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by

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On the Regularity of Lumped Nonlinear Dynamics in Banach Spaces

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Abstract

We study the regularity properties of the lumping problem for differential equations in Banach spaces, namely the projection of dynamics by a reduction operator onto a reduced state space in which a self-contained dynamical description exists. We study dynamics generated by a nonlinear operator F and a linear and bounded reduction operator M. We first show, using quotient space methods, that the reduced operator is C^1, provided that F itself is C^1 in the original state space. We further prove that a particular lumping relation holds between the Fréchet differentials of F and the reduced operator. In this way, by smoothness, the linearization principle applies and it is possible to use results from linear theory to study the local behavior of the system.

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1. Introduction

Consider the differential equation defined on a Banach space X,

$$\begin{cases} \dot{x}(t) = F(x(t)) \\ x(0) = x_0, \end{cases} \tag{1}$$

with $F : \mathscr{D}(F) \subseteq X \to X$. We assume that the dynamics (1) is well defined, in the sense that for every $x_0 \in \mathscr{D}(F)$ there exists a unique solution. In addition, consider a linear bounded map $M : X \to Y$, where Y is another Banach space. We view the operator M as a *reduction* of the state space: it is surjective but not an isomorphism. The question of interest is whether the variable $y = Mx$ also satisfies a well-posed and self-contained linear dynamics on Y, say

$$\dot{y}(t) = \widehat{F}y(t), \quad y = Mx,$$

for some $\widehat{F}$. If this is the case, then we refer to M as a *reduction* or *lumping* operator.

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Definition 1. The system (1) is said to be *exactly lumpable* by the operator M if there exists an operator $\widehat{F} : Y \to Y$ such that the following diagram commutes

$$\begin{array}{ccc} Y & \xrightarrow{\widehat{F}} & Y \\ \uparrow M & & \uparrow M \\ X & \xrightarrow{F} & X \end{array} \tag{2}$$

that is,

$$MF = \widehat{F}M. \tag{3}$$

The term *lumping* originates from chemical reaction systems, where the aim is to aggregate all the species involved in the reaction into a few groups, called *lumps* of chemical reagents [18, 17, 16, 12]. A similar concept of *aggregation* of states has been used in the theory of Markov chains, where the question is whether the newly-formed aggregates also admit a Markovian description for the state transitions [9, 11, 6]. In [10, 14] lumpability was studied in the context of ecological modelling. There is also a connection to control theory, given in [8]. Diagram 2, however, is more general, as the operator M can also represent other types of reduction, for example projections or averages. It can also be seen as describing a *multi-level* system, whose dynamics is described by F at the micro level and by $\widehat{F}$ at the macro level. The notion of lumping is also related to the *conjugacy*: two systems are conjugate if M is not only surjective but also invertible. However, conjugacy implies that the orbits of the original systems are mapped homeomorphically into orbits of the new system; so there is no reduction of dynamics. More closely related, but also less well studied, is the concept of semi-conjugacy [5, 4], where the assumption of the invertibility of M is relaxed, although in this case the interpretation of Diagram 2 is different: In the context of semi-conjugacy both operators F and $\widehat{F}$ are given and the problem is showing the existence of a surjection M, while in the case of lumping one starts from a given M and asks whether an $\widehat{F}$ exists so that Diagram 2 commutes.

Previous work on lumpability in infinite dimensional spaces was carried out for bounded operators by Coxson [8], and by Rózsa and Tóth in the context of Hilbert spaces [15]. In particular, the analysis in [8] requires the existence of a continuous pseudoinverse of the lumping operator (see Definition 5 below). However, as will be detailed in Section 2, the existence of a pseudoinverse requires some restrictive hypotheses on M; specifically, a topological complement for $\ker(M)$ is assumed to exist. In [2], the present authors have used a different method that does not resort to pseudoinverses, analyzed the lumpability of linear systems in general Banach spaces, and further extended the results to unbounded operators. For the case when the original dynamics is well-posed, in the sense of the Hille and Yosida Theorem, i.e., when F is the infinitesimal generator of a C_0-semigroup of linear operators $\{T(t)\}_{t\geq 0}$, conditions have been derived conditions for the reduced operator $\widehat{F}$ to exist and to generate again a C_0-semigroup on the reduced state space [2]. In particular, a necessary and sufficient condition for lumpability turns out to be the invariance of $\ker(M)$ under the whole semigroup, that is

$$\ker(M) \subseteq \ker(MT(t)), \quad t \geq 0.$$

Since the semigroup of solutions is in general not known *a priori*, necessary and sufficient conditions for lumpability are also given directly on the infinitesimal generator [2] (see Theorem 4 in the next section).

Lumpability of *nonlinear* systems presents further challenges. For semigroups of nonlinear operators, the differentiability of $t \mapsto T(t)x$ is not automatically guaranteed even if x belongs to the domain of the infinitesimal generator. The aim of the present paper is to obtain some regularity properties of the nonlinear reduced map. A particular outcome is the justification of the linearization principle in a neighborhood of a given point. In this way, we locally obtain a lumping of linear systems, for which lumpability has been studied, as mentioned in the above paragraph [8, 15, 2]. In particular, we show that if F is sufficiently smooth, the reduced operator $\widehat{F}$ is Fréchet differentiable on Y; in fact, it is a C^1 operator. We further prove that a lumping relation similar to (3) holds for the differentials of $\widehat{F}$ and F respectively. By the smoothness of the reduced map, one can then approximate the behaviour of the reduced system locally via linearization, and then study the lumping between the two differential operators, which are indeed linear and bounded. On the other hand, some other properties such as contractivity are not necessarily preserved after lumping (see Remark 2).

It is not hard to see that a necessary and sufficient condition for the reduced operator $\widehat{F}$ to be well-defined is that for all $x_1, x_2 \in \mathscr{D}(F)$,

$$Mx_1 = Mx_2 \quad \Rightarrow \quad MF(x_1) = MF(x_2). \tag{4}$$

In this case we say that F *preserves the fibers* of M, in the sense of *level sets*: if two points belong to the same level set of M, then also their images through F belong to the same level set. We can then define $\widehat{F}$ by

$$\widehat{F}(y) := MF(x), \qquad y = Mx. \tag{5}$$

By (4), if $Mx_1 = Mx_2$, then also $MF(x_1) = MF(x_2)$, and definition (5) is well-posed.

Since $\widehat{F}$ is defined as an operator on Y, proving its smoothness without making use of a pseudoinverse operator for M is non-trivial. Indeed, if a pseudoinverse $\overline{M}$ exists (i.e. ker(M) is complemented), then one can write $\widehat{F}(y) := MF(\overline{M}y)$. In this case the smoothness of $\widehat{F}$ follows from the boundedness of M and $\overline{M}$. But if ker(M) is not complemented in X, then we are not able to express x in terms of the reduced variable y. To prove the regularity of $\widehat{F}$ we will exploit the properties of some particular operators on quotient Banach spaces.

In the following, it will be a standing assumption that $F(0) = 0$. In this way, $\widehat{F}(0) = 0$ and 0 is an equilibrium point for both the original and the reduced system. In the next section we recall some basic definitions about differentiability in Banach spaces and strongly continuous semigroups of operators (for more details on nonlinear functional analysis, we refer to [3]). The main regularity results are presented in Section 3. We state the implications for linearization and local behavior in Section 4 and conclude with an example in Section 5.

2. Preliminaries

In this section we review some background results from functional analysis, starting with differentiability concepts in Banach spaces.

Definition 2. Let X and Y be two Banach spaces. A function $F : X \to Y$ is said to be *Gâteaux differentiable* at a point $x \in X$ if the following limit exists for every $h \in X$:

$$\frac{d}{dt}(F(x+th))_{|t=0} = \lim_{t\to 0} \frac{F(x+th) - F(x)}{t} =: DF_x(h)$$

and, for x fixed, DF_x is a bounded linear operator on X.

The Gâteaux differential is a generalization of the classical directional derivative (with the additional condition that the directional derivative be a linear operator acting on the directions). A stronger notion of derivative is obtained if it is required that the convergence in the above limit is uniform in h, when h belongs to the unit ball of X.

Definition 3. A function $F : X \to Y$ is said to be *Fréchet differentiable* at $x \in X$ if there exists a linear and bounded operator $\mathscr{D}_x \in \mathscr{B}(X,Y)$ such that

$$\lim_{h\to 0} \frac{F(x+h) - F(x) - \mathscr{D}_x(h)}{\|h\|},$$

or equivalently,

$$F(x+h) = F(x) + \mathscr{D}_x h + o(\|h\|). \tag{6}$$

In particular, (6) implies the possibility of approximating F by its linearization in a neighborhood of a point at which it is Fréchet differentiable. If F is Fréchet differentiable, then it is also Gâteaux differentiable and the two differentials coincide: $\mathscr{D}_x(h) = DF_x(h)$ for all $h \in X$. For this reason we will always use the notation DF_x for the derivative of a Fréchet differentiable function at the point x. If A is a linear operator then its Fréchet differential clearly coincides with A itself. Furthermore, the following criteria can be used to verify the Fréchet differentiability of a function.

Proposition 1. *If $F : X \to Y$ is Gâteaux differentiable in $x \in X$ and the Gâteaux derivative DF is continuous from X to $\mathscr{B}(X,Y)$, then F is also Fréchet differentiable (and, in this case, it is said to be a C^1 function).*

Most existing works on lumpability rely on pseudoinverses of operators, whose existence is related to topologically complemented subspaces. We briefly recall the relevant notions.

Definition 4 (Complemented subspace)**.** A closed subspace X_1 of a Banach space X is said to be *complemented* in X if and only if there exists a closed subspace X_2 such that

$$X = X_1 \oplus X_2,$$

where $X_1 \oplus X_2$ denotes the topological direct sum of X_1 and X_2. In this case, X_2 is called a *topological complement* for X_2.

It can be shown that every finite dimensional subspace has a topological complement. Furthermore, if H is a Hilbert space, then every closed subspace $Y \subset H$ is complemented. Indeed, the orthogonal complement $Y^\perp$ (i.e. $\langle y, y^\perp \rangle = 0$ for every $y \in Y$, $y^\perp \in Y^\perp$) is a closed subspace of H and we have $H = Y \oplus Y^\perp$. A famous theorem due to Lindenstrauss and Tzafriri asserts that the converse is true as well [13]. More precisely, if $(X, \|\cdot\|)$ is a Banach space such that every closed subspace is complemented, then $\|\cdot\|$ is induced by a scalar product, i.e. $(X, \|\cdot\|)$ is a Hilbert space. A known example of a non-complemented subspace in a Banach space is $c_0(\mathbb{Z}) \subset l^\infty(\mathbb{Z})$, i.e., the closed subspace of null sequences in the Banach space of the bounded sequences.

Definition 5 (Pseudoinverse)**.** A pseudoinverse of $A \in \mathscr{B}(X, Y)$ is any operator $\overline{A} \in \mathscr{B}(Y, X)$ such that $A\overline{A}A = A$.

Proposition 2. *Let A be a linear operator between Banach spaces. Then the following are equivalent:*

(i) A admits a bounded pseudoinverse $\overline{A} : Y \to X$;

(ii) $\ker(A)$ and $\operatorname{ran}(A)$ are complemented subspaces in X and Y respectively;

(iii) there exist continuous projections P and Q such that $\operatorname{ran}(P) = \ker(A)$ and $\operatorname{ran}(Q) = \operatorname{ran}(A)$ respectively.

Clearly, not every linear and bounded operator has a pseudoinverse (see [1] for details about this problem); therefore, a more general method is needed for the analysis of lumping in Banach spaces. The approach taken in [2] uses the concept of strongly continuous semigroups to overcome the restrictions of pseudoinverses: Let X be a Banach space. A one-parameter family of bounded operators $\{T(t)\}_{t\geq 0}$ in $\mathscr{B}(X)$ is called a *strongly continuous semigroup* if

1. $T(0) = I$,
2. $T(t + s) = T(t)T(s) \quad \forall t, s \geq 0$,
3. The map $t \mapsto T(t)x \in X$ is continuous for every $x \in X$.

The last property is called *strong continuity* as it corresponds to the continuity of the map $t \mapsto T(t) \in \mathscr{B}(X)$ when $\mathscr{B}(X)$ is endowed with the strong operator topology, (i.e., $T_n \to T$ iff $\lim_{n\to+\infty} \|T_n x - Tx\| = 0$, $\forall x \in X$).

The *generator* of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ is the closed and densely defined operator $A : \mathscr{D}(A) \subset X \to X$ defined by $Ax = \lim_{h\to 0^+} \frac{1}{h}(T(h)x - x)$ on the domain

$$\mathscr{D}(A) = \left\{ x \in X : \lim_{h\to 0^+} \frac{1}{h}(T(h)x - x) \in X \right\}.$$

If the generator is a bounded operator, i.e. $\mathscr{D}(A) = X$, then the semigroup is said to be *uniformly continuous.* A strongly continuous semigroup $T(t)$ is characterized by a real number $\omega(T)$ called the *growth bound of the semigroup*, defined as

$$\omega(T) = \inf \left\{ \omega_0 \in \mathbb{R} : \exists C > 0 \text{ with } ||T(t)|| \leq Ce^{\omega_0 t} \ \ \forall t > 0 \right\}.$$

The growth bound is linked to the spectral properties of the generator A; in fact, it can be shown that $\sup_{\lambda\in\sigma(A)}\{\operatorname{Re}(\lambda)\} \leq \omega(T)$, where $\sigma(A)$ denotes the spectrum of A.

Theorem 3. *The dynamics associated with a linear operator A is well posed if and only if A is the generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on X, and in that case for every $u_0 \in \mathscr{D}(A)$ the unique classical solution is given by $t \mapsto T(t)u_0$.*

The next result gives the necessary and sufficient conditions for lumpability of linear systems from the point of view of the infinitesimal generator.

Theorem 4 ([2]). *The system associated with A is exactly lumpable by the linear, bounded and surjective operator $M : X \to Y$ if and only if the following two conditions hold:*

1. $A\left(\mathrm{Ker}(M) \cap \mathscr{D}(A)\right) \subset \mathrm{Ker}(M)$*;*
2. there exists $\lambda > \omega$ such that $(\lambda I - A)$ is surjective from $\mathrm{Ker}(M) \cap \mathscr{D}(A)$ to $\mathrm{Ker}(M)$.

We note that these linear results are still useful in the nonlinear case, because if the nonlinear map F is differentiable, then its differential is a linear and bounded operator. In particular, it generates a uniformly continuous semigroup on X.

3. Regularity of the operator $\widehat{F}$

We now turn to the question of which regularity properties of the nonlinear function F are preserved by lumping. We first prove that continuity is maintained.

Proposition 5. *Let $F : X \to X$ be an everywhere-defined map from a Banach space X to itself satisfying condition (4). If F is continuous, then the map $\widehat{F} : Y \to Y$ defined in (5) is also continuous.*

Proof. Let $\mathscr{A} \subset Y$ be an open set. We will show that $\widehat{F}^{-1}(\mathscr{A})$ is also an open set in Y. To this end, we write

$$M^{-1}\widehat{F}^{-1}(\mathscr{A}) = (\widehat{F} \circ M)^{-1}(\mathscr{A}) = (MF)^{-1}(\mathscr{A}).$$

Since M is linear and bounded and F is continuous, $(MF)^{-1}(\mathscr{A})$ is an open set in X, so that $M^{-1}\widehat{F}^{-1}(\mathscr{A})$ is open. Given that M is surjective, we obtain

$$M(M^{-1}\widehat{F}^{-1}(\mathscr{A})) = \widehat{F}^{-1}(\mathscr{A}),$$

which is an open set in Y because M is an open map by the Banach-Schauder Theorem (i.e., it maps open sets into open sets). □

Remark 1. Note that the proposition above holds also in the case the domain $\mathscr{D}(F)$ of F is a proper subset of X. In this case F is continuous if and only if, for any open set $\mathscr{A} \cap X$, $F^{-1}(\mathscr{A})$ is open with respect to the subspace topology induced by X on $\mathscr{D}(F)$, (i.e it can be written as $\mathscr{B} \cap \mathscr{D}(F)$ for some open set $\mathscr{B}$ in X). Using the notation as above,

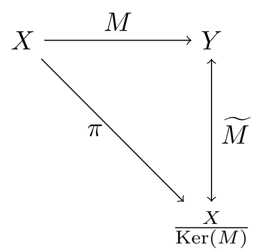
$$(MF)^{-1}(\mathscr{A}) = \mathscr{D}(F) \cap \mathscr{B}, \quad \mathscr{B} \text{ open in } X.$$

Then,

$$M(M^{-1}\widehat{F}^{-1}(\mathscr{A})) = M(\mathscr{D}(F) \cap \mathscr{B}) \subset M\mathscr{D}(F) \cap \mathscr{D},$$

where $\mathscr{D}$ is the open set $M(\mathscr{B})$. But since MF is continuous, $M(\mathscr{D}(F) \cap \mathscr{B})$ is open in Y: writing it as $M(\mathscr{D}(F) \cap \mathscr{B}) \cap \mathscr{D}$, it is clear that it is open with respect to the subspace topology. It follows that $\widehat{F}^{-1}(\mathscr{A})$ is also open with respect to the subspace topology on $\mathscr{D}(\widehat{F})$.

In the following, we make use of two particular operators defined on the quotient Banach space $\frac{X}{\ker(M)}$. First, we consider the following diagram



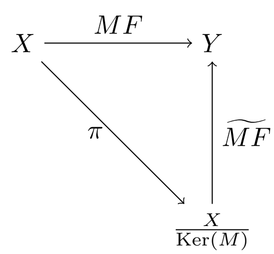
(7)

Define the operator $\widetilde{M} : \frac{X}{\mathrm{Ker}(M)} \to Y$ by $\widetilde{M}[x] := M(x)$. This definition is well posed, in the sense that it does not depend on the choice of the particular element in the equivalence class. Furthermore, since $[x] = [x - m]$ $\forall m \in \ker(M)$,

$$\begin{aligned} \|\widetilde{M}[x]\| &= \inf_{m \in \ker(M)} \|\widetilde{M}[x - m]\| = \inf_{m \in \ker(M)} \|M(x - m)\| \\ &\leq \inf_{m \in \ker(M)} \|M\| \, \|x - m\| = \|M\| \, \|[x]\|, \end{aligned}$$

which shows that $\widetilde{M}$ is bounded. Moreover, $\widetilde{M}$ is a bijective operator between Banach spaces. By the Banach-Shauder Theorem, being an open bijection, $\widetilde{M}$ is also an homeomorphism. In particular, $\widetilde{M}^{-1}$ is a bounded operator on Y.

Next, we look at the diagram



(8)

where the operator $\widetilde{MF} : \frac{X}{\mathrm{Ker}(M)} \to Y$ is defined by

$$\widetilde{MF}[x] := MF(x). \tag{9}$$

This operator is well defined even if F is nonlinear, provided that it satisfies condition (4). In particular, if F is linear and bounded, $\widetilde{MF}$ is itself a linear and bounded operator [2]. These two operators will be fundamental in the proof of the next results.

Recall that an operator F is Lipschitz continuous if there exists a constant $K \in \mathbb{R}^+$ such that $\|F(x_1) - F(x_2)\| \leq K\|x_1 - x_2\|$ for every $x_1, x_2 \in X$. In particular, if $\|F(x_1) - F(x_2)\| \leq \|x_1 - x_2\|$ $\forall x_1, x_2 \in X$, then F is called *contractive.*

Proposition 6. *Let $F : X \to X$ be an everywhere-defined map from a Banach space X to itself satisfying condition (4). If F is Lipschitz continuous, then the map $\widehat{F} : Y \to Y$ defined as in (5) is also Lipschitz continuous.*

Proof. Consider Diagram 8. Since F preserves the fibers of M, the map $\widetilde{MF}$ given in (9) is well defined. For any two elements $[x_1], [x_2] \in \frac{X}{\mathrm{Ker}(M)}$, we have

$$\begin{aligned}
&\|\widetilde{MF}([x_1]) - \widetilde{MF}([x_2])\| = \inf_{m\in \mathrm{Ker(M)}} \|\widetilde{MF}([x_1 - m]) - \widetilde{MF}([x_2])\| \\
&= \inf_{m\in \mathrm{Ker(M)}} \|MF(x_1 - m) - MF(x_2)\| \leq \inf_{m\in \mathrm{Ker(M)}} \|M\| \, \|F(x_1 - m) - F(x_2)\| \\
&\leq \inf_{m\in \mathrm{Ker(M)}} K \, \|M\| \, \|(x_1 - m) - x_2\| = K \, \|M\| \, \|[x_1] - [x_2]\|.
\end{aligned}$$

Thus $\widetilde{MF}$ is Lipschitz with Lipschitz constant $K\,\|M\|$. Now, if $y_1 = Mx_1$ and $y_2 = Mx_2$ are points in Y, then

$$\begin{aligned}
\|\widehat{F}(y_1) - \widehat{F}(y_2)\| = \|MF(x_1) - MF(x_2)\| &= \|\widetilde{MF}([x_1]) - \widetilde{MF}([x_2])\| \\
&\leq K \, \|M\| \, \|[x_1] - [x_2]\| = K \, \|M\| \, \|\widetilde{M}^{-1} y_1 - \widetilde{M}^{-1} y_2\| \\
&\leq K \, \|M\| \, \|\widetilde{M}^{-1}\| \, \|y_1 - y_2\|,
\end{aligned}$$

i.e. $\widehat{F}$ is Lipschitz continuous with Lipschitz constant equal to $K\,\|M\|\,\|\widetilde{M}^{-1}\|$. ☐

Remark 2 (Contractivity)**.** Observe that, by definition of $\widetilde{M}$ and of the equivalence class $[x]$,

$$\begin{aligned}
\|\widetilde{M}\| = \sup_{\|[x]\|\leq 1} \|\widetilde{M}[x]\| &= \sup_{\|[x]\|\leq 1} \inf_{m\in \mathrm{Ker(M)}} \|M(x - m)\| \\
&\leq \sup_{\|[x]\|\leq 1} \inf_{m\in \mathrm{Ker(M)}} \|M\| \, \|x - m\| \leq \sup_{\|[x]\|\leq 1} \|M\| \|[x]\| \leq \|M\|.
\end{aligned}$$

On the other hand, since the quotient map π is a contraction operator, one can write

$$\begin{aligned}
\|M\| = \sup_{\|x\|\leq 1} \|Mx\| &= \sup_{\|x\|\leq 1} \|\widetilde{M}[x]\| \\
&\leq \sup_{\|x\|\leq 1} \|\widetilde{M}\| \, \|\pi(x)\| \leq \sup_{\|x\|\leq 1} \|\widetilde{M}\| \|(x)\| \leq \|\widetilde{M}\|.
\end{aligned}$$

Thus $\|M\| = \|\widetilde{M}\|$. Let $\widehat{K}$ denote the Lipschitz constant of $\widehat{F}$. Then $\widehat{K} = K\,\|M\|\,\|\widetilde{M}^{-1}\| = K\,\|\widetilde{M}\|\,\|\widetilde{M}^{-1}\|$. Since in general $\|\widetilde{M}^{-1}\|\,\|\widetilde{M}\| \geq 1$, we have $\widehat{K} \geq K$. Therefore $\widehat{F}$ need not be a contractive operator even if $K < 1$, unless additional conditions are imposed on the lumping operator M, such as $\|M\|\|\widetilde{M}^{-1}\| = 1$.

We next prove that the function $\widehat{F}$ preserves the smoothness of F under suitable hypotheses. First, suppose that F is Gâteaux differentiable on the whole X, with DF_x denoting the Gâteaux derivative at the point $x \in X$. We then claim that $\widehat{F}$ is also Gâteaux differentiable on Y. Indeed, if $y = Mx$ and $z = Mh$,

$$\begin{aligned}
\lim_{t\to 0} \frac{\widehat{F}(y + tz) - \widehat{F}(y)}{t} &= \lim_{t\to 0} \frac{MF(x + th) - MF(x)}{t} \\
&= M \left(\lim_{t\to 0} \frac{F(x + th) - F(x)}{t} \right) = MDF_x(h).
\end{aligned}$$

Hence the Gâteaux derivative of $\widehat{F}$ can be defined as

$$D\widehat{F}_y(z) := MDF_x(h),$$

for every $y = Mx$ and $z = Mh$. If y is fixed, then $D\widehat{F}_y$ is a linear and bounded operator from Y to itself.

Proposition 7. *Suppose that F is an everywhere-defined operator on X satisfying condition (4). Assume further that F is C^1, i.e. DF is a continuous operator from X to $\mathscr{B}(X)$. Then the reduced operator $\widehat{F}$ defined in (5) is also C^1 on Y.*

Proof. In the following we will use the notation A for a linear operator, to distinguish from the nonlinear function F. Consider the following subspace of $\mathscr{B}(X)$:

$$\widetilde{\mathscr{B}}(X) := \{A \in \mathscr{B}(X) \text{ such that } \ker(M) \subset \ker(MA)\}.$$

This is the space of all linear and bounded operators A such that the reduced operator $\widehat{A}y := MAx$, $y = Mx$ is well-defined and belongs to $\mathscr{B}(Y)$ (see [2]). It is easy to verify that $\widetilde{\mathscr{B}}(X)$ is a linear space containing 0 and the identity map I. Moreover, it is a closed subspace of $\mathscr{B}(X)$. Indeed, given A_n such that $A_n \to A$ in $\mathscr{B}(X)$, $A_n \in \widetilde{\mathscr{B}}(X)$, and x_1, x_2 such that $Mx_1 = Mx_2$, we have

$$\begin{aligned}\|MAx_1 - MAx_2\| &\leq \|MAx_1 - MA_nx_1\| + \|MA_nx_1 - MA_nx_2\| + \|MA_nx_2 - MAx_2\| \\ &= \|MAx_1 - MA_nx_1\| + \|MA_nx_2 - MAx_2\|,\end{aligned}$$

because $MA_nx_1 = MA_nx_2$ for all $n \in \mathbb{N}$. Letting $n \to +\infty$, we obtain $\|MAx_1 - MAx_2\| = 0$. Hence, $MAx_1 = MAx_2$, i.e., $A \in \widetilde{\mathscr{B}}(X)$. Now define the following linear operator between Banach spaces:

$$\mathscr{M} : \widetilde{\mathscr{B}}(X) \to \mathscr{B}(Y), \quad \mathscr{M}(A) := \widehat{A},$$

where $\widehat{A}y = MAx$ for all $y = Mx \in Y$. We will first prove that this operator is continuous, and then show the continuity of $D\widehat{F}$ by the continuity of $\mathscr{M}$.

Consider Diagram 8 and the operator $\widetilde{MA}[x] := MAx$ from $\frac{X}{\ker(M)}$ to Y, which is well defined, linear, and bounded. Suppose that $A_n \to A$ in $\mathscr{B}(X)$. Then,

$$\begin{aligned}\|\widetilde{MA_n}[x] - \widetilde{MA}[x]\| &= \|MA_n(x-m) - \widetilde{MA}(x-m)\| \\ &= \inf_{m\in\ker(M)} \|MA_n(x-m) - \widetilde{MA}(x-m)\| \\ &\leq \inf_{m\in\ker(M)} \|MA_n - MA\|\|x-m\| = \|MA_n - MA\|\|[x]\|,\end{aligned}$$

so that

$$\sup_{\|[x]\|\leq 1} \|\widetilde{MA_n}[x] - \widetilde{MA}[x]\| \leq \|MA_n - MA\|.$$

Letting $n \to +\infty$, we obtain $\sup_{\|[x]\|\leq 1}\|\widetilde{MA_n}[x] - \widetilde{MA}[x]\| \to 0$. Now, using the properties of the operator $\widetilde{M}$ (see Diagram 7),

$$\begin{aligned}
&\|\mathscr{M}A_n - \mathscr{M}A\|_{\mathscr{B}(Y)} = \sup_{\|y\|\leq 1} \|\widehat{A}_n - \widehat{A}\| = \sup_{\|y\|\leq 1} \|MA_n x - MAx\| \\
&= \sup_{\|y\|\leq 1} \|\widetilde{MA_n}[x] - \widetilde{MA}[x]\| \leq \sup_{\|y\|\leq 1} \|\widetilde{MA_n} - \widetilde{MA}\| \|\widetilde{M}^{-1} y\| \\
&\leq \sup_{\|y\|\leq 1} \|\widetilde{MA_n} - \widetilde{MA}\| \|\widetilde{M}^{-1}\| \|y\| \leq \|\widetilde{MA_n} - \widetilde{MA}\| \|\widetilde{M}^{-1}\|.
\end{aligned}$$

Since the norm $\|\widetilde{MA_n} - \widetilde{MA}\|$ (which is the operator norm in $\mathscr{B}(\frac{X}{\ker(M)}, Y)$ tends to zero for $n \to +\infty$, we have that $\mathscr{M}A_n$ converges to $\mathscr{M}A$ in $\mathscr{B}(Y)$. This implies that $\mathscr{M}$ is a linear and bounded operator from $\mathscr{B}(X)$ to $\mathscr{B}(Y)$.

Now, we know that $\widehat{F}$ is at least Gâteaux differentiable with Gâteaux derivative $D\widehat{F}_y(z) = MDF_x(h)$, for every $y = Mx$ and $z = Mh$. Furthermore, DF is also the Fréchet differential of F because F is C^1. By definition it is easily seen that

$$D\widehat{F} \circ M = \mathscr{M}DF$$

as operators from X to $\mathscr{B}(Y)$. To show that $D\widehat{F}$ is continuous from Y to $\mathscr{B}(Y)$, we take an open set $\mathscr{A} \subset \mathscr{B}(Y)$ and write

$$M^{-1}(D\widehat{F}^{-1})(\mathscr{A}) = (D\widehat{F} \circ M)(\mathscr{A})^{-1} = (\mathscr{M}DF)^{-1}(\mathscr{A}),$$

which is an open set in X because $\mathscr{M}$ and DF are continuous. Since M is surjective and open,

$$D\widehat{F}^{-1}(\mathscr{A}) = MM^{-1}(D\widehat{F}^{-1})(\mathscr{A}) = M(\mathscr{M}DF)^{-1}(\mathscr{A}),$$

which is an open set in Y. By the continuity of the map $y \mapsto D\widehat{F}_y$, it follows that $\widehat{F}$ is C^1 and $D\widehat{F}$ is its Fréchet differential. □

Remark 3. Note that Proposition 7 still holds if F is defined on a proper subset $\mathscr{D}(F) \subset X$, provided that F is C^1 on its domain. Indeed, even in this case DF_x is a bounded operator on X for $x \in \mathscr{D}(F)$. We have $D\widehat{F} \circ M = \mathscr{M}DF$ as operators from $\mathscr{D}(F)$ to $\mathscr{B}(Y)$. In this case, for every open set $\mathscr{A} \subset \mathscr{B}(Y)$, $D\widehat{F}^{-1}(\mathscr{A})$ is open with respect to the subspace topology on $\mathscr{D}(\widehat{F})$. Thus, $\widehat{F}$ is C^1 on $M\mathscr{D}(F)$.

4. Linearization and local lumping

We consider an application of the results obtained in the previous section about regularity of the reduced map. Consider a point $x_0 \in X$ in which F is C^1, and the ball $\mathscr{B}_\alpha(x_0)$ centered in x_0 with ray $\alpha > 0$. Denote $y_0 := Mx_0$. Since M is an open map, the following property is well known: For every $\alpha > 0$ there exists $\beta > 0$ such that

$$\mathscr{B}_\beta(Mx_0) \subset M\mathscr{B}_\alpha(x_0).$$

In other words, for α fixed, one can find $\beta > 0$ such that all the points $y \in \mathscr{B}_\beta(y_0)$ can be written as $y = Mx$, with $x \in \mathscr{B}_\alpha(x_0)$. By virtue of Proposition 7, one can write the following linearization for the reduced operator $\widehat{F}$:

$$\widehat{F}(y_0 + y) = \widehat{F}(y_0) + D\widehat{F}_{y_0} y + o(\|y\|),$$

that is,

$$\widehat{F}(y_0 + y) = \widehat{F}(y_0) + MD\widehat{F}_{x_0} x + o(\|y\|).$$

Now take $x_0 = 0$. Since 0 is an equilibrium for the system by standing assumption, the linearization around 0 becomes

$$\widehat{F}(y) = MDF_0 x + o(\|y\|). \tag{10}$$

Choose $\alpha \ll 1$, and $\gamma > 0$ such that

$$\gamma < \beta, \;\; \gamma \ll 1.$$

For all $y \in \mathscr{B}_\beta(0)$, $\widehat{F}$ can be approximated by MDF_0 using (10).

We have proved in the previous section that the lumping relation holds between DF and $D\widehat{F}$:

$$D\widehat{F}_0(y) = MDF_0(x),$$

and $\ker(M)$ is DF_0-invariant. Hence, looking at F and $\widehat{F}$ in $\mathscr{B}_\alpha(0) \subset X$ and $\mathscr{B}_\gamma(0) \subset Y$ respectively, we are dealing with the lumping of linear operators. In particular, it has been proved that stability of equilibria is preserved by lumping [16, 15]. Note that both DF_0 and $D\widehat{F}_0$ generate well-posed dynamics since they are linear and bounded operators. Denote by $T(t)$ and $\widehat{T}(t)$, respectively, the uniformly continuous semigroups generated by DF_0 and $D\widehat{F}_0$. It is proved in [2] that $\ker(M)$ is $T(t)$-invariant and the lumping relation

$$MT(t) = \widehat{T}(t)M$$

holds on X. In particular, the *growth bound* $\widehat{\omega}$ of the semigroup $\widehat{T}(t)$ is always less or equal than the growth bound ω of $T(t)$ [15], and, by boundedness of the operators involved,

$$\sup_{\lambda \in \sigma(\widehat{A})} \{\mathrm{Re}(\lambda)\} = \omega(\widehat{T}) \leq \sup_{\lambda \in \sigma(A)} \{\mathrm{Re}(\lambda)\} = \omega(T).$$

It is well known that a semigroup $T(t)$ is exponentially stable if and only if $\omega < 0$.

Using the linearized stability theorem in Banach spaces (see, e.g., [7]), one can study the local stability of the zero equilibrium for the nonlinear system associated with $\widehat{F}$ by looking at the growth bound of the semigroup generated by $D\widehat{F}_0$ (which is indeed linear). By the lumping we obtained, this growth bound can be estimated by the growth bound of the semigroup generated by DF_0. In particular, if 0 is exponentially stable for the system associated with DF_0, then it is locally exponentially stable for the nonlinear system associated with $\widehat{F}$.

5. An example with a nonlinear composition operator

Consider the Banach space $X := \mathscr{B}(\mathbb{R})$ of continuous, bounded, real-valued functions from $\mathbb{R}$ to itself with the supremum norm, $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$. Define the following composition operator:

$$F(f)(x) := \phi(f(x)), \tag{11}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a C^2 function. Since f is bounded and ϕ is smooth, the operator F is continuous from X to itself. We will show that F is Fréchet differentiable with Fréchet differential $DF : X \to \mathscr{B}(X)$, defined as

$$DF_f[h](x) := \phi'(f(x))\, h(x), \qquad \forall f, h \in X,\, x \in \mathbb{R}.$$

Since ϕ is C^2, we can write:

$$\phi(x+y) = \phi(x) + \phi'(x)y + O(y^2).$$

Then,

$$\begin{aligned}
&\lim_{t\to 0}\left(\sup_{x\in\mathbb{R}}\left|\frac{F(f+th)(x) - F(f)(x)}{t} - \phi'(f(x))\, h(x)\right|\right)\\
&= \lim_{t\to 0}\left(\sup_{x\in\mathbb{R}}\frac{1}{t}\left|\phi(f(x)) + \phi'(f(x))th(x) + O(t^2h(x)^2) - \phi(f(x)) - t\phi'(f(x))\, h(x)\right|\right)\\
&\leq \lim_{t\to 0} C\, t\|h\|^2 = 0.
\end{aligned}$$

It follows that $\phi'(f(x))\, h(x)$ is the Gâteaux differential of F evaluated at f and acting in the direction h. It is also the Fréchet differential since

$$\lim_{h\to 0}\frac{\|F(f+h) - F(f) - \phi'(f)h\|}{\|h\|} \leq \lim_{h\to 0}\frac{C\|h\|^2}{\|h\|} = 0.$$

Moreover, F is C^1, because the map $X \ni f \to DF_f \in \mathscr{B}(X)$ is continuous.

Now, given a set of points $y_1, \ldots, y_n$ in $\mathbb{R}$, consider the following lumping operator

$$Mf = \begin{pmatrix} f(y_1) \\ \vdots \\ f(y_n) \end{pmatrix}, \quad M : X \to \mathbb{R}^n.$$

The operator F preserves the fibers of M since

$$\begin{pmatrix} g(y_1) \\ \vdots \\ g(y_n) \end{pmatrix} = \begin{pmatrix} f(y_1) \\ \vdots \\ f(y_n) \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \phi(g(y_1)) \\ \vdots \\ \phi(g(y_n)) \end{pmatrix} = \begin{pmatrix} \phi(f(y_1)) \\ \vdots \\ \phi(f(y_n)) \end{pmatrix}.$$

Therefore the reduced operator $\widehat{F}$ exists and is well defined. Applying the foregoing theory, we can linearize the lumped system without calculating $\widehat{F}$ explicitly. Indeed, by Proposition 7, we know that the reduced operator is C^1 and the following lumping relation holds between the Fréchet differentials:

$$D\widehat{F}_g(z) = MDF_f(h), \quad \forall\, g = Mf,\, z = Mh.$$

Thus $D\widehat{F}_g$ acts as

$$D\widehat{F}_g(z) = M\phi'(f(x))\,h(x) = \begin{pmatrix} \phi'(f(y_1))h(y_1) \\ \vdots \\ \phi'(f(y_n))h(y_n) \end{pmatrix},$$

where

$$g = \begin{pmatrix} f(y_1) \\ \vdots \\ f(y_n) \end{pmatrix}, \qquad z = \begin{pmatrix} h(y_1) \\ \vdots \\ h(y_n) \end{pmatrix}.$$

For a vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, $D\widehat{F}_v$ is the $n \times n$ diagonal matrix given by

$$D\widehat{F}_v := \begin{pmatrix} \phi'(v_1) & 0 & \ldots & 0 \\ 0 & \phi'(v_2) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \phi'(v_n) \end{pmatrix}.$$

This finite-dimensional operator represents the linearization of the reduced system associated to $\widehat{F}$.

Note that in this particular example one can verify the smoothness of $\widehat{F}$ by explicit calculation. Indeed, it is easy to see that

$$\widehat{F}\left[\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}\right] = \begin{pmatrix} \phi(v_1) \\ \vdots \\ \phi(v_n) \end{pmatrix},$$

and its regularity can be deduced from the regularity of ϕ. However, in general $\widehat{F}$ is defined by formula (5) only in an implicit way. Since we don't have an inverse map to obtain the original state variable from the lumped variable, there may be cases where $\widehat{F}$ is difficult to compute. In such situations, the results of this paper help deduce and exploit the smoothness of the reduced dynamics without computing it directly.

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References

[1] Y. Abramovich and C. Aliprantis. *An Invitation to Operator Theory.* Number v. 1 in Graduate studies in mathematics. American Mathematical Society, 2002.

[2] F. Atay and L. Roncoroni. Exact lumpability of linear evolution equations in Banach spaces. Preprint: http://www.mis.mpg.de/publications/preprints/2013/prepr2013-109.html.

[3] Y. Benyamini and J. Lindenstrauss. *Geometric Nonlinear Functional Analysis*. American Mathematical Society colloquium publications. American Mathematical Soc., 1998.

[4] L. Block, J. Keesling, and D. Ledis. Semi-conjugacies and inverse limit spaces. *Journal of Difference Equations and Applications*, 18(4):627–645, 2012-04-01T00:00:00.

[5] E. M. Bollt and J. D. Skufca. On comparing dynamical systems by defective conjugacy: A symbolic dynamics interpretation of commuter functions. *Physica D: Nonlinear Phenomena*, 239(10):579–590, 2010.

[6] P. Buchholz. Exact and ordinary lumpability in finite Markov chains. *J. Appl. Probab.*, 31(1):59–75, 1994.

[7] F. M. Ceragioli. The linearization method of stability theory in banach spaces. *Rendiconti del Seminario Matematico*, 54:167–177, 1996.

[8] P. Coxson. Lumpability and observability of linear systems. *Journal of Mathematical Analysis and Applications*, 99(2):435–446, 1984.

[9] L. Gurvits and J. Ledoux. Markov property for a function of a Markov chain: A linear algebra approach. *Linear Algebra and its Applications*, 404:85–117, 2005.

[10] Y. Iwasa, V. Andreasen, and S. Levin. Aggregation in model ecosystems. i. perfect aggregation. *Ecological Modelling*, 37(3):287–302, 1987.

[11] J. Ledoux. On weak lumpability of denumerable Markov chains. *Statist. Probab. Lett.*, 25(4):329–339.

[12] G. Li and H. Rabitz. A general analysis of approximate lumping in chemical kinetics. *Chemical Engineering Science*, 45(4):977–1002, 1990.

[13] J. Lindenstrauss and L. Tzafriri. On the complemented subspaces problem. *Israel Journal of Mathematics*, 9(2):263–269, 1971.

[14] N. K. Luckyanov, Y. M. Svirezhev, and O. V. Voronkova. Aggregation of variables in simulation models of water ecosystems. *Ecological Modelling*, 18(3):235–240, 1983.

[15] Z. Rózsa and J. Tóth. Exact linear lumping in abstract spaces. In *Proceedings of the 7th Colloquium on the Qualitative Theory of Differential Equations*, volume 7, pages No. 21, 20 pp. (electronic). Electron. J. Qual. Theory Differ. Equ., Szeged, 2004.

[16] A. Tomlin, G. Li, H. Rabitz, and J. Tóth. The effect of lumping and expanding on kinetic differential equations. *SIAM Journal on Applied Mathematics*, 57(6):1531–1556, 1997.

[17] J. Wei and J. Kuo. Lumping analysis in monomolecular reaction systems. Analysis of approximately lumpable system. *Industrial & Engineering Chemistry Fundamentals*, 8(1):124–133, 1969.

[18] J. Wei and J. Kuo. Lumping analysis in monomolecular reaction systems. Analysis of the exactly lumpable system. *Industrial & Engineering Chemistry Fundamentals*, 8(1):114–123, 1969.