Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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by

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Preprint no.: 44 2015

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We present a complete set of local unitary invariants for generic multi-qubit systems which gives necessary and sufficient conditions for two states being local unitary equivalent. These invariants are canonical polynomial functions in terms of the generalized Bloch representation of the quantum states. In particular, we prove that there are at most 12 polynomial local unitary invariants for two-qubit states and at most 90 polynomials for three-qubit states. Comparison with Makhlin's 18 local unitary invariants is given for two-quibit systems.

PACS numbers: 03.67.-a, 02.20.Hj, 03.65.-w

Local unitary equivalence is a foundational concept in quantum entanglement and quantum information, as it provides the key symmetry in classifying quantum entangled states of physical systems [1]. Two quantum states are of the same nature in implementing quantum information processing if they are equivalent under a local unitary (LU) transformation, and many crucial properties such as the degree of entanglement [6, 7], maximal violations of Bell inequalities [8–11], and the teleportation fidelity [12, 13] remain invariant under LU transformations. Moreover, quantum entanglement in multipartite qubits has also figured prominently in many quantum information processing such as one-way quantum computing, quantum error correction and quantum secret sharing [2–5]. For this reason, it has been a key problem to find a complete and operational procedure to distinguish two quantum states under LU transformations.

In [14], Makhlin presented a complete set of 18 polynomial LU invariants for classifying two-qubit states. There are numerous results on LU invariants for three qubits states [15], some general mixed states [16–19, 24], tripartite pure and mixed states [20]. A theoretical method to reduce the problem to pure n-qubit states was proposed in [21], and later generalized to arbitrary dimensions in [22]. From a different viewpoint, [23] gave a procedure to find the LU operator for multi-qubits using the core tensor method. Very recently a method to judge LU equivalence for multi-qubits [26] was also proposed and more generally SLOCC invariants for multi-partite states are found [27]. Nevertheless, it remains a wild problem to find a complete set of invariants to answer the LU question except for two qubit cases. Even for two partite cases it is also desirable to find an alternative set of invariants to judge LU equivalence, as the original Makhlin invariants contain some nontrivial tensor vectors.

In this article, we propose a brand new method to quantify polynomial LU invariants for multi-qubit systems in an operational way. For the special case of two-qubit systems, our method is more efficient and needs fewer invariants than that in [14] in general. In fact, we show that many invariants in [14] are consequences of other invariants, and there are at most 12 invariants to determine the LU equivalence for two-qubit states. We prove for the first time that there are at most 90 invariants for generic mixed 3-qubit states. We also propose an operational method to derive a list of polynomial invariants for generic multi-qubit states. We remark that the invariants can not be derived from [23] as the latter aimed to compute the LU operator for two equivalent multi-qubits, while our current work takes a different strategy to seek a complete set of polynomial invariants.

We start our discussion to express an N-qubit state ρ in terms of Pauli matrices σ_α, $\alpha = 1, 2, 3$,

$$\begin{aligned}\rho \;=\; & \frac{1}{2^N} I^{\otimes N} + \sum_{j_1=1}^{N} \sum_{\alpha_1=1}^{3} T_{j_1}^{\alpha_1}\, \sigma_{\alpha_1}^{j_1} \\ & + \sum_{1\le j_1<j_2\le N} \sum_{\alpha_1,\alpha_2=1}^{3} T_{j_1 j_2}^{\alpha_1\alpha_2}\, \sigma_{\alpha_1}^{j_1}\sigma_{\alpha_2}^{j_2} + \cdots \\ & + \sum_{1\le j_1<\cdots<j_M\le N} \sum_{\alpha_1,\cdots,\alpha_M=1}^{3} T_{j_1 j_2\cdots j_M}^{\alpha_1\alpha_2\cdots\alpha_M}\, \sigma_{\alpha_1}^{j_1}\cdots\sigma_{\alpha_M}^{j_M} \\ & + \cdots + \sum_{\alpha_1,\alpha_2,\cdots,\alpha_N=1}^{3} T_{12\cdots N}^{\alpha_1\alpha_2\cdots\alpha_N}\, \sigma_{\alpha_1}^{1}\sigma_{\alpha_2}^{2}\cdots\sigma_{\alpha_N}^{N},\end{aligned} \qquad (1)$$

where I is the 2×2 identity matrix, $\sigma_{\alpha_k}^{j_k} = I\otimes I\otimes\cdots\otimes \sigma_{\alpha_k}\otimes I\otimes\cdots\otimes I$ with σ_{α_k} at the j_k-th position and

$$T_{j_1 j_2\cdots j_M}^{\alpha_1\alpha_2\cdots\alpha_M} = \frac{1}{2^N}\mathrm{Tr}[\rho\, \sigma_{\alpha_1}^{j_1}\sigma_{\alpha_2}^{j_2}\cdots\sigma_{\alpha_M}^{j_M}], \qquad M \le N, \qquad (2)$$

are real coefficients. In particular, $T_j = (T_j^1, T_j^2, T_j^3)$, $j = 1, ..., N$, are three dimensional vectors, $T_{jk} = (T_{jk}^{\alpha_1\alpha_2})$, $1 \le j < k \le N$, are 3×3 matrices. Generally, $T_{j_1 j_2\cdots j_M} = (T_{j_1 j_2\cdots j_M}^{\alpha_1\alpha_2\cdots\alpha_M})$ are tensors.

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Let ρ and ρ' be two N-qubit mixed states. They are called local unitary equivalent if

$$\rho' = (U_1 \otimes ... \otimes U_N)\rho(U_1 \otimes ... \otimes U_N)^\dagger \tag{3}$$

for some unitary operators $U_i \in SU(2)$, $i = 1, 2, ..., N$, where $\dagger$ denotes transpose and conjugate.

Lemma 1 *Two mixed states ρ and ρ' are local unitary equivalent if and only if there are special orthogonal matrices $O_1, \cdots, O_N \in SO(3)$ such that*

$$(O_{j_1} \otimes \cdots \otimes O_{j_k})T_{j_1\cdots j_k} = T'_{j_1\cdots j_k} \tag{4}$$

for any $1 \le j_1 < \cdots < j_k \le N$, $k = 1, 2, ..., N$.

Proof. The group $SU(2)$ acts on the real vector space spanned by σ_i, $i = 1, 2, 3$ via [27]:

$$U_i \sigma_k U_i^\dagger = \sum_{j=1}^{3} O_{kl}\sigma_l, \tag{5}$$

where $O = (O_{kl})$ belongs to $SO(3)$. From (1), (3) and (5) one gets the tensor relation (4). Note that this action realizes the well-known double-covering map $SU(2) \to SO(3)$. The sufficiency then follows from the fact that $SU(2)$ is the universal double covering of $SO(3)$. ∎

Two-qubit states: To derive explicitly the invariants under the transformation (3), we first consider the two-qubit case. From (1) a two-qubit state is given by the 3-dimensional real column vectors T_1, T_2, and the real 3×3-matrix T_{12}. Two states ρ and ρ' are local unitary equivalent if and if there are $SO(3)$ operators O_1 and O_2 such that

$$\begin{aligned} T_1' &= O_1 T_1, \quad T_2' = O_2 T_2, \\ T_{12}' &= (O_1 \otimes O_2)T_{12} = O_1 T_{12} O_2^t, \end{aligned} \tag{6}$$

where t denotes the transpose of a matrix.

We introduce the following sets of 3-dimensional real column vectors:

$$\langle \mathcal{O}_1 \rangle = \{T_1, T_{12}T_2, T_{12}T_{12}^t T_1, T_{12}T_{12}^t T_{12}T_2, \cdots\} \subset \mathbf{R}^3,$$
$$\langle \mathcal{O}_2 \rangle = \{T_2, T_{12}^t T_1, T_{12}^t T_{12}T_2, T_{12}^t T_{12}T_{12}^t T_1, \cdots\} \subset \mathbf{R}^3,$$

which are respectively generated by the $(T_{12}T_{12}^t)$-orbit of $\{T_1, T_{12}T_2\}$ and the $(T_{12}^t T_{12})$-orbit of $\{T_2, T_{12}^t T_1\}$. Here (g) denotes the cyclic group generated by g. By the Cayley-Hamilton theorem the minimal polynomials of $T_{12}T_{12}^t$ and $T_{12}^t T_{12}$ have degree ≤ 3, therefore it is enough to use elements in the orbits up to the quadratic powers. It is straightforward to verify that all the vectors in $\langle \mathcal{O}_1 \rangle$ are transformed to $O_1\langle \mathcal{O}_1 \rangle$ under the transformation (6), while all the vectors in $\langle \mathcal{O}_2 \rangle$ are transformed into $O_2\langle \mathcal{O}_2 \rangle$. Moreover, there are at most three linear independent vectors in $\langle \mathcal{O}_i \rangle$, $dim\langle O_i \rangle \le 3$, $i = 1, 2$. We say that a two-qubit state is *generic* if $dim\langle O_1 \rangle = dim\langle O_2 \rangle = 3$. For simplicity, we only deal with generic cases in the following. The non-generic (degenerate) cases can be studied in details too, see remarks after the proof of Theorem 1.

Let $\{\mu_1, \cdots, \mu_6\}$ and $\{\nu_1, \cdots, \nu_6\}$ denote the first six (spanning) vectors in $\langle \mathcal{O}_1 \rangle$ and $\langle \mathcal{O}_2 \rangle$, respectively. We first give a general result using all the spanning vectors.

Theorem 1 *Two generic two-qubit states are local unitary equivalent if and only if they have the same values of the following invariant polynomials:*

$$\begin{aligned} &\langle \mu_i, \mu_j \rangle, \quad \langle \nu_i, \nu_j \rangle, \quad i \le j = 1, 2, \cdots, 6. \\ &tr(T_{12}T_{12}^t)^\alpha, \quad \alpha = 1, 2, 3. \end{aligned} \tag{7}$$

Proof. By using the relations in (6), it is direct to verify that the quantities given in (7) are invariants under local unitary transformations.

For generic states, the matrix $T_{12}T_{12}^t$ is nonsingular, so is $T_{12}^t T_{12}$ by trace property. Thus T_{12} and T_{12}^t are also nonsingular. We notice that $\langle \mathcal{O}_1 \rangle \xrightarrow{T_{12}^t} \langle \mathcal{O}_2 \rangle$ and $\langle \mathcal{O}_2 \rangle \xrightarrow{T_{12}} \langle \mathcal{O}_1 \rangle$ as subsets or subspaces, therefore $\langle \mathcal{O}_1 \rangle \simeq \langle \mathcal{O}_2 \rangle = \mathbf{R}^3$ for generic states, and a basis of $\mathbf{R}^3$ can be pared down from the vectors of $\langle \mathcal{O}_1 \rangle$ or $\langle \mathcal{O}_2 \rangle$ by assumption.

Assuming that two generic two-qubit states ρ and ρ' have the same values of the invariant polynomials (7), namely, the inner products of any two vectors in $\langle \mathcal{O}_i \rangle$ are invariant under $\rho \to \rho'$, one has that there must exist an orthogonal matrix O_i such that

$$O_i\langle \mathcal{O}_i \rangle = \langle \mathcal{O}_i' \rangle.$$

In particular $O_i T_i = T_i'$. Then we can build the following commutative diagram:

$$\begin{array}{ccc} \langle \mathcal{O}_1 \rangle & \xrightarrow{O_1} & \langle \mathcal{O}_1' \rangle \\ \downarrow T_{12}^t & & \downarrow T_{12}'^t \\ \langle \mathcal{O}_2 \rangle & \xrightarrow{O_2} & \langle \mathcal{O}_2' \rangle \end{array}$$

Consequently $T_{12}'^t O_1 = O_2 T_{12}^t$ in $End(\mathbf{R}^3)$, or $T_{12}' = O_1 T_{12} O_2^t$. Therefore, ρ and ρ' are local unitary equivalent. ∎

Remark. In the above discussions we are only concerned with the generic case. For degenerate cases, one needs to analyze case by case. For instance, let us consider the case $T_1 = T_2 = 0$. In this case, $dim\langle O_i \rangle = 0$, $i = 1, 2$. The only invariants left are $tr(T_{12}T_{12}^t)^\alpha$, $\alpha = 1, 2, 3$. Note that $p_\alpha = tr(T_{12}T_{12}^t)^\alpha = \sum_{i=1}^{3} \lambda_i^\alpha$ is the αth-power sum of the eigenvalues of $T_{12}T_{12}^t$. A well-known result of symmetric functions implies that p_α is an algebraic function of p_1, p_2, and p_3. For example, $p_3 = p_1^3 - 3p_1p_2 + 6p_1^2$ in the two variable case. Hence $tr(T_{12}T_{12}^t)^\alpha$ are invariants for any $\alpha \ge 1$. By [28] if two states ρ and ρ' have the same values of $tr(T_{12}T_{12}^t)^\alpha$, there exists a unitary matrix U such that $T_{12}'T_{12}'^t = UT_{12}T_{12}^t U^\dagger$, which means that $T_{12}T_{12}^t$ and $T_{12}'T_{12}'^t$ have identical eigenvalues. Both $T_{12}T_{12}^t$ and $T_{12}'T_{12}'^t$ are similar to $diag(\lambda_1, \lambda_2, \lambda_3)$. Then there exists an $O_1 \in SO(3)$ such that $T_{12}'T_{12}'^t = O_1 T_{12}T_{12}^t O_1^t$. Similarly there exists O_2 such that $T_{12}'^t T_{12}' = O_2 T_{12}^t T_{12} O_2^t$.

Subsequently $T'_{12} = O_1 T_{12} O_2^t$ for some O_1 and O_2, so ρ and ρ' are local unitary equivalent.

We now sharpen the result of Theorem 1. Since there are at most three linearly independent 3-dimensional vectors of μ_i and ν_i in (7) respectively, one can apply Theorem 1 to the basis vectors. The standard Gaussian elimination on the matrix $[\mu_1, \cdots, \mu_6]$ can pare down the column vectors into a basis $\{\mu_{i_1}, \mu_{i_2}, \mu_{i_3}\}$ of $\langle O_1 \rangle$, where $\{i_1, i_2, i_3\} \subset \{1, 2, \cdots, 6\}$. This means that the number of independent invariants that are used to judge the local unitary equivalence of two generic two-qubit states is at most 15 in general (instead of 33 as is Theorem 1). In fact, further analysis can reduce the number to at most 12 polynomial invariants.

Theorem 2 *Two generic two-qubit states are local unitary equivalent if and only if they have the same values for the following 12 invariants:*

$$\langle T_1, (T_{12}T_{12}^t)^\beta T_1 \rangle, \qquad \langle T_2, (T_{12}^t T_{12})^\beta T_2 \rangle, \tag{8}$$

$$\langle T_1, (T_{12}T_{12}^t)^\beta \, T_{12} T_2 \rangle, \quad \beta = 0, 1, 2, \tag{9}$$

$$tr(T_{12}T_{12}^t)^\alpha, \quad \alpha = 1, 2, 3. \tag{10}$$

Proof. The set $\langle \mathcal{O}_1 \rangle$ is a union of two orbits $(T_{12}T_{12}^t) \cdot T_1$ and $(T_{12}T_{12}^t) \cdot T_{12}T_2$. The independent inner products given in Theorem 1 are $\langle T_1, (T_{12}^t T_{12})^\beta T_1 \rangle$, $\langle T_1, (T_{12}T_{12}^t)^\beta T_{12} T_2 \rangle$ for $\beta = 0, 1, 2, 3$, due to the Cayley-Hamilton theorem and the fact that $\langle T_{12} u, v \rangle = \langle u, T_{12}^t v \rangle$ for any vectors u, v (T_{12} ia a real matrix). Similarly the orbit $\langle \mathcal{O}_2 \rangle$ will only contribute the remaining independent inner products $\langle T_2, (T_{12}^t T_{12})^\beta T_2 \rangle$, $\beta = 0, 1, 2, 3$.

We claim that the 3 invariants with $\beta = 3$ are not needed if the traces (10) are known. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of $T_{12}T_{12}^t$, and denote the trace powers by $p_\alpha = tr(T_{12}T_{12}^t)^\alpha = \lambda_1^\alpha + \lambda_2^\alpha + \lambda_3^\alpha$. The Cayley-Hamilton theorem says that

$$(T_{12}T_{12}^t)^3 = e_1 (T_{12}T_{12}^t)^2 - e_2 (T_{12}T_{12}^t) + e_3 I, \tag{11}$$

where $e_i = e_i(\lambda_1, \lambda_2, \lambda_3)$ are the elementary symmetric polynomials. By the fundamental theorem of symmetric polynomials, the e_i can be expressed as classical polynomials in the traces p_α (i.e. classical invariant polynomials of the density matrix):

$$e_1 = p_1, \quad e_2 = \frac{1}{2}(p_1^2 - p_2), \tag{12}$$

$$e_3 = \frac{1}{6}(p_1^3 - 3p_2 p_1 + 2p_3). \tag{13}$$

Plugging (11) into the three invariants $\langle T_1, (T_{12}^t T_{12})^3 T_1 \rangle$ etc., we see that they are given by linear combinations of the invariants (8-9) with coefficients fixed by classical invariant polynomials of the density matrices, so they are redundant. ∎

As we commented above if we use the Gaussian elimination we also worry about just 12 invariants. i.e., if we add $\beta = 3$ in the first set of invariants (8-9) for the 3 basis vectors we can waive the trace identities. Hence the total number of invariants is at most 12 either way. We still include the trace identities (10) for the sake of general (non-generic) cases.

Multi-qubit case: To simplify presentation, we introduce the following notation: $T_{ij} = T_{ji}^t$. We say that a word of T_i, T_{ij} is admissible if the adjacent subindices match. For example, $T_{12}T_2, T_{12}T_{21}T_1T_{12}$ are admissible ones.

We first consider the three-qubit case to present our general results. In this case, corresponding to (1), a quantum state has the form:

$$\rho = \frac{1}{8} I + \sum_{i=1}^{3} T_i \, \sigma^{(i)} + \sum_{i<j}^{3} T_{ij} \, \sigma^{(i)} \sigma^{(j)} + T_{123} \, \sigma^{(1)} \sigma^{(2)} \sigma^{(3)}. \tag{14}$$

If two states ρ and ρ' are local unitary equivalent, then there are orthogonal matrices $O_i \in SO(3)$ such that

$$T_1' = O_1 T_1, \quad T_2' = O_2 T_2, \quad T_3' = O_3 T_3, \tag{15}$$

$$T_{12}' = O_1 T_{12} O_2^t, \; T_{13}' = O_1 T_{12} O_3^t, \; T_{23}' = O_2 T_{23} O_3^t, \tag{16}$$

$$T_{123}' = (O_1 \otimes O_2 \otimes O_3) T_{123}. \tag{17}$$

It is known [16] that the last relation (17) is equivalent to either of the following two relations:

$$T_{123}' = O_1 T_{123} (O_2 \otimes O_3)^t, \; T_{123}' = (O_1 \otimes O_2) T_{123} O_3^t. \tag{18}$$

Here T_{123} is understood as the bipartition $T_{1|23}$ (resp. $T_{12|3}$) in the first (resp. 2nd) equation of (18). To state our results we introduce two subsets of vectors:

$$\langle \mathcal{O}_1 \rangle_{1|23} = \{T_1, T_{123}(T_{23}T_{23}^t)^\beta T_{23}, T_{123}T_{123}^t T_1, T_{123}T_{123}^t T_{123}(T_{23}T_{23}^t)^\beta T_{23}, (T_{123}T_{123}^t)^2 T_1, \cdots \} \subset \mathbf{R}^3,$$

$$\langle \mathcal{O}_2 \otimes \mathcal{O}_3 \rangle_{1|23} = \{T_{23}, T_{123}^t T_1, T_{123}^t T_{123}(T_{23}T_{23}^t)^\beta T_{23}, T_{123}^t T_{123} T_{123}^t T_1, (T_{23}T_{23}^t)^\beta T_{23}, \cdots \} \subset \mathbf{R}^9 \simeq \mathbf{R}^3 \otimes \mathbf{R}^3,$$

where $\beta = 0, \cdots, 3$, which are respectively the $(T_{123}T_{123}^t)$-orbit of $\{T_1, T_{123}(T_{23}T_{23}^t)^\beta T_{23} | \, \beta = 0, 1, 2, 3\}$

and the $(T_{123}^t T_{123})$-orbit of $\{(T_{23}T_{23}^t)^\beta T_{23}, T_{123}^t T_1 | \beta = 0,1,2,3\}$. Here T_{23} is taken as its (column) vector realignment in $\mathbf{R}^9$ and T_{123} is folded as a 3×9-matrix, by viewing T_{123} as the bipartition $1|23$ and T_{123}^t is the transpose with respect to such partition. As before we also use the same symbols for the corresponding real subspaces.

Similarly, by permuting the indices we define $\langle \mathcal{O}_2\rangle := \langle \mathcal{O}_2\rangle_{2|31}$ and $\langle \mathcal{O}_3\rangle := \langle \mathcal{O}_3\rangle_{3|12}$ to be the $(T_{231}T_{231}^t)$-orbit of $\{T_2, T_{231}(T_{31}T_{31}^t)^\beta T_{31} | \beta = 0,1,2,3\}$ and the $(T_{123}^t T_{123})$-orbit of $\{T_3, T_{312}(T_{12}T_{12}^t)^\beta T_{12} | \beta = 0,1,2,3\}$ respectively. Here the 3×9-matrix T_{231} (resp. T_{312}) is the realignment of T_{123} with respect to the partition of $\{123\}$ into $\{2|31\}$ (resp. $\{3|12\}$). Let $\langle \mathcal{O}_1\rangle = \{\mu_1,\mu_2,\mu_3\}$, $\langle \mathcal{O}_2\rangle = \{\nu_1,\nu_2,\nu_3\}$ and $\langle \mathcal{O}_3\rangle = \{\lambda_1,\lambda_2,\lambda_3\}$; $\langle \mathcal{O}_2\otimes\mathcal{O}_3\rangle_{1|23} = \{\alpha_1,\alpha_2,\ldots,\alpha_9\}$, $\langle \mathcal{O}_3\otimes\mathcal{O}_1\rangle_{2|31} = \{\beta_1,\beta_2,\ldots,\beta_9\}$, and $\langle \mathcal{O}_1\otimes\mathcal{O}_2\rangle_{3|12} = \{\gamma_1,\gamma_2,\ldots,\gamma_9\}$.

Theorem 3 *A three-qubit state ρ is local unitary equivalent to a three-qubit state ρ' if and only if the respective invariant polynomials are equal:*

$$\begin{aligned}
&\langle \mu_i,\mu_j\rangle = \langle \mu_i',\mu_j'\rangle, \quad \langle \nu_i,\nu_j\rangle = \langle \nu_i',\nu_j'\rangle,\\
&\langle \lambda_i,\lambda_j\rangle = \langle \lambda_i',\lambda_j'\rangle, \quad 1\le i\le j\le 3\\
&\langle \alpha_k,\alpha_l\rangle = \langle \alpha_k',\alpha_l'\rangle, \quad 1\le k\le l\le 9\\
&\langle \beta_k,\beta_l\rangle = \langle \beta_k',\beta_l'\rangle, \quad 1\le k\le l\le 9\\
&\langle \gamma_k,\gamma_l\rangle = \langle \gamma_k',\gamma_l'\rangle, \quad 1\le k\le l\le 9
\end{aligned} \tag{19}$$

Proof. By the result of two-qubit case, the invariance of inner products of vectors in $\langle \mathcal{O}_i\rangle$ implies the existence of orthogonal matrices O_i, $i=1,2,3$ such that Eqs. (15-16) hold. Thus we are left to show that the orthogonal matrices O_i also satisfy Eq. (17) or equivalently Eq. (18).

We use a similar method of Theorem 1 to show this by viewing the three-qubit state ρ as a bi-partite one on $\mathbf{C}^3\otimes\mathbf{C}^9$ and partition the hyper-matrix T_{123} as a rectangular matrix $T_{1|23}$. Then the 3×9-matrix T_{123} maps the subset $\langle \mathcal{O}_2\otimes\mathcal{O}_3\rangle_{1|23}$ into the subset $\langle \mathcal{O}_1\rangle_{1|23}$ by left multiplication.

We have already seen that there exists an orthogonal matrix O_i such that

$$O_i\langle \mathcal{O}_i\rangle = \langle \mathcal{O}_i'\rangle.$$

and Eqs. (16) hold. Then we can directly verify that the following diagram is commutative:

$$\begin{array}{ccc}
\langle \mathcal{O}_2\otimes\mathcal{O}_3\rangle & \xrightarrow{O_2\otimes O_3} & \langle \mathcal{O}_2'\otimes\mathcal{O}_3'\rangle\\
\Big\downarrow T_{123} & & \Big\downarrow T_{123}'\\
\langle \mathcal{O}_1\rangle_{1|23} & \xrightarrow{O_1} & \langle \mathcal{O}_1'\rangle_{1|23}
\end{array}$$

Consequently $T_{123}'(O_2\otimes O_3) = O_1 T_{123}$ in $End(\mathbf{R}^3\otimes\mathbf{R}^3)$, or $T_{123}' = O_1 T_{123}(O_2\otimes O_3)^t$. ∎

The following result shows that there are at most 90 invariants to judge LU equivalence for two three-qubit states.

Theorem 4 *Two generic three-qubit states are local unitary equivalent if and only if they have the same values of the following invariants:*

$$\begin{aligned}
&\langle T_1,(T_{12}T_{12}^t)^\alpha T_1\rangle, \quad \langle T_2,(T_{12}^t T_{12})^\alpha T_2\rangle,\\
&\langle T_1,(T_{12}T_{12}^t)^\alpha\, T_{12}T_2\rangle,\\
&tr(T_{12}T_{12}^t)^\beta, \quad tr(T_{13}T_{13}^t)^\beta, \quad tr(T_{23}T_{23}^t)^\beta,\\
&\langle T_1,(T_{1|23}T_{1|23}^t)^k T_1\rangle, \quad \langle T_2,(T_{2|31}T_{2|31}^t)^k T_2\rangle,\\
&\langle T_3,(T_{3|12}T_{3|12}^t)^k T_3\rangle; \quad \langle T_{23},(T_{1|23}^t T_{1|23})^k T_{23}\rangle,\\
&\langle T_{23},(T_{1|23}^t T_{123})^k T_{1|23}^t T_1\rangle,\\
&tr(T_{1|23}^t T_{1|23})^l, \quad tr(T_{2|31}^t T_{2|31})^l, \quad tr(T_{3|12}^t T_{3|12})^l.
\end{aligned} \tag{20}$$

where $\alpha = 0,1,2$; $\beta = 1,2,3$ and $k = 0,1,\cdots,8$; $l = 1,\cdots,9$.

The above criteria can be generalized to multi-qubits. Define $\langle \mathcal{O}_i\rangle = \langle T_i, T_{n\cdots n1\cdots i-1|i}T_n,\cdots\rangle \subset \mathbf{R}^3$ as the $(T_{\hat{i}i}^t T_{\hat{i}i})$-orbit, where $\hat{i} = 1\cdots\hat{i}\cdots n$ means the index i is absent. In general for any strict sequence $\mathbf{i} = (i_1\cdots i_k)$ (i.e. distinct i_j's), we define the $(T_{\hat{\mathbf{i}}\mathbf{i}}^t T_{\hat{\mathbf{i}}\mathbf{i}})$-orbit $\langle O_{\mathbf{i}}\rangle = \langle T_{\mathbf{i}},\cdots\rangle$, where the admissible generating words have only i when crossing out redundant strings. e.g., $T_{312}T_{12}T_1$ is a word of indices $3,1$ when crossing out 12. Then we have the following result. Let $\langle \mathcal{O}_1\rangle = \{\mu_1,\mu_2,\ldots,\mu_m\}$, $\langle \mathcal{O}_2\rangle = \{\nu_1,\nu_2,\ldots,\nu_m\},\cdots$, $\langle \mathcal{O}_N\rangle = \{\lambda_1,\lambda_2,\ldots,\lambda_m\}$, and more generally, for any strict sequence $\mathbf{i}$, let $\langle O_{\mathbf{i}}\rangle = \{\tau_1,\cdots,\tau_n\}$, where $n = n(\mathbf{i})$. Let's list these $\langle O_{\mathbf{i}}\rangle$ as $\langle O_{\mathbf{i}_j}\rangle = \{\tau_1^{(j)},\cdots,\tau_{m_j}^{(j)}\}$, $j = 1,\cdots,M$.

Theorem 5 *Two generic multi-qubit states ρ and ρ' are local unitary equivalent if and only if the respective invariant polynomials are equal:*

$$\begin{aligned}
\langle \tau_i^{(1)},\tau_j^{(1)}\rangle &= \langle \tau_i^{(1)'},\tau_j^{(1)'}\rangle,\cdots\\
\langle \tau_i^{(M)},\tau_j^{(M)}\rangle &= \langle \tau_i^{(M)'},\tau_j^{(M)'}\rangle,
\end{aligned} \tag{21}$$

where each pair of indices (i,j) are such that $1\le i,j\le m(\mathbf{i})$ for the sequences $\mathbf{i}_1,\cdots,\mathbf{i}_M$.

Proof. We use induction on n to reduce the problem to $(n-1)$-partite qubits. Note that for any sequence $\mathbf{i}$ of indices for n-partite state, we can view the elements in $\langle O_{\mathbf{i}}\rangle$ as $\langle O_{\mathbf{i}'}\otimes O_j\rangle$ where $\mathbf{i}'$ is obtained by realignment of the Block matrix with respect to the index j, and $\mathbf{i}'$ is obtained from $\mathbf{i}$ after the realignment. Then we can use the similar commutative diagram

$$\begin{array}{ccc}
\langle \mathcal{O}_2\otimes\mathcal{O}_{3\cdots n}\rangle & \xrightarrow{O_2\otimes O_{3\cdots n}} & \langle \mathcal{O}_2'\otimes\mathcal{O}_{3\cdots n}'\rangle\\
\Big\downarrow T_{1\cdots n} & & \Big\downarrow T_{1\cdots n}'\\
\langle \mathcal{O}_1\rangle_{1|2\cdots n} & \xrightarrow{O_1} & \langle \mathcal{O}_1'\rangle_{1|2\cdots n}
\end{array}$$

to get $T_{12\cdots n}'(O_2\otimes O_{3\cdots n}) = O_1 T_{12\cdots n}$ in $End(\mathbf{R}^3\otimes\mathbf{R}^{3(n-2)})$, or $T_{12\cdots n}' = O_1 T_{12\cdots n}(O_2\otimes O_{3\cdots n})^t$. Here $O_{3\cdots n}$

is an orthogonal matrix in the bigger orthogonal group. Then we use the induction to argue further for the matrix $T_{1|2\cdots n}$ viewed as a reduced matrix for $(n-1)$-partite state to get the final result. ■

Conclusions and Remarks: It is a basic and fundamental question to classify quantum states under local unitary operations. The problem has been figured out in [21, 22] for pure multipartite quantum states. However, it is much more difficult to classify mixed quantum states under LU transformations. Operational methods have been presented only for non-degenerate bipartite states. Although the authors in [25] have shown that the problem of mixed states can be reduced to one of pure states in terms of the purification of mixed states mathematically, the protocol is far from being operational. We have provided an operational way to verify and classify quantum states by using the generalized Bloch representation in terms of the generators of $SU(2)$. We remark that [23] gives a practical procedure to compute the LU operator for two equivalent multi-qubits, but it can not derive the polynomial invariants from the procedure, as it is based on a different strategy. In our current approach we set our goal to write down a set of simple invariants with which two states can be easily checked if they are LU equivalent. Since the coefficients (tensors) in the representation can be determined directly by measuring some local quantum mechanical observables-Pauli operators, the method is experimentally feasible. Our criterion is both sufficient and necessary for generic multi-qubit quantum systems, thus gives rise to a complete classification of multi-qubit generic quantum states under LU transformations.

Acknowledgments The work is supported by NSFC (11105226, 11271138, 11275131), CSC, Simons Foundation 198129, Humboldt Foundation, the Fundamental Research Funds for the Central Universities (12CX04079A, 24720122013), and Research Award Fund for outstanding young scientists of Shandong Province (BS2012DX045).

[1] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information. Cambridge: Cambridge University Press, (2000).
[2] D. Gottesman, Ph.D. thesis, Caltech [arXiv:quant-ph/9705052].
[3] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. 86, 5188 (2001).
[4] L. Amico et al., Rev. Mod. Phys. 80, 517 (2008); F. Verstraete et al., Adv. Phys. 57, 143 (2008), and references therein.
[5] R. Horodecki et al., Rev. Mod. Phys. 81, 865 (2009).
[6] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A 54, 3824 (1996); C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, ibid. 53, 2046 (1996); V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. 78, 2275 (1997).
[7] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[8] R. Horodecki, P. Horodecki, and M. Horodecki, Phys. Lett. A 200, 340 (1995).
[9] R. F. Werner and M. M. Wolf, Phys. Rev. A 64, 032112 (2001).
[10] M. Zukowski and C. Brukner, Phys. Rev. Lett. 88, 210401 (2002).
[11] M. Li and S. M. Fei, Phys. Rev. A 86, 052119 (2012).
[12] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A, 60, 1888 (1999).
[13] S. Albeverio, S. M. Fei, and W. L. Yang, Phys. Rev. A, 66, 012301(2002).
[14] Y. Makhlin, Quant. Info. Proc. 1, 243 (2002).
[15] N. Linden, S. Popescu, and A. Sudbery, Phys. Rev. Lett. 83, 243 (1999).
[16] T.-G. Zhang, M.-J. Zhao, M. Li, S.-M. Fei, and X. Li-Jost: Phys. Rev. A 88 042304 (2013).
[17] S. Albeverio, S. M. Fei, P. Parashar, and W. L. Yang, Phys. Rev. A 68, 010303 (2003).
[18] S. Albeverio, S. M. Fei, and D.Goswami, Phys. Lett. A 340, 37 (2005).
[19] B. Z. Sun, S. M. Fei, X. Q. Li-Jost, and Z. X. Wang, J. Phys. A 39, L43-L47 (2006).
[20] S. Albeverio, L. Cattaneo, S. M. Fei, and X. H. Wang, Int. J. Quant. Inform. 3, 603 (2005).
[21] B. Kraus, Phys. Rev. Lett. 104, 020504 (2010); Phys. Rev. A 82, 032121 (2010).
[22] B. Liu, J. L. Li, X. Li, and C. F. Qiao, Phys. Rev. Lett.108, 050501 (2012).
[23] M. Li, T. Zhang, S. M. Fei, X. Li-Jost, and N. Jing: Phys. Rev. A 89, 062325 (2014).
[24] C. Zhou, T. G. Zhang, S. M. Fei, N. Jing, and X. Li-Jost, Phys. Rev. A 86(R), 010303 (2012).
[25] J. L. Li and C. F. Qiao, J. Phys. A 46, 075301 (2013).
[26] A. M. Martins: Phys. Rev. A 91, 042308 (2015).
[27] N. Jing, M. Li, X. Li-Jost, T. Zhang, and S. Fei, J. Phys. A 47, no.21, 215303 (2014).
[28] W. Specht: Jber. Deutsch Math. Verein. 50, 19 (1940).