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**Regularity theory for 2-dimensional almost
minimal currents I: Lipschitz approximation**

by

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REGULARITY THEORY FOR 2-DIMENSIONAL ALMOST MINIMAL CURRENTS I: LIPSCHITZ APPROXIMATION

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ABSTRACT. We construct Lipschitz Q -valued functions which approximate carefully integral currents when their cylindrical excess is small and they are almost minimizing in a suitable sense. This result is used in two subsequent works to prove the discreteness of the singular set for the following three classes of 2-dimensional integral currents: area minimizing in Riemannian manifolds, semicalibrated and spherical cross sections of 3-dimensional area minimizing cones.

This paper is the second in a series of works aimed at establishing an optimal regularity theory for 2-dimensional integral currents which are almost minimizing in a suitable sense. Building upon the monumental work of Almgren [1], Chang in [4] established that 2-dimensional area-minimizing currents in Riemannian manifolds are classical minimal surfaces, namely they are regular (in the interior) except for a discrete set of branching singularities. The argument of Chang is however not entirely complete since a key starting point of his analysis, the existence of the so-called “branched center manifold”, is only sketched in the appendix of [4] and requires the understanding (and a suitable modification) of the most involved portion of the monograph [1].

An alternative proof of Chang’s theorem has been found by Rivière and Tian in [15] for the special case of J -holomorphic curves. Later on the approach of Rivière and Tian has been generalized by Bellettini and Rivière in [3] to handle a case which is not covered by [4], namely that of special Legendrian cycles in S^5 (see also [2] for a further generalization).

Meanwhile the first and second author revisited Almgren’s theory giving a much shorter version of his program for proving that area-minimizing currents are regular up to a set of Hausdorff codimension 2, cf. [5, 6, 7, 8, 9]. In this note and its companion papers [10, 11] we build upon the latter works in order to give a complete regularity theory which includes both the theorems of Chang and Bellettini-Rivière as special cases. In order to be more precise, we introduce the following terminology (cf. [12, Definition 0.3]).

Definition 0.1. Let $\Sigma \subset \mathbb{R}^{m+n}$ be a C^2 submanifold and $U \subset \mathbb{R}^{m+n}$ an open set.

- (a) An m -dimensional integral current T with finite mass and $\text{spt}(T) \subset \Sigma \cap U$ is area-minimizing in $\Sigma \cap U$ if $\mathbf{M}(T + \partial S) \geq \mathbf{M}(T)$ for any $m + 1$ -dimensional integral current S with $\text{spt}(S) \subset \subset \Sigma \cap U$.
- (b) A semicalibration (in Σ) is a C^1 m -form ω on Σ such that $\|\omega_x\|_c \leq 1$ at every $x \in \Sigma$, where $\|\cdot\|_c$ denotes the comass norm on $\Lambda^m T_x \Sigma$. An m -dimensional integral current T with $\text{spt}(T) \subset \Sigma$ is *semicalibrated* by ω if $\omega_x(\vec{T}) = 1$ for $\|T\|$ -a.e. x .

- (c) An m -dimensional integral current T supported in $\partial\mathbf{B}_R(p) \subset \mathbb{R}^{m+n}$ is a *spherical cross-section of an area-minimizing cone* if $p \ast T$ is area-minimizing.

In what follows, given an integer rectifiable current T , we denote by $\text{Reg}(T)$ the subset of $\text{spt}(T) \setminus \text{spt}(\partial T)$ consisting of those points x for which there is a neighborhood U such that $T \llcorner U$ is a (constant multiple of) a regular submanifold. Correspondingly, $\text{Sing}(T)$ is the set $\text{spt}(T) \setminus (\text{spt}(\partial T) \cup \text{Reg}(T))$. Observe that $\text{Reg}(T)$ is relatively open in $\text{spt}(T) \setminus \text{spt}(\partial T)$ and thus $\text{Sing}(T)$ is relatively closed. The main result of this and the works [10, 11] is then the following

Theorem 0.2. *Let Σ and ω be as in Definition 0.1, let T be as in (a), (b) or (c) and assume in addition that $m = 2$, that Σ is of class $C^{3,\alpha}$ and ω of class $C^{2,\alpha}$ for some positive α . Then $\text{Sing}(T)$ is discrete.*

Clearly Chang's result is covered by case (a). As for the case of special Lagrangian cycles considered by Bellettini and Rivière in [3] observe that they form a special subclass of both (b) and (c). Indeed these cycles arise as spherical cross-sections of 3-dimensional special lagrangian cones: as such they are then spherical cross sections of area-minimizing cones but they are also semicalibrated by a specific smooth form on \mathbb{S}^5 .

Following the Almgren-Chang program, Theorem 0.2 will be established through a suitable “blow-up argument” which requires several tools. The first important tool is the theory of multiple-valued functions, for which we will use the results and terminology of the papers [5, 6]. The second tool is a suitable approximation result for area-minimizing currents with graphs of multiple valued functions. The one needed to carry out the proof in case (a) is already contained in [7]. However the latter paper does not cover the cases (b) and (c): therefore the purpose of this note is to extend the theorems in [7] to these cases. Moreover, since the corresponding theorems can be proved for all dimensions m without any additional effort, in this note we will state all the results in such generality

The final tool is the so-called “center manifold”: this will be constructed in [10], whereas the final argument for Theorem 0.2 will then be given in [11]. We note in passing that Theorem 0.2 uses heavily the uniqueness of tangent cones for T . This result is a, by now classical, theorem of White for area-minimizing 2-dimensional currents in the euclidean space, cf. [17]. Chang extended it to case (a) in the appendix of [4], whereas Pumberger and Rivière covered case (b) in [14]. A general derivation of these results for a wide class of almost minimizers has been given in [12]: the theorems in there cover, in particular, all the cases of Definition 0.1.

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1. NOTATION AND STATEMENT OF THE MAIN THEOREM

We start recalling here several geometric facts which have been shown in [12]. The first is given by the following technical lemma.

Lemma 1.1 (Lemma 1.1 in [12]). *Let $k \in \mathbb{N} \setminus \{0\}$, $\varepsilon_0 \in [0, 1]$, $\Sigma \subset \mathbb{R}^{m+n}$ be a C^{k+1, ε_0} $(m + \bar{n})$ -dimensional submanifold, $V \subset \mathbb{R}^{m+n}$ an open subset and ω a C^{k, ε_0} m -form on $V \cap \Sigma$. If T is a cycle in $V \cap \Sigma$ semicalibrated by ω , then T is semicalibrated in V by a C^{k, ε_0} form $\tilde{\omega}$.*

For this reason, without loss of generality we will from now on consider semicalibrated currents directly in the Euclidean space. The ambient manifold Σ will then play a role only in case (a) of Definition 0.1. Next we recall

Proposition 1.2 (Proposition 1.2 in [12]). *Let T be as in Definition 0.1 (b) (in which case we assume $\Sigma = \mathbb{R}^{m+n}$) or (c). Then there is a constant Ω such that*

$$\mathbf{M}(T) \leq \mathbf{M}(T + \partial S) + \Omega \mathbf{M}(S) \quad \forall S \in \mathbf{I}_{m+1}(\mathbb{R}^{m+n}) \quad \text{with compact support.} \quad (1.1)$$

$\Omega \leq \|d\omega\|_0$ in case (b) and $\Omega \leq (m+1)R^{-1}$ in case (c).

Moreover, if $\chi \in C_c^\infty(\mathbb{R}^{m+n} \setminus \text{spt}(\partial T), \mathbb{R}^{m+n})$, we have

$$\delta T(\chi) = T(d\omega \lrcorner \chi) \quad \text{in case (b),} \quad (1.2)$$

$$\delta T(\chi) = \int mR^{-1} x \cdot \chi(x) d\|T\|(x) \quad \text{in case (c).} \quad (1.3)$$

In the case of Riemannian minimizers we introduce a further relevant quantity, the maximum norm of the second fundamental form of Σ , which according to the notation of the papers [5, 6, 7] will be denoted by \mathbf{A} . Therefore we have

$$\mathbf{A} := \sup_{x \in \Sigma} |A_\Sigma(x)|. \quad (1.4)$$

For the notation concerning submanifolds $\Sigma \subset \mathbb{R}^{m+n}$ we refer to [6, Section 1]. With $\mathbf{B}_r(p)$ and $B_r(x)$ we denote, respectively, the open ball with radius r and center p in \mathbb{R}^{m+n} and the open ball with radius r and center x in \mathbb{R}^m . $\mathbf{C}_r(x)$ will always denote the cylinder $B_r(x) \times \mathbb{R}^n$ and the point x will be omitted when it is the origin. In fact, by a slight abuse of notation, we will often treat the center x as a point in \mathbb{R}^{m+n} , avoiding the correct, but more cumbersome, $(x, 0)$. e_i will denote the unit vectors in the standard basis, π_0 the (oriented) plane $\mathbb{R}^m \times \{0\}$ and $\vec{\pi}_0$ the m -vector $e_1 \wedge \dots \wedge e_m$ orienting it. We denote by \mathbf{p} and \mathbf{p}^\perp the orthogonal projections onto, respectively, π_0 and its orthogonal complement π_0^\perp . In some cases we need orthogonal projections onto other planes π and their orthogonal complements π^\perp , for which we use the notation \mathbf{p}_π and \mathbf{p}_π^\perp . For what concerns integral currents we use the definitions and the notation of [16]. We isolate the main assumption of our approximation theorem in the following

Assumption 1.3. In case (a) $\Sigma \subset \mathbb{R}^{m+n}$ is a C^2 submanifold of dimension $m + \bar{n} = m + n - l$, which is the graph of an entire function $\Psi : \mathbb{R}^{m+\bar{n}} \rightarrow \mathbb{R}^l$ and satisfies the bounds

$$\|D\Psi\|_0 \leq c_0 \quad \text{and} \quad \mathbf{A} := \|A_\Sigma\|_0 \leq c_0, \quad (1.5)$$

where c_0 is a positive (small) dimensional constant. ω is a C^1 m -form. T is an integral current of dimension m with bounded support. Moreover it satisfies one of the three conditions (a), (b) or (c) in Definition 0.1. In particular in case (a) we have $\text{spt}(T) \subset \Sigma$

and T is area-minimizing in Σ . In case (b) we assume $\Sigma = \mathbb{R}^{m+n}$ and T is semicalibrated by ω . In case (c) we have that Σ coincides with a portion of $\partial\mathbf{B}_R(p)$, which is the graph of a map $\Psi : \Omega \rightarrow \mathbb{R}$ satisfying (1.5), for some $\Omega \subset \mathbb{R}^{m+n-1}$. Finally, for some open cylinder $\mathbf{C}_{4r}(x)$ (with $r \leq 1$) and some positive integer Q ,

$$\mathbf{p}_\#T = Q \llbracket B_{4r}(x) \rrbracket \quad \text{and} \quad \partial T \llcorner \mathbf{C}_{4r}(x) = 0. \quad (1.6)$$

Definition 1.4 (Excess measure). For a current T as in Assumption 1.3 we define the *cylindrical excess* $\mathbf{E}(T, \mathbf{C}_r(x))$, the *excess measure* \mathbf{e}_T and its *density* \mathbf{d}_T :

$$\begin{aligned} \mathbf{E}(T, \mathbf{C}_r(x)) &:= \frac{\|T\|(\mathbf{C}_r(x))}{\omega_m r^m} - Q, \\ \mathbf{e}_T(A) &:= \|T\|(A \times \mathbb{R}^n) - Q|A| \quad \text{for every Borel } A \subset B_r(x), \\ \mathbf{d}_T(y) &:= \limsup_{s \rightarrow 0} \frac{\mathbf{e}_T(B_s(y))}{\omega_m s^m} = \limsup_{s \rightarrow 0} \mathbf{E}(T, \mathbf{C}_s(y)), \end{aligned}$$

where ω_m is the measure of the m -dimensional unit ball (the subscripts T will be omitted if clear from the context).

The main theorem of the paper is then the following approximation result (for the notation concerning multiple valued functions and their graphs we refer to [5, 6, 7]).

Theorem 1.5. *There exist constants $M, C_{21}, \beta_0, \varepsilon_{21} > 0$ (depending on m, n, \bar{n}, Q) with the following property. Assume that T satisfies Assumption 1.3 in the cylinder $\mathbf{C}_{4r}(x)$ and $E = \mathbf{E}(T, \mathbf{C}_{4r}(x)) < \varepsilon_{21}$. Then, there exist a map $f : B_r(x) \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$, with $\{x\} \times \text{spt}(f(x)) \subset \Sigma$ for every x , and a closed set $K \subset B_r(x)$ such that*

$$\text{Lip}(f) \leq C_{21} E^{\beta_0} + C_{21} \Omega r \quad \text{in case (a) and (c)}, \quad (1.7)$$

$$\text{Lip}(f) \leq C_{21} E^{\beta_0} \quad \text{in case (b)} \quad (1.8)$$

$$\mathbf{G}_f \llcorner (K \times \mathbb{R}^n) = T \llcorner (K \times \mathbb{R}^n) \quad \text{and} \quad |B_r(x) \setminus K| \leq C_{21} E^{\beta_0} (E + r^2 \Omega^2) r^m, \quad (1.9)$$

$$\left| \|T\|(\mathbf{C}_r(x)) - Q \omega_m r^m - \frac{1}{2} \int_{B_r(x)} |Df|^2 \right| \leq C_{21} E^{\beta_0} (E + r^2 \Omega^2) r^m, \quad (1.10)$$

where $\Omega = \mathbf{A}$ in case (a). If in addition $\mathbf{h}(T, \mathbf{C}_{4r}(x)) := \sup\{|\mathbf{p}^\perp(x) - \mathbf{p}^\perp(y)| : x, y \in \text{spt}(T) \cap \mathbf{C}_{4r}(x)\} \leq r$, then

$$\text{osc}(f) \leq C_{21} \mathbf{h}(T, \mathbf{C}_{4r}(x)) + C_{21} (E^{1/2} + r \Omega) r \quad \text{in case (a) and (c)}, \quad (1.11)$$

$$\text{osc}(f) \leq C_{21} \mathbf{h}(T, \mathbf{C}_{4r}(x)) + C_{21} r E^{1/2} \quad \text{in case (b)}. \quad (1.12)$$

First of all we observe that the case of area minimizing currents in a Riemannian manifold (case (a) of Definition 0.1) is already covered by [7, Theorem 1.4] (note that the estimate [7, (1.4)], which corresponds to (1.7), misses the term $r\Omega = r\mathbf{A}$: this is however only a typo), so we only need to prove Theorem 1.5 for semicalibrated currents and for spherical cross sections of area minimizing cones. Secondly, our proof does not exploit all the structure implied by the conditions (b) and (c) in Definition 0.1: without any additional effort we achieve in fact the following more general version of the corresponding portions of Theorem 1.5.

Definition 1.6 (Ω -minimality). A current with compact support satisfying condition (1.1) will be called Ω -minimal.

Proposition 1.7. *There exist constants $M, C_{21}, \beta_0, \varepsilon_{21} > 0$ (depending on m, n, Q) with the following property. Assume that $T \in \mathbf{I}_m(\mathbb{R}^{m+n})$ is Ω -minimal, it satisfies (1.6) in the cylinder $\mathbf{C}_{4r}(x)$ and $E = \mathbf{E}(T, \mathbf{C}_{4r}(x)) < \varepsilon_{21}$. Then, there exist a map $f: B_r(x) \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$ and a closed set $K \subset B_r(x)$ satisfying (1.7), (1.9) and (1.10). If in addition $\mathbf{h}(T, \mathbf{C}_{4r}(x)) \leq r$, then (1.12) also holds.*

The rest of the paper is devoted to prove Proposition 1.7. This will be achieved in four sections. In the first we recall some results from [7], which can be applied in our case without any modification. In the second we improve upon the almost minimality condition under the assumption that the cylindrical excess is small: this section contains, indeed, the most significant new ideas compared to [7]. In the two subsequent sections we modify accordingly the computations of [7] to prove Proposition 1.7. Observe that the graph of the map of Proposition 1.7 is not supported in $\Sigma = \partial\mathbf{B}_R(p)$ in case (c). We will thus need to show how this last requirement can be achieved respecting the estimates claimed in Theorem 1.5: this will be accomplished in the very last section. From now on constants which depend only upon m, n and Q will be called *dimensional constants*.

2. PRELIMINARIES

All the results stated in this section are proved in [7] in the more general case of m -dimensional currents which minimize the area in a Riemannian manifold. However since as noted before we will need them only in the euclidean setting, we shall restrict ourselves to this easier case. Here we adopt the notation and terminology on Q -valued maps of the papers [5, 6, 7]: we therefore refer the reader to these works for all the terms and symbols used in what follows.

The first result is a procedure to approximate general integral currents without boundary. To state it we need the following definition.

Definition 2.1 (Maximal function of the excess measure). Given a current T as in Assumption 1.3 we introduce the “non-centered” maximal function of \mathbf{e}_T :

$$\mathbf{me}_T(y) := \sup_{y \in B_{s/2}(w) \subset B_{4r}(x)} \frac{\mathbf{e}_T(B_s(w))}{\omega_m s^m} = \sup_{y \in B_{s/2}(w) \subset B_{4r}(x)} \mathbf{E}(T, \mathbf{C}_s(w)).$$

Notice that with respect to [7, Definition 2.1], we define the Maximal function taking the supremum over balls of radius $s/2$ and not s . This is just a technicality which allows to construct the Lipschitz approximation of the next Proposition in the ball of radius $7r/2$.

Proposition 2.2 (Lipschitz approximation; cf. [7, Proposition 2.2]). *There exists a constant $C_{22}(m, n, Q) > 0$ with the following property. Let T be as in Proposition 1.7 in the cylinder $\mathbf{C}_{4s}(x)$. Set $E = \mathbf{E}(T, \mathbf{C}_{4r}(x))$, let $0 < \delta < 1$ be such that*

$$r_0 := 16 \sqrt[m]{\frac{E}{\delta}} < 1,$$

and define $K := \{\mathbf{me}_T < \delta\} \cap B_{7r/2}(x)$. Then, there is $u \in \text{Lip}(B_{7r/2}(x), \mathcal{A}_Q(\mathbb{R}^n))$ such that

$$\begin{aligned} \text{Lip}(u) &\leq C_{22} \delta^{1/2}, \\ \mathbf{G}_{u \llcorner} (K \times \mathbb{R}^n) &= T \llcorner (K \times \mathbb{R}^n), \\ |B_s(x) \setminus K| &\leq \frac{10^m}{\delta} \mathbf{e}_T(\{\mathbf{me}_T > 2^{-m} \delta\} \cap B_{s+r_0s}(x)) \quad \forall s \leq \frac{7r}{2}. \end{aligned} \quad (2.1)$$

When $\delta = E^{2\beta}$, the map u given by the proposition will be called E^β -Lipschitz approximation of T in $\mathbf{C}_{7r/2}(x)$.

The second result deals with the construction of suitable competitors (in energetic terms) for a sequence of $W^{1,2}$ Q -valued functions whose Dirichlet energy is uniformly bounded. Before stating it we need a definition and a concentration compactness lemma.

Definition 2.3 (Translating sheets). Let $\Omega \subset \mathbb{R}^m$ be a bounded open set. A sequence of maps $\{h_l\}_{l \in \mathbb{N}} \subset W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ is called a sequence of *translating sheets* if there are:

- (a) integers $J \geq 1$ and $Q_1, \dots, Q_J \geq 1$ satisfying $\sum_{j=1}^J Q_j = Q$,
- (b) vectors $y_l^j \in \mathbb{R}^n$ (for $j \in \{1, \dots, J\}$ and $l \in \mathbb{N}$) with

$$\lim_{l \rightarrow \infty} |y_l^j - y_l^i| = +\infty \quad \forall i \neq j, \quad (2.2)$$

- (c) and maps $\zeta^j \in W^{1,2}(\Omega, \mathcal{A}_{Q_j})$ for $j \in \{1, \dots, J\}$,

such that $h_l = \sum_{j=1}^J \llbracket \tau_{y_l^j} \circ \zeta^j \rrbracket$ (see [5, Section 3.3.3] for the notation).

Translating sheets are a useful device to recover a suitable ‘‘compactness statement’’ for sequences of Q -valued maps with equi-bounded energy.

Proposition 2.4 (Concentration compactness; cf. [7, Proposition 3.3]). *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set and $(g_l)_{l \in \mathbb{N}} \subset W^{1,2}(\Omega, \mathcal{A}_Q)$ a sequence of functions with $\sup_l \int_\Omega |Dg_l|^2 < \infty$. Then, there exist a subsequence (not relabeled) and a sequence of translating sheets h_l such that $\|\mathcal{G}(g_l, h_l)\|_{L^2} \rightarrow 0$ and the following inequalities hold for every open $\Omega' \subset \Omega$ and any sequence of measurable sets J_l with $|J_l| \rightarrow 0$:*

$$\liminf_{l \rightarrow +\infty} \left(\int_{\Omega' \setminus J_l} |Dg_l|^2 - \int_{\Omega'} |Dh_l|^2 \right) \geq 0 \quad (2.3)$$

$$\limsup_{l \rightarrow +\infty} \int_\Omega (|Dg_l| - |Dh_l|)^2 \leq \limsup_{l \rightarrow \infty} \int_\Omega (|Dg_l|^2 - |Dh_l|^2). \quad (2.4)$$

They are also the key tool in the construction of competitors for the energy.

Proposition 2.5 (Construction of a competitor; cf. [7, Proposition 3.4]). *Consider two radii $1 \leq r_0 < r_1 < 4$ and maps $g_l, h_l \in W^{1,2}(B_{r_1}, \mathcal{A}_Q(\mathbb{R}^n))$ such that $\{h_l\}_l$ is a sequence of translating sheets,*

$$\sup_l \text{Dir}(g_l, B_{r_1}) < +\infty \quad \text{and} \quad \|\mathcal{G}(g_l, h_l)\|_{L^2(B_{r_1} \setminus B_{r_0})} \rightarrow 0.$$

For every $\eta > 0$, there exist $r \in]r_0, r_1[$, a subsequence of $\{g_l\}_l$ (not relabeled) and functions $H_l \in W^{1,2}(B_{r_1}, \mathcal{A}_Q(\mathbb{R}^n))$ such that $H_l|_{B_{r_1} \setminus B_r} = g_l|_{B_{r_1} \setminus B_r}$ and $\text{Dir}(H_l, B_{r_1}) \leq \text{Dir}(h_l, B_{r_1}) + \eta$.

In addition, there is a dimensional constant C_{23} and a constant $C^*(\eta)$ (depending also on the two sequences, but not on l) such that

$$\text{Lip}(H_l) \leq C^*(\eta) (\text{Lip}(g_l) + 1), \quad (2.5)$$

$$\|\mathcal{G}(H_l, h_l)\|_{L^2(B_r)} \leq C_{23} \text{Dir}(g_l, B_r) + C_{23} \text{Dir}(H_l, B_r), \quad (2.6)$$

$$\|\boldsymbol{\eta} \circ H_l\|_{L^1(B_{r_1})} \leq C_{23} \|\boldsymbol{\eta} \circ g_l\|_{L^1(B_{r_1})} + C_{23} \|\boldsymbol{\eta} \circ h_l\|_{L^1(B_{r_1})}. \quad (2.7)$$

The third result we recall here is about the higher integrability of Dir-minimizing Q -valued function.

Theorem 2.6 (Higher integrability of Dir-minimizers; cf. [7, Theorem 5.1]). *There exists $p_1 > 2$ such that, for every $\Omega' \subset \subset \Omega \subset \mathbb{R}^2$ open domains, there is a constant $C_{24} > 0$ such that*

$$\|Du\|_{L^{p_1}(\Omega')} \leq C_{24} \|Du\|_{L^2(\Omega)} \quad \text{for every Dir-minimizing } u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n)). \quad (2.8)$$

The fourth and final result of this section is the existence of a special projection from \mathbb{R}^N to \mathcal{Q} , which avoids loss of energy when composed with a $W^{1,2}$ Q -valued map.

Proposition 2.7 (Cf. [7, Proposition 6.2]). *For every $n, Q \in \mathbb{N} \setminus \{0\}$ there are geometric constants $\delta_0, C_{24} > 0$ with the following property. For every $\delta \in]0, \delta_0[$ there is $\boldsymbol{\rho}_\delta^* : \mathbb{R}^{N(Q,n)} \rightarrow \mathcal{Q} = \boldsymbol{\xi}(\mathcal{A}_Q(\mathbb{R}^n))$ such that $|\boldsymbol{\rho}_\delta^*(P) - P| \leq C_{24} \delta^{8-nQ}$ for all $P \in \mathcal{Q}$ and, for every $u \in W^{1,2}(\Omega, \mathbb{R}^N)$, it holds*

$$\int |D(\boldsymbol{\rho}_\delta^* \circ u)|^2 \leq \left(1 + C_{24} \delta^{8-nQ-1}\right) \int_{\{\text{dist}(u, \mathcal{Q}) \leq \delta^{nQ+1}\}} |Du|^2 + C_{24} \int_{\{\text{dist}(u, \mathcal{Q}) > \delta^{nQ+1}\}} |Du|^2. \quad (2.9)$$

3. HOMOTOPY LEMMA

Before proving the main Lipschitz approximation theorem we need a lemma which estimates carefully the difference in mass between an Ω -almost minimizer and a competitor in terms of a power of the excess and the constant Ω . The key idea is to choose the surface S in (1.1) to be an homotopy between the E^β approximation of T and that of S .

Lemma 3.1 (Homotopy Lemma). *Let T be an Ω -almost minimizer which satisfies (1.6). There are positive dimensional constants ε_{22} and C_{25} such that, if $E = \mathbf{E}(T, \mathbf{C}_{4r}(x)) \leq \varepsilon_{22}$, then the following holds. For every $R \in \mathbf{I}_m(\mathbf{C}_{3r}(x))$ such that $\partial R = \partial(T \llcorner \mathbf{C}_{3r}(x))$, we have*

$$\|T\|(\mathbf{C}_{3r}(x)) \leq \mathbf{M}(R) + C_{25} r^{m+1} \Omega E^{1/2}. \quad (3.1)$$

Moreover, let $\beta \leq \frac{1}{2m}$, $s \in]r, 2r[$, $R = \mathbf{G}_g \llcorner \mathbf{C}_s(x)$ for some Lipschitz map $g : B_s \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$ with $\text{Lip}(g) \leq 1$ and f be the E^β -approximation of T in \mathbf{C}_{3r} . If $f = g$ on ∂B_s and $P \in \mathbf{I}_m(\mathbb{R}^{m+n})$ is such that $\partial P = \partial((T - \mathbf{G}_f) \llcorner \mathbf{C}_s)$, then

$$\|T\|(\mathbf{C}_s(x)) \leq \mathbf{M}(\mathbf{G}_g) + \mathbf{M}(P) + C_{25} \Omega \left(E^{3/4} r^{m+1} + (\mathbf{M}(P))^{1+1/m} + \int_{B_s(x)} \mathcal{G}(f, g) \right). \quad (3.2)$$

Proof. We will first show (3.1): in fact (3.2) follows easily from a portion of the same argument, as it will be highlighted at the end.

Without loss of generality we assume $x = 0$. If $\|T\|(\mathbf{C}_{3r}) \leq \mathbf{M}(R)$ then there is nothing to prove. Hence we can suppose

$$\mathbf{M}(R) \leq \|T\|(\mathbf{C}_{3r}). \quad (3.3)$$

Define the current $R' \in \mathbf{I}_m(\mathbf{C}_{4r})$ by $R' := R + T \llcorner (\mathbf{C}_{4r} \setminus \mathbf{C}_{3r})$. Observe that $\partial(T - R') = 0$. So $\partial(\mathbf{p}_\#(T - R')) = 0$. On the other hand $\mathbf{p}_\#(T - R') = k \llbracket B_{4r} \rrbracket$ for some constant k and thus we conclude $\mathbf{p}_\#(T - R') = 0$. Therefore R' satisfies (1.6). Moreover we notice that, thanks to (3.3), the cylindrical excess of R' enjoys the following bound:

$$\mathbf{E}(R', C_{4r}) = \frac{\mathbf{M}(R')}{\omega_m(4r)^m} - Q \stackrel{(3.3)}{\leq} \frac{\mathbf{M}(T)}{\omega_m(4r)^m} - Q = \mathbf{E}(T, C_{4r}) =: E.$$

Let $f, h: B_{7r/2} \rightarrow \mathcal{A}_Q(\mathbb{R}^n)$ be the E^β -Lipschitz approximations of T and R' respectively, in the cylinders $\mathbf{C}_{7r/2}$ (the exponent β will be chosen to equal $\frac{1}{2m}$ in the proof of (3.1), but we choose to keep the symbol β to highlight the changes needed in order to achieve (3.2)). Then there exist sets $K_T, K_{R'} \subset B_{7r/2}(x)$ such that $T \llcorner (K_T \times \mathbb{R}^n) = \mathbf{G}_f \llcorner (K_T \times \mathbb{R}^n)$ and $R' \llcorner (K_{R'} \times \mathbb{R}^n) = \mathbf{G}_h \llcorner (K_{R'} \times \mathbb{R}^n)$, fulfilling the following estimates:

$$\mathbf{M}((T - \mathbf{G}_f) \llcorner C_{7r/2}) \leq C_{21} r^m E^{1-2\beta} \quad \text{and} \quad \mathbf{M}((R' - \mathbf{G}_h) \llcorner C_{7r/2}) \leq C_{21} r^m E^{1-2\beta}, \quad (3.4)$$

$$|B_{7r/2} \setminus K_T| \leq C_{21} r^m E^{1-2\beta} \quad \text{and} \quad |B_{7r/2} \setminus K_{R'}| \leq C_{21} r^m E^{1-2\beta}, \quad (3.5)$$

$$\text{Lip}(f) \leq C_{21} E^\beta \quad \text{and} \quad \text{Lip}(h) \leq C_{21} E^\beta. \quad (3.6)$$

Next we set $K := K_T \cap K_{R'}$ and we notice that by (3.5)

$$|B_{7r/2} \setminus K| \leq C_{21} r^m E^{1-2\beta}. \quad (3.7)$$

Let $|\cdot|$ be the function $|(x, y)| := |x|_2$ for every $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, where $|x|_2$ is the euclidean norm of the vector x . By the slicing theory, (3.4), (3.7) and Fubini's Theorem there exists $s \in (3r, 7/2r)$ such that

$$\mathbf{M}(\langle T - \mathbf{G}_f, |\cdot|, s \rangle) + \mathbf{M}(\langle R' - \mathbf{G}_h, |\cdot|, s \rangle) \leq C_{21} r^{m-1} E^{1-2\beta} \quad (3.8)$$

and

$$|\partial B_s \setminus K| \leq C_{21} r^{m-1} E^{1-2\beta}. \quad (3.9)$$

By the Isoperimetric Inequality, there exists $P_T, P_R \in \mathbf{I}_m(\mathbb{R}^{m+n})$ such that

$$\partial P_T = \langle T - \mathbf{G}_f, |\cdot|, s \rangle \quad \partial P_R = \langle R' - \mathbf{G}_h, |\cdot|, s \rangle$$

and

$$\begin{aligned} \mathbf{M}(P_T) + \mathbf{M}(P_R) &\leq C(\mathbf{M}(\langle T - \mathbf{G}_f, |\cdot|, s \rangle))^{m/(m-1)} + C(\mathbf{M}(\langle R' - \mathbf{G}_h, |\cdot|, s \rangle))^{m/(m-1)} \\ &\leq C r^m E^{m(1-2\beta)/(m-1)}. \end{aligned}$$

Choosing $\beta = \frac{1}{2m}$, we can conclude that

$$\partial((T - \mathbf{G}_f) \llcorner \mathbf{C}_s) = \partial P_T \quad \partial((R' - \mathbf{G}_h) \llcorner \mathbf{C}_s) = \partial P_R \quad (3.10)$$

with

$$\mathbf{M}(P_T) + \mathbf{M}(P_R) \leq Cr^m E. \quad (3.11)$$

Next consider the functions

$$f' := \xi \circ f: B_{7r/2} \rightarrow \mathcal{Q} \subset \mathbb{R}^{N(Q,n)} \quad \text{and} \quad h' := \xi \circ h: B_{7r/2} \rightarrow \mathcal{Q} \subset \mathbb{R}^{N(Q,n)}.$$

and the homotopy between them, defined by

$$\tilde{H}(x, t): [0, 1] \times B_{7r/2}(x) \ni (t, x) \rightarrow (x, tf'(x) + (1-t)h'(x)) \in \mathbb{R}^m \times \mathbb{R}^N.$$

Consider the Lipschitz map

$$\phi: \mathbb{R}^m \times \mathbb{R}^N \ni (x, y) \rightarrow (x, \xi^{-1}(\rho(y))) \in \mathbb{R}^m \times \mathcal{A}_Q(\mathbb{R}^n)$$

and define $H := \phi \circ \tilde{H}$. H can be seen as a Q -valued map $H: B_{2r} \times [0, 1] \rightarrow \mathcal{A}_Q(\mathbb{R}^{m+n})$. Without changing notation for H we restrict it to $[0, 1] \times B_s$ and following the notation of [6, Definition 1.3] we define $S := \mathbf{T}_H$. If we set $G := H|_{[0,1] \times \partial B_s}$ we can use [6, Theorem 2.1] to conclude that

$$\partial S = (\mathbf{G}_f - \mathbf{G}_h) \llcorner \mathbf{C}_s + \mathbf{T}_G = (\mathbf{G}_f - \mathbf{G}_h) \llcorner \mathbf{C}_s + P, \quad (3.12)$$

where $P := \mathbf{T}_G$. We now want to estimate $\mathbf{M}(S)$ and $\mathbf{M}(P)$ and we will do it using the Q -valued area formula in [6, Lemma 1.9]. We start with $\mathbf{M}(S)$. We fix a point of differentiability p where $DH = \sum \llbracket DH_i \rrbracket$. On $[0, 1] \times B_s$ we use the coordinates (t, x) and on the target space \mathbb{R}^{m+n} the coordinates (x, y) . Let $p = (t_0, x_0)$. It is then obvious that the matrix DH_i can be decomposed as

$$DH_i(p) = \begin{pmatrix} I_{m \times m} & 0_{m \times 1} \\ A_{n \times m} & v_{n \times 1} \end{pmatrix}$$

where the matrices A and v can be bound using the following observation. If we consider the map $t \mapsto \Phi(t) := H(x_0, t)$ and $x \mapsto \Lambda(x) := H(t_0, x)$, we then have $|v| \leq C \text{Lip}(\Phi)$ and $|A| \leq C \text{Lip}(\Lambda)$, where the constant C depends only on n and Q . On the other hand, it is easy to see that $\text{Lip}(\Phi) \leq C \mathcal{G}(f(x_0), h(x_0))$ and $\text{Lip}(\Lambda) \leq C(\text{Lip}(h) + \text{Lip}(f)) \leq E^\beta = E^{1/2m}$. Thus we can estimate

$$\mathbf{J}H_i := \sqrt{\det(DH_i^* \cdot DH_i)} \leq C \mathcal{G}(f(x_0), g(x_0)).$$

Using [6, Lemma 1.9] we then conclude

$$\mathbf{M}(S) \leq C \int_{B_s} \mathcal{G}(f, h)$$

and, arguing in a similar fashion,

$$\mathbf{M}(P) \leq C \int_{\partial B_s} \mathcal{G}(f, h).$$

Observe that f and h coincide, respectively, with the slices of the currents T and R' on any $x_0 \in K$. On the other hand, $s > 3r$ and $T \llcorner \mathbf{C}_{4r} \setminus \mathbf{C}_{3r} = R' \llcorner \mathbf{C}_{4r} \setminus \mathbf{C}_{3r}$. We thus conclude

that $h = f$ on $K \cap \partial B_s$. Let $x \in \partial B_s \setminus K$. By (3.9), there exists $x_0 \in K \cap \partial B_s$ such that $|x - x_0| \leq CrE^{(1-2\beta)/(m-1)} = CrE^{2\beta}$ (recall that $\beta = \frac{1}{2m}$). Thus

$$\mathcal{G}(f(x), h(x)) \leq (\text{Lip}(f) + \text{Lip}(h)) |x - x_0| \leq CrE^{3\beta},$$

and so we conclude

$$\mathbf{M}(P) \leq C \int_{\partial B_s} \mathcal{G}(f, h) \leq CrE^{3\beta} |\partial B_s \setminus K| \leq Cr^m E^{1+\beta} \leq Cr^m E. \quad (3.13)$$

On the other hand, we recall that, by a standard variant of the Poincaré inequality,

$$\begin{aligned} \int_{B_s} \mathcal{G}(f, h) &\leq Cr \|\mathcal{G}(f, h)\|_{L^1(\partial B_s)} + Cr \|D(\mathcal{G}(f, h))\|_{L^1(B_s)} \\ &\stackrel{(3.13)}{\leq} Cr^{m+1} E + Cr^{1+m/2} \left(\int (|Df|^2 + |Dh|^2) \right)^{1/2} \leq Cr^{m+1} E^{1/2}. \end{aligned} \quad (3.14)$$

Thus,

$$(\mathbf{G}_f - \mathbf{G}_h) \llcorner \mathbf{C}_s = \partial S + P \quad (3.15)$$

with

$$\mathbf{M}(P) \leq Cr^m E \quad \text{and} \quad \mathbf{M}(S) \leq Cr^{m+1} E^{1/2}. \quad (3.16)$$

Now observe that

$$0 = \partial(T - R') = \partial((\mathbf{G}_f - \mathbf{G}_h) \llcorner \mathbf{C}_s) + \partial(P_T - P_R) = \partial\partial S + \partial P + \partial(P_T - P_R).$$

Hence, $\partial(P + P_T - P_R) = 0$ and, by the isoperimetric inequality, there is an S' with $\mathbf{M}(S') \leq Cr^{m+1} E^{1+1/m}$ and $\partial S' = P + P_T - P_R$. Additionally, again using the isoperimetric inequality, there are currents S_T and S_R such that

$$\begin{aligned} \partial S_T &= (T - \mathbf{G}_f) \llcorner \mathbf{C}_s - P_T \\ \partial S_R &= (R' - \mathbf{G}_h) \llcorner \mathbf{C}_s - P_R \end{aligned}$$

and

$$\begin{aligned} \mathbf{M}(S_T) &\leq C (\|T - \mathbf{G}_f\|(\mathbf{C}_s) + \mathbf{M}(P_T))^{(m+1)/m} \leq CE^{3/4} r^{m+1} \\ \mathbf{M}(S_R) &\leq C (\|T - \mathbf{G}_h\|(\mathbf{C}_s) + \mathbf{M}(P_R))^{(m+1)/m} \leq CE^{3/4} r^{m+1}. \end{aligned}$$

In the latter inequalities we have used $\|T - \mathbf{G}_h\|(\mathbf{C}_s) + \|T - \mathbf{G}_f\|(\mathbf{C}_s) \leq CE^{1-2\beta} r^m = CE^{(m-1)/m} r^m$: in particular $(1 - 2\beta)(m + 1)/m = 1 - 1/m^2 \geq 3/4$; observe that this estimate is valid even if $\beta < 1/(2m)$ and explains the exponent of E in the third summand of the right hand side of (3.2).

Thus, setting $S'' = S + S_T - S_R + S'$ we finally achieve $(T - R') \llcorner \mathbf{C}_s = \partial S''$ and $\mathbf{M}(S'') \leq Cr^{m+1} E^{1/2}$. Recalling that $s > 3r$ and that $R' = R + T \llcorner (\mathbf{C}_{4r} \setminus \mathbf{C}_{3r})$ we conclude $\partial S'' = (T - R) \llcorner \mathbf{C}_{3r}$. Applying now the Ω -minimality of T we conclude

$$\|T\|(\mathbf{C}_{3r}) \leq \mathbf{M}(R) + C_{25} r^{m+1} \Omega E^{1/2}.$$

For the proof of (3.2) we conclude with the same computations, except that this time $f = g$ on ∂B_s and the current R is already given by $\mathbf{G}_g \llcorner \mathbf{C}$. The modifications to the argument are then straightforward, given the remark of the previous paragraph. \square

4. HARMONIC APPROXIMATION AND GRADIENT L^p ESTIMATES

In this and in the next section we follow largely [7] with minor modifications: on the one hand we have the additional Ω -error terms, but on the other hand the ambient Riemannian manifold is the euclidean space. Thus the arguments are somewhat less technical.

4.1. Harmonic Approximation. In this subsection we prove that if T is an almost minimizer then its E^β -Lipschitz approximation is close to a Dir-minimizing function w . This comes with an $o(E)$ -improvement of the estimates in Proposition 2.2.

Remark 4.1. There exists a dimensional constant $c > 0$ such that, if $E \leq c$, then the E^β -Lipschitz approximation satisfies the following estimates:

$$\text{Lip}(f) \leq C E^\beta, \quad (4.1)$$

$$\int_{B_{3s}(x)} |Df|^2 \leq C E s^m. \quad (4.2)$$

Indeed (4.1) follows from Proposition 2.2, while (4.2) follows from the Taylor expansion of the mass of \mathbf{G}_u :

$$\mathbf{M}(\mathbf{G}_f) = Q |B_{3s}(x)| + \int_{B_{3s}(x)} \frac{|Df|^2}{2} + \int_{B_{3s}(x)} \sum_i R(Df_i),$$

where $R : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$ is a C^1 function satisfying $|R(D)| = |D|^3 L(D)$ for some positive function L such that $L(0) = 0$ and $\text{Lip}(L) \leq C$ (cp. [6, Corollary 3.3]). Indeed, for E sufficiently small we have

$$\int_{B_{3s}(x)} \sum_i R(Df_i) \leq C E^{2\beta} \int_{B_{3s}(x)} |Df|^2 < \frac{1}{4} \int_{B_{3s}(x)} |Df|^2,$$

and therefore, since $T \llcorner (K \times \mathbb{R}^n) = \mathbf{G}_f \llcorner (K \times \mathbb{R}^n)$,

$$\begin{aligned} \int_{B_{3s}(x)} |Df|^2 &\leq C (\mathbf{M}(\mathbf{G}_f \llcorner \mathbf{C}_{3s}(x)) - Q \omega_m (3s)^m) \\ &\leq C (\mathbf{M}(T \llcorner (K \times \mathbb{R}^n)) - Q \omega_m (3s)^m) + C \mathbf{M}(\mathbf{G}_f \llcorner (B_{3s}(x) \setminus K) \times \mathbb{R}^n) \\ &\leq C (\mathbf{M}(T \llcorner \mathbf{C}_{3s}(x)) - Q \omega_m (3s)^m) + C E^{2\beta} |B_{3s}(x) \setminus K| \leq C E s^m. \end{aligned}$$

Theorem 4.2 (First harmonic approximation). *For every $\eta_1, \delta > 0$ and every $\beta \in (0, \frac{1}{2m})$, there exists a constant $\varepsilon_{23} > 0$ with the following property. Let T be an Ω -almost minimizer which satisfies Assumption 1.3 in $\mathbf{C}_{4s}(x)$. If $E = \mathbf{E}(T, \mathbf{C}_{4s}(x)) \leq \varepsilon_{23}$ and $s\Omega \leq \varepsilon_{23} E^{1/2}$, then the E^β -Lipschitz approximation f in $\mathbf{C}_{3s}(x)$ satisfies*

$$\int_{B_{2s}(x) \setminus K} |Df|^2 \leq \eta_1 E \omega_m (4s)^m = \eta_1 \mathbf{e}_T(B_{4s}(x)). \quad (4.3)$$

Moreover, there exists a Dir-minimizing function w such that

$$s^{-2} \int_{B_{2s}(x)} \mathcal{G}(f, w)^2 + \int_{B_{2s}(x)} (|Df| - |Dw|)^2 \leq \eta_1 E \omega_m (4s)^m = \eta_1 \mathbf{e}_T(B_{4s}(x)), \quad (4.4)$$

$$\int_{B_{2s}(x)} |D(\boldsymbol{\eta} \circ f) - D(\boldsymbol{\eta} \circ w)|^2 \leq \eta_1 E \omega_m (4s)^m = \eta_1 \mathbf{e}_T(B_{4s}(x)). \quad (4.5)$$

Proof of Theorem 4.2. By rescaling and translating, it is not restrictive to assume that $x = 0$ and $s = 1$. We proceed by contradiction. Assume there exist a constant $c_1 > 0$, a sequence of positive real numbers $(\varepsilon_l)_l$, a sequence of $\boldsymbol{\Omega}_l$ -minimal currents $(T_l)_{l \in \mathbb{N}}$ and corresponding E_l^β -Lipschitz approximations $(f_l)_{l \in \mathbb{N}}$ such that

$$E_l := \mathbf{E}(T_l, \mathbf{C}_4) \leq \varepsilon_l \rightarrow 0, \quad \boldsymbol{\Omega}_l \leq \varepsilon_l E_l^{1/2} \quad \text{and} \quad \int_{B_2 \setminus K_l} |Df_l|^2 \geq c_1 E_l, \quad (4.6)$$

where $K_l := \{x \in B_3 : \mathbf{m}_{T_l}(x) < E_l^{2\beta}\}$. Set $\Gamma_l := \{x \in B_4 : \mathbf{m}_{T_l}(x) \leq 2^{-m} E_l^{2\beta}\}$ and observe that $\Gamma_l \cap B_3 \subset K_l$. From Proposition 2.2, it follows that

$$\text{Lip}(f_l) \leq C_{22} E_l^\beta, \quad (4.7)$$

$$|B_r \setminus K_l| \leq C_{22} E_l^{-2\beta} \mathbf{e}_T(B_{r+r_0(l)} \setminus \Gamma_l) \quad \text{for every } r \leq 3, \quad (4.8)$$

where $r_0(l) = 16 E_l^{(1-2\beta)/m} < \frac{1}{2}$. Then, (4.6), (4.7) and (4.8) give

$$c_1 E_l \leq \int_{B_2 \setminus K_l} |Df_l|^2 \leq C_{22} \mathbf{e}_{T_l}(B_s \setminus \Gamma_l) \quad \forall s \in [\frac{5}{2}, 3].$$

Setting $c_2 := c_1/(2C_{22})$, we have $2c_2 E_l \leq \mathbf{e}_{T_l}(B_s \setminus \Gamma_l) = \mathbf{e}_{T_l}(B_s) - \mathbf{e}_{T_l}(B_s \cap \Gamma_l)$, thus leading to

$$\mathbf{e}_{T_l}(\Gamma_l \cap B_s) \leq \mathbf{e}_{T_l}(B_s) - 2c_2 E_l, \quad (4.9)$$

for l large enough. Next observe that $\omega_m 4^m E_l = \mathbf{e}_{T_l}(B_4) \geq \mathbf{e}_{T_l}(B_s)$. Therefore, by the Taylor expansion in [6, Corollary 3.3], (4.9) and $E_l \downarrow 0$, it follows that, for every $s \in [5/2, 3]$,

$$\begin{aligned} \int_{\Gamma_l \cap B_s} \frac{|Df_l|^2}{2} &\leq (1 + C E_l^{2\beta}) \mathbf{e}_{T_l}(\Gamma_l \cap B_s) \\ &\leq (1 + C E_l^{2\beta}) (\mathbf{e}_{T_l}(B_s) - 2c_2 E_l) \leq \mathbf{e}_{T_l}(B_s) - c_2 E_l. \end{aligned} \quad (4.10)$$

Our aim is to show that (4.10) contradicts the $\boldsymbol{\Omega}$ -almost minimizing property (1.1) of T_l . To construct a competitor consider $g_l := E_l^{-1/2} f_l$. Observe that from the estimates of Remark 4.1, we easily infer $\text{Dir}(f_l, B_3) \leq C E_l$. Hence, $\sup_l \text{Dir}(g_l, B_3) < \infty$. Since $|B_3 \setminus \Gamma_l| \rightarrow 0$, by Proposition 2.4 we can find a subsequence (not relabelled) of translating sheets h_l satisfying (2.3) - (2.4) and $\|\mathcal{G}(g_l, h_l)\|_{L^2(B_3)} \rightarrow 0$. In particular, we are in the position to apply Proposition 2.5 to g_l and h_l , with $r_0 = \frac{5}{2}$, $r_1 = 3$ and $\eta = \frac{c_2}{2}$, and find

$r \in (\frac{5}{2}, 3)$ and competitor functions H_l satisfying $H_l|_{B_3 \setminus B_r} = g_l|_{B_3 \setminus B_r}$,

$$\text{Dir}(H_l, B_r) \leq \text{Dir}(g_l, B_r \cap \Gamma_l) + \frac{c_2}{2}, \quad (4.11)$$

$$\text{Lip}(H_l) \leq C^* E_l^{\beta-1/2} \quad (4.12)$$

$$\|\mathcal{G}(H_l, g_l)\|_{L^2(B_r)} \leq C_{23} \text{Dir}(g_l, B_r) + C_{23} \text{Dir}(H_l, B_r) \leq M < \infty. \quad (4.13)$$

Note that (4.12) follows from (2.5) observing that $E_l^{\beta-1/2} \uparrow \infty$: thus C^* depends on c_2 and the two chosen sequences, but not on l . From now on, although this and similar constants are not dimensional, we will keep denoting them by C , with the understanding that they do not depend on l . Note that, from (4.7) and (4.8), one gets

$$\begin{aligned} \|T_l - \mathbf{G}_{f_l}\|(\mathbf{C}_3) &= \|T_l\|(B_3 \setminus K_l) \times \mathbb{R}^n + \|\mathbf{G}_{f_l}\|((B_3 \setminus K_l) \times \mathbb{R}^n) \\ &\leq Q|B_3 \setminus K_l| + E_l + Q|B_3 \setminus K_l| + C|B_3 \setminus K_l| \text{Lip}(f_l)^2 \\ &\leq E_l + C E_l^{1-2\beta} \leq C E_l^{1-2\beta}. \end{aligned} \quad (4.14)$$

Consider the function $\varphi(z, y) = |z|$ and the slice $\langle T_l - \mathbf{G}_{f_l}, \varphi, r \rangle$. For every l , there exists $r_l \in (r, 3)$ such that $\mathbf{M}(\langle T_l - \mathbf{G}_{f_l}, \varphi, r_l \rangle) \leq C E_l^{1-2\beta}$.

Let now $u_l := E_l^{1/2} H_l|_{B_{r_l}}$, and consider the current $Z_l := \mathbf{G}_{u_l} \llcorner \mathbf{C}_{r_l}$. Since $u_l|_{\partial B_{r_l}} = f_l|_{\partial B_{r_l}}$, one gets $\partial Z_l = \langle \mathbf{G}_{f_l}, \varphi, r_l \rangle$ and, hence, $\mathbf{M}(\partial(T_l \llcorner \mathbf{C}_{r_l} - Z_l)) \leq C E_l^{1-2\beta}$. By the Isoperimetric Inequality there is an integral current R_l such that

$$\partial R_l = \partial(T_l \llcorner \mathbf{C}_{r_l} - Z_l) \quad \text{and} \quad \mathbf{M}(R_l) \leq C E_l^{m(1-2\beta)/(m-1)}.$$

Set $S_l = T_l \llcorner (\mathbf{C}_4 \setminus \mathbf{C}_{r_l}) + Z_l + R_l$. Notice that $\partial S_l = \partial T_l$. We assume from now on $\beta < \frac{1}{2m}$ and we let γ be such that $1 < 1 + \gamma \leq m(1 - 2\beta)/(m - 1) > 1$. We want to compare the mass of S_l with that of T_l to achieve a contradiction in the limit for $l \rightarrow \infty$.

$$\int_{B_{r_l}} |Du_l|^2 - \int_{B_r \cap \Gamma_l} |Df_l|^2 = \text{Dir}(B_{r_l}, u_l) - \text{Dir}(B_{r_l} \cap \Gamma_l, f_l) \stackrel{(4.11)}{\leq} \frac{c_2}{2} E_l$$

where the factor E_l in the last inequality comes from the renormalizations $u_l = E_l^{1/2} H_l$ and $f_l = E_l^{1/2} g_l$. By possibly changing γ so that $2\beta \geq \gamma$, we can then write

$$\begin{aligned} \mathbf{M}(S_l) - \mathbf{M}(T_l) &\leq \mathbf{M}(Z_l) + C \mathbf{M}(R_l) - \mathbf{M}(T_l \llcorner \mathbf{C}_r) \\ &\leq Q|B_r| + \int_{B_r} \frac{|Du_l|^2}{2} + C E_l^{1+\gamma} - Q|B_r| - \mathbf{e}_{T_l}(B_r) \\ &\leq \int_{B_r \cap \Gamma_l} \frac{|Df_l|^2}{2} + \frac{c_2}{2} E_l + C E_l^{1+\gamma} - \mathbf{e}_{T_l}(B_r) \\ &\stackrel{(4.10)}{\leq} -\frac{c_2 E_l}{4} + C E_l^{1+\beta} + C E_l^{1+\gamma}. \end{aligned} \quad (4.15)$$

Hence,

$$\mathbf{M}(S_l) < \mathbf{M}(T_l) \quad \text{for } l \text{ large enough.} \quad (4.16)$$

This would be already a contradiction if T were area-minimizing. In our case, by (3.1) of Lemma 3.1 we have the upper bound

$$\mathbf{M}(S_l) - \mathbf{M}(T_l) \geq -C_{25}\Omega_l E_l^{1/2} \geq -C_{25}\varepsilon_l E_l.$$

Combining this inequality with (4.15) we obtain

$$\frac{c_2 E_l}{4} \leq C E_l^{1+\gamma} + C \varepsilon_l E_l$$

which for E_l, ε_l sufficiently small (and hence for l large enough) provides the desired contradiction.

For what concerns (4.4), we argue similarly. Let $(T_l)_l$ be a sequence with vanishing $E_l := \mathbf{E}(T_l, \mathbf{C}_4)$, contradicting the second part of the statement and perform the same analysis as before. Up to subsequences, one of the following statement must be false:

- (i) $\lim_l \int_{B_2} |Dg_l|^2 = \int_{B_2} |Dh_{l_0}|^2$, for any l_0 (recall that $\int_{B_2} |Dh_l|^2$ is constant);
- (ii) h_l is Dir-minimizing in B_2 .

If (i) is false, then there is a positive constant c_2 such that, for every $r \in [5/2, 3]$,

$$\int_{B_r} \frac{|Dh_l|^2}{2} \leq \int_{B_r} \frac{|Dg_l|^2}{2} - c_2 \leq \frac{\mathbf{e}_{T_l}(B_r)}{E_l} - \frac{c_2}{2},$$

for l large enough (where the last inequality is again an effect of the Taylor expansion of Remark 4.1. Therefore we can argue exactly as in the proof of (4.3) (using h_l instead of H_l to construct the competitors) and reach a contradiction. If (ii) is false, then h_l is not Dir-minimizing in $B_{5/2}$. This implies that one of the ζ^j in the translating sheets h_l is not Dir-minimizing in B_2 . Indeed, in the opposite case, by [5, Theorem 3.9], $\|\mathcal{G}(\zeta^j, Q \llbracket 0 \rrbracket)\|_{C^0(B_2)} < \infty$ and, since $h_l = \sum_i \llbracket \tau_{y_i^i} \circ \zeta^i \rrbracket$ and $|y_i^i - y_j^j| \rightarrow \infty$ for $i \neq j$, by the maximum principle of [5, Proposition 3.5], h_l would be Dir-minimizing. Thus, we can find a competitor $\hat{\zeta}^j$ for some ζ^j with less energy in the ball B_2 . So the functions $F_l = \sum_j \llbracket \tau_{y_l^j} \circ \hat{\zeta}^j \rrbracket$ satisfy, for any $r \in [5/2, 3]$,

$$\int_{B_r} \frac{|DF_l|^2}{2} \leq \int_{B_r} \frac{|Dh_l|^2}{2} - c_2 \leq \lim_l \int_{B_r} \frac{|Dg_l|^2}{2} - 2c_2 \leq \frac{\mathbf{e}_T(B_r)}{E_l} - \frac{c_2}{2},$$

provided l is large enough (where $c_2 > 0$ is a constant independent of r and l). On the other hand $F_l = h_l$ on $B_3 \setminus B_{5/2}$ and therefore $\|\mathcal{G}(F_l, g_l)\|_{L^2(B_3 \setminus B_{5/2})} \rightarrow 0$. We then argue as above with F_l in place of H_l and reach a contradiction in this case as well. \square

4.2. Improved excess estimate. The higher integrability of the Dir-minimizing functions and the harmonic approximation lead to the following estimate, which we call “weak” since we will improve it in the next section with Theorem 5.1.

Proposition 4.3 (Weak excess estimate). *For every $\eta_2 > 0$, there exist $\varepsilon_{24}, C_{26} > 0$ with the following property. Let T be an Ω -almost minimizer and assume it satisfies (1.6) in $\mathbf{C}_{4s}(x)$. If $E = \mathbf{E}(T, \mathbf{C}_{4s}(x)) \leq \varepsilon_{24}$, then*

$$\mathbf{e}_T(A) \leq \eta_2 \mathbf{e}_T(B_{4s}(x)) + C_{26} \Omega^2 s^{m+2}, \quad (4.17)$$

for every $A \subset B_s(x)$ Borel with $|A| \leq \varepsilon_{24}|B_s(x)|$ (C_{26} depends only on η_2, m, n and Q).

Proof. Without loss of generality, we can assume $s = 1$ and $x = 0$. We distinguish the two regimes: $\hat{\varepsilon}^2 E \leq \Omega^2$ and $\Omega^2 \leq \hat{\varepsilon}^2 E$, where $\hat{\varepsilon} \leq \varepsilon_{24}$ is a parameter whose choice will be specified later. In the former, clearly $\mathbf{e}_T(A) \leq C E \leq C \Omega^2$. In the latter, we let f be the $E^{1/4m}$ -Lipschitz approximation of T in \mathbf{C}_3 . By a Fubini-type argument as the ones already used in the previous sections, we find a radius $r \in (1, 2)$ and a current P with $\mathbf{M}(P) \leq C E^{1+\gamma}$ and $\partial((T - \mathbf{G}_f) \llcorner \mathbf{C}_r) = \partial P$ for some $\gamma(m) > 0$. We can thus apply Lemma 3.1 to $R = \mathbf{G}_f \llcorner \mathbf{C}_r + P + T \llcorner (\mathbf{C}_3 \setminus \mathbf{C}_r)$. Recalling the Taylor expansion in [6, Corollary 3.3], we have

$$\begin{aligned} \|T\|(\mathbf{C}_r) &\leq \mathbf{M}(R \llcorner \mathbf{C}_r) + C \Omega E^{1/2} \leq \|\mathbf{G}_f\|(\mathbf{C}_r) + C \hat{\varepsilon} E + C E^{1+\gamma} \\ &\leq Q |B_r| + \int_{B_r} \frac{|Df|^2}{2} + C \hat{\varepsilon} E + C E^{1+\gamma}, \end{aligned} \quad (4.18)$$

for some positive γ (possibly smaller than the previous one). On the other hand, using again the Taylor expansion for the part of the current which coincides with the graph of f , we deduce as well that

$$\begin{aligned} \|T\|(\mathbf{C}_r) &= \|T\|((B_r \setminus K) \times \mathbb{R}^n) + \|T\|((B_r \cap K) \times \mathbb{R}^n) \\ &\geq \|T\|((B_r \setminus K) \times \mathbb{R}^n) + Q |B_r \cap K| + \int_{B_r \cap K} \frac{|Df|^2}{2} - C E^{1+\gamma}. \end{aligned} \quad (4.19)$$

Subtracting (4.19) from (4.18), we deduce

$$\mathbf{e}_T(B_r \setminus K) \leq \int_{B_r \setminus K} \frac{|Df|^2}{2} + C \hat{\varepsilon} E + C E^{1+\gamma}, \quad (4.20)$$

where the constant C is independent of $\hat{\varepsilon}$. If ε_{24} is chosen small enough, we infer from (4.20) and (4.3) in Theorem 4.2 that

$$\mathbf{e}_T(B_r \setminus K) \leq \eta \mathbf{e}_T(B_4) + C E^{1+\gamma}, \quad (4.21)$$

for a suitable $\eta = \hat{\varepsilon}/2C$ to be specified later. Let now $A \subset B_1$ be such that $|A| \leq \varepsilon_{24} \omega_m$. Combining (4.21) with the Taylor expansion, we have

$$\mathbf{e}_T(A) \leq \mathbf{e}_T(A \setminus K) + \int_A \frac{|Df|^2}{2} + C E^{1+\gamma} \leq \int_A \frac{|Df|^2}{2} + \eta \mathbf{e}_T(B_4) + C E^{1+\gamma}. \quad (4.22)$$

If ε_{24} is small enough, we can again use Theorem 4.2 and Theorem 2.6 in (4.22) to get, for a Dir-minimizing w ,

$$\mathbf{e}_T(A) \stackrel{(4.4)}{\leq} \int_A \frac{|Dw|^2}{2} + 2\eta \mathbf{e}_T(B_4) + C E^{1+\gamma} \leq (C_{24}|A|^{1-2/p_1} + 2\eta) \mathbf{e}_T(B_4) + C E^{1+\gamma}. \quad (4.23)$$

Hence, if ε_{24} and η are suitably chosen, (4.17) follows from (4.23). \square

4.3. Gradient L^p estimate. The density \mathbf{d} of the excess measure is naturally an L^1 function. We prove here that for Ω -almost minimizer this function is in fact L^p , for some $p > 1$.

Theorem 4.4 (Gradient L^p estimate). *There exist constants $p_2 > 1$ and $C, \varepsilon_{25} > 0$ (depending on n, Q) with the following property. Assume T satisfies (1.6) in the cylinder \mathbf{C}_4 . If T is an Ω -almost minimizer and $E = \mathbf{E}(T, \mathbf{C}_4) < \varepsilon_{25}$, then*

$$\int_{\{\mathbf{d} \leq 1\} \cap B_2} \mathbf{d}^{p_2} \leq C E^{p_2-1} (E + \Omega^2). \quad (4.24)$$

Proof. We assume without loss of generality that $E > 0$ and divide the proof into two steps.

Step 1. There exist constants $\gamma \geq 2^m$ and $\varrho > 0$ such that, for every $c \in [1, (\gamma E)^{-1}]$ and $s \in [2, 4]$ with $\bar{s} = s + 2c^{-1/m} \leq 4$, we have

$$\int_{\{\gamma c E \leq \mathbf{d} \leq 1\} \cap B_s} \mathbf{d} \leq \gamma^{-\varrho} \int_{\{\frac{cE}{\gamma} \leq \mathbf{d} \leq 1\} \cap B_s} \mathbf{d} + C c^{-2/m} \Omega^2. \quad (4.25)$$

In order to prove it, let N_B be the constant in Besicovich's covering theorem [13, Section 1.5.2] and choose $N \in \mathbb{N}$ so large that $N_B < 2^{N-1}$. Let ε_{24} be as in Proposition 4.3 when we choose $\eta_2 = 2^{-2m-N}$, and set

$$\gamma = \max\{2^m, \varepsilon_{24}^{-1}\} \quad \text{and} \quad \varrho = \min\left\{-\log_\gamma(N_B/2^{N-1}), \frac{1}{4}\right\}.$$

Let c and s be any real numbers as above. For almost every $x \in \{\gamma c E \leq \mathbf{d} \leq 1\} \cap B_s$, there exists r_x such that

$$\mathbf{E}(T, \mathbf{C}_{4r_x}(x)) \leq cE \quad \text{and} \quad \mathbf{E}(T, \mathbf{C}_t(x)) \geq cE \quad \forall t \in]0, 4r_x[. \quad (4.26)$$

Indeed, since $\mathbf{d}(x) = \lim_{r \rightarrow 0} \mathbf{E}(T, \mathbf{C}_r(x)) \geq \gamma c E \geq 2^2 c E$ and

$$\mathbf{E}(T, \mathbf{C}_t(x)) = \frac{\mathbf{e}_T(B_t(x))}{\omega_m t^m} \leq \frac{4^m E}{t^m} \leq cE \quad \text{for } t \geq \frac{4}{\sqrt[m]{c}},$$

we just choose $4r_x = \min\{t \leq 4/\sqrt[m]{c} : \mathbf{E}(T, \mathbf{C}_t(x)) \leq cE\}$. Note also that $r_x \leq 1/\sqrt[m]{c}$. Consider the current T in $\mathbf{C}_{4r_x}(x)$. Setting $A = \{\gamma c E \leq \mathbf{d}\} \cap B_{4r_x}(x)$, we have that

$$\mathbf{E}(T, \mathbf{C}_{4r_x}(x)) \leq cE \leq \frac{E}{\gamma E} \leq \varepsilon_{24} \quad \text{and} \quad |A| \leq \frac{cE |B_{4r_x}(x)|}{\gamma c E} \leq \varepsilon_{24} |B_{4r_x}(x)|.$$

Hence, we can apply Proposition 4.3 to $T \llcorner \mathbf{C}_{4r_x}(x)$ to get

$$\begin{aligned} & \int_{B_{r_x}(x) \cap \{\gamma c E \leq \mathbf{d} \leq 1\}} \mathbf{d} \leq \int_A \mathbf{d} \leq \mathbf{e}_T(A) \leq 2^{-2m-N} \mathbf{e}_T(B_{4r_x}(x)) + C r_x^{m+2} \Omega^2 \\ & \leq 2^{-2m-N} (4r_x)^m \omega_m \mathbf{E}(T, \mathbf{C}_{4r_x}(x)) + C r_x^{m+2} \Omega^2 \stackrel{(4.26)}{\leq} 2^{-N} \mathbf{e}_T(B_{r_x}(x)) + C r_x^{m+2} \Omega^2. \end{aligned} \quad (4.27)$$

Thus,

$$\begin{aligned}
\mathbf{e}_T(B_{r_x}(x)) &= \int_{B_{r_x}(x) \cap \{\mathbf{d} > 1\}} \mathbf{d} + \int_{B_{r_x}(x) \cap \{\frac{cE}{\gamma} \leq \mathbf{d} \leq 1\}} \mathbf{d} + \int_{B_{r_x}(x) \cap \{\mathbf{d} < \frac{cE}{\gamma}\}} \mathbf{d} \\
&\leq \int_A \mathbf{d} + \int_{B_{r_x}(x) \cap \{\frac{cE}{\gamma} \leq \mathbf{d} \leq 1\}} \mathbf{d} + \frac{cE}{\gamma} \omega_m r_x^m \\
&\stackrel{(4.26), (4.27)}{\leq} (2^{-N} + \gamma^{-1}) \mathbf{e}_T(B_{r_x}(x)) + C r_x^{m+2} \Omega^2 + \int_{B_{r_x}(x) \cap \{\frac{cE}{\gamma} \leq \mathbf{d} \leq 1\}} \mathbf{d}. \quad (4.28)
\end{aligned}$$

Therefore, recalling that $\gamma \geq 2^m \geq 4$, from (4.27) and (4.28) we infer:

$$\begin{aligned}
\int_{B_{r_x}(x) \cap \{\gamma c E \leq \mathbf{d} \leq 1\}} \mathbf{d} &\leq \frac{2^{-N}}{1 - 2^{-N} - \gamma^{-1}} \int_{B_{r_x}(x) \cap \{\frac{cE}{\gamma} \leq \mathbf{d} \leq 1\}} \mathbf{d} + C r_x^{m+2} \Omega^2 \\
&\leq 2^{-N+1} \int_{B_{r_x}(x) \cap \{\frac{cE}{\gamma} \leq \mathbf{d} \leq 1\}} \mathbf{d} + C r_x^{m+2} \Omega^2.
\end{aligned}$$

By Besicovich's covering theorem, we choose N_B families of disjoint balls $\bar{B}_{r_x}(x)$ whose union covers $\{\gamma c E \leq \mathbf{d} \leq 1\} \cap B_s$ and, since as already noticed $r_x \leq 1/\sqrt[m]{c}$ for every x , we conclude:

$$\int_{\{\gamma c E \leq \mathbf{d} \leq 1\} \cap B_s} \mathbf{d} \leq N_B 2^{-N+1} \int_{\{\frac{cE}{\gamma} \leq \mathbf{d} \leq 1\} \cap B_{s+2/\sqrt[m]{c}}} \mathbf{d} + C c^{-2/m} \Omega^2,$$

which, for the above defined ϱ , implies (4.25).

Step 2. We iterate (4.25) in order to conclude (4.24). Denote by L the largest integer smaller than $2^{-1} \log_\gamma(E^{-1} - 1)$, $s_L = 2$ and recursively $s_k = s_{k+1} + 2\gamma^{-2k/m}$ for $k \in \{L, L-1, \dots, 1\}$. Notice that, since $\gamma \geq 2^m$, $s_k < 4$ for every k . Thus, we can apply (4.25) with $c = \gamma^{2k}$, $s = s_k$ and $\bar{s} = s_{k-1}$ to conclude

$$\int_{\{\gamma^{2k+1} E \leq \mathbf{d} \leq 1\} \cap B_{s_k}} \mathbf{d} \leq \gamma^{-\varrho} \int_{\{\gamma^{2k-1} E \leq \mathbf{d} \leq 1\} \cap B_{s_{k-1}}} \mathbf{d} + C \gamma^{-4k/m} \Omega^2 \quad \forall k \in \{1, \dots, L\}.$$

In particular, iterating this estimate we get

$$\int_{\{\gamma^{2k+1} E \leq \mathbf{d} \leq 1\} \cap B_2} \mathbf{d} \leq \gamma^{-k\varrho} \int_{\{\gamma E \leq \mathbf{d} \leq 1\} \cap B_{s_0}} \mathbf{d} + C \Omega^2 \sum_{l=0}^{k-1} \gamma^{-\left(\frac{4(k-l)}{m} + l\varrho\right)}. \quad (4.29)$$

Set $A_0 = \{\mathbf{d} < \gamma E\}$, $A_k = \{\gamma^{2k-1} E \leq \mathbf{d} < \gamma^{2k+1} E\}$ for $k = 1, \dots, L$, and $A_{L+1} = \{\gamma^{2L+1} E \leq \mathbf{d} \leq 1\}$. Since $\cup A_k = \{\mathbf{d} \leq 1\}$, for $p_2 < 1 + \frac{\varrho}{2} \leq 1 + \frac{1}{2}$, we conclude:

$$\begin{aligned} \int_{B_2 \cap \{\mathbf{d} \leq 1\}} \mathbf{d}^{p_2} &= \sum_{k=0}^{L+1} \int_{A_k \cap B_2} \mathbf{d}^{p_2} \leq \sum_k \gamma^{(2k+1)(p_2-1)} E^{p_2-1} \int_{A_k \cap B_2} \mathbf{d} \\ &\stackrel{(4.29)}{\leq} C \sum_k \gamma^{k(2(p_2-1)-\varrho)} E^{p_2} + C \sum_k \sum_{l=0}^{k-1} \gamma^{k(2(p_2-1)-4/m)+l(4/m-\varrho)} E^{p_2-1} \Omega^2 \\ &\leq C E^{p_2} + C \sum_k \gamma^{k(2(p_2-1)-\varrho)} \Omega^2. \end{aligned} \quad \square$$

5. STRONG EXCESS ESTIMATE AND PROOF OF PROPOSITION 1.7

5.1. Almgren's strong excess estimate. Thanks to the higher integrability of Theorem 4.4, we can control the excess where $\mathbf{d} \leq 1$. To control it outside this region, we will need the following estimate.

Theorem 5.1 (Almgren's strong excess estimate). *There are constants $\varepsilon_{21}, \gamma_2, C_{27} > 0$ (depending on n, Q) with the following property. Assume T satisfies Assumption 1.3 in \mathbf{C}_4 and is Ω almost minimizing. If $E = \mathbf{E}(T, \mathbf{C}_4) < \varepsilon_{21}$, then*

$$\mathbf{e}_T(A) \leq C_{27} (E^{\gamma_2} + |A|^{\gamma_3}) (E + \Omega^2) \quad \text{for every Borel } A \subset B_1. \quad (5.1)$$

5.2. Regularization by convolution. Theorem 5.1 will be proved using a suitable competitor constructed via convolution of the E^{β_1} -Lipschitz approximation. The precise claim is contained in the following proposition.

Proposition 5.2. *Let $\beta_1 \in (0, \frac{1}{2m})$ and T be an Ω -almost minimizing current satisfying (1.6) in \mathbf{C}_4 . Let f be its E^{β_1} -Lipschitz approximation. Then, there exist constants $\gamma_3, C_{28} > 0$ and a subset of radii $B \subset [1, 2]$ with $|B| > 1/2$ with the following properties. For every $\sigma \in B$, there exists a Q -valued function $g \in \text{Lip}(B_\sigma, \mathcal{A}_Q)$ such that*

$$g|_{\partial B_\sigma} = f|_{\partial B_\sigma}, \quad \text{Lip}(g) \leq C_{28} E^{\beta_1}$$

and

$$\int_{B_\sigma} |Dg|^2 \leq \int_{B_\sigma \cap K} |Df|^2 + C_{28} E^{1+\gamma_3}. \quad (5.2)$$

Proof. Since $|Df|^2 \leq C \mathbf{d}_T \leq C E^{2\beta_1} \leq 1$ on K , by Theorem 4.4 there exists $q_2 = 2p_2 > 2$ such that

$$\| |Df| \|_{L^{q_2}(K \cap B_2)}^2 \leq C E^{1-1/p_1} (E + \Omega^2)^{1/p_1} \leq C (E + \Omega^2). \quad (5.3)$$

Given two (vector-valued) functions h_1 and h_2 and two radii $0 < s < r$, we denote by $\text{lin}(h_1, h_2)$ the linear interpolation in $B_r \setminus \bar{B}_s$ between $h_1|_{\partial B_r}$ and $h_2|_{\partial B_s}$. More precisely, if $(\theta, t) \in \mathbb{S}^{m-1} \times [0, \infty)$ are spherical coordinates, then

$$\text{lin}(h_1, h_2)(\theta, t) = \frac{r-t}{r-s} h_2(\theta, s) + \frac{t-s}{r-s} h_1(\theta, r).$$

Next, let $\delta > 0$ and $\varepsilon > 0$ be two parameters and let $1 < r_1 < r_2 < r_3 < 2$ be three radii, all to be chosen later. To keep the notation simple, we will write $\boldsymbol{\rho}^*$ in place of $\boldsymbol{\rho}_\delta^*$, where $\boldsymbol{\rho}_\delta^*$ is the map of Proposition 2.7. Let $\varphi \in C_c^\infty(B_1)$ be a standard (nonnegative!) mollifier. We set $f' := \boldsymbol{\xi} \circ f$. Recall the map $\boldsymbol{\rho}$ of [5, Theorem 2.1] and define:

$$g' := \begin{cases} \sqrt{E} \boldsymbol{\rho} \circ \text{lin} \left(\frac{f'}{\sqrt{E}}, \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} \right) \right) & \text{in } B_{r_3} \setminus B_{r_2}, \\ \sqrt{E} \boldsymbol{\rho} \circ \text{lin} \left(\boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} \right), \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon \right) \right) & \text{in } B_{r_2} \setminus B_{r_1}, \\ \sqrt{E} \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon \right) & \text{in } B_{r_1}. \end{cases} \quad (5.4)$$

Finally set $g := \boldsymbol{\xi}^{-1} \circ g'$. We claim that, for $\sigma := r_3$ in a suitable set $B \subset [1, 2]$ with $|B| > 1/2$, we can choose $r_2 = r_3 - s$ and $r_1 = r_2 - s$ so that g satisfies the conclusion of the proposition. Some computations will be simplified taking into account that our choice of the parameters will imply the following inequalities:

$$\delta^{2 \cdot 8^{-nQ}} \leq s, \quad \varepsilon \leq s \quad \text{and} \quad E^{1-2\beta_1} \leq \varepsilon^m. \quad (5.5)$$

We start noticing that clearly $g|_{\partial B_{r_3}} = f|_{\partial B_{r_3}}$. Moreover we have $\text{Lip}(g) \leq CE^{\beta_1}$, indeed

$$\begin{cases} \text{Lip}(g) \leq C \text{Lip}(f' * \varphi_\varepsilon) \leq C \text{Lip}(f) \leq CE^{\beta_1} & \text{in } B_{r_1}, \\ \text{Lip}(g) \leq C \text{Lip}(f') + C \frac{\|f' - f' * \varphi_\varepsilon\|_{L^\infty}}{s} \leq C(1 + \frac{\varepsilon}{s}) \text{Lip}(f') \leq CE^{\beta_1} & \text{in } B_{r_2} \setminus B_{r_1}, \\ \text{Lip}(g) \leq C \text{Lip}(f') + CE^{1/2} \frac{\delta^{8^{-nQ}}}{s} \leq CE^{\beta_1} + CE^{1/2} \leq CE^{\beta_1} & \text{in } B_{r_3} \setminus B_{r_2}. \end{cases}$$

In the second inequality of the last line we have used that, since \mathcal{Q} is a cone, $E^{-1/2} f'(x) \in \mathcal{Q}$ for every x : therefore $|\boldsymbol{\rho}^*(f'/E^{1/2}) - f'/E^{1/2}| \leq C\delta^{8^{-nQ}}$. We pass now to estimate the Dirichlet energy of g .

Step 1. Energy in $B_{r_3} \setminus B_{r_2}$. By Proposition 2.7, $|\boldsymbol{\rho}^*(P) - P| \leq C_{24} \delta^{8^{-nQ}}$ for all $P \in \mathcal{Q}$. Thus, elementary estimates on the linear interpolation give

$$\begin{aligned} \int_{B_{r_3} \setminus B_{r_2}} |Dg|^2 &\leq \frac{CE}{(r_3 - r_2)^2} \int_{B_{r_3} \setminus B_{r_2}} \left| \frac{f'}{\sqrt{E}} - \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} \right) \right|^2 + C \int_{B_{r_3} \setminus B_{r_2}} |Df'|^2 \\ &\quad + C \int_{B_{r_3} \setminus B_{r_2}} |D(\boldsymbol{\rho}^* \circ f')|^2 \leq C \int_{B_{r_3} \setminus B_{r_2}} |Df|^2 + CE s^{-1} \delta^{2 \cdot 8^{-nQ}}. \end{aligned} \quad (5.6)$$

Step 2. Energy in $B_{r_2} \setminus B_{r_1}$. Here, using the same interpolation inequality and a standard estimate on convolutions of $W^{1,2}$ functions, we get

$$\begin{aligned} \int_{B_{r_2} \setminus B_{r_1}} |Dg|^2 &\leq C \int_{B_{r_2} \setminus B_{r_1}} |Df|^2 + \frac{C}{(r_2 - r_1)^2} \int_{B_{r_2} \setminus B_{r_1}} |f' - \varphi_\varepsilon * f'|^2 \\ &\leq C \int_{B_{r_2} \setminus B_{r_1}} |Df|^2 + C \varepsilon^2 s^{-2} \int_{B_3} |Df'|^2 = C \int_{B_{r_2} \setminus B_{r_1}} |Df|^2 + C \varepsilon^2 E s^{-2}. \end{aligned} \quad (5.7)$$

Step 3. Energy in B_{r_1} . Define $Z := \left\{ \text{dist} \left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon, \mathcal{Q} \right) > \delta^{nQ+1} \right\}$ and use (2.9) to get

$$\int_{B_{r_1}} |Dg|^2 \leq \left(1 + C \delta^{8-nQ-1} \right) \int_{B_{r_1} \setminus Z} |D(f' * \varphi_\varepsilon)|^2 + C \int_Z |D(f' * \varphi_\varepsilon)|^2 =: I_1 + I_2. \quad (5.8)$$

We consider I_1 and I_2 separately. For I_1 we first observe the elementary inequality

$$\begin{aligned} \|(Df' * \varphi_\varepsilon)\|_{L^2}^2 &\leq \| |Df'| * \varphi_\varepsilon \|_{L^2}^2 \leq \| (|Df'| \mathbf{1}_K) * \varphi_\varepsilon \|_{L^2}^2 + \| (|Df'| \mathbf{1}_{K^c}) * \varphi_\varepsilon \|_{L^2}^2 \\ &\quad + 2 \| (|Df'| \mathbf{1}_K) * \varphi_\varepsilon \|_{L^2} \| (|Df'| \mathbf{1}_{K^c}) * \varphi_\varepsilon \|_{L^2}, \end{aligned} \quad (5.9)$$

where K^c is the complement of K in B_3 . Recalling $r_1 + \varepsilon \leq r_1 + s \leq r_2$ we estimate the first summand in (5.9) as follows:

$$\| (|Df'| \mathbf{1}_K) * \varphi_\varepsilon \|_{L^2(B_{r_1})}^2 \leq \int_{B_{r_1+\varepsilon}} (|Df'| \mathbf{1}_K)^2 \leq \int_{B_{r_3} \cap K} |Df'|^2. \quad (5.10)$$

To treat the other terms recall that $\text{Lip}(f) \leq C E^{\beta_1}$ and $|K^c| \leq C E^{1-2\beta_1}$:

$$\| (|Df'| \mathbf{1}_{K^c}) * \varphi_\varepsilon \|_{L^2(B_{r_1})}^2 \leq C E^{2\beta_1} \| \mathbf{1}_{K^c} * \varphi_\varepsilon \|_{L^2}^2 \leq C E^{2\beta_1} \| \mathbf{1}_{K^c} \|_{L^1}^2 \| \varphi_\varepsilon \|_{L^2}^2 \leq \frac{C E^{2-2\beta_1}}{\varepsilon^m}. \quad (5.11)$$

Putting (5.10) and (5.11) in (5.9) and recalling $E^{1-2\beta_1} \geq \varepsilon^m$ and $\int |Df'|^2 \leq CE$, we get

$$I_1 \leq \int_{B_{r_2} \cap K} |Df'|^2 + C \delta^{8-nQ-1} E + C \varepsilon^{-m/2} E^{3/2-\beta_1}. \quad (5.12)$$

For what concerns I_2 , first we argue as for I_1 , splitting in K and K^c , to deduce that

$$I_2 \leq C \int_Z ((|Df'| \mathbf{1}_K) * \varphi_\varepsilon)^2 + C \varepsilon^{-m/2} E^{3/2-\beta_1}. \quad (5.13)$$

Then, regarding the first summand in (5.13), we note that

$$|Z| \delta^{2nQ+2} \leq \int_{B_{r_1}} \left| \frac{f'}{\sqrt{E}} * \varphi_\varepsilon - \frac{f'}{\sqrt{E}} \right|^2 \leq C \varepsilon^2. \quad (5.14)$$

Recalling that $q_2 = 2p_2 > 2$, we use (5.3) to obtain

$$\begin{aligned} \int_Z ((|Df'| \mathbf{1}_K) * \varphi_\varepsilon)^2 &\leq |Z|^{\frac{p_1-1}{p_1}} \| (|Df'| \mathbf{1}_K) * \varphi_\varepsilon \|_{L^{q_2}}^2 \leq C \left(\frac{\varepsilon}{\delta^{nQ+1}} \right)^{\frac{2(p_1-1)}{p_1}} \| |Df'| \|_{L^{q_2}(K)}^2 \\ &\leq C \left(\frac{\varepsilon}{\delta^{nQ+1}} \right)^{\frac{2(p_1-1)}{p_1}} (E + \Omega^2). \end{aligned} \quad (5.15)$$

Gathering all the estimates together, (5.8), (5.12), (5.13) and (5.15) give

$$\int_{B_{r_1}} |Dg|^2 \leq \int_{B_{r_2} \cap K} |Df'|^2 + C \left(E \delta^{8-nQ-1} + \frac{E^{3/2-\beta_1}}{\varepsilon^{m/2}} + (E + \Omega^2) \left(\frac{\varepsilon}{\delta^{nQ+1}} \right)^{\frac{2(p_1-1)}{p_1}} \right). \quad (5.16)$$

Final estimate. Summing (5.6), (5.7) and (5.16) (and recalling $\varepsilon < s$), we conclude

$$\begin{aligned} \int_{B_{r_3}} |Dg|^2 &\leq \int_{B_{r_1} \cap K} |Df|^2 + C \int_{B_{r_1+3s} \setminus B_{r_1}} |Df|^2 \\ &\quad + C E \left(\frac{\varepsilon^2}{s^2} + \frac{\delta^{2 \cdot 8^{-Q}}}{s} + \frac{E^{1/2-\beta_1}}{\varepsilon^{m/2}} + (1 + \Omega^2 E^{-1}) \left(\frac{\varepsilon}{\delta^{nQ+1}} \right)^{\frac{2(p_1-2)}{p_1}} \right). \end{aligned}$$

We set $\varepsilon = E^a$, $\delta = E^b$ and $s = E^c$, where

$$a = \frac{1-2\beta_1}{2m}, \quad b = \frac{1-2\beta_1}{4m(nQ+1)} \quad \text{and} \quad c = \frac{1-2\beta_1}{8^{nQ} 4m(nQ+1)}.$$

This choice respects (5.5). Assume E is small enough so that $s \leq \frac{1}{8}$. Now, if $C > 0$ is a sufficiently large constant, there is a set $B' \subset [1, \frac{7}{8}]$ with $|B'| > 1/2$ such that,

$$\int_{B_{r_1+3s} \setminus B_{r_1}} |Df|^2 \leq C s \int_{B_2} |Df|^2 \leq C E^{1+c} \quad \text{for every } r_1 \in B'.$$

For $\sigma = r_3 \in B = s + B'$ we then conclude, for some $\gamma(\beta_1, n, N, Q) > 0$,

$$\int_{B_\sigma} |Dg|^2 \leq \int_{B_\sigma \cap K} |Df|^2 + C E^{1+\gamma}. \quad \square$$

5.3. Proof of Theorem 5.1. Using the isoperimetric inequality and a slicing argument, we find a radius $\sigma \in (1, 2)$ for which Proposition 5.2 applies and such that there is $P \in \mathbf{I}_m(\mathbb{R}^{m+n})$ with $\partial P = \partial((T - \mathbf{G}_f) \llcorner \mathbf{C}_s)$ and $\mathbf{M}(P) \leq C E^{1+\gamma}$. We can therefore apply Lemma 3.1 to conclude that

$$\|T\|(\mathbf{C}_\sigma) \leq \|\mathbf{G}_g\|(\mathbf{C}_\sigma) + C \Omega \int_{B_\sigma} \mathbf{G}(g, f) + C E^{1+\gamma}. \quad (5.17)$$

In order to estimate $\int_{B_\sigma} \mathbf{G}(g, f)$, we recall how g is constructed, and in particular, using the notation of the previous section

$$\begin{aligned} \int_{B_\sigma} \mathcal{G}(f, g) &\leq C \underbrace{\int_{B_\sigma \setminus B_{\sigma-s}} \left| f' - \sqrt{E} \boldsymbol{\rho} \circ \text{lin} \left(\frac{f'}{\sqrt{E}}, \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} \right) \right) \right|}_{I_1} + \\ &\quad + C \underbrace{\int_{B_{\sigma-s} \setminus B_{\sigma-2s}} \left| f' - \sqrt{E} \boldsymbol{\rho} \circ \text{lin} \left(\boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} \right), \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon \right) \right) \right|}_{I_2} + \\ &\quad + C \underbrace{\int_{B_{\sigma-2s}} \left| f' - \sqrt{E} \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} * \varphi_\varepsilon \right) \right|}_{I_3}. \end{aligned}$$

We will estimate I_1, I_2, I_3 separately. Recall that $\boldsymbol{\rho} \circ f' = f'$, $\boldsymbol{\rho}$ is Lipschitz and moreover $\lambda \boldsymbol{\rho}(P) = \boldsymbol{\rho}(\lambda P)$, for every $\lambda > 0, P \in \mathcal{Q}$, since \mathcal{Q} is a cone.

$$\begin{aligned} I_1 &\leq C \int_{\sigma-s}^{\sigma} \int_{\partial B_t} \sqrt{E} \left| \frac{f'}{\sqrt{E}} - \frac{t+s-\sigma}{s} \frac{f'}{\sqrt{E}} - \frac{\sigma-t}{s} \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} \right) \right| dt \\ &= C \sqrt{E} \int_{\sigma-s}^{\sigma} \frac{\sigma-t}{s} \int_{\partial B_t} \left| \frac{f'}{\sqrt{E}} - \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} \right) \right| dt \leq C \sqrt{E} \delta^{8-n_Q} |B_{\sigma} \setminus B_{\sigma-s}| \leq CE^{1/2+c} \end{aligned}$$

where we used $|B_{\sigma} \setminus B_{\sigma-s}| \leq Cs \leq CE^c$. We next bound I_2 .

$$\begin{aligned} I_2 &\leq C \sqrt{E} \int_{\sigma-2s}^{\sigma-s} \int_{\partial B_t} \left| \frac{f'}{\sqrt{E}} - \frac{t+2s-\sigma}{s} \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} \right) - \frac{\sigma-s-t}{s} \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} * \varphi_{\varepsilon} \right) \right| \\ &\leq C \sqrt{E} \int_{\sigma-2s}^{\sigma-s} \int_{\partial B_t} \left(\left| \frac{f'}{\sqrt{E}} - \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} \right) \right| + \frac{\sigma-s-t}{s} \left| \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} \right) - \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} * \varphi_{\varepsilon} \right) \right| \right) dt \\ &\leq CE^{1/2+c} + C \int_{B_{\sigma-s} \setminus B_{\sigma-2s}} |f' - f' * \varphi_{\varepsilon}| \end{aligned}$$

where we have used the fact that $\boldsymbol{\rho}^*$ is Lipschitz. The estimate for I_3 is then

$$\begin{aligned} I_3 &\leq C \sqrt{E} \int_{B_{\sigma-2s}} \left(\left| \frac{f'}{\sqrt{E}} - \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} \right) \right| + \left| \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} \right) - \boldsymbol{\rho}^* \left(\frac{f'}{\sqrt{E}} * \varphi_{\varepsilon} \right) \right| \right) \\ &\leq CE^{1/2+c} + C \int_{B_{\sigma-2s}} |f' - f' * \varphi_{\varepsilon}|. \end{aligned}$$

We therefore achieve the estimate

$$I_2 + I_3 \leq CE^{1/2+c} + \int_{B_{\sigma-s}} |f' - f' * \varphi_{\varepsilon}|$$

and to conclude, we compute

$$\begin{aligned} \int_{B_{\sigma-s}} |f' - f' * \varphi_{\varepsilon}| &\leq \int_{B_{\sigma-s}} \int_{B_{\varepsilon}} \varphi_{\varepsilon}(x) |f'(y-x) - f'(y)| dy dx \\ &\leq \int_{B_{\sigma-s}} \int_{B_{\varepsilon}} \int_0^1 \varphi_{\varepsilon}(x) |Df'(y-tx) \cdot x| dt dy dx \\ &\leq \int_0^1 \int_{B_{\varepsilon}} \varphi_{\varepsilon}(x) \varepsilon \int_{B_{\sigma-s}} |Df(y-tx)| dy dx dt \leq \varepsilon \|Df\|_{L^1(B_{\sigma})} \leq CE^{1/2+a}, \end{aligned}$$

(where we have used the fact that $\varepsilon \leq s$). Putting everything together we conclude that

$$\mathbf{M}(S) \leq CE^{1/2+\gamma}$$

for a suitable $\gamma > 0$. Then, from (5.17), the Taylor expansion for $\mathbf{M}(\mathbf{G}_g)$ and Proposition 5.2 we achieve

$$\|T\|(\mathbf{C}_{\sigma}) \leq Q |B_{\sigma}| + \int_{B_{\sigma} \cap K} \frac{|Df|^2}{2} + CE^{\gamma}(E + \Omega^2). \quad (5.18)$$

On the other hand, by the Taylor's expansion in [6, Corollary 3.3],

$$\begin{aligned} \|T\|(\mathbf{C}_s) &= \|T\|((B_s \setminus K) \times \mathbb{R}^n) + \|\mathbf{G}_f\|((B_s \cap K) \times \mathbb{R}^n) \\ &\geq \|T\|((B_s \setminus K) \times \mathbb{R}^n) + Q |K \cap B_s| + \int_{K \cap B_s} \frac{|Df|^2}{2} - C E^{1+\gamma}. \end{aligned} \quad (5.19)$$

Hence, from (5.18) and (5.19), we get $\mathbf{e}_T(B_s \setminus K) \leq C E^\gamma (E + \Omega^2)$.

This is enough to conclude the proof. Indeed, let $A \subset B_1$ be a Borel set. Using the higher integrability of $|Df|$ in K (and therefore possibly selecting a smaller $\gamma > 0$) we get

$$\begin{aligned} \mathbf{e}_T(A) &\leq \mathbf{e}_T(A \cap K) + \mathbf{e}_T(A \setminus K) \leq \int_{A \cap K} \frac{|Df|^2}{2} + C E^{1+\gamma} + C E^\gamma (E + \Omega^2) \\ &\leq C |A \cap K|^{\frac{p_1-1}{p_1}} \left(\int_{A \cap K} |Df|^{q_2} \right)^{2/q_2} + C E^{1+\gamma} + C E^\gamma (E + \Omega^2) \\ &\leq C |A|^{\frac{p_1-1}{p_1}} (E + \Omega^2) + C E^\gamma (E + \Omega^2) + C E^{1+\gamma}. \end{aligned}$$

5.4. Proof of Proposition 1.7. As usual we assume, w.l.o.g., $r = 1$ and $x = 0$. Choose $\beta_2 < \min\{\frac{1}{2m}, \frac{\gamma_3}{2(1+\gamma_3)}\}$, where γ_3 is the constant in Theorem 5.1. Let f be the E^{β_2} -Lipschitz approximation of T . Clearly (1.7) follows directly from Proposition 2.2 if $\gamma_1 < \beta_2$. Set next $A := \{\mathbf{m}_T > 2^{-m} E^{2\beta_2}\} \cap B_{9/8}$. By Proposition 2.2, $|A| \leq C E^{1-2\beta_2}$. Apply estimate (5.1) to A to conclude:

$$|B_1 \setminus K| \leq C E^{-2\beta_2} \mathbf{e}_T(A) \leq C E^{\gamma_3 - 2\beta_2(1+\gamma_3)} (E + \Omega^2).$$

By our choice of γ_3 and β_2 , this gives (1.9) for some positive β_0 . Finally, set $S = \mathbf{G}_f$. Recalling the strong Almgren's estimate (5.1) and the Taylor expansion in [6, Corollary 3.3], we conclude:

$$\begin{aligned} \left| \|T\|(\mathbf{C}_1) - Q \omega_m - \int_{B_1} \frac{|Df|^2}{2} \right| &\leq \mathbf{e}_T(B_1 \setminus K) + \mathbf{e}_S(B_1 \setminus K) + \left| \mathbf{e}_S(B_1) - \int_{B_1} \frac{|Df|^2}{2} \right| \\ &\leq C E^{\gamma_3} (E + \Omega^2) + C |B_1 \setminus K| + C \text{Lip}(f)^2 \int_{B_1} |Df|^2 \leq C E^{\gamma_1} (E + \Omega^2). \end{aligned}$$

The L^∞ bound follows from Proposition 2.2.

6. PROOF OF THEOREM 1.5

As already mentioned, case (a) is contained in [7]. Note also that case (b) follows directly from Proposition 1.7. It remains to handle case (c), because the graph of the map f given by Proposition 1.7 is not necessarily contained in Σ . We show here how to modify it in such a way to fulfill the requirements of Theorem 1.5.

We assume that Ψ is a function whose graph coincides with Σ (the connected component of $\partial \mathbf{B}_R(p) \cap \mathbf{C}_{4r}(x)$ containing $\text{spt}(T)$) and arguing as in [7, Remark 1.5] we can assume that $\|\Psi_0\| \leq C E^{1/2} r + C \Omega r^2$, $\|D\Psi\|_0 \leq C E^{1/2} + C \Omega r$ and $\|D^2\Psi\|_0 \leq C \Omega$. The domain of Ψ is a subset of $B_{4r}(x) \times \mathbb{R}^{n-1}$. Let now $f = \sum_i \llbracket f_i \rrbracket$ be the function given by Proposition

1.7 and let $\bar{f} = \sum_i \llbracket \bar{f}_i \rrbracket$, where $\bar{f}_i(y)$ gives the first $n - 1$ coordinates of $f_i(y)$. Observe that on the set K we necessarily have

$$f(y) = \sum_i \llbracket (\bar{f}_i(y), \Psi(y, \bar{f}_i(y))) \rrbracket .$$

We then can extend \bar{f} to $B_r(x) \setminus K$ with $\text{Lip}(\bar{f}) \leq C\text{Lip}(f)$ and $\text{osc}(\bar{f}) \leq C\text{osc}(f)$ and hence define $\hat{f}(y) = \sum_i \llbracket (\bar{f}_i(y), \Psi(y, \bar{f}_i(y))) \rrbracket$ for every $y \in B_r(x)$ (it must be shown that $(y, \bar{f}_i(y))$ belongs to the domain of definition of Ψ , but this follows easily from the smallness of $\text{osc}(\bar{f})$). Obviously $f = \hat{f}$ on K . On the other hand it is straightforward to check that

$$\text{Lip}(\hat{f}) \leq C\text{Lip}(\bar{f}) + C(\text{Lip}(\bar{f}) + 1)\|D\Psi_0\| \leq CE_0^\beta + C\Omega r \quad (6.1)$$

$$\text{osc}(\hat{f}) \leq C\text{osc}(f) + \|\Psi\|_0 \leq C\mathbf{h}(T, \mathbf{C}_{4r}(x)) + C(E^{1/2} + \Omega r)r . \quad (6.2)$$

In addition we conclude

$$\left| \int_{B_r(x)} |Df|^2 - \int_{B_r(x)} |D\hat{f}|^2 \right| \leq (\text{Lip}(f)^2 + \text{Lip}(\hat{f})^2)|B_r(x) \setminus K| \leq C|K| .$$

Thus the estimates in Proposition 1.7 complete the proof.

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