Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

The Saturation of Several Universal Inequalities in Information-Processing

by

Lin Zhang, Junde Wu, and Shao-Ming Fei

Preprint no.: 38 2015

The Saturation of Several Universal Inequalities in Information-Processing

Lin Zhang[a*], Junde Wu[b†], Shao-Ming Fei[c]

[a]*Institute of Mathematics, Hangzhou Dianzi University, Hangzhou 310018, PR China*

[b]*Department of Mathematics, Zhejiang University, Hangzhou 310027, PR China*

[c]*School of Mathematics of Sciences, Capital Normal University, Beijing 100048, PR China*

Abstract

Strong subadditivity inequality of quantum entropy, proved by Lieb and Ruskai, is a powerful tool in quantum information theory. The fact that the strong subadditivity inequality saturated only by so-called Markov states is obtained in the recent literature [P. Hayden *et al.*, Commun. Math. Phys. **246**, 359 (2004).].

In this paper, we give a characterization of another equivalent version for strong subadditivity inequality. We discuss the coherent information saturating its upper bound as well. A necessary and sufficient condition for this saturation is derived. The possible applications are discussed.

1 Introduction and preliminaries

Let $\mathcal{H}$ be a finite dimensional complex Hilbert space. A *quantum state* ρ on $\mathcal{H}$ is a positive semi-definite operator of trace one, in particular, for each unit vector $|\psi\rangle \in \mathcal{H}$,

*E-mail: godyalin@163.com; linyz@hdu.edu.cn

†Corresponding author. E-mail: wjd@zju.edu.cn

the operator $\rho = |\psi\rangle\langle\psi|$ is said to be a *pure state*. The set of all quantum states on $\mathcal{H}$ is denoted by $\mathrm{D}(\mathcal{H})$. For each quantum state $\rho \in \mathrm{D}(\mathcal{H})$, its von Neumann entropy is defined by $\mathrm{S}(\rho) = -\operatorname{Tr}(\rho \log_2 \rho)$.

The celebrated strong subadditivity (SSA) inequality of quantum entropy, proved by Lie and Ruskai in [1],

$$\mathrm{S}(\rho_{ABC}) + \mathrm{S}(\rho_B) \leqslant \mathrm{S}(\rho_{AB}) + \mathrm{S}(\rho_{BC}), \tag{1.1}$$

is a very powerful tool in quantum information theory. Let a reference system D be introduced to purify the tripartite state ρ_{ABC} such that ρ_{ABCD} be a pure state with $\rho_{ABC} = \operatorname{Tr}_D(\rho_{ABCD})$. An equivalent version of SSA can be described as follows [9]:

$$\mathrm{S}(\rho_D) + \mathrm{S}(\rho_B) \leqslant \mathrm{S}(\rho_{AB}) + \mathrm{S}(\rho_{AD}). \tag{1.2}$$

In fact, SSA plays an essential role in almost every nontrivial insight in quantum information theory, for instance, as some direct consequences of SSA, the data processing inequality, the well-known Holevo bound [2], etc. Moreover, SSA connects with the monotonicity of relative entropy under quantum channels [3].

The condition for the equality to be hold is an interesting and important subject. An important result about the structure of states which satisfy Eq. (1.1) with equality was obtained in [4]:

Proposition 1.1 ([4]). *A state $\rho_{ABC} \in \mathrm{D}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ saturating the strong subadditivity inequality, i.e.,*

$$\mathrm{S}(\rho_{AB}) + \mathrm{S}(\rho_{BC}) = \mathrm{S}(\rho_{ABC}) + \mathrm{S}(\rho_B) \tag{1.3}$$

if and only if there is a decomposition of system B as

$$\mathcal{H}_B = \bigoplus_j \mathcal{H}_{b_j^L} \otimes \mathcal{H}_{b_j^R} \tag{1.4}$$

into a direct (orthogonal) sum of tensor products, such that

$$\rho_{ABC} = \bigoplus_j \lambda_j \rho_{Ab_j^L} \otimes \rho_{b_j^R C}, \tag{1.5}$$

where $\rho_{Ab_j^L} \in \mathrm{D}\left(\mathcal{H}_A \otimes \mathcal{H}_{b_j^L}\right)$ and $\rho_{b_j^R C} \in \mathrm{D}\left(\mathcal{H}_{b_j^R} \otimes \mathcal{H}_C\right)$, and $\{\lambda_j\}$ is a probability distribution.

An analogous characterization of the structures of states which satisfy Eq. (1.2) with equality is highly desirable. We investigate the sufficient and necessary conditions for saturating the inequality (1.2) in the following.

2 Main result and applications

A *quantum operation* Φ on $\mathcal{H}$ is a trace-preserving completely positive linear mapping defined on the set $\mathrm{D}(\mathcal{H})$. It follows from ([5, Prop. 5.2 and Cor. 5.5]) that there exists linear operators $\{K_\mu\}_\mu$ on $\mathcal{H}$ such that $\sum_\mu K_\mu^\dagger K_\mu = \mathbb{1}$ and $\Phi = \sum_\mu \mathrm{Ad}_{K_\mu}$, namely, for each quantum state ρ, we have the Kraus representation

$$\Phi(\rho) = \sum_\mu K_\mu \rho K_\mu^\dagger.$$

The corresponding complementary channel $\widehat{\Phi}$ for Φ is defined as

$$\widehat{\Phi}(\rho) = \sum_{\mu,\nu} \mathrm{Tr}\left(K_\mu \rho K_\nu^\dagger\right) |\mu\rangle\langle\nu|.$$

Let $\mathcal{E} = \{(p_\mu, \rho_\mu)\}$ be a quantum ensemble on $\mathcal{H}$, that is, each $\rho_\mu \in \mathrm{D}(\mathcal{H})$, $p_\mu > 0$, and $\sum_\mu p_\mu = 1$. The *Holevo quantity* of the quantum ensemble $\{(p_\mu, \rho_\mu)\}$ is defined by:

$$\chi\{(p_\mu, \rho_\mu)\} = \mathrm{S}(\sum_\mu p_\mu \rho_\mu) - \sum_\mu p_\mu \mathrm{S}(\rho_\mu). \tag{2.1}$$

2.1 The saturation of an equivalent version of SSA

Theorem 2.1. *Let $\sigma_{ABC} \in \mathrm{D}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$. Then*

$$\mathrm{S}(\sigma_A) + \mathrm{S}(\sigma_C) = \mathrm{S}(\sigma_{AB}) + \mathrm{S}(\sigma_{CB}) \tag{2.2}$$

if and only if there are two decompositions of system A and C, respectively, as

$$\mathcal{H}_A = \bigoplus_{i=1}^{K_A} \mathcal{H}_{a_i^L} \otimes \mathcal{H}_{a_i^R} \quad \textit{and} \quad \mathcal{H}_C = \bigoplus_{j=1}^{K_C} \mathcal{H}_{c_j^L} \otimes \mathcal{H}_{c_j^R} \tag{2.3}$$

such that

$$\sigma_{ABC} = \bigoplus_{i,j} \mu_{ij} \sigma_{a_i^L B c_j^L} \otimes \sigma_{a_i^R c_j^R}, \tag{2.4}$$

where $\sigma_{a_i^L B c_j^L} \equiv |\psi\rangle\langle\psi|_{a_i^L B c_j^L} \in \mathrm{D}\left(\mathcal{H}_{a_i^L} \otimes \mathcal{H}_B \otimes \mathcal{H}_{c_j^L}\right)$, $\sigma_{a_i^R c_j^R} \in \mathrm{D}\left(\mathcal{H}_{a_i^R} \otimes \mathcal{H}_{c_j^R}\right)$ *and* $\{\mu_{ij}\}$ *is a joint probability distribution.*

Proof. We introduce a reference system D such that σ_{ABCD} is a purification of σ_{ABC}. Thus Eq. (2.2) can be rewritten as

$$\mathrm{S}(\sigma_A) + \mathrm{S}(\sigma_C) = \mathrm{S}(\sigma_{CD}) + \mathrm{S}(\sigma_{AD}). \tag{2.5}$$

It can be seen that, when the systems A and C are fixed, the set of equations (2.2) and (2.5) keeps invariant under the exchange of the systems B and D. Analogously, we have

$$\begin{aligned} \mathrm{S}(\sigma_A) + \mathrm{S}(\sigma_{ABD}) &= \mathrm{S}(\sigma_{AB}) + \mathrm{S}(\sigma_{AD}), && (2.6)\\ \mathrm{S}(\sigma_{CBD}) + \mathrm{S}(\sigma_C) &= \mathrm{S}(\sigma_{CD}) + \mathrm{S}(\sigma_{CB}). && (2.7) \end{aligned}$$

Again, when the systems B and D are fixed, Eq. (2.6) and Eq. (2.7) are transformed mutually under the exchange of the systems A and C.

From Proposition 1.1 there are two decompositions of A and C, respectively,

$$\mathcal{H}_A = \bigoplus_{i=1}^{K_A} \mathcal{H}_{a_i^L} \otimes \mathcal{H}_{a_i^R} \quad \text{and} \quad \mathcal{H}_C = \bigoplus_{j=1}^{K_C} \mathcal{H}_{c_j^L} \otimes \mathcal{H}_{c_j^R} \tag{2.8}$$

such that

$$\sigma_{ABD} = \bigoplus_i p_i \sigma_{a_i^L B} \otimes \sigma_{a_i^R D} \quad \text{and} \quad \sigma_{BCD} = \bigoplus_j q_j \sigma_{B c_j^L} \otimes \sigma_{c_j^R D}. \tag{2.9}$$

Thus σ_{ABC} must be of the form:

$$\sigma_{ABC} = \bigoplus_{i,j} \mu_{ij} \sigma^{(ij)}_{a_i^L B c_j^L} \otimes \sigma^{(ij)}_{a_i^R c_j^R},$$

where

$$\mathrm{S}\left(\sigma^{(ij)}_{a_i^L B}\right) + \mathrm{S}\left(\sigma^{(ij)}_{B c_j^L}\right) = \mathrm{S}\left(\sigma^{(ij)}_{a_i^L}\right) + \mathrm{S}\left(\sigma^{(ij)}_{c_j^L}\right), \quad \forall\, i, j.$$

Without loss of generality, we assume that the systems a_i^L and c_j^L can not be decomposed like the $\mathcal{H}_A$ and $\mathcal{H}_C$, respectively. Therefore $\sigma_{a_i^L B c_j^L}$ must be a pure state, which implies that

$$\sigma_{a_i^L B c_j^L} \equiv |\psi\rangle\langle\psi|_{a_i^L B c_j^L}.$$

If the state σ_{ABC} is of the form Eq. (2.4), then it is easy to check that Eq. (2.2) holds. □

Remark 2.2. A simple example for Eq. (2.2) is a pure tripartite states σ_{ABC}. In the above proof, based on this point, by employing Proposition 1.1, we obtain that there must exist a decomposition of σ_{ABC} such that its substates are locally pure states.

2.2 The saturation of Araki-Lieb inequality

The following proposition can be seen as a characterization of the saturation of Araki-Lieb inequality:

$$|\mathrm{S}(\omega_B) - \mathrm{S}(\omega_C)| \leqslant \mathrm{S}(\omega_{BC}). \tag{2.10}$$

Proposition 2.3 ([6])**.** *Let $\omega_{BC} \in \mathrm{D}(\mathcal{H}_B \otimes \mathcal{H}_C)$. The reduced states are $\omega_B = \mathrm{Tr}_C(\omega_{BC}), \omega_C = \mathrm{Tr}_B(\omega_{BC})$, respectively. Then $\mathrm{S}(\omega_{BC}) = \mathrm{S}(\omega_B) - \mathrm{S}(\omega_C)$ if and only if*

(i) $\mathcal{H}_B$ can be factorized into the form $\mathcal{H}_B = \mathcal{H}_L \otimes \mathcal{H}_R$,

(ii) $\omega_{BC} = \omega_L \otimes |\psi\rangle\langle\psi|_{RC}$ for $|\psi\rangle_{RC} \in \mathcal{H}_R \otimes \mathcal{H}_C$.

Remark 2.4. The result in Proposition 2.3 is employed to study the saturation of the upper bound of quantum discord in [7]. Later on, Carlen gives an elementary proof about this result in [8].

The *Coherent information* defined by

$$I_c(\rho, \Phi) \stackrel{\text{def}}{=} \mathrm{S}(\Phi(\rho)) - \mathrm{S}(\widehat{\Phi}(\rho)) \tag{2.11}$$

is a useful concept in quantum information theory. It connects with other notions in data processing. For instance, the fundamental problem in quantum error correction is to determine when the effect of a quantum channel Φ defined on $\mathcal{H}$ acting on half of a pure entangled state can be perfectly reversed.

In general, we see from SSA that

$$I_c(\rho, \Phi) \leqslant \mathrm{S}(\rho). \tag{2.12}$$

In what follows, we use Proposition 2.3 to study the coherent information saturating its upper bound. A necessary and sufficient condition for this saturation can be easily derived.

Proposition 2.5. *Let $\rho \in \mathrm{D}(\mathcal{H})$ and Φ be a quantum channel defined over $\mathcal{H}$. The coherent information, defined by $I_c(\rho, \Phi)$ achieves its maximum, that is, $I_c(\rho, \Phi) = \mathrm{S}(\rho)$ if and only if the following statements holds:*

(i) the underlying Hilbert space can be decomposed as: $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$;

(ii) the output state of the quantum channel Φ is a product state: $\Phi(\rho) = \rho_L \otimes \rho_R$, where $\rho_L \in \mathrm{D}(\mathcal{H}_L), \rho_R \in \mathrm{D}(\mathcal{H}_R)$.

Proof. Note that

$$\mathrm{S}(\widehat{\Phi}(\rho)) = \mathrm{S}((\mathbb{1}_A \otimes \Phi)(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho|)), \tag{2.13}$$

where $|\mathbf{u}_\rho\rangle$ is a purification of ρ in a larger Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_B \equiv \mathcal{H}$. It was shown that there exists a quantum channel Ψ (see [4]) such that

$$I_c(\rho, \Phi) = \mathrm{S}(\rho) \Longleftrightarrow (\mathbb{1}_A \otimes \Psi \circ \Phi)(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho|) = |\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho|. \tag{2.14}$$

From the *Stinespring dilation theorem* (see [5]), we may assume that

$$\Phi(\rho) = \mathrm{Tr}_C \left(U(\rho \otimes |0\rangle\langle 0|)U^\dagger \right), \quad U \in \mathrm{U}(\mathcal{H}_B \otimes \mathcal{H}_C), \ \ |0\rangle \in \mathcal{H}_C,$$

which indicates that

$$\begin{aligned} \mathbb{1}_A \otimes \Phi(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho|) &= \mathrm{Tr}_C((\mathbb{1}_A \otimes U)(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho| \otimes |0\rangle\langle 0|)(\mathbb{1}_A \otimes U)^\dagger) \\ &= \mathrm{Tr}_C(|\Omega\rangle\langle\Omega|), \end{aligned} \tag{2.15}$$

where $|\Omega\rangle \stackrel{\text{def}}{=} (\mathbb{1}_A \otimes U)(|\mathbf{u}_\rho\rangle \otimes |0\rangle)$. Now

$$|\Omega\rangle\langle\Omega| = (\mathbb{1}_A \otimes U)(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho| \otimes |0\rangle\langle 0|)(\mathbb{1}_A \otimes U)^\dagger$$

is a tripartite state onc $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, it follows that

$$\begin{aligned} \mathrm{Tr}_C(|\Omega\rangle\langle\Omega|) &= \mathbb{1}_A \otimes \Phi(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho|) \equiv \Omega_{AB}, \\ \mathrm{Tr}_A(|\Omega\rangle\langle\Omega|) &= U(\rho \otimes |0\rangle\langle 0|)U^\dagger \equiv \Omega_{BC}, \\ \mathrm{Tr}_{AC}(|\Omega\rangle\langle\Omega|) &= \Phi(\rho) \equiv \Omega_B, \end{aligned}$$

where $\Omega_{ABC} \equiv |\Omega\rangle\langle\Omega|$. From the above expressions, we have

$$\begin{aligned} \mathrm{S}(\Omega_{ABC}) &= 0, \\ \mathrm{S}(\Omega_B) &= \mathrm{S}(\Phi(\rho)) \\ \mathrm{S}(\Omega_{BC}) &= \mathrm{S}(\rho), \\ \mathrm{S}(\Omega_{AB}) &= \mathrm{S}((\mathbb{1}_A \otimes \Phi)(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho|)). \end{aligned}$$

Apparently, $I_c(\rho, \Phi) = \mathrm{S}(\rho) \Longleftrightarrow \mathrm{S}(\Phi(\rho)) = \mathrm{S}((\mathbb{1}_A \otimes \Phi)(|\mathbf{u}_\rho\rangle\langle\mathbf{u}_\rho|)) + \mathrm{S}(\rho)$, that is,

$$\begin{aligned} I_c(\rho, \Phi) = \mathrm{S}(\rho) &\Longleftrightarrow \mathrm{S}(\Omega_B) = \mathrm{S}(\Omega_{AB}) + \mathrm{S}(\Omega_{BC}) \\ &\Longleftrightarrow \mathrm{S}(\Omega_B) - \mathrm{S}(\Omega_C) = \mathrm{S}(\Omega_{BC}). \end{aligned}$$

It follows from Proposition 2.3 that this equation holds if and only if

(i) $\mathcal{H}_B$ can be factorized into the form $\mathcal{H}_B = \mathcal{H}_L \otimes \mathcal{H}_R$,

(ii) $\Omega_{BC} = \rho_L \otimes |\psi\rangle\langle\psi|_{RC}$ for $|\psi\rangle_{RC} \in \mathcal{H}_R \otimes \mathcal{H}_C$.

That is,

$$\Phi(\rho) = \mathrm{Tr}_C \left(\rho_L \otimes |\psi\rangle\langle\psi|_{RC}\right) = \rho_L \otimes \rho_R.$$

This indicates that the coherent information arrives at its maximal value if and only if the output state of the quantum channel Φ is a product state. $\square$

2.3 Applications

In the following, we make a theoretical analysis of Roga's result concerning universal bound for Holevo information [2].

Consider a state ρ, a quantum channel Φ, and the image of ρ under Φ:

$$\rho' = \Phi(\rho) = \sum_\mu K_\mu \rho K_\mu^\dagger. \tag{2.16}$$

The complementary channel produces a *correlation matrix*

$$\widehat{\Phi}(\rho) = \sum_{\mu,\nu} \mathrm{Tr}\left(K_\mu \rho K_\nu^\dagger\right) |\mu\rangle\langle\nu|. \tag{2.17}$$

Denote

$$q_\mu = \mathrm{Tr}\left(K_\mu \rho K_\mu^\dagger\right) \quad \text{and} \quad \rho'_\mu = q_\mu^{-1} K_\mu \rho K_\mu^\dagger$$

so that

$$\rho' = \sum_\mu q_\mu \rho'_\mu.$$

Then the Holevo information is bounded by the *exchange entropy*:

$$\chi(\{q_\mu, \rho'_\mu\}) \leqslant \mathrm{S}\left(\widehat{\Phi}(\rho)\right). \tag{2.18}$$

Moreover, the *average entropy* is bounded by the entropy of the initial state:

$$\sum_\mu q_\mu \mathrm{S}(\rho'_\mu) \leqslant \mathrm{S}(\rho). \tag{2.19}$$

In order to make an analysis of the saturations in Eq. (2.18) and Eq. (2.19), let us revisit the original proof of these inequalities. The authors in [2] introduced a tripartite state,

$$\omega_{ABC} \stackrel{\text{def}}{=} \sum_{\mu,\nu} \left(K_\mu \rho K_\nu^\dagger\right)_A \otimes |\mu\rangle\langle\nu|_B \otimes |\mu\rangle\langle\nu|_C. \tag{2.20}$$

From the above expression, we see that ω_{ABC} is symmetric with respect to BC and $\mathcal{H}_B = \mathcal{H}_C$. It is convenient to introduce the notation $A_{\mu\nu} = K_\mu \rho K_\nu^\dagger$, so that $q_\mu = \mathrm{Tr}\left(A_{\mu\mu}\right)$ and $\rho'_\mu = q_\mu^{-1} A_{\mu\mu}$. Since

$$\mathrm{S}(\omega_{BC}) = \mathrm{S}(\widehat{\Phi}(\rho)), \quad \mathrm{S}(\omega_A) = \mathrm{S}(\sum_\mu q_\mu \rho'_\mu), \quad \sum_\mu q_\mu \mathrm{S}(\rho'_\mu) = \mathrm{S}(\omega_{AC}) - \mathrm{S}(\omega_B),$$

one has

$$\chi(\{q_\mu, \rho'_\mu\}) = \mathrm{S}\left(\widehat{\Phi}(\rho)\right) \Longleftrightarrow \mathrm{S}(\omega_A) + \mathrm{S}(\omega_C) = \mathrm{S}(\omega_{AB}) + \mathrm{S}(\omega_{BC}).$$

This amounts to say, by Theorem 2.1, that

$$\omega_{ABC} = \sum_{\mu,\nu} \left(K_\mu \rho K_\nu^\dagger\right)_A \otimes |\mu\rangle\langle\nu|_B \otimes |\mu\rangle\langle\nu|_C = \bigoplus_{i,j} p_{ij} \omega^{(ij)}_{a_i^L B c_j^L} \otimes \omega^{(ij)}_{a_i^R c_j^R}, \tag{2.21}$$

where each $\omega^{(ij)}_{a_i^L B c_j^L}$ is a pure state. Since both B and C are identical, it follows that

$$\sum_\mu K_\mu \rho K_\mu^\dagger \otimes |\mu\rangle\langle\mu| = \bigoplus_{i,j} p_{ij} \omega^{(ij)}_{a_i^L c_j^L} \otimes \omega^{(ij)}_{a_i^R c_j^R},$$

which implies that

$$\Phi(\rho) = \omega_A = \bigoplus_{i,j} p_{ij} \omega^{(ij)}_{a_i^L} \otimes \omega^{(ij)}_{a_i^R}.$$

Moreover

$$\sum_\mu q_\mu \mathrm{S}(\rho'_\mu) = \mathrm{S}(\rho) \Longleftrightarrow \mathrm{S}(\omega_{AC}) - \mathrm{S}(\omega_B) = \mathrm{S}(\omega_{ABC}) = \mathrm{S}(\omega_{AB}) - \mathrm{S}(\omega_C).$$

This amounts to say, by Proposition 2.3, that

$$\omega_{ABC} = \omega_L \otimes |\psi\rangle\langle\psi|_{RC}, \quad \omega_{ACB} = \omega_{\hat{L}} \otimes |\psi\rangle\langle\psi|_{\hat{R}B},$$

which implies that

$$\omega_A = \omega_L = \omega_{\hat{L}}, \quad \omega_R = \omega_B, \quad \omega_C = \omega_{\hat{R}}.$$

Furthermore, $|\psi\rangle\langle\psi|_{BC}$ is a symmetric state on $\mathcal{H}_B \otimes \mathcal{H}_C$ with $\mathcal{H}_B = \mathcal{H}_C$. Now we have

$$|\psi\rangle\langle\psi|_{BC} = \sum_{\mu,\nu} \mathrm{Tr}\left(K_\mu \rho K_\nu^\dagger\right) |\mu\rangle\langle\nu| \otimes |\mu\rangle\langle\nu|.$$

This indicates that $\sum_{\mu,\nu} \mathrm{Tr}\left(K_\mu \rho K_\nu^\dagger\right) |\mu\rangle\langle\nu| \equiv \widehat{\Phi}(\rho)$ is still a pure state. Therefore

$$\sum_{\mu,\nu} K_\mu \rho K_\nu^\dagger \otimes |\mu\rangle\langle\nu| = \Phi(\rho) \otimes \widehat{\Phi}(\rho). \tag{2.22}$$

Let $\mathrm{Tr}\left(K_\mu \rho K_\nu^\dagger\right) = \lambda_\mu \bar{\lambda}_\nu$ for complex numbers λ_μ. Then $\sum_\mu \left|\lambda_\mu\right|^2 = 1$. Now we can infer from Eq. (2.22) that

$$\Phi(\rho) = \left(\lambda_\mu^{-1} K_\mu\right) \rho \left(\lambda_\nu^{-1} K_\nu\right)^\dagger = \left(\lambda_\nu^{-1} K_\nu\right) \rho \left(\lambda_\mu^{-1} K_\mu\right)^\dagger \quad \forall \mu, \nu$$

or

$$K_\mu \rho K_\nu^\dagger = \lambda_\mu \bar{\lambda}_\nu \Phi(\rho),$$

which implies that

$$\rho = (\sum_\mu K_\mu^\dagger K_\mu) \rho (\sum_\nu K_\nu^\dagger K_\nu) = \sum_{\mu,\nu} K_\mu^\dagger (K_\mu \rho K_\nu^\dagger) K_\nu \tag{2.23}$$

$$= (\sum_\mu \lambda_\mu K_\mu^\dagger) \Phi(\rho) (\sum_\nu \lambda_\nu K_\nu^\dagger)^\dagger \equiv M \Phi(\rho) M^\dagger, \tag{2.24}$$

where $M \stackrel{\text{def}}{=} \sum_\mu \lambda_\mu K_\mu^\dagger$. From Eq. (2.22), we can see that

$$\mathrm{S}(\rho) = \mathrm{S}(\Phi(\rho)) + \mathrm{S}(\widehat{\Phi}(\rho)) = \mathrm{S}(\Phi(\rho))$$

as $\widehat{\Phi}(\rho)$ is a pure state. Furthermore, we can have that $I_c(\rho, \Phi) = \mathrm{S}(\rho)$, which shows that

$$\Phi(\rho) = \rho_L \otimes \rho_R.$$

Remark 2.6. In the above process, the output of the complementary channel is a pure state. Without loss of generality, we assume that the environment starts in a pure state, this implies that the complementary channel is an unitary channel. Form the basic properties of quantum channel, we obtain that the equality in the inequality (2.19) holds if and only if the quantum channel is a unitary one.

Acknowledgement

We want to express our heartfelt thanks to Wojciech Roga for useful comments. This project is supported by Natural Science Foundations of China (11171301, 10771191, 10471124 and 11275131) and Natural Science Foundation of Zhejiang Province of China (Y6090105). LZ is also grateful for funding from Hangzhou Dianzi University (Grant No. KYS075612038).

References

[1] E.H. Lieb and M.B. Ruskai. *Proof of the strong subadditivity of quantum-mechanical entropy.* J. Math. Phys. **14**, 1938-1941 (1973).

[2] W. Roga, M. Fannes, and K. Życzkowski. *Universal bounds for the Holevo quantity, coherent information, and the Jensen-Shannon divergence.* Phys. Rev. Lett. **105**, 040505 (2010).

[3] G. Lindblda. *Completely positive maps and entropy inequalities.* Commun. Math. Phys. **40**, 147-151 (1975).

[4] P. Hayden, R. Jozsa, D. Petz and A. Winter. *Structure of states which satisfy strong subadditivity of quantum entropy with equality.* Commun. Math. Phys. **246**, 359-374 (2004).

[5] J. Watrous, Theory of Quantum Information, University of Waterloo, Waterloo (2008). See `http://www.cs.uwaterloo.ca/~watrous/quant-info/`

[6] L. Zhang and J. Wu. *On Conjectures of Classical and Quantum Correlations in Bipartite States.* J. Phys. A: Math. Theor. **45**, 025301(2012).

[7] Z. Xi, X. Lu, X. Wang, and Y. Li. *Necessary and sufficient condition for saturating the upper bound of quantum discord.* Phys. Rev. A **85**, 032109 (2012).

[8] E.A. Carlen and E.H. Lieb. *Bounds for entanglement via an extension of strong subadditivity of entropy.* Lett. Math. Phys. **101**, 1-11 (2012).

[9] M. Nielsen and I. Chuang. *Quantum Computation and Quantum Information*. Cambridge Series on Information and the Natural Sciences. Cambridge University Press, 2000.