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by

Jun Zhang, Yuan-Hong Tao, Hua Nan, and Shao-Ming Fei

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Construction of Mutually Unbiased Bases in $\mathbb{C}^d \otimes \mathbb{C}^{2^l d'}$

Jun Zhang [1] , Yuan-Hong Tao [1,2], Hua Nan [1*], Shao-Ming Fei [2]

1. Department of Mathematics, College of Sciences, Yanbian University, Yanji 133002, China

2. School of Mathematical Sciences, Capital Normal University, Beijing, 100048, China

Abstract: We study mutually unbiased bases in $\mathbb{C}^d \otimes \mathbb{C}^{2^l d'}$. A systematic way of constructing mutually unbiased maximally entangled bases (MUMEBs) in $\mathbb{C}^d \otimes \mathbb{C}^{2^l d'} (l \in \mathbb{Z}^+)$ from MUMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'} (d' = kd, k \in \mathbb{Z}^+)$, and a general approach to construct mutually unbiased unextendible maximally entangled bases (MUUMEBs) in $\mathbb{C}^d \otimes \mathbb{C}^{2^l d'} (l \in \mathbb{Z}^+)$ from MUUMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'} (d' = kd + r, 0 < r < d)$ have been presented. Detailed examples are given.

Keywords: mutually unbiased bases; maximally entangled states; unextendible maximally entangled basis

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I. Introduction

Mutually unbiased bases (MUBs) [1, 2] have attracted much attention ever since the concept had been introduced. They play the central roles in the formulation of the discrete Wigner function [3, 4], cryptographic protocols [5, 6], quantum error correction codes [7] and quantum state tomography [8, 9]. MUBs also have applications in increasing the security of the ping-pong protocol [10] and in the solutions to the mean king problem [11, 12].

Let $\mathcal{B}_1 = \{|\phi_i\rangle\}_{i=1}^d$ and $\mathcal{B}_2 = \{|\psi_i\rangle\}_{i=1}^d$ be two orthonormal bases of a d-dimensional complex vector space $\mathbb{C}^d$. $\mathcal{B}_1$ and $\mathcal{B}_2$ are said to be mutually unbiased bases (MUBs) if and only if

$$|\langle\phi_i|\psi_j\rangle| = \frac{1}{\sqrt{d}}, \quad \forall i, j = 1, 2, \ldots, d.$$

It has been shown that there are $d + 1$ MUBs when d is a prime power [8], and there are many useful results [13]. However, for general d, the maximum number of MUBs is still unknown, let alone the detailed constructions of the MUBs.

When one considers MUBs in tensor spaces, the problem becomes more interesting and complicated: the MUBs can be product basis [14], maximally entangled basis [15], unextendible product basis [16], and unextendible maximally entangled basis [17] and so on.

*Corresponding author : Hua Nan, E-mail: nanhua@ybu.edu.cn

The entangled connection between mutually unbiased bases and another essential feature of quantum mechanics, quantum entanglement were studied in [18]. The quantum entanglement is an essential feature of quantum mechanics. Its importance has been demonstrated in various quantum information processing such as teleportation [19], superdense coding [20, 21] etc. Particularly, the maximally entangled states play very important roles in many quantum information processing tasks [22].

A state $|\psi\rangle$ is said to be a $d \otimes d'$ $(d' > d)$ maximally entangled if for an arbitrary given orthonormal complete basis $\{|i_A\rangle\}$ of subsystem A, there exists an orthonormal basis $\{|i'_B\rangle\}$ of subsystem B such that $|\psi\rangle$ can be written as $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i_A\rangle \otimes |i'_B\rangle$ [23]. A basis constituted by maximally entangled states is called a maximally entangled basis (MEB).

A set of states $\{|\phi_i\rangle \in \mathbb{C}^d \otimes \mathbb{C}^{d'} : i = 1, 2, \ldots, n, n < dd'\}$ is said to be an n-member unextendible maximally entangled bases (UMEBs) if (i) $|\phi_i\rangle, i = 1, 2, \ldots, n$, are maximally entangled; (ii) $\langle\phi_i|\phi_j\rangle = \delta_{ij}$; (iii) if $\langle\phi_i|\psi\rangle = 0$ for all $i = 1, 2, \ldots, n$, then $|\psi\rangle$ cannot be maximally entangled.

In [17] the authors first time considered the mutually unbiased bases in which all the bases are unextendible maximally entangled ones. And a systematic way of constructing a set of d^2 orthonormal maximally entangled states in $\mathbb{C}^d \bigotimes \mathbb{C}^{d'}$ $(\frac{d'}{2} < d < d')$ is provided. Necessary conditions of constructing a pair of mutually unbiased unextendible maximally entangled bases (MUUMEBs) in $\mathbb{C}^2 \bigotimes \mathbb{C}^3$ are derived in [24]. In [25, 26], UMEBs in $\mathbb{C}^d \bigotimes \mathbb{C}^{d'}$ with $d' = dq + r$, $0 < r < d$, have been constructed. UMEBs in $\mathbb{C}^d \bigotimes \mathbb{C}^d$ have been investigated in [27].

In this paper, we study MUBs in arbitrary bipartite spaces $\mathbb{C}^d \otimes \mathbb{C}^{2^l d'}$, $l \in \mathbb{Z}$, on condition that a pair of MUBs are given in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ $(d' = kd + r, k, r \in \mathbb{Z})$. In section II we investigate the case $d' = kd$. We show that once a pair of MUMEBs are given in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, there is a systematic method of constructing mutually unbiased MEBs (MUMEBs) in $\mathbb{C}^d \otimes \mathbb{C}^{2^l d'}$. For the case $d' = kd + r$, $0 < r < d$, we put forward a systematic method to construct mutually unbiased UMEBs (MUUMEBs) in $\mathbb{C}^d \otimes \mathbb{C}^{2^l d'}$ from MUUMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ in section III. Discussions and conclusions are given in section IV.

II. MUMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{2^l d'}$ $(d' = kd)$.

Let $\{|p\rangle\}_{p=0}^{d-1}$ be the orthonormal bases in $\mathbb{C}^d$. Consider a set of unitary operators, which forms a basis of the operator space on $\mathbb{C}^d$:

$$U_{n,m} = \sum_{p=0}^{d-1} \xi_d^{np} |p \oplus m\rangle\langle p|, \qquad n, m = 0, 1, \ldots, d-1,$$

where $\xi_d = e^{\frac{2\pi\sqrt{-1}}{d}}$, and $p \oplus m$ denotes $(p + m)$ mod d. Let $\{|p'\rangle\}_{p=0}^{d'-1}$ and $\{\mu'_p\}_{p=0}^{d'-1}$ denote two orthonormal bases of $\mathbb{C}^{d'}$. In bipartite Hilbert space $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ $(d' = kd)$, we can construct two MEBs in the following way:

$$|\phi_{n,m}^t\rangle = (U_{n,m} \otimes I_{d'})|\phi^t\rangle, \qquad t = 0, 1, \ldots, k-1; \quad n, m = 0, 1, \ldots, d-1, \tag{1}$$

$$|\psi^t_{n,m}\rangle = (U_{n,m} \otimes I_{d'})|\psi^t\rangle, \qquad t = 0, 1, \ldots, k-1; \quad n, m = 0, 1, \ldots, d-1, \tag{2}$$

where $|\phi^t\rangle = \frac{1}{\sqrt{d}} \sum_{p=0}^{d-1} |p\rangle|(p+dt)'\rangle$, $|\psi^t\rangle = \frac{1}{\sqrt{d}} \sum_{p=0}^{d-1} |p\rangle|\mu'_{(p+dt)}\rangle$, $(t = 0, 1, \ldots, k-1)$. Then (1) and (2) are mutually unbiased if and only if

$$\left|\langle\phi^{t_1}_{n_1,m_1}|\psi^{t_2}_{n_2,m_2}\rangle\right| = \frac{1}{\sqrt{dd'}},\ n_1, n_2, m_1, m_2 = 0, 1, \ldots, d-1;\ t_1, t_2 = 0, 1, \cdots, k-1. \tag{3}$$

Moreover, there must be a unitary transition matrix A such that

$$(|\mu'_0\rangle, |\mu'_1\rangle, \cdots, |\mu'_{d'-1}\rangle)^T = A(|0'\rangle, |1'\rangle, \cdots, |(d'-1)'\rangle)^T. \tag{4}$$

For the simple case $d' = d$, there is a method to get the transition matrix A in (4) for the $\mathbb{C}^{d'}$. We choose the dephased form of such vector $v = \frac{1}{\sqrt{d}}(1, e^{i\alpha_1}, \cdots, e^{i\alpha_{d-1}})$ in $\mathbb{C}^{d'}(d' = d)$ with real parameters $\alpha_p \in [0, 2\pi]$, $p = 1, 2, \cdots, d-1$, where $i = \sqrt{-1}$. The vector v is mutually unbiased to $\frac{1}{\sqrt{d}} \sum\limits_{p=0}^{d-1} \xi_d^{np}|p'\rangle$ $(n = 0, 1, \cdots, d-1)$ if

$$\left|\sum_{p=0}^{d-1} \xi_d^{np} e^{i\alpha_p}\right| = \sqrt{d},\ n = 0, 1, \cdots, d-1, \tag{5}$$

defining $\alpha_0 = 1$. The solutions of these equations can provide unit vectors which can be to compose the bases $|\mu'_p\rangle$ $(p = 0, 1, \cdots, d-1)$ in (4). And from the formula (4), we know that v is the row of the matrix A. In [28], the solutions of (5) are given when d equals 2 to 5. For general case $d' = kd, k \geq 1$, we have that (3) are valid if and only if $A = (a_{s,t})$ $(s, t = 0, 1, \cdots, d-1)$ satisfies

$$\left|\sum_{p=0}^{d-1} \xi_d^{np} a_{p+dj,\pi^l(p)+di}\right| = \frac{1}{\sqrt{k}}, \quad i, j = 0, 1, \cdots, k-1;\ n, l = 0, 1, \cdots, d-1, \tag{6}$$

where π denotes the permutation of $\{0, 1, \cdots, d-1\}$ defined by $\pi = \begin{pmatrix} 0 & 1 & \cdots & d-1 \\ 1 & 2 & \cdots & 0 \end{pmatrix}$.

Let us first consider the case $d = d' = 3$. Set $|\phi_0\rangle = \frac{1}{\sqrt{3}}(|0\rangle|0'\rangle + |1\rangle|1'\rangle + |2\rangle|2'\rangle)$. The first MEB in $\mathbb{C}^3 \otimes \mathbb{C}^3$ can be chosen to be

$$|\phi_{n,m}\rangle = (U_{n,m} \otimes I_3)|\phi_0\rangle, \qquad n, m = 0, 1, 2. \tag{7}$$

By(5),we can get a unitary transition matrix

$$A = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{pmatrix}, \quad \omega = \frac{-1 + \sqrt{3}i}{2},$$

so we get an alternative orthonormal basis $\{|\varepsilon'_j\rangle\}_{j=0}^2$ in $\mathbb{C}^3$,

$$(|\varepsilon'_0\rangle, |\varepsilon'_1\rangle, |\varepsilon'_2\rangle)^T = A(|0'\rangle, |1'\rangle, |2'\rangle)^T.$$

Let $|\psi_0\rangle = \frac{1}{\sqrt{3}}(|0\rangle|\varepsilon_0^{'}\rangle + |1\rangle|\varepsilon_1'\rangle + |2\rangle|\varepsilon_2'\rangle)$. We get the second MEB in $\mathbb{C}^3 \otimes \mathbb{C}^3$,

$$|\psi_{n,m}\rangle = (U_{n,m} \otimes I_3)|\psi_0\rangle, \qquad n, m = 0, 1, 2. \tag{8}$$

It is easy to check that (7) and (8) are mutually unbiased.

Based on the MUMEBs in $\mathbb{C}^3 \otimes \mathbb{C}^3$, we construct the MUMEBs in $\mathbb{C}^3 \otimes \mathbb{C}^{12}$. Let

$$|\phi_{nm}^s\rangle = \frac{1}{\sqrt{3}} \sum_{p=0}^{2} w^{np} |p \oplus m\rangle |(3s+p)'\rangle, \quad s = 0, 1, 2, 3, \tag{9}$$

$$|\psi_{nm}^s\rangle = \frac{1}{\sqrt{3}} \sum_{p=0}^{2} w^{np} |p \oplus m\rangle |\nu_{3s+p}'\rangle, \qquad s = 0, 1, 2, 3, \tag{10}$$

where

$$\begin{pmatrix} |\nu_0'\rangle \\ |\nu_1'\rangle \\ \vdots \\ |\nu_{11}'\rangle \end{pmatrix} = \frac{1}{2} \begin{pmatrix} A & A & A & A \\ A & -A & A & -A \\ A & A & -A & -A \\ A & -A & -A & A \end{pmatrix} \begin{pmatrix} |0'\rangle \\ |1'\rangle \\ \vdots \\ |11'\rangle \end{pmatrix}.$$

It is straightforward to verify that (9) and (10) are MUMEBs in $\mathbb{C}^3 \otimes \mathbb{C}^{12}$.

The above construction of MUMEB can be generalized to $\mathbb{C}^d \otimes \mathbb{C}^{2^l d'} (l \in \mathbb{Z}^+)$. Namely, based on the MUMEBs (1)and (2) in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, we construct the MUMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{2^l d'}$. Let $\{B_n\}$ be a sequence of matrices which satisfy the following recurrence relation:

$$B_n = \frac{1}{\sqrt{2}} \begin{pmatrix} B_{n-1} & B_{n-1} \\ B_{n-1} & -B_{n-1} \end{pmatrix},$$

where $B_0 = A$. Since

$$B_1^\dagger B_1 = \frac{1}{2} \begin{pmatrix} 2B_0^\dagger B_0 & 0 \\ 0 & 2B_0^\dagger B_0 \end{pmatrix} = \begin{pmatrix} I & \\ & I \end{pmatrix} = I,$$

all $\{B_n\}$ are unitary matrices. Hence, we can get an alternative orthonormal basis $\{|\nu_p'\rangle\}_{p=0}^{2^l d'-1}$ of $\mathbb{C}^{2^l d'} (d' = kd,\ l, k \in \mathbb{Z}^+)$ from the orthonormal basis $\{|p'\rangle\}_{p=0}^{2^l d'-1}$,

$$(|\nu_0'\rangle, |\nu_1'\rangle, \cdots, |\nu_{2^l d'-1}'\rangle)^T = B_l(|0'\rangle, |1'\rangle, \cdots, |(2^l d'-1)'\rangle)^T.$$

Let

$$|\phi^{st}\rangle = \frac{1}{\sqrt{d}} \sum_{p=0}^{d-1} |p\rangle |(d's + dt + p)'\rangle, \qquad |\psi^{st}\rangle = \frac{1}{\sqrt{d}} \sum_{p=0}^{d-1} |p\rangle |\nu_{d's+dt+p}'\rangle,$$

and

$$|\phi_{n,m}^{s,t}\rangle \ = (U_{n,m} \otimes I_{2^l d'})|\phi^{st}\rangle, \tag{11}$$

$$|\psi_{n,m}^{s,t}\rangle \ = (U_{n,m} \otimes I_{2^l d'})|\psi^{st}\rangle, \tag{12}$$

where $n, m = 0, 1, \cdots, d-1;\ s = 0, 1, \cdots, 2^l - 1;\ t = 0, 1, \cdots, k-1$. Obviously, $\{\phi_{n,m}^{s,t}\}$ and $\{\psi_{n,m}^{s,t}\}$ are two MEBs in $\mathbb{C}^d \otimes \mathbb{C}^{2^l d'}$. For the case $\mathbb{C}^d \otimes \mathbb{C}^{2d'}$ $(l = 1)$, according to (3) we have

$$\left|\langle \phi_{n_1,m_1}^{s_1,t_1} | \psi_{n_2,m_2}^{s_2,t_2} \rangle\right| = \left| \frac{1}{\sqrt{2}} \langle \phi_{n_1,m_1}^{t_1} | \psi_{n_2,m_2}^{t_2} \rangle \right| = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{dd'}},$$

where $n_1, n_2, m_1, m_2 = 0, 1, \ldots, d-1;\ s_1, s_2 = 0, 1; t_1, t_2 = 0, 1, \cdots, k-1$. While in the space $\mathbb{C}^d \otimes \mathbb{C}^{2^l d'}$, we also have

$$\left|\langle \phi_{n_1,m_1}^{s_1,t_1} | \psi_{n_2,m_2}^{s_2,t_2} \rangle\right| = \left| \frac{1}{\sqrt{2^l}} \langle \phi_{n_1,m_1}^{t_1} | \psi_{n_2,m_2}^{t_2} \rangle \right| = \frac{1}{\sqrt{2^l}} \frac{1}{\sqrt{dd'}},$$

where $n_1, n_2, m_1, m_2 = 0, 1, \ldots, d-1;\ s_1, s_2 = 0, 1, \cdots, 2^l - 1; t_1, t_2 = 0, 1, \cdots, k-1$. Therefore by induction we have the following conclusion:

Theorem 1. Giving that (1) and (2) are MUMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ $(d' = kd)$, the maximally entangled bases (11) and (12) are MUMEBs of $\mathbb{C}^d \otimes \mathbb{C}^{2^l d'}$.

III. MUUMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{2^l d'} (d' = kd + r, 0 < r < d)$

In [17] and [25], a systematic way of constructing kd^2-member UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ has been introduced. Let $\{|p'\rangle\}_{p=0}^{d'-1}$ and $\{\mu_p'\}_{p=0}^{d'-1}$ denote two orthonormal bases of $\mathbb{C}^{d'}$, and $|\phi^t\rangle = \frac{1}{\sqrt{d}} \sum_{p=0}^{d-1} |p\rangle |(dt+p)'\rangle$, $|\psi^t\rangle = \frac{1}{\sqrt{d}} \sum_{p=0}^{d-1} |p\rangle |\mu'_{(p+dt)}\rangle$ $(t = 0, 1, \cdots, k-1)$. Then

$$|\phi_{n,m}^t\rangle = (U_{n,m} \otimes I_{d'}) |\phi^t\rangle, \qquad n, m = 0, 1, \cdots, d-1;\ t = 0, 1, \cdots k-1, \tag{13}$$

and

$$|\psi_{n,m}^t\rangle = (U_{n,m} \otimes I_{d'}) |\psi^t\rangle, \qquad n, m = 0, 1, \cdots, d-1;\ \ t = 0, 1, \cdots k-1, \tag{14}$$

are two kd^2-member UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'} (d' = kd + r, 0 < r < d)$.

Suppose that $\{|a_i\rangle\}$ and $\{|b_i\rangle\}$ are two alternative orthonormal bases in $\mathbb{C}^d$. Set

$$|\phi_{ji}\rangle = |a_i\rangle \otimes |j'\rangle, \quad j = kd, \cdots, d'-1;\ i = 0, 1, \cdots, d-1, \tag{15}$$

$$|\psi_{ji}\rangle = |b_i\rangle \otimes |\mu_j'\rangle, \quad j = kd, \cdots, d'-1;\ i = 0, 1, \cdots, d-1, \tag{16}$$

such that (13) together with the rd orthonormal product state (15) constitute a complete UMEB $|\Phi_{n,m}^t\rangle$ in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, and (14) together with the rd orthonormal product state(16) constitute another complete UMEB $|\Psi_{n,m}^t\rangle$ in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, and these two UMEBs are mutually unbiased. They satisfy

$$\left|\langle \Phi_{n_1,m_1}^{t_1} | \Psi_{n_2,m_2}^{t_2} \rangle\right| = \left|\langle \Phi_{j_1,i_1} | \Psi_{n_2,m_2}^{t_2} \rangle\right| = \left|\langle \Phi_{n_1,m_1}^{t_1} | \Psi_{j_2,i_2} \rangle\right| = \frac{1}{\sqrt{dd'}}. \tag{17}$$

There should exist a unitary transition matrix D such that

$$(|\mu_0'\rangle, |\mu_1'\rangle, \cdots, |\mu_{d'-1}'\rangle)^T = D(|0'\rangle, |1'\rangle, \cdots, |((d'-1)')\rangle)^T.$$

In $\mathbb{C}^{d'}, d' = kd + r$ $(0 < r < d)$, the transition matrix D is not easy to find. In Ref[17, 18, 25], the authors provided the transition matrices for $\mathbb{C}^2 \otimes \mathbb{C}^3$, $\mathbb{C}^3 \otimes \mathbb{C}^4$ and

$\mathbb{C}^2 \otimes \mathbb{C}^6$. In other bipartite system, constructing the transition matrices is still an open problem. For low dimension, it can be found by using the method in [24].

Similar approach in section II can be used to construct MUUMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{2^l d'}$, providing that a pair of MUUMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ are given as above. Suppose that $\{|p'\rangle\}_{p=0}^{2^l d'-1}$ and $\{\varepsilon'_p\}_{p=0}^{2^l d'-1}$ are two orthonormal bases of $\mathbb{C}^{2^l d'}$ with

$$(|\varepsilon'_0\rangle, |\varepsilon'_1\rangle, \cdots, |\varepsilon'_{2^l d'-1}\rangle)^T = D_l(|0'\rangle, |1'\rangle, \cdots, |(2^l d'-1)'\rangle)^T,$$

where

$$D_l = \frac{1}{\sqrt{2}} \begin{pmatrix} D_{l-1} & D_{l-1} \\ D_{l-1} & -D_{l-1} \end{pmatrix}$$

with $D_0 = D$.

Now we construct the following UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{2^l d'}$ $(2^l r < d)$:

$$\begin{cases} |\Phi^{s,t}_{n,m}\rangle = (U_{n,m} \otimes I_{2^l d'})|\phi^{st}\rangle, \\ |\Phi^{s}_{j,i}\rangle = |a_i\rangle \otimes |(d's+j)'\rangle, \end{cases} \tag{18}$$

$$\begin{cases} |\Psi^{s,t}_{n,m}\rangle = (U_{n,m} \otimes I_{2^l d'})|\psi^{st}\rangle, \\ |\Psi^{s}_{j,i}\rangle = |b_i\rangle \otimes |\varepsilon'_{d's+j}\rangle, \end{cases} \tag{19}$$

where $n, m = 0, 1, \cdots, d-1$; $s = 0, 1, \cdots, 2^l-1$; $t = 0, 1, \cdots, k-1$, $j = kd, \cdots, d'-1$; $i = 0, 1, \cdots, d-1$, and

$$|\phi^{st}\rangle = \frac{1}{\sqrt{d}} \sum_{p=0}^{d-1} |p\rangle |(d's+dt+p)'\rangle, \quad |\psi^{st}\rangle = \frac{1}{\sqrt{d}} \sum_{p=0}^{d-1} |p\rangle |\varepsilon'_{d's+dt+p}\rangle.$$

The next conclusion can be obtained:

Theorem 2. The UMEBs (18) and (19) in bipartite spaces $\mathbb{C}^d \otimes \mathbb{C}^{2^l d'}$ $(2^l r < d)$ are mutually unbiased, providing that the complete MUUMEBs $|\Phi^t_{n,m}\rangle$ given by (13), (15) and $|\Psi^t_{n,m}\rangle$ given by (14), (16) in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ $(d' = kd + r, 0 < r < d)$ are mutually unbiased UMEBs.

Proof. Similar to the verification of theorem 1, according to (17) we have

$$\left|\langle \Phi^{s_1,t_1}_{n_1,m_1} | \Psi^{s_2,t_2}_{n_2,m_2}\rangle\right| = \left|\frac{1}{\sqrt{2^l}} \langle \Phi^{t_1}_{n_1,m_1} | \Psi^{t_2}_{n_2,m_2}\rangle\right| = \frac{1}{\sqrt{2^l}} \frac{1}{\sqrt{dd'}},$$

$$\left|\langle \Phi^{s_1}_{j_1,i_1} | \Psi^{s_2,t_2}_{n_2,m_2}\rangle\right| = \left|\frac{1}{\sqrt{2^l}} \langle \Phi_{j_1,i_1} | \Psi^{t_2}_{n_2,m_2}\rangle\right| = \frac{1}{\sqrt{2^l}} \frac{1}{\sqrt{dd'}},$$

$$\left|\langle \Phi^{s_1,t_1}_{n_1,m_1} | \Psi^{s_2}_{n_2,m_2}\rangle\right| = \left|\frac{1}{\sqrt{2^l}} \langle \Phi^{t_1}_{n_1,m_1} | \Psi_{j_2,i_2}\rangle\right| = \frac{1}{\sqrt{2^l}} \frac{1}{\sqrt{dd'}}.$$

Let V denotes the subspace spanned by $\{|\Phi^{s,t}_{n,m}\rangle,\ n,m=0,1,\ldots,d-1;\ t=0,1,\cdots,2^l-1\}$. Then $Dim(V)=2^ld^2$, and $Dim(V^{\perp})=2^ldd'-2^ld^2=2^lrd$. Here the condition $2^lr<d$ is necessary, otherwise, there may exist maximally entangled state $|\Psi\rangle\in V^{\perp}$ such that $\langle\Phi^t_{nm}|\Psi\rangle=0$. ∎

Example. Let us give a detailed example of MUUMEBs in $\mathbb{C}^3\otimes\mathbb{C}^8$.

Two sets of complete MUUMEBs $\{|\phi_{n,m}\rangle\}$ and $\{|\psi_{n,m}\rangle\}$ $(n=0,1,2,3;\ m=0,1,2)$ in $\mathbb{C}^3\otimes\mathbb{C}^4$ were constructed in [25]:

$$
\begin{aligned}
|\phi_{nm}\rangle &= \frac{1}{\sqrt{3}}\sum_{j=0}^{2}\omega^{nj}\ |j\oplus m\rangle|j'\rangle, \quad n,m=0,1,2,\\
|\phi_{30}\rangle &= \frac{1}{\sqrt{3}}\left(|0\rangle+|1\rangle+|2\rangle\right)|3'\rangle,\\
|\phi_{31}\rangle &= \frac{1}{\sqrt{3}}\left(|0\rangle+\omega|1\rangle+\bar{\omega}|2\rangle\right)|3'\rangle,\\
|\phi_{32}\rangle &= \frac{1}{\sqrt{3}}\left(|0\rangle+\bar{\omega}|1\rangle+\omega|2\rangle\right)|3'\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
|\psi_{nm}\rangle &= \frac{1}{\sqrt{3}}\sum_{j=0}^{2}\ \omega^{nj}|j\oplus m\rangle|\mu'_j\rangle, \quad n,m=0,1,2,\\
|\psi_{3,0}\rangle &= \frac{1}{\sqrt{3}}\left(|0\rangle+\omega|1\rangle+\omega|2\rangle\right)|\mu'_3\rangle,\\
|\psi_{3,1}\rangle &= \frac{1}{\sqrt{3}}\left(\omega|0\rangle+|1\rangle+\omega|2\rangle\right)|\mu'_3\rangle,\\
|\psi_{3,2}\rangle &= \frac{1}{\sqrt{3}}\left(\omega|0\rangle+\omega|1\rangle+|2\rangle\right)|\mu'_3\rangle,
\end{aligned}
$$

where $(|\mu'_0\rangle),|\mu'_1\rangle,|\mu'_2\rangle,|\mu'_3\rangle))^T=D(|0'\rangle,|1'\rangle,|2'\rangle,|3'\rangle)^T$,

$$
D=\frac{1}{2}\left(\begin{array}{cccc}
1 & \omega & -\overline{\omega} & \omega\\
-\overline{\omega} & \omega & \overline{\omega} & 1\\
\overline{\omega} & -1 & \omega & 1\\
\overline{\omega} & \omega & \overline{\omega} & -1
\end{array}\right)
$$

with $\omega=\frac{-1+\sqrt{3}i}{2}$.

From these MUUMEBs in $\mathbb{C}^3\otimes\mathbb{C}^4$ we can construct two sets of complete MUUMEBs

$\{\Phi^t_{nm}\}$, $\{\Psi^t_{nm}\}$ in $\mathbb{C}^3\otimes\mathbb{C}^8$ as follows:

$$
\begin{aligned}
|\Phi^t_{nm}\rangle &= \frac{1}{\sqrt{3}}\sum_{j=0}^{2}\omega^{nj}\ |j\oplus m\rangle|(4t+j)'\rangle,\\
|\Phi^t_{30}\rangle &= \frac{1}{\sqrt{3}}\left(|0\rangle+|1\rangle+|2\rangle\right)|(4t+3)'\rangle,\\
|\Phi^t_{31}\rangle &= \frac{1}{\sqrt{3}}\left(|0\rangle+\omega|1\rangle+\bar{\omega}|2\rangle\right)|(4t+3)'\rangle,\\
|\Phi^t_{32}\rangle &= \frac{1}{\sqrt{3}}\left(|0\rangle+\bar{\omega}|1\rangle+\omega|2\rangle\right)|(4t+3)'\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
|\Psi^t_{nm}\rangle &= \frac{1}{\sqrt{3}}\sum_{j=0}^{2}\omega^{nj}\ |j\oplus m\rangle|\nu'_{(4t+j)}\rangle,\\
|\Psi^t_{30}\rangle &= \frac{1}{\sqrt{3}}\left(|0\rangle+\omega|1\rangle+\omega|2\rangle\right)|\nu'_{(4t+3)}\rangle,\\
|\Psi^t_{31}\rangle &= \frac{1}{\sqrt{3}}\left(\omega|0\rangle+|1\rangle+\omega|2\rangle\right)|\nu'_{(4t+3)}\rangle,\\
|\Psi^t_{32}\rangle &= \frac{1}{\sqrt{3}}\left(\omega|0\rangle+\omega|1\rangle+|2\rangle\right)|\nu'_{(4t+3)}\rangle,
\end{aligned}
$$

where $n,m=0,1,2;\ t=0,1,$ and

$$
\left(|\nu'_0\rangle,|\nu'_1\rangle,\cdots,|\nu'_7\rangle\right)^T=\frac{1}{\sqrt{2}}\begin{pmatrix} D & D\\ D & -D\end{pmatrix}\left(|0'\rangle,|1'\rangle,\cdots,|7'\rangle\right)^T.
$$

It is direct to verify that

$$
\left|\langle\Phi^{t_1}_{n_1,m_1}|\Psi^{t_2}_{n_2,m_2}\rangle\right|=\left|\frac{1}{\sqrt{2}}\langle\phi_{n_1,m_1}|\psi_{n_2,m_2}\rangle\right|=\frac{1}{\sqrt{24}},
$$

where $n_1,n_2=0,1,2,3$, $m_1,m_2=0,1,2$, $t_1,t_2=0,1$.

V. Discussions and Conclusion

The prerequisite of the methods presented in this paper is the assumption that there exists the unitary transition matrix for the $\mathbb{C}^{d'}(d'=kd,\text{or } d'=kd+r, 0<r<d)$ system. In part II and part III, we give the way of approaching the transition matrix A in $\mathbb{C}^{d'}(d'=kd)$, and discuss the transition matrix in $\mathbb{C}^{d'}(d'=kd+r, 0<r<d')$ with low dimension.

The construction of mutually unbiased bases with maximally entangled bases and unextendible maximally entangled bases are open problems. Partial solutions have been given for low dimensional cases. The construction of mutually unbiased unextendible maximally entangled bases is more complicated. We have presented a way to construct MUMEBs and MUUMEBS in higher dimensions from the corresponding MUMEBs and MUUMEBS in low dimensions: MUMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{2^l d'} (l \in \mathbb{Z}^+)$ from MUMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'} (d' = kd, k \in \mathbb{Z}^+)$, and MUUMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{2^l d'} (l \in \mathbb{Z}^+)$ from MUUMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'} (d' = kd + r, 0 < r < d)$.

There might be several UMEBs with different numbers in a bipartite space [26]. For a given UMEB in a space $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, the vectors that are supplement to the maximally entangled basic vectors may not be product ones. In this paper, we have only concerned the case that supplementary basic vectors are all product states. It would be desirable that our approach can be similarly applied in constructing MUMEBs for the case that the basic vectors in the supplement space of UMEBs are not product states in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$.

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