname{sign}(z_1)\,z\,G(z).$$

To that purpose, we claim that for any characteristic function $\chi\in\{0,1\}$,

$$\frac{1}{\sqrt{h}}\int\eta\left|\sqrt{h}\,\nabla G_h*\chi-\left(\chi\,K_h*(1-\chi)+(1-\chi)K_h*\chi\right)\right|dx\lesssim\varepsilon^2+\frac{\sqrt{h}}{r^2}+\frac{\sqrt{h}}{r}E_h(\chi).\qquad(80)$$

Here,

$$\varepsilon^2:=\inf_{\chi^*}\left\{\frac{1}{\sqrt{h}}\int\eta\left[\chi\,G_h*(1-\chi)+(1-\chi)\,G_h*\chi\right]dx\right.$$
$$\left.-\frac{1}{\sqrt{h}}\int\eta\left[\chi^*\,G_h*(1-\chi^*)+(1-\chi^*)\,G_h*\chi^*\right]dx+\frac{1}{r}\int_{B_{2r}}|\chi-\chi^*|\,dx\right\},$$

where the infimum is taken over all half spaces $\chi^* = \mathbf{1}_{x_1>\lambda}$ in direction e_1. Using this inequality for $\chi_i^{K(l-1)}$ and χ_i^{Kl}, $i=3$ and with $-K$ instead of K for $i=2$, we can substitute those two summands and the error is estimated as desired; for instance the error for χ_2 is controlled by

$$\begin{aligned}
&\frac{1}{\alpha}\|\xi\|_\infty \tau \sum_{l=0}^{L} \frac{1}{\sqrt{h}} \int \eta \left| \sqrt{h}\, \nabla G_h * \chi_2^{Kl} - \left[\chi_2^{Kl}\, K_h * \left(1-\chi_2^{Kl}\right) + \left(1-\chi_2^{Kl}\right) K_h * \chi_2^{Kl} \right] \right| dx \\
&\lesssim \frac{1}{\alpha}\|\xi\|_\infty \tau \sum_{l=0}^{L} \varepsilon_1^2(\chi^{Kl}) + o(1), \quad \text{as } h \to 0.
\end{aligned}$$

Argument for (80): By measuring length in terms of $\sqrt{h}$, we may assume that $h=1$. Since $\int \nabla G\, dz = 0$ and $\nabla G(z) = -zG(z)$, using the identities $u = |u| - 2u_-$ and $u = -|u| + 2u_+$,

$$\begin{aligned}
\nabla G * \chi &= \int z\, G(z)\, (\chi(x+z) - \chi(x))\, dz \\
&= \int_{\{z_1>0\}} K(z)\, |\chi(x+z) - \chi(x)|\, dz - 2 \int_{\{z_1>0\}} z\, G(z)\, (\chi(x+z) - \chi(x))_-\, dz \\
&\quad + \int_{\{z_1<0\}} K(z)\, |\chi(x+z) - \chi(x)|\, dz - 2 \int_{\{z_1<0\}} z\, G(z)\, (\chi(x+z) - \chi(x))_+\, dz.
\end{aligned}$$

Using $|\chi_1 - \chi_2| = (1-\chi_1)\chi_2 + \chi_1(1-\chi_2)$ for $\chi_1,\, \chi_2 \in \{0,1\}$, this implies the pointwise identity

$$\nabla G * \chi = \chi\, K * (1-\chi) + (1-\chi) K * \chi - 2 \int_{\{z_1 \lessgtr 0\}} \operatorname{sign}(z_1)\, z\, G(z)\, (\chi(x+z) - \chi(x))_\pm\, dz,$$

where the last term stands for the sum of the two integrals. Integration w.r.t. $\eta\, dx$ now yields:

$$\begin{aligned}
&\int \eta \Big| \nabla G * \chi - (\chi K * (1-\chi) + (1-\chi) K * \chi) \Big| dx \\
&\lesssim \int_{\{z_1 \lessgtr 0\}} |z| G(z) \int \eta(x)\, (\chi(x+z) - \chi(x))_\pm\, dx\, dz.
\end{aligned}$$

As in the argument for (72), we can follow the lines from (77) on so that (71) yields (80).

Step 2: Rough estimate for minority phase $i=1$. By a manipulation as in the proof of Lemma 4.6 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left| \tau \sum_{l=1}^{L} \int \frac{\chi_1^{Kl} - \chi_1^{K(l-1)}}{\tau}\, \xi(l\tau) \cdot \left(-\sqrt{h}\, \nabla G_h\right) * \left(\chi_1^{K(l-1)} + \chi_1^{Kl}\right) dx \right| \\
&\leq \|\xi\|_\infty \left(\sum_{l=1}^{L} \int \eta \left| G_{h/2} * \left(\chi_1^{Kl} - \chi_1^{K(l-1)}\right) \right|^2 dx \right)^{1/2} \left(\sum_{l=0}^{L} \int \eta \left| \sqrt{h} \nabla G_{h/2} * \chi_1^{Kl} \right|^2 dx \right)^{1/2} + o(1),
\end{aligned}$$

as $h \to 0$. Note that for any characteristic function $\chi \in \{0,1\}$, since $\int \nabla G(z)\,dz = 0$,

$$\begin{aligned}
\frac{1}{\sqrt{h}} \int \eta \left|\sqrt{h}\nabla G_{h/2} * \chi\right|^2 dx &\lesssim \frac{1}{\sqrt{h}} \int \eta \left|\sqrt{h}\nabla G_{h/2} * \chi\right| dx \\
&\lesssim \frac{1}{\sqrt{h}} \int \left|\sqrt{h}\nabla G_{h/2}(z)\right| \int \eta(x)\, |\chi(x+z) - \chi(x)|\, dx\, dz \\
&\lesssim \frac{1}{\sqrt{h}} \int G_h(z) \int \eta(x)\, |\chi(x+z) - \chi(x)|\, dx\, dz \\
&= \frac{1}{\sqrt{h}} \int \eta \left[(1-\chi)\, G_h * \chi + \chi\, G_h * (1-\chi)\right] dx.
\end{aligned}$$

Treating the metric term as in the proof of Lemma 4.6 with the triangle inequality and Jensen's inequality afterwards, we obtain the bound

$$\|\xi\|_\infty \left(\iint \eta\, d\mu_h\right)^{1/2} \left(\tau \sum_{l=1}^{L} \varepsilon_1^2(\chi^{Kl})\right)^{1/2} + o(1) \leq \|\xi\|_\infty \left(\frac{\tau}{\alpha} \sum_{l=1}^{L} \varepsilon_1^2(\chi^{Kl}) + \alpha \iint \eta\, d\mu_h\right) + o(1).$$

Step 3: An identity for K. We claim that for any two characteristic functions χ, $\tilde{\chi} \in \{0,1\}$, we have the pointwise identity

$$\begin{aligned}
&(\chi - \tilde{\chi}) \big(\chi\, K_h * (1-\chi) + (1-\chi) K_h * \chi + \tilde{\chi}\, K_h * (1-\tilde{\chi}) + (1-\tilde{\chi}) K_h * \tilde{\chi}\big) \\
&\quad = 2c_0\, e_1\, (\chi - \tilde{\chi}) - |\chi - \tilde{\chi}|\, K_h * (\chi - \tilde{\chi})\,.
\end{aligned}$$

Indeed, by scaling, we may w.l.o.g. assume $h = 1$ and start with

$$\begin{aligned}
&(\chi - \tilde{\chi})\, \tilde{\chi}\, K * (1-\tilde{\chi}) + (\chi - \tilde{\chi})\, (1-\tilde{\chi})\, K * \tilde{\chi} \\
&= (\chi - 1)\, \tilde{\chi} \left(\int K - K * \tilde{\chi}\right) + \chi\, (1-\tilde{\chi})\, K * \tilde{\chi} \\
&= (\chi - 1)\, \tilde{\chi} \left(\int K\right) + \big((1-\chi)\, \tilde{\chi} + \chi\, (1-\tilde{\chi})\big) K * \tilde{\chi} \\
&= (\chi - 1)\, \tilde{\chi} \left(\int K\right) + |\chi - \tilde{\chi}|\, K * \tilde{\chi}.
\end{aligned}$$

Exchanging the roles of χ and $\tilde{\chi}$, one obtains for the second part

$$\begin{aligned}
&(\chi - \tilde{\chi})\, \chi\, K * (1-\chi) + (\chi - \tilde{\chi})\, (1-\chi)\, K * \chi \\
&\qquad = -\, (\tilde{\chi} - 1)\ \chi \left(\int K\right) - |\chi - \tilde{\chi}|\, K * \chi.
\end{aligned}$$

Using the factorization property of G and the symmetry $\int z' G^{d-1}(z')dz' = 0$, one computes that for any vector $\xi \in \mathbb{R}^d$

$$
\begin{aligned}
\xi \cdot \int K = & \int \operatorname{sign}(z_1) \int \left(\xi_1\, z_1 + \xi' \cdot z'\right) G^{d-1}(z')\, dz'\, G^1(z_1)\, dz_1 \\
=& \xi_1 \int |z_1|\, G^1(z_1)\, dz_1 = 2\,\xi_1 \int_0^\infty z_1\, G^1(z_1)\, dz_1 \\
=& 2\,\xi_1 \int_0^\infty -\frac{d}{dz_1} G^1(z_1)\, dz_1 = 2\,\xi_1 G^1(0) = 2\,\xi_1 \frac{1}{\sqrt{2\pi}} = 2c_0\xi_1.
\end{aligned}
$$

Hence, the identity follows from $(\chi-1)\tilde\chi - (\tilde\chi - 1)\chi = \chi - \tilde\chi$.

Step 4: Conclusion. Applying Steps 1 and 2, using the identity in Step 3 for the remaining two terms involving Phases 2, 3, we end up with the right-hand side of the claim. The error is controlled by

$$
\|\xi\|_\infty \left[\frac{1}{\alpha}\tau \sum_{l=1}^{L} \varepsilon_1^2(\chi^{Kl}) + \frac{1}{\alpha}\tau \sum_{l=1}^{L} \frac{1}{\sqrt{h}} \int \eta^2 \left|\chi^{Kl} + \chi^{K(l-1)}\right| |K_h| * \left|\chi^{Kl} - \chi^{K(l-1)}\right| dx \right] + o(1),
$$

as $h \to 0$. Note that $|K| = k$, where k is the kernel defined in the statement of the lemma. It remains to argue that η can be equally distributed on both copies of $\left|\chi^{Kl} - \chi^{K(l-1)}\right|$. For this, note that for $u = \left|\chi^{Kl} - \chi^{K(l-1)}\right| \in [0,1]$,

$$
\begin{aligned}
& \frac{1}{\sqrt{h}} \left| \int \eta^2 u\, k_h * u\, dx - \int \eta u\, k_h * (\eta u)\, dx \right| \\
& \leq \frac{1}{\sqrt{h}} \int k_h(z) \int \eta(x) u(x) u(x+z)\, |\eta(x+z) - \eta(x)|\, dx\, dz \\
& \leq \|\nabla \eta\|_\infty \int \frac{|z|}{\sqrt{h}} k_h(z)\, dz \int \eta u\, dx \\
& \lesssim \frac{1}{r} \int u\, dx.
\end{aligned}
$$

Thus, in our case, we can use Lemma 2.7 and bound the error by

$$
\frac{1}{\alpha} \|\xi\|_\infty \frac{1}{r} \tau \sum_{l=1}^{L} \int \left|\chi^{Kl} - \chi^{K(l-1)}\right| dx \lesssim \frac{1}{\alpha^{1/2}} \|\xi\|_\infty \frac{1}{r} E_0 T h^{1/4} = o(1).
$$

□

The following lemma deals with the error term in the foregoing lemma and brings it into the standard form.

Lemma 4.8. *Under the hypotheses of Lemma 4.6, we have*

$$\begin{aligned}
&\frac{1}{\alpha}\tau\sum_{l=1}^{L}\frac{1}{\sqrt{h}}\int \eta\left|\chi^{Kl}-\chi^{K(l-1)}\right|k_h*\left(\eta\left|\chi^{Kl}-\chi^{K(l-1)}\right|\right)dx\\
&\lesssim\frac{1}{\alpha}\tau\sum_{l=1}^{L}\left[\varepsilon_1^2(\chi^{Kl-1})+\varepsilon_1^2(\chi^{Kl})\right]+\alpha^{1/3}\iint\eta\,d\mu_h+\alpha^{1/3}r^{d-1}T+o(1),
\end{aligned}$$

as $h\to 0$.

Proof. First, we note that it is enough to prove the following similar statement for fixed time:

$$\frac{1}{\sqrt{h}}\int \eta\left|\chi^{K}-\chi^{0}\right|k_h*\left(\eta\left|\chi^{K}-\chi^{0}\right|\right)dx\lesssim\varepsilon_1^2+\frac{\alpha^{1/3}}{\sqrt{h}}\int_0^\tau\int\eta\,d\mu_h+\alpha^{4/3}r^{d-1}+o(1), \tag{81}$$

where similar to the notation in Corollary 4.5, we write $\varepsilon_1^2:=\varepsilon_1^2(\chi^K)+\varepsilon_1^2(\chi^{K-1})+\varepsilon_1^2(\chi^0)$. Indeed, dividing by α and summing this estimate over mesoscopic time blocks yields the desired bound. In the proof of (81), we will exploit the convolution in the normal direction e_1 in Step 1, which will allow us in Step 2 to make use of the quadratic structure of this term.

Step 1: We can estimate the kernel k by a kernel that factorizes in two kernels k^1, k' in normal- and tangential direction, respectively, which are of the form

$$\begin{aligned}
k^1(z_1):=&(1+z_1^2)^{1/2}\,G^1(z_1),\\
k'(z'):=&(1+|z'|^2)^{1/2}\,G^{d-1}(z').
\end{aligned}$$

Let us still denote the kernel by k. We have

$$k_h*\left(\eta|\chi^K-\chi^0|\right)\le\sup_{x_1}\left\{k_h'*'k_h^1*_1\left(\eta|\chi^K-\chi^0|\right)\right\}\le k_h'*'\sup_{x_1}\left\{k_h^1*_1\left(\eta|\chi^K-\chi^0|\right)\right\}.$$

The second factor in the right-hand side convolution can be estimated in two ways:

$$\begin{aligned}
\sup_{x_1}\left\{k_h^1*_1\left(\eta|\chi^K-\chi^0|\right)\right\}\le&\min\left\{\int k_h^1\,dz_1\sup_{x_1}\left(\eta|\chi^K-\chi^0|\right),\left(\sup_{x_1}k_h^1\right)\int\eta\left|\chi^K-\chi^0\right|dx_1\right\}\\
\lesssim&\min\left\{1,\frac{1}{\sqrt{h}}\int\eta\left|\chi^K-\chi^0\right|dx_1\right\}.
\end{aligned}$$

Therefore, we obtain a quadratic term with two copies of $\frac{1}{\sqrt{h}}\int\eta\left|\chi^K-\chi^0\right|dx_1$:

$$\begin{aligned}
&\frac{1}{\sqrt{h}}\int\eta\left|\chi^K-\chi^0\right|\,k_h*\left(\eta\left|\chi^K-\chi^0\right|\right)dx\\
&\quad\lesssim\int\left(\frac{1}{\sqrt{h}}\int\eta\left|\chi^K-\chi^0\right|dx_1\right)k_h'*'\left(1\wedge\frac{1}{\sqrt{h}}\int\left|\chi^K-\chi^0\right|dx_1\right)dx'.
\end{aligned}$$

Step 2: Now we use Corollary 4.5 before integration in x' for the majority phases. We adopt the notation $\varepsilon^2(x')$ from there (with K playing the role of n) and set

$$\alpha^2(x') := \frac{1}{\sqrt{h}} \int \eta \left|G_h * \left(\chi^{K-1} - \chi^{-1}\right)\right|^2 dx_1 + \frac{1}{\sqrt{h}} \int \eta \left|G_h * \left(\chi^{K} - \chi^{0}\right)\right|^2 dx_1.$$

As in Step 4 of the proof of Lemma 4.6, we optimize the right-hand side (this time before integration in x') w.r.t. s. Then we have for $i = 2, 3$,

$$\begin{aligned}
&\int \left(\frac{1}{\sqrt{h}} \int \eta \left|\chi_i^K - \chi_i^0\right| dx_1\right) k_h' *' \left(1 \wedge \frac{1}{\sqrt{h}} \int \left|\chi^K - \chi^0\right| dx_1\right) dx' \\
&\lesssim \int \left(\varepsilon^2(x') + \alpha^2(x') + \alpha^{2/3}(x')\right) \\
&\qquad\qquad k_h' *' \left(1 \wedge \left(\varepsilon^2(x') + \alpha^2(x') + \alpha^{2/3}(x') + \frac{1}{\sqrt{h}} \int \eta \left|\chi_1^K - \chi_1^0\right| dx_1\right)\right) dx' \\
&\lesssim \int \left(\varepsilon^2(x') + \alpha^2(x') + \alpha^{2/3}(x')\right) \\
&\qquad\qquad k_h' *' \left(1 \wedge \left(\varepsilon^{4/3}(x') + \alpha^{2/3}(x') + \left(\frac{1}{\sqrt{h}} \int \eta \left|\chi_1^K - \chi_1^0\right| dx_1\right)^{2/3}\right)\right) dx'.
\end{aligned}$$

For the first two summands in the first factor, $\varepsilon^2(x') + \alpha^2(x')$, we use the 1 in the minimum on the right and obtain

$$\int \left(\varepsilon^2(x') + \alpha^2(x')\right) k_h' *' 1 \, dx' \lesssim \int \left(\varepsilon^2(x') + \alpha^2(x')\right) dx'.$$

For the last summand on the left, $\alpha^{2/3}(x')$, we use the second term in the minimum for the pairing. Then by Hölder's inequality and the L^p-convolution, we have

$$\begin{aligned}
&\int \alpha^{2/3}(x') \, k_h' *' \left(\varepsilon^{4/3}(x') + \alpha^{2/3}(x') + \left(\frac{1}{\sqrt{h}} \int \eta \left|\chi_1^K - \chi_1^0\right| dx_1\right)^{2/3}\right) dx' \\
&\lesssim \left(\int \alpha^2(x') \, dx'\right)^{1/3} \left[\left(\int \varepsilon^2(x') \, dx'\right)^{2/3} + \left(\frac{1}{\sqrt{h}} \int \eta \left|\chi_1^K - \chi_1^0\right| dx\right)^{2/3}\right] + \int \alpha^{4/3}(x') \, dx'.
\end{aligned}$$

Using Hölder's inequality against $\mathbf{1}_B$ for the last term and Young's inequality, this is controlled by

$$\int \varepsilon^2(x') \, dx' + \int \alpha^2(x') \, dx' + \frac{1}{\sqrt{h}} \int \eta \left|\chi_1^K - \chi_1^0\right| dx + \alpha^{4/3} r^{d-1} + \frac{1}{\alpha^{2/3}} \int \alpha^2(x') \, dx'.$$

Note that for the minority phase χ_1 we clearly have

$$\int \left(\frac{1}{\sqrt{h}} \int \eta \left|\chi_1^K - \chi_1^0\right| dx_1\right) k_h' *' \left(1 \wedge \frac{1}{\sqrt{h}} \int \left|\chi^K - \chi^0\right| dx_1\right) dx' \lesssim \frac{1}{\sqrt{h}} \int \eta \left|\chi_1^K - \chi_1^0\right| dx$$

by using the 1 in the minimum on the right. Therefore, using Corollary 4.5 for the minority phase χ_1, we obtain

$$\frac{1}{\sqrt{h}}\int \eta\left|\chi^K-\chi^0\right| k_h * \left(\eta\left|\chi^K-\chi^0\right|\right)dx$$
$$\lesssim \varepsilon_1^2+\alpha^{4/3}r^{d-1}+\frac{1}{\alpha^{2/3}}\int \alpha^2(x')\,dx'+\frac{1}{r}\int \eta\left|\chi^K-\chi^0\right|dx+(1+E_0)\frac{\sqrt{h}}{r^2}.$$

Iterating the triangle inequality for the metric term as in Step 4 of the proof of Lemma 4.6, we have

$$\int \alpha^2(x')\,dx' \le \frac{\alpha}{\sqrt{h}}\int_0^\tau\int \eta\,d\mu_h.$$

Using Lemma 2.7 for the term measuring the L^1-distance of χ^0 and χ^K, we obtain (81). □

If there is just a small amount of the boundaries inside a ball B, we can estimate roughly.

Lemma 4.9. *In the situation as in Proposition 4.1, we have*

$$\left|\sum_{i=1}^3\int_0^T\int(\nabla\cdot\xi-\nu_i\cdot\nabla\xi\,\nu_i)\left|\nabla\chi_i\right|dt\right|+\left|\sum_{i=1}^3\int_0^T\int\xi\cdot\nu_iV_i\left|\nabla\chi_i\right|dt\right|$$
$$\lesssim \|\xi\|_\infty\left[\sum_{i=1}^3\int_0^T\int\eta\left(\frac{1}{\alpha}+\alpha\,V_i^2\right)\left|\nabla\chi_i\right|dt+\alpha\iint\eta\,d\mu\right].$$

Proof. Thanks to the convergence assumption (8), we can apply Proposition 3.1. Using the Euler-Lagrange equation for χ^n, we can identify the first term on the left-hand side as the limit of the first variation of the dissipation functional as $h\to0$. Following Step 1 of the proof of Lemma 4.6 and then estimating directly as in Step 3, but for ξ instead of η, we obtain

$$\sum_{i=1}^3\int_0^T\int(\nabla\cdot\xi-\nu_i\cdot\nabla\xi\,\nu_i)\left|\nabla\chi_i\right|dt$$
$$=2\lim_{h\to0}\sum_{i=1}^3\sum_{n=1}^N\int\bar\xi^n\cdot\left[\left(\sqrt{h}\nabla G_{h/2}\right)*\chi_i^n\right]\left[G_{h/2}*(\chi_i^n-\chi_i^{n-1})\right]dx.$$

Using the Cauchy-Schwarz inequality, we have

$$\left|\sum_{i=1}^3\sum_{n=1}^N\int\bar\xi^n\cdot\left[\left(\sqrt{h}\nabla G_{h/2}\right)*\chi_i^n\right]\left[G_{h/2}*(\chi_i^n-\chi_i^{n-1})\right]dx\right|$$
$$\lesssim\|\xi\|_\infty\left(\sum_{n=1}^N\frac{1}{\sqrt{h}}\int\eta\left|G_{h/2}*(\chi^n-\chi^{n-1})\right|^2dx\right)^{1/2}\left(h\sum_{n=1}^N\frac{1}{\sqrt{h}}\int\eta\left|\sqrt{h}\nabla G_{h/2}*\chi^n\right|^2dx\right)^{1/2}.$$

The first right-hand side factor is bounded by $\iint\eta\,d\mu_h$. For the second right-hand side factor, note that since $\int \nabla G(z)\,dz = 0$, we have for any $\chi \in \{0,1\}$,

$$\begin{aligned}
\int \eta\, |\nabla G * \chi|^2\, dx &\lesssim \int \eta\, |\nabla G * \chi|\, dx \\
&\le \int |\nabla G(z)| \int \eta(x)\, |\chi(x+z) - \chi(x)|\, dx\, dz \\
&\lesssim \int G_2(z) \int \eta(x)\, |\chi(x+z) - \chi(x)|\, dx\, dz \\
&= \int \eta\, [(1-\chi)\, G_2 * \chi + \chi\, G_2 * (1-\chi)]\, dx
\end{aligned}$$

Using this for χ_i^n and introducing the parameter h again, we can bound the second term by

$$\sum_{i=1}^3 h \sum_{n=1}^N \frac{1}{\sqrt{h}} \int \eta\, [(1-\chi)\, G_2 * \chi + \chi\, G_2 * (1-\chi)]\, dx \to 2c_0 \sum_{i=1}^3 \int_0^T \int \eta\, |\nabla \chi_i|\, dt,$$

as $h \to 0$, where we used Remark 2.9. Thus, using Young's inequality, we have

$$\left| \sum_{i=1}^3 \int_0^T \int (\nabla \cdot \xi - \nu_i \cdot \nabla \xi\, \nu_i)\, |\nabla \chi_i|\, dt \right| \lesssim \|\xi\|_\infty \left(\frac{1}{\alpha} \sum_{i=1}^3 \int_0^T \int \eta\, |\nabla \chi_i|\, dt + \alpha \iint \eta\, d\mu_h \right).$$

To estimate the second term in the lemma, note that by Young's inequality we have

$$|\xi \cdot \nu_i\, V_i| \le \|\xi\|_\infty \eta \left(\frac{1}{\alpha} V_i^2 + \alpha \right).$$

Integrating w.r.t. $|\nabla \chi_i|\, dt$ yields

$$\left| \int_0^T \int \xi \cdot \nu_i\, V_i\, |\nabla \chi_i|\, dt \right| \le \|\xi\|_\infty \left(\int_0^T \int \eta\, V_i^2\, |\nabla \chi_i|\, dt + \int_0^T \int \eta\, |\nabla \chi_i|\, dt \right),$$

which concludes the proof. □

5 Convergence

In Section 3, we identified the limit of the first variation of the energy; in Section 4, we identified the limit of first variation of the metric term up to an error that measures the local approximability by a half space. In this section, we show by soft arguments from Geometric Measure Theory that this error can be made arbitrarily small. Before that, we will state the main ingredients of the proof here.

Definition 5.1. Given $r > 0$, we define the covering

$$\mathscr{B}_r := \{B_r(i) \colon i \in \mathscr{L}_r\}$$

of $[0,\Lambda)^d$, where $\mathscr{L}_r = [0,\Lambda)^d \cap \frac{r}{\sqrt{d}}\mathbb{Z}^d$ is a regular grid of midpoints on $[0,\Lambda)^d$. By construction, for each $n \geq 1$ and each $r > 0$, the covering

$$\{B_{nr}(i) \colon i \in \mathscr{L}_r\} \quad \text{is locally finite,} \tag{82}$$

in the sense that for each point in $[0,\Lambda)^d$, the number of balls containing this point is bounded by a constant $c(d,n)$ which is independent of r. For given $\delta > 0$ and $\chi : [0,\Lambda)^d \to \{0,1\} \in BV$, we define $\mathscr{B}_{r,\delta}$ to be the subset of $\mathscr{B}_r$ consisting of all balls B such that the following two conditions hold:

$$\inf_{\nu^*} \int \eta_{2B} \left|\nu - \nu^*\right|^2 |\nabla \chi| \leq \delta r^{d-1} \quad \text{and} \tag{83}$$

$$\int_{2B} |\nabla \chi| \geq \frac{1}{2} \omega_{d-1} (2r)^{d-1}, \tag{84}$$

where η_{2B} is a cut-off for $2B$.

Lemma 5.2 (Approximation of the normal)**.** *For every $\varepsilon > 0$ and $\chi : [0,\Lambda)^d \to \{0,1\}$, there exists an $r_0 > 0$ such that for all $r \leq r_0$ there exist unit vectors $\nu_B \in S^{d-1}$ such that*

$$\sum_{B \in \mathscr{B}_r} \frac{1}{2} \int \eta_{2B} \left|\nu - \nu_B\right|^2 |\nabla \chi| \lesssim \varepsilon^2 \int |\nabla \chi| \,.$$

The following lemma will be used to control the error terms obtained in Section 4 on the "bad" balls $B \in \mathscr{B}_r - \mathscr{B}_{r,\delta}$.

Lemma 5.3. *For any $\delta > 0$ and any $\chi \colon [0,\Lambda)^d \to \{0,1\} \in BV$, we have*

$$\lim_{r \to 0} \sum_{B \in \mathscr{B}_r - \mathscr{B}_{r,\delta}} \int_{2B} |\nabla \chi| = 0.$$

In a rescaled version, the following lemma can be used to control the error terms on the "good“ balls $B \in \mathscr{B}_{r,\delta}$.

Lemma 5.4. *Let η be a radially symmetric cut-off for the unit ball B. Then for any $\varepsilon > 0$ there exists $\delta = \delta(d,\varepsilon) > 0$ such that for any $\chi \colon [0,\Lambda)^d \to \{0,1\}$ with*

$$\int \eta \left|\nu - e_1\right|^2 |\nabla \chi| \leq \delta^2 \tag{85}$$

there exists a half space χ^* in direction e_1 such that

$$\left| \int_B \left(|\nabla \chi| - |\nabla \chi^*| \right) \right| \leq \varepsilon^2, \quad \int_B |\chi - \chi^*| \, dx \leq \varepsilon^2. \tag{86}$$

Lemma 5.5. *Let η be a cut-off for the unit ball B. Then for any $\varepsilon > 0$ there exists $\delta = \delta(d,\varepsilon) > 0$ such that for any $\chi \colon [0,\Lambda)^d \to \{0,1\}^3$ with $\chi_1 + \chi_2 + \chi_3 = 1$, the following statement holds:*

Whenever we can approximate each normal separately, i.e.

$$\sum_{i=1}^{3} \inf_{\nu_i^*} \frac{1}{2} \int \eta \, |\nu_i - \nu_i^*|^2 \, |\nabla \chi_i| \leq \delta^2,$$

then we can do so with one normal $\nu^* \in S^{d-1}$ *and its inverse* $-\nu^*$*:*

$$\min_{1 \leq i \leq 3} \inf_{\nu^*} \left\{ \int_B |\nabla \chi_i| + \frac{1}{2} \int_B |\nu_{i+1} - \nu^*|^2 \, |\nabla \chi_{i+1}| + \frac{1}{2} \int_B |\nu_{i+2} + \nu^*|^2 \, |\nabla \chi_{i+2}| \right\} \leq \varepsilon^2,$$

where $i+1$, $i+2$ *are to be understood mod 3.*

5.1 Proof of Theorem 1.3

Using Proposition 4.1 and the lemmas from above, we can give the proof of the main result. The proof consists of three steps:

1. Post-processing Propositions 3.1 and 4.1, using the Euler-Lagrange equation (31) and by making the half space time-dependent,
2. Estimates for fixed time and
3. Integration in time.

Proof of Theorem 1.3. Step 1: Post-processing Propositions 3.1 and 4.1.

Let us first link the results we obtained in Sections 3 and 4. For any fixed vector $\nu^* \in S^{d-1}$ and any test function $\xi \in C_0^\infty((0,T) \times B, \mathbb{R}^d)$ we claim

$$\begin{aligned}
&\left| \sum_{i=1}^{3} \int_0^T \int \left(\nabla \cdot \xi_B - \nu_i \cdot \nabla \xi_B \, \nu_i + 2 \, \xi_B \cdot \nu_i \, V_i \right) |\nabla \chi_i| \, dt \right| \qquad (87)\\
&\lesssim \|\xi\|_\infty \left[\left(\min_{1 \leq i \leq 3} \int_0^T \left(\frac{1}{\alpha} \mathscr{E}_i^2(\nu^*, t) + \alpha^{1/3} r^{d-1} \right) dt \right) \wedge \left(\frac{1}{\alpha} \sum_{i=1}^{3} \int_0^T \int_B |\nabla \chi_i| \, dt \right) \right.\\
&\qquad \left. + \, \alpha^{1/3} \iint \eta_B \, d\mu + \alpha \sum_{i=1}^{3} \int_0^T \int \eta_B \, V_i^2 \, |\nabla \chi_i| \, dt \right],
\end{aligned}$$

where $\mathscr{E}_1^2$ is defined via

$$\begin{aligned}
\mathscr{E}_1^2(\nu^*, t) := &\int \eta_{2B} \, |\nabla \chi_1(t)| + \int \eta_{2B} \, |\nu_2(t) - \nu^*|^2 \, |\nabla \chi_2(t)| + \int \eta_{2B} \, |\nu_3(t) + \nu^*|^2 \, |\nabla \chi_3(t)| \\
&+ \inf_{\chi^*} \left\{ \left| \int \eta_B \left(|\nabla \chi_2(t)| - |\nabla \chi^*| \right) \right| + \left| \int \eta_B \left(|\nabla \chi_3(t)| - |\nabla \chi^*| \right) \right| \right.\\
&\qquad \left. + \frac{1}{r} \int_{2B} |\chi_2(t) - \chi^*| \, dx + \frac{1}{r} \int_{2B} |\chi_3(t) - (1 - \chi^*)| \, dx \right\}.
\end{aligned}$$

The infimum is taken over all half spaces $\chi^* = \mathbf{1}_{\{x\cdot\nu^*>\lambda\}}$ in direction ν^*. For $i=2,3$, $\mathscr{E}_i^2$ is a similar expression with cyclically exchanged roles of the phases.
Indeed, by symmetry, may assume w.l.o.g. that the minimum over i on the right-hand side of (87) is realized for $i=1$. The Euler-Lagrange equation (31) of the minimizing movement interpretation (5) links Proposition 3.1 with the metric term:

$$\lim_{h\to 0}\int_0^T -\delta E_h(\,\cdot\, - \chi^h(t-h))(\chi^h(t),\xi_B(t))\,dt = -c_0\sum_{i=1}^3\int_0^T\int (\nabla\cdot\xi_B - \nu_i\cdot\nabla\xi_B\,\nu_i)\,|\nabla\chi_i|\,dt.$$

Using Proposition 4.1 yields (87).
Now let $\xi\in C_0^\infty((0,T)\times[0,\Lambda)^d,\mathbb{R}^d)$ be given. First, we localize ξ in space according to the covering $\mathscr{B}_r$ from Definition 5.1. To do so, we introduce a subordinate partition of unity $\{\varphi_B\}_{B\in\mathscr{B}_r}$ and set $\xi_B := \varphi_B\xi$. Then $\xi=\sum_{B\in\mathscr{B}_r}\xi_B$, $\xi_B\in C_0^\infty(B)$ and $\|\xi_B\|_\infty\le\|\xi\|_\infty$. Given a radially symmetric and radially non-increasing cut-off η of $B_1(0)$ in $B_2(0)$, for each ball B in the covering, we can construct a cut-off η_B of B in $2B$ by shifting and rescaling. Given any measurable function $\nu^*\colon(0,T)\to S^{d-1}$ and any $\alpha\in(0,1)$ we claim

$$\begin{aligned}
&\left|\sum_{i=1}^3\int_0^T\int (\nabla\cdot\xi_B - \nu_i\cdot\nabla\xi_B\,\nu_i + 2\,\xi_B\cdot\nu_i\,V_i)\,|\nabla\chi_i|\,dt\right| \qquad (88)\\
&\lesssim \|\xi\|_\infty\left[\int_0^T\left(\frac{1}{\alpha}\mathscr{E}_B^2(\nu^*(t),t)+\alpha^{1/3}r^{d-1}\right)\wedge\left(\frac{1}{\alpha}\sum_{i=1}^3\int_B|\nabla\chi_i|\right)dt + \alpha^{1/3}\iint\eta\,d\mu\right.\\
&\qquad\left. +\alpha\sum_{i=1}^3\int_0^T\int\eta\,V_i^2\,|\nabla\chi_i|\,dt\right],
\end{aligned}$$

where $\mathscr{E}_B^2(\nu^*,t) := \min_{1\le i\le 3}\mathscr{E}_i^2(\nu^*,t)$ for $\nu^*\in S^{d-1}$.
We give the argument for (88) in two steps. In the first step, starting from (87) for ξ_B playing the role of ξ, we symmetrize the second left-hand side term. In the second step, we approximate the measurable function $\nu^*\colon(0,T)\to S^{d-1}$ by piecewise constant functions to show that (87) together with the symmetrization implies (88).
We start with the first step. To symmetrize the second term on the left-hand side of (63), we note

$$\begin{aligned}
&\left|\sum_{i=1}^3\int\xi_B\cdot\nu_i\,V_i\,|\nabla\chi_i| - \left(\int\xi_B\cdot\nu^*\,V_2\,|\nabla\chi_2| - \int\xi_B\cdot\nu^*\,V_3\,|\nabla\chi_3|\right)\right|\\
&\le\|\xi\|_\infty\left(\int_B|V_1|\,|\nabla\chi_1| + \int_B|\nu_2-\nu^*|\,|V_2|\,|\nabla\chi_2| + \int_B|\nu_3+\nu^*|\,|V_3|\,|\nabla\chi_3|\right).
\end{aligned}$$

After integration in time, we can estimate the terms on the right-hand side with the Cauchy-Schwarz inequality and Young's inequality. We have on the one hand for the majority phases, e.g. for $i=2$:

$$\int_0^T \int_B |\nu_2 - \nu^*| \, |V_2| \, |\nabla \chi_2| \, dt \le \left(\int_0^T \int_B |\nu_2 - \nu^*|^2 \, |\nabla \chi_2| \, dt \right)^{1/2} \left(\int_0^T \int_B V_2^2 \, |\nabla \chi_2| \, dt \right)^{1/2}$$
$$\lesssim \frac{1}{\alpha} \int_0^T \int_B |\nu_2 - \nu^*|^2 \, |\nabla \chi_2| \, dt + \alpha \int_0^T \int_B V_2^2 \, |\nabla \chi_2| \, dt.$$

and on the other hand for the minority phase $i=1$:

$$\int_0^T \int_B |V_1| \, |\nabla \chi_1| \, dt \lesssim \frac{1}{\alpha} \int_0^T \int_B |\nabla \chi_1| \, dt + \alpha \int_0^T \int_B V_1^2 \, |\nabla \chi_1| \, dt.$$

Here, we see, why we needed to introduce extra terms in $\mathscr{E}_1$ compared to the terms that were already present in the definition of $\mathscr{E}_1$ in Section 4. These different terms are sometimes called *tilt-excess* and *excess energy*, respectively. Therefore, Proposition 4.1 applied on ξ_B implies (87).

Now we give the argument that (87) implies (88). We approximate the measurable function ν^* in time by a piecewise constant function. Let $0 = T_0 < \cdots < T_M = T$ denote the partition of $(0,T)$ such that the approximation ν_M^* of ν^* is constant on each interval $[T_{m-1}, T_m)$ Since the measures on the left-hand side are absolutely continuous in time, we can approximate ξ_B by vector fields which vanish at the points T_m and both, the curvature and the velocity term converge. Therefore, we can apply (87) on each time interval (T_{m-1}, T_m). Lebesgue's dominated convergence gives us the convergence of the integral on the right-hand side and thus (88) holds.

Step 2: Estimates for fixed time. Let $t \in (0,T)$ be fixed. We will omit the argument t in the following. Let $\varepsilon > 0$ and let $\delta = \delta(\varepsilon)$ (to be determined later). Let $\mathscr{B}_{r,\delta}$ be defined as the set of good balls in the lattice:

$$\mathscr{B}_{r,\delta} := \left\{ B \in \mathscr{B}_r \colon \sum_{i=1}^3 \inf_{\nu^*} \int \eta_{2B} \, |\nu_i - \nu^*|^2 \, |\nabla \chi_i| \le \delta r^{d-1} \text{ and } \sum_{i=1}^3 \int_{2B} |\nabla \chi_i| \ge \frac{1}{2} \omega_{d-1} (2r)^{d-1} \right\}.$$

For $B \in \mathscr{B}_{r,\delta}$, and $i = 1,2,3$, we denote by ν_{B_i} the vector ν^* for which the infimum is attained, so that

$$\sum_{i=1}^3 \frac{1}{2} \int \eta_{2B} \, |\nu_i - \nu_{B,i}|^2 \, |\nabla \chi_i| \le \delta r^{d-1}.$$

By a rescaling and since η is radially symmetric, we can upgrade Lemma 5.5, so that for given $\gamma > 0$, we can find $\delta = \delta(d,\gamma) > 0$ (independent of χ) and $\nu_B \in S^{d-1}$, such that

$$\min_{1 \le i \le 3} \left\{ \int \eta_B \, |\nabla \chi_i| + \frac{1}{2} \int \eta_B \, |\nu_{i+1} - \nu_B|^2 \, |\nabla \chi_{i+1}| + \frac{1}{2} \int \eta_B \, |\nu_{i+2} + \nu_B|^2 \, |\nabla \chi_{i+2}| \right\} \le \gamma r^{d-1}.$$

Rescaling Lemma 5.4, we can define $\gamma = \gamma(\varepsilon) > 0$ and a half space χ^* in direction ν_B, such that

$$\mathscr{E}_B^2(\nu_B, t) \leq \varepsilon^2 r^{d-1}.$$

These two steps give us the dependence of δ on ε. Using the lower bound on the perimeters on $B \in \mathscr{B}_{r,\delta}(t)$, we obtain

$$\sum_{B\in\mathscr{B}_{r,\delta}} \left(\frac{1}{\alpha}\mathscr{E}_B^2(\nu_B,t) + \alpha^{1/3} r^{d-1}\right) \lesssim \sum_{B\in\mathscr{B}_{r,\delta}} \left(\frac{1}{\alpha}\varepsilon^2 + \alpha^{1/3}\right) r^{d-1} \lesssim \left(\frac{1}{\alpha}\varepsilon^2 + \alpha^{1/3}\right) \sum_{i=1}^{3} \int |\nabla \chi_i| \,.$$

Note that for the balls $B \in \mathscr{B}_r - \mathscr{B}_{r,\delta}$, we have by Lemma 5.3:

$$\sum_{B\in\mathscr{B}_r-\mathscr{B}_{r,\delta}} \sum_{i=1}^{3} \int_B |\nabla \chi_i| \to 0, \quad \text{as } r \to 0. \tag{89}$$

The speed of convergence depends on χ and ε (through δ).

Step 3: Integration in time. Using Lebesgue's dominated convergence theorem, we can integrate the pointwise-in-time estimates of Step 2. Recalling the decomposition $\xi = \sum_B \xi_B$ and using the finite overlap (82), we have

$$\begin{aligned}
&\left|\sum_{i=1}^{3} \int_0^T \int (\nabla\cdot\xi - \nu_i\cdot\nabla\xi\,\nu_i + 2\,\xi\cdot\nu_i\,V_i)\,|\nabla\chi_i|\,dt\right| \\
&\lesssim \sum_{B\in\mathscr{B}_r} \left|\sum_{i=1}^{3} \int_0^T \int (\nabla\cdot\xi_B - \nu_i\cdot\nabla\xi_B\,\nu_i + 2\,\xi_B\cdot\nu_i\,V_i)\,|\nabla\chi_i|\,dt\right| \\
&\lesssim \|\xi\|_\infty \Bigg[\left(\frac{1}{\alpha}\varepsilon^2 + \alpha^{1/3}\right) \int_0^T \sum_{i=1}^{3} \int |\nabla\chi_i|\,dt + \int_0^T \sum_{B\in\mathscr{B}_r-\mathscr{B}_{r,\delta}(t)} \frac{1}{\alpha} \sum_{i=1}^{3} \int_B |\nabla\chi_i|\,dt \\
&\qquad\qquad + \alpha^{1/3} \iint d\mu + \alpha \int_0^T \int V_i^2\,|\nabla\chi_i|\,dt\Bigg].
\end{aligned}$$

We use the energy-dissipation estimate (9) for the first term. By Lebesgue's dominated convergence and (89), the second term vanishes as $r \to 0$. Since the measure μ is finite and by Lemma 2.10, we can handle the last two terms. Thus we obtain

$$\left|\sum_{i=1}^{3} \int_0^T \int (\nabla\cdot\xi - \nu_i\cdot\nabla\xi\,\nu_i + 2\,\xi\cdot\nu_i\,V_i)\,|\nabla\chi_i|\,dt\right| \lesssim \|\xi\|_\infty \left(\frac{1}{\alpha}\varepsilon^2 E_0 T + \alpha^{1/3}(1+T)E_0\right).$$

Taking first the limit ε to zero and then α to zerp yields (6), which concludes the proof of Theorem 1.3. $\square$

5.2 Proofs of the lemmas

Proof of Lemma 5.2. Let $\varepsilon > 0$ be given and w.l.o.g. $\int |\nabla\chi| > 0$. Since the normal ν is measurable, we can approximate it by a continuous vector field $\tilde{\nu}\colon [0,\Lambda)^d \to \overline{B}$ in the sense that

$$\sum_{B\in\mathscr{B}_r} \frac{1}{2}\int_B |\nu-\tilde{\nu}|^2\,|\nabla\chi| \lesssim \int |\nu-\tilde{\nu}|\,|\nabla\chi| \le \varepsilon^2\int|\nabla\chi|\,,$$

where we have used the finite overlap property (82). Since $\tilde{\nu}$ is continuous, we can find $r_0 > 0$ and for any $r \le r_0$, vectors $\tilde{\nu}_B$ with $|\tilde{\nu}_B| \le 1$ such that

$$\sum_{B\in\mathscr{B}_r} \frac{1}{2}\int_B |\tilde{\nu}-\tilde{\nu}_B|^2\,|\nabla\chi| \le \varepsilon^2\int|\nabla\chi|\,.$$

The only missing step is to argue that we can also choose $\nu_B \in S^{d-1}$. If $|\tilde{\nu}_B| \ge 1/2$, this is clear because then $|\nu - \tilde{\nu}_B/|\tilde{\nu}_B|| \le 2\,|\nu-\tilde{\nu}_B|$. If $|\tilde{\nu}_B| \le 1/2$, we have the easy estimate

$$|\nu-\tilde{\nu}_B| \ge \frac{1}{2} \ge \frac{1}{4}\left(|\nu|+|\nu_B|\right) \ge \frac{1}{4}\,|\nu-\nu_B|$$

for any $\nu_B \in S^{d-1}$. □

Proof of Lemma 5.3. Let $\varepsilon, \delta > 0$ be arbitrary. Note that a ball in $\mathscr{B}_r - \mathscr{B}_{r,\delta}$ satisfies

$$\inf_{\nu^*}\int_{2B} |\nu-\nu^*|^2\,|\nabla\chi| \ge \delta r^{d-1} \quad\text{or} \tag{90}$$

$$\int_{2B} |\nabla\chi| \le \frac{1}{2}\omega_{d-1}r^{d-1}. \tag{91}$$

Step 1: Balls satisfying (90). By Lemma 5.2, for any $\gamma > 0$, to be chosen later, there exists $r_0 = r_0(\gamma,\delta,\chi) > 0$, such that for every $r \le r_0$ we can find vectors $\nu_B \in S^{d-1}$ such that

$$\sum_{B\in\mathscr{B}_r}\int_{2B} |\nu-\nu_B|^2\,|\nabla\chi| \lesssim \gamma\delta\int|\nabla\chi|\,. \tag{92}$$

Thus we have

$$\#\left\{B\colon \int_{2B}|\nu-\nu_B|^2\,|\nabla\chi| \ge \delta r^{d-1}\right\} \le \sum_B \frac{1}{\delta r^{d-1}}\int_{2B}|\nu-\nu_B|^2\,|\nabla\chi| \overset{(92)}{\lesssim} \frac{\gamma}{r^{d-1}}\int|\nabla\chi|\,. \tag{93}$$

Using that the covering is locally finite and De Giorgi's structure result, we have

$$\sum_{B:(90)}\int_{2B}|\nabla\chi| \lesssim \int_{\bigcup_{(90)}2B}|\nabla\chi| = \mathscr{H}^{d-1}\left(\partial^*\Omega \cap \bigcup_{(90)} 2B\right).$$

Since $\partial^*\Omega$ is rectifiable, we can find Lipschitz graphs Γ_n such that $\partial^*\Omega \subset \bigcup_{n=1}^\infty \Gamma_n$. Therefore,

$$\mathscr{H}^{d-1}\left(\partial^*\Omega \cap \bigcup_{(90)} 2B\right) \leq \sum_{n=1}^N \mathscr{H}^{d-1}\left(\Gamma_n \cap \bigcup_{(90)} 2B\right) + \mathscr{H}^{d-1}\left(\partial^*\Omega - \bigcup_{n\leq N} \Gamma_n\right).$$

Note that for any ball B

$$\mathscr{H}^{d-1}\left(\Gamma_n \cap 2B\right) \lesssim \left(1 + \mathrm{Lip}\ \Gamma_n\right) r^{d-1}$$

and thus

$$\mathscr{H}^{d-1}\left(\Gamma_n \cap \bigcup_{(90)} 2B\right) \leq \sum_{B:(90)} \mathscr{H}^{d-1}\left(\Gamma_n \cap 2B\right) \lesssim \left(1 + \max_{n\leq N} \mathrm{Lip}\ \Gamma_n\right) r^{d-1} \#\left\{B\colon (90)\right\}.$$

Using (93), we have

$$\sum_{B:(90)} \int_{2B} |\nabla\chi| \lesssim \left(1 + \max_{n\leq N} \mathrm{Lip}\ \Gamma_n\right) \gamma \int |\nabla\chi| + \mathscr{H}^{d-1}\left(\partial^*\Omega - \bigcup_{n\leq N} \Gamma_n\right).$$

Now, choose N large enough such that

$$\mathscr{H}^{d-1}\left(\partial^*\Omega - \bigcup_{n\leq N} \Gamma_n\right) \leq \varepsilon^2.$$

Then, choose $\gamma > 0$ small enough, such that

$$\left(1 + \max_{n\leq N} \mathrm{Lip}\ \Gamma_n\right) \gamma \int |\nabla\chi| \leq \varepsilon^2.$$

Step 2: Balls satisfying (91). By De Giorgi's structure theorem (Theorem 4.4 in [16]), we may restrict to balls B which in addition satisfy $\partial^*\Omega \cap 2B \neq \emptyset$ and pick $x \in \partial^*\Omega \cap 2B$. Note that since B has radius r we have

$$B_{2r}(x) \subset 4B \subset B_{6r}(x).$$

Therefore, if (91) holds,

$$\int_{B_{2r}(x)} |\nabla\chi| \leq \int_{4B} |\nabla\chi| \leq \frac{1}{2}\omega_{d-1}(2r)^{d-1}.$$

For $x \in \partial^*\Omega$ we have

$$\liminf_{r\to 0} \frac{1}{r^{d-1}} \int_{B_r(x)} |\nabla\chi| \geq \omega_{d-1}$$

and thus in particular

$$\mathbf{1}\left(\left\{x \in \partial^*\Omega\colon \int\limits_{B_r(x)} |\nabla\chi| \leq \frac{1}{2}\omega_{d-1}r^{d-1}\right\}\right) \to 0$$

pointwise as $r \to 0$. By De Giorgi's structure theorem (Theorem 4.4 in [16]), the finite overlap and Lebesgue's dominated convergence theorem, we thus have

$$\sum_{B:(91)} \int\limits_{2B} |\nabla\chi| \lesssim \mathscr{H}^{d-1}\left(\partial^*\Omega \cap \bigcup_{B:(91)} 2B\right) \to 0$$

as $r \to 0$. □

Proof of Lemma 5.4. Let us first prove that for any χ satisfying (85), we have

$$(1-\delta)\int \eta\,|\nabla\chi| \leq \left|\int \chi\nabla\eta\,dx\right| + \delta. \tag{94}$$

Indeed, we have

$$\left|\int \eta\,\nu\,|\nabla\chi|\right| \geq \left|\int \eta\,e_1\,|\nabla\chi|\right| - \left|\int \eta\,(\nu - e_1)\,|\nabla\chi|\right| = \int \eta\,|\nabla\chi| - \left|\int \eta\,(\nu - e_1)\,|\nabla\chi|\right|$$

and using the Cauchy-Schwarz inequality, (85) and Young's inequality

$$\begin{aligned}\left|\int \eta\,(\nu - e_1)\,|\nabla\chi|\right| &\leq \int \eta\,|\nu - e_1|\,|\nabla\chi| \\ &\leq \left(\int \eta\,|\nabla\chi|\right)^{1/2}\left(\int \eta\,|\nu - e_1|^2\,|\nabla\chi|^2\right)^{1/2} \\ &\leq \delta\left(\int \eta\,|\nabla\chi|\right)^{1/2} \leq \delta + \delta\int \eta\,|\nabla\chi|\,.\end{aligned}$$

Thus,

$$\left|\int \eta\,\nu\,|\nabla\chi|\right| \geq (1-\delta)\int \eta\,|\nabla\chi| - \delta,$$

which is (94).

Now we give an indirect argument. Suppose there exists an $\varepsilon > 0$ and a sequence $\{\chi_n\}_n$ such that

$$\int \eta\,|\nu_n - e_1|^2\,|\nabla\chi_n| \leq \frac{1}{n^2} \tag{95}$$

while for all half spaces χ^* in direction e_1,

$$\int\limits_B |\nabla\chi_n| \geq \varepsilon^2 + \int\limits_B |\nabla\chi^*|, \quad \int\limits_B |\nabla\chi^*| \geq \varepsilon^2 + \int\limits_B |\nabla\chi_n|, \quad \text{or} \quad \int\limits_B |\chi_n - \chi^*|\,dx \geq \varepsilon^2. \tag{96}$$

By (95), we can use (94) for χ_n and obtain:

$$\int \eta\,|\nabla \chi_n| \le \frac{1}{1-1/n}\left(\int |\nabla \eta|\,dx + \frac{1}{n}\right) \quad \text{stays bounded as } n\to\infty.$$

Therefore, after passage to a subsequence and a diagonal argument to exhaust the open ball $\{\eta>0\}$, we find χ such that

$$\chi_n \to \chi \quad \text{pointwise a.e. on } \{\eta>0\}. \tag{97}$$

By (95) we have

$$2\int \eta\,|\nabla\chi_n| - 2\int \nabla\eta\cdot e_1\,\chi_n\,dx = \int \eta\,|\nu_n - e_1|^2\,|\nabla\chi_n| \le \frac{1}{n^2}\to 0.$$

Since the first term on the left-hand side is lower semi-continuous and the second one is continuous, we can pass to the limit and obtain

$$\int \eta\,|\nu-e_1|^2\,|\nabla\chi| = 2\int \eta\,|\nabla\chi| - 2\int \nabla\eta\cdot e_1\,\chi\,dx \le 0.$$

Hence

$$\nu = e_1 \quad |\nabla\chi|\text{-a.e. in } \{\eta>0\}.$$

A mollification argument shows that there exists a half space χ^* in direction e_1 such that

$$\chi = \chi^* \quad \text{a.e. in } \{\eta>0\}.$$

Because of (97), this rules out

$$\int_B |\chi_n - \chi^*| \ge \varepsilon^2$$

on the one hand. On the other hand, by lower semi-continuity of the perimeter, also

$$\int_B |\nabla\chi^*| \ge \varepsilon^2 + \int_B |\nabla\chi_n|$$

is ruled out. To obtain a contradiction also w.r.t. the first statement in (96), let $\tilde\eta\le\eta$ be a cut-off for B in $(1+\delta)B$. Since (94) holds also for $\tilde\eta$ instead of η, we have

$$\begin{aligned}\varepsilon^2 + \int_B |\nabla\chi^*| &\overset{(96)}{\le} \int_B |\nabla\chi_n| \le \int \tilde\eta\,|\nabla\chi_n| \overset{(94)}{\le} \frac{1}{1-1/n}\left(\left|\int \chi_n\,\nabla\tilde\eta\,dx\right| + \frac{1}{n}\right)\\ &\overset{(97)}{\to} \left|\int \chi^*\,\nabla\tilde\eta\,dx\right| = \left|\int \tilde\eta\,\nabla\chi^*\right| \le \int_{(1+\delta)B} |\nabla\chi^*|\,.\end{aligned}$$

Since χ^* is a half space and therefore has no mass on ∂B, we have

$$\int_{(1+\delta)B} |\nabla\chi^*| \to \int_B |\nabla\chi^*|, \quad \text{as } \delta \to 0,$$

which is a contradiction. $\square$

Proof of Lemma 5.5. We give an indirect argument. Assume there exists a sequence of characteristic functions $\{\chi^n\}_n$ with $\chi_1^n + \chi_2^n + \chi_3^n = 1$ a.e., a number $\varepsilon > 0$ such that we can find approximate normals $\nu_i^{*n} \in S^{d-1}$ with

$$\sum_{i=1}^{3} \frac{1}{2} \int \eta \, |\nu_i^n - \nu_i^{*n}|^2 \, |\nabla\chi_i^n| \leq \frac{1}{n^2}$$

while for all $\nu^* \in S^{d-1}$, $n \in \mathbb{N}$ and $i = 1, 2, 3$, we have

$$\int_B |\nabla\chi_i^n| + \frac{1}{2} \int_B \left|\nu_{i+1}^n - \nu^*\right|^2 \left|\nabla\chi_{i+1}^n\right| + \frac{1}{2} \int_B \left|\nu_{i+2}^n + \nu^*\right|^2 \left|\nabla\chi_{i+2}^n\right| \geq \varepsilon^2. \tag{98}$$

Since S^{d-1} is compact, we can find vectors $\nu^* \in S^{d-1}$, such that, after passing to a subsequence if necessary, $\nu_i^{*n} \to \nu_i^*$ as $n \to \infty$. Following the lines of the proof of Lemma 5.4, we find

$$\int \eta \, |\nabla\chi_i^n| \leq \frac{1}{1 - 1/n} \left(\int |\nabla\eta| \, dx + \frac{1}{n} \right) \quad \text{stays bounded as } n \to \infty$$

so that there exist $\chi_i \in \{0, 1\}$ with

$$\chi_i^n \to \chi_i \quad \text{pointwise a.e. on } \{\eta > 0\} \tag{99}$$

and

$$\frac{1}{2} \int \eta \, |\nu_i - \nu_i^*|^2 \, |\nabla\chi_i| \leq \liminf_{n\to\infty} \frac{1}{2} \int \eta \, |\nu_i^n - \nu_i^{*n}|^2 \, |\nabla\chi_i^n| = 0.$$

Therefore, $\nu_i = \nu_i^*$ $|\nabla\chi_i|$- a.e. and each $\chi_i = \chi_i^*$ is a half space in direction ν_i^*. Continuing in our setting now, we note that the condition $\chi_1^n + \chi_2^n + \chi_3^n = 1$ carries over to the limit: $\chi_1^* + \chi_2^* + \chi_3^* = 1$. Therefore there exists an $i \in \{1, 2, 3\}$ (w.l.o.g. $i = 1$) such that $\chi_1^* = 0$ in B. Then the other two half spaces are complementary, $\chi_2^* = (1 - \chi_3^*)$ and in particular $\nu_2^* = -\nu_3^* =: \nu^*$. As in the proof of Lemma 5.4, we have

$$\int_B |\nabla\chi_i^n| \to \int_B |\nabla\chi_i^*|.$$

Together with (99), we can take the limit $n \to \infty$ in (98) and obtain

$$\int_B |\nabla\chi_1^*| + \frac{1}{2} \int_B |\nu_2^* - \nu^*|^2 \, |\nabla\chi_2^*| + \frac{1}{2} \int_B |\nu_3^* + \nu^*|^2 \, |\nabla\chi_3^*| \geq \varepsilon^2,$$

which is a contradiction since the left-hand side vanishes by construction. □

Bibliography

[1] Giovanni Alberti and Giovanni Bellettini. "A nonlocal anisotropic model for phase transitions". In: *Mathematische Annalen* 310.3 (1998), pp. 527–560.

[2] Fred Almgren, Jean E. Taylor, and Lihe Wang. "Curvature-driven flows: a variational approach". In: *SIAM Journal on Control and Optimization* 31.2 (1993), pp. 387–438.

[3] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Gradient flows: in metric spaces and in the space of probability measures.* Birkhäuser, 2008.

[4] Guy Barles and Christine Georgelin. "A simple proof of convergence for an approximation scheme for computing motions by mean curvature". In: *SIAM Journal on Numerical Analysis* 32.2 (1995), pp. 484–500.

[5] Eric Bonnetier, Elie Bretin, and Antonin Chambolle. "Consistency result for a non monotone scheme for anisotropic mean curvature flow". In: *Interfaces and Free Boundaries* 14.1 (2012), pp. 1–35.

[6] Kenneth A Brakke. *The motion of a surface by its mean curvature.* Vol. 20. Princeton University Press Princeton, 1978.

[7] Lia Bronsard and Fernando Reitich. "On three-phase boundary motion and the singular limit of a vector-valued Ginzburg-Landau equation". In: *Archive for Rational Mechanics and Analysis* 124.4 (1993), pp. 355–379.

[8] Antonin Chambolle and Matteo Novaga. "Convergence of an algorithm for the anisotropic and crystalline mean curvature flow". In: *SIAM Journal on Mathematical Analysis* 37.6 (2006), pp. 1978–1987.

[9] Ennio De Giorgi. "New problems on minimizing movements". In: *Boundary Value Problems for PDE and Applications* (1993), pp. 91–98.

[10] Matt Elsey, Selim Esedoglu, and Peter Smereka. "Diffusion generated motion for grain growth in two and three dimensions". In: *Journal of Computational Physics* 228.21 (2009), pp. 8015–8033.

[11] Matt Elsey, Selim Esedoglu, and Peter Smereka. "Large-scale simulation of normal grain growth via diffusion-generated motion". In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science* 467.2126 (2011), pp. 381–401.

[12] Matt Elsey, Selim Esedoglu, and Peter Smereka. "Large-scale simulations and parameter study for a simple recrystallization model". In: *Philosophical Magazine* 91.11 (2011), pp. 1607–1642.

[13] Selim Esedoglu and Felix Otto. "Threshold dynamics for networks with arbitrary surface tensions". In: *Communications on Pure and Applied Mathematics* (2014).

[14] Lawrence C. Evans. "Convergence of an algorithm for mean curvature motion". In: *Indiana University Mathematics Journal* 42.2 (1993), pp. 533–557.

[15] Lawrence C. Evans and Joel Spruck. “Motion of level sets by mean curvature I”. In: *Journal of Differential Geometry* 33.3 (1991), pp. 635–681.

[16] Enrico Giusti. *Minimal surfaces and functions of bounded variation.* 80. Birkhäuser, 1984.

[17] Tom Ilmanen. “Convergence of the Allen-Cahn equation to Brakkes motion by mean curvature”. In: *Journal of Differential Geometry* 38.2 (1993), pp. 417–461.

[18] Hitoshi Ishii, Gabriel E. Pires, and Panagiotis E. Souganidis. “Threshold dynamics type approximation schemes for propagating fronts”. In: *Journal of the Mathematical Society of Japan* 51.2 (1999), pp. 267–308.

[19] Richard Jordan, David Kinderlehrer, and Felix Otto. “The variational formulation of the Fokker–Planck equation”. In: *SIAM Journal on Mathematical Analysis* 29.1 (1998), pp. 1–17.

[20] Tim Laux and Drew Swartz. *Convergence of a thresholding scheme for volume-preserving mean-curvature flow.* In preparation.

[21] Stephan Luckhaus and Luciano Modica. “The Gibbs-Thompson relation within the gradient theory of phase transitions”. In: *Archive for Rational Mechanics and Analysis* 107.1 (1989), pp. 71–83.

[22] Stephan Luckhaus and Thomas Sturzenhecker. “Implicit time discretization for the mean curvature flow equation”. In: *Calculus of variations and partial differential equations* 3.2 (1995), pp. 253–271.

[23] Carlo Mantegazza, Matteo Novaga, and Vincenzo Maria Tortorelli. “Motion by curvature of planar networks”. In: *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie V* 3.2 (2004), pp. 235–324.

[24] Barry Merriman, James K. Bence, and Stanley J. Osher. *Diffusion generated motion by mean curvature.* Department of Mathematics, University of California, Los Angeles, 1992.

[25] Barry Merriman, James K. Bence, and Stanley J. Osher. “Motion of multiple junctions: A level set approach”. In: *Journal of Computational Physics* 112.2 (1994), pp. 334–363.

[26] Peter W. Michor and David Mumford. “Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms”. In: *Documenta Math* 10 (2005), pp. 217–245.

[27] Michele Miranda et al. “Short-time heat flow and functions of bounded variation in R N”. In: *Annales-Faculte des Sciences Toulouse Mathematiques.* Vol. 16. 1. Université Paul Sabatier. 2007, p. 125.

[28] Luciano Modica. “The gradient theory of phase transitions and the minimal interface criterion”. In: *Archive for Rational Mechanics and Analysis* 98.2 (1987), pp. 123–142.

[29] Luciano Modica and Stefano Mortola. “Un esempio di Gamma–convergenza”. In: *Bolletino della Unione Matematica Itialana B (5)* 14.1 (1977), pp. 285–299.

[30] Luca Mugnai, Christian Seis, and Emanuele Spadaro. “Global solutions to the volume-preserving mean-curvature flow”. In: *arXiv preprint arXiv:1502.07232* (2015).

[31] W. T. Read and William B. Shockley. "Dislocation models of crystal grain boundaries". In: *Physical Review* 78.3 (1950), p. 275.

[32] Yu G. Reshetnyak. "General theorems on semicontinuity and on convergence with a functional". In: *Siberian Mathematical Journal* 8.5 (1967), pp. 801–816.

[33] Steven J. Ruuth and Brian T. R. Wetton. "A simple scheme for volume-preserving motion by mean curvature". In: *Journal of Scientific Computing* 19.1-3 (2003), pp. 373–384.

[34] Etienne Sandier and Sylvia Serfaty. "Gamma-convergence of gradient flows with applications to Ginzburg-Landau". In: *Communications on Pure and Applied Mathematics* 57.12 (2004), pp. 1627–1672.

[35] Robert I Saye and James A Sethian. "The Voronoi implicit interface method for computing multiphase physics". In: *Proceedings of the National Academy of Sciences* 108.49 (2011), pp. 19498–19503.

[36] Oliver Schnürer et al. "Evolution of convex lens-shaped networks under the curve shortening flow". In: *Transactions of the American Mathematical Society* 363.5 (2011), pp. 2265–2294.