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From Quantum Discord and Quantum Entanglement to Local Hidden Variable Models

by

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Abstract

We revisit the quantum discord, quantum entanglement and local hidden variable models in quantum mechanics, and present a kind of understanding of quantum states from the view of correlations given by the probability distributions of local measurements outcomes.

The correlations among the subsystems of a multipartite system play significant roles in many information processing tasks and physical processes. The types of correlations existed in a system depend on the state of the system. The evolution of the states results in the evolution of the correlations. It is of importance to learn and classify the states according to the correlations contained in a system. Physically, to get information from a system one needs to measure the system. Hence the kind of correlations in a state can be determined in terms of the probability distributions of the measurement outcomes. By revisiting the quantum discord, quantum entanglement and local hidden variable models in quantum mechanics, we attend to present a kind of understanding of quantum states from the view of correlations.

Let us begin with the classical correlations between two random variables. If X is a random variable which has value x with probability $p(x)$, then the information content $S(X)$ of X is defined to be the information you would gain if you learned the value of X, given by the Shannon entropy,

$$S(X) = -\sum_x p(x)\, \log_2 p(x). \tag{1}$$

S is a function of the probability distribution of the values of X. The information is maximum when our prior knowledge of X is minimum. If X can take N different values, the information content (or entropy) of X is maximized when the probability distribution p is

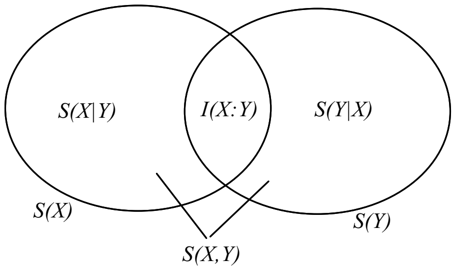

FIG. 1: The mutual information $I(X : Y)$ is given by overlap between $S(X)$ and $S(Y)$, $I(X : Y) = S(X) + S(Y) - S(X, Y)$. Whenever the value of X (resp. Y) is learned, the information left in Y (resp. X) is $S(Y|X)$ (resp. $S(X|Y)$).

flat, with every $p(x) = 1/N$. And the maximum information which could in principle be stored by a variable which can take on N different values is $\log_2(N)$. When X can take two values with equal probability: $S(X) = 1$. A two-valued or binary variable thus can contain one unit of information. This unit is called a bit. The two values of a bit are typically written as the binary digits 0 and 1.

If there are two correlated random variables X and Y, with probability $p(y|x)$ that $Y = y$ given that $X = x$. The conditional entropy $S(Y|X)$ is defined by

$$S(Y|X) = -\sum_x p(x) \sum_y p(y|x) \log p(y|x) = -\sum_x \sum_y p(x, y) \log p(y|x),$$

where $p(x, y) = p(x)p(y|x)$ is the probability that $X = x$ and $Y = y$. $S(Y|X)$ is a measure of how much information on average would remain in Y if we were to learn X. $S(Y|X) \leq S(Y)$ always and $S(Y|X) \neq S(X|Y)$ usually. Then we have the so called mutual information,

$$I(X : Y) = \sum_x \sum_y p(x, y) \log \frac{p(x, y)}{p(x)p(y)} = S(Y) - S(Y|X) = S(X) - S(X|Y).$$

It is a measure of how much X and Y contain information about each other. If X and Y are independent then $p(x, y) = p(x)\, p(y)$, and $I(X : Y) = 0$. Let $S(X, Y)$ be the information content of X and Y together (the information we would gain if, initially knowing neither, we learned the value of both X and Y). Then one has $S(X, Y) = S(X) + S(Y) - I(X : Y)$, see Fig. 1.

The important thing here is that whenever one measures the variable Y (resp. X), the information contained in Y (resp. X) will be completely learned. Namely, all the mutual information $I(X:Y)$ will be learned and the information left in $S(X,Y)$ is just $S(X|Y)$ (resp. $S(Y|X)$).

However, this is not always the case in quantum systems. Let H_A and H_B be n-dimensional vector spaces. Let $\rho \in H_A \otimes H_B$ denote the density matrix of a composite bipartite system A and B, and $\rho^{A(B)} = Tr_{B(A)}(\rho)$ the reduced density matrices. The quantum mutual information is defined by

$$\mathcal{I}(\rho) = S(\rho^A) + S(\rho^B) - S(\rho), \tag{2}$$

where $S(\rho) = -Tr(\rho \log_2 \rho)$ is the Von Neuman entropy. Generally, a minimal amount of information will be left after measuring one of the subsystems. This amount is described by the so called quantum discord, as introduced by Oliver and Zurek [1–3].

Let B_k be a set of one-dimensional project measurement performed on subsystem B, the conditional density operator ρ_k associated with the measurement result k is $\rho_k = (I \otimes B_k)\rho(I \otimes B_k)/p_k$, where $p_k = tr(I \otimes B_k)\rho(I \otimes B_k)$, I is the identity operator on the subsystem A. The quantum conditional entropy with respect to this measurement is given by $S(\rho|\{B_k\}) = \sum_k p_k S(\rho_k)$, and the associated quantum mutual information is given by $\mathcal{I}(\rho|\{B_k\}) = S(\rho^A) - S(\rho|\{B_k\})$. Classical correlation is defined as the superior of $\mathcal{I}(\rho|\{B_k\})$ over all possible von Neumann measurement B_k, $\mathcal{C}(\rho) = \sup_{\{B_k\}} I(\rho|\{B_k\})$. Quantum discord is then given by the difference of mutual information $\mathcal{I}(\rho)$ and the classical correlation $\mathcal{C}(\rho)$,

$$\mathcal{Q}(\rho) = \mathcal{I}(\rho) - \mathcal{C}(\rho). \tag{3}$$

Generally it is a challenging problem to compute the quantum correlation $\mathcal{Q}(\rho)$. Analytically formulae of $\mathcal{Q}(\rho)$ can be obtained only for some special quantum states like Bell-diagonal states, X-type states [4]. If, instead of the projective measurements, one considers general positive operator valued measurements (POVMs), the problem becomes even more complicated. It has been shown that the quantum discord is required for some information processing like assisted optimal state discrimination [5].

The quantum discord is zero for classical-classical correlated states of the form $\rho_{cc} = \sum_i p_i |e_i\rangle\langle e_i| \otimes |f_i\rangle\langle f_i|$, where $0 \leq p_i \leq 1$, $\sum_i p_i = 1$, $|e_i\rangle$ and $|f_i\rangle$ are orthogonal bases of subsystems A and B respectively. The classical-quantum correlated states

$\rho_{cq} = \sum_i p_i\,|e_i\rangle\langle e_i| \otimes \rho_i^B$ and quantum-classical correlated states $\rho_{qc} = \sum_i p_i\,\rho_i^A \otimes |f_i\rangle\langle f_i|$, have non-zero quantum correlations by either measuring the subsystems B or A, where ρ_i^A and ρ_i^B are density matrices associated to subsystems A and B respectively. The quantum-quantum correlated states ρ_{qq},

$$\rho_{qq} = \sum_i p_i\,\rho_i^A \otimes \rho_i^B, \tag{4}$$

is of particular sense. As long as ρ_{qq} can not been written in the form of ρ_{cc}, it has non-zero quantum correlations. However, the probability distributions of the measurement outcomes from measuring the subsystem A are independent of the probability distributions of the measurement outcomes from measuring the subsystem B. From the view of quantum entanglement, the state ρ_{qq} is separable.

Quantum entanglement [6] is of special importance in quantum-information processing and is responsible for many quantum tasks such as quantum teleportation [7, 8], dense coding [9], swapping [10, 11], error correction [12, 13] and remote state preparation [14, 15]. Generally, a state $\rho \in H_A \otimes H_B$ has a form

$$\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|, \tag{5}$$

where $|\psi_i\rangle$ are vectors (pure states) in $H_A \otimes H_B$, $p_i \geq 0$ and $\sum_i p_i = 1$. Distinguishing quantum entangled states (5) from the separable ones (4) is a basic and longer standing problem in the theory of quantum entanglement. One still has no general criterion to determine a mixed state is separable or not. A strong (necessary) criterion, named PPT (partial positive transposition), to recognize mixed entangled quantum state was proposed by Peres in 1996 in [16], which was shown to be also sufficient for 2×2 and 2×3 bipartite systems [17]. Another powerful operational criterion for separability is the realignment criterion [19, 20]. It demonstrates a remarkable ability in detecting the entanglement of many bound entangled states and even genuinely tripartite entanglement. Considerable efforts have been made in proposing stronger variants and multipartite generalizations for this criterion [21]. In [22–24], a separability criteria based on the local uncertainty relations (LUR) was obtained. This criterion is strictly stronger than the realignment criterion and is optimized in [25]. The covariance matrix of a quantum state is also used to study separability in [26]. It has been pointed out in [27] that the LUR criterion can be derived from the covariance matrix criterion. In [28] the author has given a criterion based on the correlation matrix of a state. The correlation matrix criterion is then shown to be independent of PPT and realignment criterion in [29]. In [30] a generalized form of the correlation matrix criterion for bipartite quantum systems [28, 29] and for multipartite quantum systems [31] has been presented, which includes the criterion based on the correlation matrix as a special case and is more powerful than the later for detecting entanglement.

To characterize the degree of entanglement, measures of entanglement are needed. The entanglement of formation (EoF) [32] is a well-defined important measure of entanglement for bipartite systems. The entanglement of formation of a pure state $|\psi\rangle \in H_A \otimes H_B$ is given by the entropy of the reduced density matrix $\rho_A = Tr_B(|\psi\rangle\langle\psi|)$, $E(|\psi\rangle) = S(\rho_A)$. For a bipartite mixed state ρ, the entanglement of formation is given by the minimum average marginal entropy of the ensemble decompositions of ρ,

$$E(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle), \tag{6}$$

for all possible ensemble realizations $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, $p_i \geq 0$ and $\sum_i p_i = 1$.

Another significant measure of quantum entanglement is the concurrence. The concurrence of a pure bipartite state $|\psi\rangle$ is given by $C(|\psi\rangle) = \sqrt{2[1 - Tr(\rho_A^2)]}$. It is extended to mixed states by the convex roof construction

$$C(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle), \tag{7}$$

for all possible ensemble realizations ρ.

Considerable effort has been made to estimate the entanglement of formation and concurrence for bipartite quantum mixed states, and their lower and upper bounds via analytical and numerical approaches. For the two-qubit case, EoF is a monotonically increasing function of the concurrence, and an analytical formula of concurrence has been derived [33]. For the general high-dimensional case, due to the extremizations involved in the computation, only a few analytic formulas have been obtained for isotropic states [34] and Werner states [35] for EoF, and for some special symmetric states [36–38] for concurrence.

Instead of analytic formulas, some progress has been made toward the lower and upper bounds of EoF and concurrence for any given mixed quantum state. In [39–43], explicit analytical lower and upper bounds of concurrence have been presented. In [39], a simple analytical lower bound of EoF has been derived. Recently new results related to the bounds of EoF have been further derived in [44, 45]. In [46] new lower and upper bounds of EoF based on the concurrence have been derived, which improve the bounds in [44, 45].

For any quantum states with non-zero entanglement, the probability distributions of the measurement outcomes from measuring the subsystem A will depend the probability distributions of the measurement outcomes from measuring the subsystem B. Nevertheless, it is still possible that the correlations between the measurement outcomes from measuring the subsystem A and from measuring the subsystem B can be described by classical probability distributions. A quantum state is said to admit a local hidden variable (LHV) model if all the measurement outcomes can be modeled as a classical random distribution over a probability space. If Alice performs a measurement P on the subsystem A with an outcome p_i and, at space-like separation, Bob performs a measurement Q on the subsystem B with an outcome q_j, then an LHV model supposes that the joint probability of getting p_i and q_j satisfies

$$\Pr(p_i, q_j|P, Q, \rho) = \int_{\Omega} \Pr(p_i|P, \lambda) \Pr(q_j|Q, \lambda)\, d\omega^{\rho}(\lambda),$$

where $d\omega^{\rho}(\lambda)$ is some distribution over a space Ω of hidden variable λ. A quantum state is called *local* if it admits an LHV model, and *nonlocal* otherwise.

Bell showed that all quantum states admitting LHV models satisfy the so-called Bell inequalities [47]: a state admits no LHV models if it violates some Bell inequalities. It is known that every pure entangled bipartite or multipartite state violates a generalized Bell inequality [48, 49]. Namely, for pure states the entanglement and the non-locality coincide. However, for mixed states the situation is more complicated. There are no general methods to judge whether a mixed state admits an LHV model or not. To find all Bell inequalities is computationally hard [50, 51]. Even for the simple two-qubit Werner states, the precise threshold value of nonlocality is still unknown.

The two-qubit Werner state is given by

$$\rho_p^W = p \left|\psi^-\right\rangle \left\langle\psi^-\right| + (1-p)I/4, \tag{8}$$

where in computational basis, $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ and $0 \leq p \leq 1$. ρ_p^W is separable if $p \leq 1/3$ [52]. It admits an LHV model for all kind of measurements for $p \leq 5/12$ [53], and admits an LHV model for projective measurements for $p \leq 0.6595$ [54].

Let P_i and Q_i, $i = 1, 2, ..., m$, be dichotomic observables with respect to the two qubits,

$P_i = \mathbf{p}_i \cdot \sigma$ and $Q_i = \mathbf{q}_i \cdot \sigma$, with $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ the Pauli matrices. $\mathbf{p}_i = (p_i^{(1)}, p_i^{(2)}, p_i^{(3)})$, $\mathbf{q}_i = (q_i^{(1)}, q_i^{(2)}, q_i^{(3)})$ are 3-dimensional real unit vectors. For any Bell operator,

$$B(M) = \sum_{i,j=1}^{m} M_{ij}\, P_i \otimes Q_j,$$

where $M \in \mathcal{M}_m(\mathbb{R})$, $\mathcal{M}_m(\mathbb{R})$ is the set of $m \times m$ real matrices, the mean value is given by

$$Tr(B(M)\rho_p^W) = p \sum_{i,j=1}^{m} M_{ij}\, \mathbf{p_i} \cdot \mathbf{q_j}.$$

Therefore, the maximal violation of the corresponding Bell inequality is given by $p\, K(3)$, $K(3)$ is the Grothendieck constant of order 3. Hence ρ_p^W admits LHV models for projective measurements if and only if $p \leq 1/K(3)$ [54, 55]. Hence the nonlocality problem of the two-qubit Werner states is reduced to estimate the value of the Grothendieck constant $K(3)$.

Let S^{d-1} be the unit sphere in $\mathbb{R}^d$, $m, d \in \mathbb{N}$. Given $M \in \mathcal{M}_m(\mathbb{R})$, we define

$$C(M) = \sup \left| \sum_{i,j=1}^{m} M_{ij}\, p_i q_j \right|,$$

where the supremum is taken over all possible assignment $p_i, q_j \in \{1, -1\}$, $1 \leq i, j \leq m$. Replacing p_i, q_j by d-dimensional unit vectors, we define

$$Q(M) = \sup_{\mathbf{p_i}, \mathbf{q_j} \in S^{d-1}} \left| \sum_{i,j=1}^{m} M_{ij}\, \mathbf{p_i} \cdot \mathbf{q_j} \right|,$$

where the supremum is taken over all d-dimensional unit vectors $\mathbf{p_i}$ and $\mathbf{q_j}$ and $\mathbf{p_i} \cdot \mathbf{q_j}$ denotes their scalar product. The Grothendieck constant of order d is defined by

$$K(d) = \sup_{m \geq 1} \sup_{\substack{M \in \mathcal{M}_m(\mathbb{R}) \\ M \neq 0}} \frac{Q(M)}{C(M)}. \tag{9}$$

It is a great challenge to evaluate the Grothendieck constant $K(d)$ for general d. Till now the only exactly known result of $K(d)$ is for $d = 2$, $K(2) = \sqrt{2}$ [56]. For $d \geq 3$, there are some lower bounds of $K(d)$. For instance, Krivine [56] showed that $K(3) \leq 1.5163$. The Clauser-Horne-Shimony-Holt (CHSH) [57] inequality implies that $K(3) \geq \sqrt{2}$. Vértesi [58] constructed Bell inequalities involving 465 settings on each qubit ($\{P_i, Q_i\}_{i=1}^m$, $m \geq 465$) to show that $K(3) \geq 1.417241$, i.e., ρ_p^W admits no LHV models for $p > 0.705596$. Recently, by generalizing the Bell operator in [58] to a continuous model with infinitely many measurement settings, we present an analytical formula in estimating the lower bounds of the Grothendieck constants. This formula is valid for Grothendieck constants of arbitrary order and improves many previously obtained bounds. From our lower bound of $K(3)$, we derived a bound of the threshold value for the nonlocality of the two-qubit Werner states, $K(3) \geq 1.41758$ [59]. Therefore ρ_p^W admit no LHV models for $p > 0.705428$. This provides the best known bound for the nonlocality of two-qubit Werner states.

From information correlation, entanglement and non-locality, the quantum states can be viewed in this way: see Fig. 2, the states between A and C have zero entanglement: the probabilities of the measurement outcomes from measuring the subsystem A are independent of the probabilities of the measurement outcomes from measuring the subsystem B. However, those separable states may be further classified as classically correlated states and quantum correlated ones, depending on the possibility to learn all the mutual information by measuring one of the subsystems. The states between C and E are all entangled. Nevertheless, they can be also divided into two subsets, depending on if all the measurement outcomes from measuring the subsystems can be modeled as a classical random distribution over a probability space. One subset admits local hidden variable models. They do not violate any Bell inequalities. Another subset does not admit any local hidden variable models and violates at least one Bell inequality. For pure states, all the points B, C and D emerge together.

Here the point B is described by the correlation measure, say, the quantum discord (3). The point C is given by the entanglement measures, like the EoF (6) and concurrence (7). The quantum discord (3), EoF (6) and concurrence (7) are well defined, although formidably difficult to compute analytically. For the point D, we do not have nicely defined quantity such that it is zero for states admitting LHV models and is greater than zero for states without LHV models. In [60] the maximal violation of CHSH inequalities has been used as a measure of non-locality.

The non-locality has no simple relations with bound entanglement and distillability. It has been shown that some bound entangled (non-distillable) states may also violate Bell inequality [61]. Moreover, some entangled states admitting LHV models can be super-activated: one can obtain violations of Bell inequalities by tensorizing a local state with itself [62]. In [63] the existence of genuine hidden nonlocality has been studied. It has been

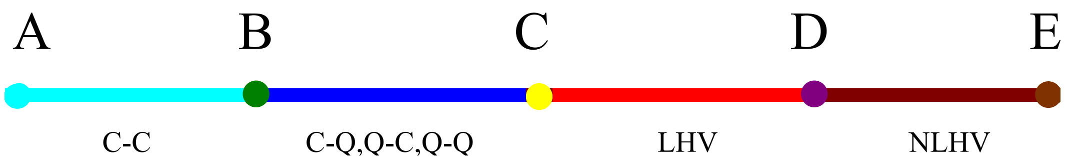

FIG. 2: States between A and B are separable and classical-classical correlated, states between B and C are separable but classical-quantum, quantum-classical, quantum-quantum correlated, states between C and D are entangled but admit LHV models, states between D and E are entangled and admit no LHV models.

shown that a class of two-qubit entangled states admitting LHV models might violate a Bell inequality after local filtering.

We have analyzed the quantum states according to quantum discord, quantum entanglement and local hidden variable models in quantum mechanics, and discussed the understanding of quantum states from the view of correlations given by the probability distributions of local measurements outcomes. In stead of the usual quantum mechanics with Hermitian Hamiltonian systems, it would be also interesting if one considers the local measurements and the corresponding state classification in pseudo Hermitian systems [64].

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