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DIRAC-HARMONIC MAPS BETWEEN RIEMANN SURFACES

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Abstract. In this paper, we consider the existence and structure of Dirac-harmonic maps between closed Riemann surfaces. Utilizing the Riemann-Roch formula, we compute the dimension of harmonic spinors along a map, based on which we prove an existence theorem for Dirac-harmonic maps between closed Riemann surfaces. We also obtain a structure theorem for Dirac-harmonic maps between two surfaces if their genera and the degree of the map satisfy a certain relation.

1. Introduction

Dirac-harmonic maps have been introduced in[4;5]. They were motivated by the supersymmetric σ-model of quantum field theory. They replace the anticommuting spinor field of that model, which takes values in a Grassmannian algebra and makes the model supersymmetric, by a commuting field. Nevertheless, they preserve important symmetries, in particular conformal invariance. Mathematically, they can be seen as an extension of the harmonic map problem as they couple a harmonic map type field with a spinor field. Since all the fields are ordinary, commuting variables, we may apply the methods of the geometric calculus of variations. A technical difficulty, however, arises from the fact that the underlying action functional is not bounded from below, in contrast to standard harmonic maps where it is nonnegative.

We now present the mathematical definitions. (M, g) is a Riemann surface with a conformal metric g and a fixed spin structure, and ΣM the spinor bundle over M, on which we chose a Hermitian metric $\langle\cdot,\cdot\rangle$. The Levi-Civita connection ∇ on ΣM is compatible with $\langle\cdot,\cdot\rangle$. Let (N, h) be a Riemannian manifold (subsequently, it will likewise be of dimension 2, that is, a Riemann surface with a conformal metric), Φ a map from M to N, and $\Phi^{-1}TN$ the pull-back bundle of TN by Φ. We also denote the metric induced from the metrics on ΣM and $\Phi^{-1}TN$ on the twisted bundle $\Sigma M\otimes\Phi^{-1}TN$ by $\langle\cdot,\cdot\rangle$. Likewise, we also denote the connection on $\Sigma M\otimes\Phi^{-1}TN$ induced from those on ΣM and $\Phi^{-1}TN$ by ∇.

A cross-section Ψ of $\Sigma M\otimes\Phi^{-1}TN$ can be locally written as $\Psi = \psi^\alpha\otimes\theta_\alpha$, where $\{\psi^\alpha\}$ are local cross-sections of ΣM, $\{\theta_\alpha\}$ are local cross-sections of $\Phi^{-1}TN$. We always use the standard summation convention.

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The *Dirac operator along the map* Φ is

$$
\begin{aligned}
\not{D}\Psi :=& e_i \cdot \nabla_{e_i}\Psi \\
=& \not{\partial}\psi^\alpha \otimes \theta_\alpha + \psi^\alpha \otimes \nabla_{e_i}\theta_\alpha,
\end{aligned}
$$

where $\{e_i\}$ is a local orthonormal frame on M, $\not{\partial} := e_i \cdot \nabla_{e_i}$ is the Dirac operator on M and $X\cdot$ is the Clifford multiplication by the vector field X on M.

The action functional of the theory is

$$
L(\Phi, \Psi) = \frac{1}{2}\int_M \left(\|\mathrm{d}\Phi\|^2 + \langle \Psi, \not{D}\Psi\rangle\right),
$$

and as mentioned, it couples the harmonic map type field Φ with the spinor field Ψ, because the Dirac operator $\not{D}$ depends on Φ. We see this coupling also from the Euler-Lagrange equations for $L(\Phi, \Psi)$ that critical points (Φ, Ψ) have to satisfy (c.f.[4]):

$$
\begin{cases}
\tau(\Phi) =& \dfrac{1}{2}\left\langle \psi^\alpha, e_i \cdot \psi^\beta\right\rangle R^N(\theta_\alpha, \theta_\beta)\Phi_*(e_i), \\
\not{D}\Psi =& 0,
\end{cases}
\tag{1.1}
$$

where $R^N(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, $\forall X, Y \in \Gamma(TN)$ stands for the curvature operator of N, and $\tau(\Phi)$ is the tension field of Φ. Therefore, solutions of (1.1) are called *Dirac-harmonic maps from M to N*.

Not every solution of (1.1) needs to be coupled, however, as either component could be trivial. When Φ is constant, Ψ satisfies the ordinary Dirac equation, and when Ψ vanishes, Φ is a harmonic map. We therefore say that a Dirac-harmonic map is uncoupled if the underlying map is harmonic. From our perspective, such solutions are trivial. A question that we shall address in this paper is when such Dirac-harmonic maps are necessarily uncoupled.

Ammann-Ginoux[1] analyzed the space of Dirac-harmonic maps by using tools from index theory, and the existence of uncoupled solutions was proved under the assumption that the α-genus $\alpha(M, \chi, [\Phi]) := \alpha(M, \chi, \Phi^{-1}TN)$ is nontrivial[1] Theorem 1.2.

On the other hand, for a real vector bundle of rank k over M, they also proved the following formula[1] Proposition 10.1

$$
\alpha(M, \chi, E) = (k + 1)\alpha(M, \chi) + \alpha(M, \chi + w_1(E)) + w_2(E)[M].
$$

If M, N are two Riemann surfaces and $\phi : M \longrightarrow N$, then

$$
\alpha(M, \chi, [\phi]) = \alpha(M, \chi) + \alpha(M, \chi) = 0
$$

since $k = 2, w_1(\phi^{-1}TN) = 0, w_2(\phi^{-1}TN) = 0$. Hence one cannot apply the theorem stated by Ammann-Ginoux[1] to get the general existence of – coupled or uncoupled – Dirac-harmonic maps between Riemann surfaces.

In this paper, we will consider Dirac-harmonic maps between Riemann surfaces M and N. For that purpose, we shall now analyze the relevant geometry of M. The spinor bundle ΣM can be identified with

$$
\Sigma M = K_M^{1/2} \oplus \Lambda^{0,1}K_M^{1/2},
$$

where $K_M^{1/2}$ is a square root of the canonical line bundle K_M of M. Choose a local conformal parameter $z = x + \sqrt{-1}y$ of M and denote the metric of M locally by $\lambda(z)\,|\mathrm{d}z|^2$. Then every spinor ψ on M can be written as

$$\psi = fs + g\mathrm{d}\bar{z} \otimes s,$$

where s is a local holomorphic section of $K_M^{1/2}$ and f, g are local complex functions. For convenience, we simplify this notation by

$$\psi = f + g\mathrm{d}\bar{z}.$$

Let N be a Riemann surface and $\Phi : M \longrightarrow N$ be a smooth map. Choose a local conformal parameter $\phi = u + \sqrt{-1}v$ of N and denote the metric of N locally by $\rho(\phi)\,|\mathrm{d}\phi|^2$. Denote the local representation of Φ by ϕ. Then the Dirac bundle $\Sigma M \otimes \Phi^{-1}TN$ can be split as follows:

$$\Sigma M \otimes \Phi^{-1}TN = \left(K_M^{1/2} \otimes \Phi^{-1}K_N^{-1}\right) \oplus \left(\Lambda^{0,1}K_M^{1/2} \otimes \Phi^{-1}K_N^{-1}\right) \oplus \left(K_M^{1/2} \otimes \Phi^{-1}\bar{K}_N^{-1}\right) \oplus \left(\Lambda^{0,1}K_M^{1/2} \otimes \Phi^{-1}\bar{K}_N^{-1}\right)$$

and we can rewrite the spinor Ψ as follows:

$$\Psi = f\partial_\phi + \mathrm{d}\bar{z} \otimes g\partial_\phi + \bar{p}\partial_{\bar{\phi}} + \mathrm{d}\bar{z} \otimes \bar{q}\partial_{\bar{\phi}}.$$

Set

$$\Theta = (f\bar{g} - \bar{p}q)\,\rho\mathrm{d}z,$$

Suppose (Φ, Ψ) is Dirac-harmonic, then Φ is harmonic if $\Theta = 0$ (see Lemma 2.1).

Our first main result is the following:

Theorem 1.1 (Existence of Dirac-harmonic maps)**.** *Let M, N be two closed Riemann surfaces and ϕ a continuous map from M to N. Then we can find metrics on M and N such that there exist a smooth map $\Phi : M \longrightarrow N$ which is homotopic to ϕ and a complex vector spaces V with complex dimension $4\left|\deg(\phi)(g_N - 1)\right|$ such that every $(\Phi, \Psi) \in V$ is Dirac-harmonic except in the case when $g_M = 1, g_N = 0, \left|\deg(\phi)\right| = 1$.*

Remark 1.1. Eells-Wood[9] and Lemaire[15] proved that there is no harmonic map from the 2-torus to the 2-sphere with degree ± 1 whatever the metrics. Moreover, L.Yang[19] proved that there is no coupled Dirac-harmonic map from the 2-torus to the 2-sphere with nontrivial degree. Hence there is no Dirac-harmonic map from the 2-torus to the 2-sphere such that the degree of the map part is ± 1 (c.f.[3]).

Remark 1.2. If $\deg(\phi) \neq 0$, then for any complex structure on N there is a complex structure on M relative to which the homotopy class of ϕ contains a holomorphic representative if and only if $\left|\deg(\phi)\right| > [\pi_1(N) : \phi_*(\pi_1(M))]$ or $\phi_* : \pi_1(M) \to \pi_1(N)$ is injective (a consequence of the work of Edmonds[7], c.f. Eells and Lemaire[8]).

We recall that by a topological theorem of H.Kneser (c.f.[9;13]), if $\deg(\Phi) \neq 0$ and $g_N \geq 2$, then $g_M - 1 \geq \left|\deg(\Phi)\right|(g_N - 1)$. In particular, $g_M \geq g_N$.

Our next main result yields a formula for the dimension of the harmonic spinor spaces along a fixed map under the following condition:

$$g_M - 1 < 2\left|\deg(\Phi)(g_N - 1)\right|. \tag{1.2}$$

Theorem 1.2. *Let M, N be two closed Riemann surfaces and Φ a smooth map form M to N. Suppose* (1.2) *holds, then the space of harmonic spinors along the map Φ is a $4\left|\deg(\Phi)(g_N - 1)\right|$ dimensional complex linear vector space.*

The first non-trivial Dirac-harmonic map was given in[5] for $M = N = S^2$, based on an explicit construction involving a harmonic map and so-called twistor-spinors on the domain manifold. More precisely, given a harmonic map $\Phi : S^2 \longrightarrow S^2$ and a twistor spinor $\eta \in \Sigma S^2$, construct a spinor Ψ along the map Φ as follows:

$$\Psi = e_1 \cdot \eta \otimes \Phi_*(e_1) + e_2 \cdot \eta \otimes \Phi_*(e_2),$$

where $\{e_1, e_2\}$ is a local orthonormal frame of S^2. Then (Φ, Ψ) is a Dirac-harmonic map. In[19], L.Yang proved that every Dirac-harmonic map between 2-spheres can be constructed in this way with η possibly having isolated singularities. He also proved a structure theorem for Dirac-harmonic maps between spheres.

Here, we shall derive a structure theorem when the target is a sphere and the domain genus satisfies an inequality.

Theorem 1.3. *Let N be a closed surface of genus 0, and let M be a closed surface satisfying*

$$1 \leq g_M < |\deg(\Phi)| + 1, \tag{1.3}$$

and let (Φ, Ψ) be a Dirac-harmonic map from M to N. Assume that (1.3) *holds. Then either*

(1) Φ is holomorphic and

$$\Psi = \lambda^{-1}\left(\partial_{\bar z} \cdot \eta \otimes \partial\Phi(\partial_z) + \partial_z \cdot \eta \otimes \bar\partial\bar\Phi(\partial_{\bar z})\right),$$

where η is a twistor spinor on M possibly with isolated singularities, or

(2) Φ is anti-holomorphic and

$$\Psi = \lambda^{-1}\left(\partial_{\bar z} \cdot \eta \otimes \partial\bar\Phi(\partial_z) + \partial_z \cdot \eta \otimes \bar\partial\Phi(\partial_{\bar z})\right),$$

where η is a twistor spinor on M possibly with isolated singularities.

2. The complex form of the Dirac-harmonic map equation

Let M be a Riemann surface and N a Riemannian manifold. Let Φ be a map from M to N and Ψ a spinor along the map Φ, i.e., a cross-section of the Dirac bundle $\Sigma M \otimes \Phi^{-1}TN$. Choose a local conformal parameter $z = x + \sqrt{-1}y$ of M and denote the metric of M locally by $\lambda(z)\,|\mathrm{d}z|^2$. Introduce

$$\partial_z = \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}\right), \quad \partial_{\bar z} = \frac{\partial}{\partial \bar z} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right),$$

and

$$\mathrm{d}z = \mathrm{d}x + \sqrt{-1}\mathrm{d}y, \quad \mathrm{d}\bar z = \mathrm{d}x - \sqrt{-1}\mathrm{d}y.$$

Now the Laplacian operator Δ is

$$\Delta = \frac{4}{\lambda}\frac{\partial^2}{\partial z \partial \bar z}.$$

Decompose the spinor bundle as $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$ with

$$\Sigma^+ M := \{\psi \in \Sigma M : \partial_{\bar{z}} \cdot \psi = 0\}, \quad \Sigma^- M := \{\psi \in \Sigma M : \partial_z \cdot \psi = 0\}.$$

Now we identify the spinor bundle ΣM with $L \oplus \Lambda^{0,1} L$, where L is a square root of the canonical bundle K_M of M, such that $\Sigma^+ M = L$ and $\Sigma^- M = \Lambda^{0,1} L$. Then every half spinor in $\Sigma^+ M$ can be written as follows (c.f.[11;12]):

$$\psi^+ = fs,$$

while every half spinor in $\Sigma^- M$ can be written as

$$\psi^- = g \mathrm{d}\bar{z} \otimes s,$$

where s is a local holomorphic section of L and f, g are local complex functions. For convenience, we omit the symbol s and simplify write the spinor ψ^+ as

$$\psi^+ = f$$

and ψ^- as

$$\psi^- = g \mathrm{d}\bar{z}.$$

Here the Clifford multiplication is defined by

$$X \cdot \psi := \sqrt{2} \left((X^{1,0})^\flat \wedge \psi - \iota_{X^{0,1}} \psi \right),$$

i.e.,

$$\partial_z \cdot f = \frac{\lambda}{\sqrt{2}} f \mathrm{d}\bar{z}, \quad \partial_{\bar{z}} \cdot f = 0, \quad \partial_z \cdot (g \mathrm{d}\bar{z}) = 0, \quad \partial_{\bar{z}} \cdot (g \mathrm{d}\bar{z}) = -\sqrt{2} g.$$

Then the Dirac operator is

$$\partial\!\!\!/ = \frac{2}{\lambda} \left(\partial_z \cdot \nabla_{\partial_{\bar{z}}} + \partial_{\bar{z}} \cdot \nabla_{\partial_z} \right),$$

where ∇ is the covariant derivative of the holomorphic line bundle L. In particular,

$$\partial\!\!\!/|_{\Sigma^+ M} = \sqrt{2} \bar{\partial}, \quad \partial\!\!\!/|_{\Sigma^- M} = \sqrt{2} \bar{\partial}^*,$$

i.e.,

$$\partial\!\!\!/ f = \sqrt{2} \bar{\partial} f, \quad \partial\!\!\!/ (g \mathrm{d}\bar{z}) = -\frac{2\sqrt{2}}{\lambda} g_z.$$

Next we consider the Dirac bundle $\Sigma M \otimes \Phi^{-1} TN$ and split this bundle as

$$\Sigma M \otimes \Phi^{-1} TN = \left(\Sigma^+ M \otimes \Phi^{-1} TN \right) \bigoplus \left(\Sigma^- M \otimes \Phi^{-1} TN \right).$$

Thus every spinor Ψ along the map Φ has the form

$$\Psi = f^\alpha \otimes \theta_\alpha + g^\alpha \mathrm{d}\bar{z} \otimes \theta_\alpha,$$

where f^α, g^α are local sections of L respectively and θ_α are local sections of $\Phi^{-1} TN$. Recall the Euler-Lagrange equations for Dirac-harmonic maps

$$\begin{cases} \tau(\Phi) = \mathcal{R}(\Phi, \Psi) := \frac{1}{2} \left(\Psi^\alpha, e_i \cdot \Psi^\beta \right) R^N(\theta_\alpha, \theta_\beta) \Phi_*(e_i), \\ D\!\!\!\!/\, \Psi = 0. \end{cases}$$

Definition 2.1. We say that a Dirac-harmonic map is uncoupled if the map part is harmonic and is coupled otherwise.

First we write the curvature term $\mathcal{R}(\Phi, \Psi)$ as follows:

$$\begin{aligned}\mathcal{R}(\Phi, \Psi) =& \frac{1}{\lambda}\operatorname{Re}\left\{\left\langle \Psi^\alpha, \partial_z \cdot \Psi^\beta \right\rangle R^N(\theta_\alpha, \theta_\beta)\Phi_*(\partial_z) + \left\langle \Psi^\alpha, \partial_{\bar z} \cdot \Psi^\beta \right\rangle R^N(\theta_\alpha, \theta_\beta)\Phi_*(\partial_{\bar z})\right\} \\ =& \frac{\sqrt{2}}{\lambda}\operatorname{Re}\left\{g^\alpha \overline{f^\beta} R^N(\theta_\alpha, \theta_\beta)\Phi_*(\partial_z) - f^\alpha \overline{g^\beta} R^N(\theta_\alpha, \theta_\beta)\Phi_*(\partial_{\bar z})\right\} \\ =& -\frac{2\sqrt{2}}{\lambda}\operatorname{Re}\left\{R^N(f, \bar g)\Phi_*(\partial_{\bar z})\right\},\end{aligned}$$

where we simply denote

$$\Psi = f + d\bar z \otimes g.$$

Introduce

$$\Theta := f^\alpha \bar g^\beta dz \otimes \theta_\alpha \wedge \theta_\beta = dz \otimes f \wedge \bar g.$$

Then

$$\mathcal{R}(\Phi, \Psi) = -\frac{2\sqrt{2}}{\lambda}\operatorname{Re}\left\{R^N(\Theta(\partial z))\Phi_*(\partial_{\bar z})\right\}.$$

Now the Euler-Lagrange equations for Dirac-harmonic maps are of the following form.

$$\begin{cases}\Phi^\alpha_{z\bar z} + \Gamma^\alpha_{\beta\gamma}(\Phi)\Phi^\beta_z\Phi^\gamma_{\bar z} + \frac{\sqrt{2}}{4}R^\alpha_{\beta\gamma\delta}(\Phi)\left(\Phi^\beta_{\bar z} f^\gamma \bar g^\delta - \Phi^\beta_z g^\gamma \bar f^\delta\right) = 0, \\ f^\alpha_{\bar z} + \Gamma^\alpha_{\beta\gamma}(\Phi)\Phi^\beta_{\bar z} f^\gamma = 0, \\ g^\alpha_z + \Gamma^\alpha_{\beta\gamma}(\Phi)\Phi^\beta_z g^\gamma = 0.\end{cases}$$

The following Lemma is obvious.

Lemma 2.1. *Any Dirac-harmonic map is uncoupled if the associated Θ is trivial.*

Moreover, we have

Lemma 2.2. *If Ψ is harmonic, i.e., $\not{D}\Psi = 0$, then Θ is harmonic.*

Proof. Choose θ_α such that $\nabla\theta_\alpha = 0$ at the considered point. Then at the considered point $\Gamma^\alpha_{\beta\gamma} = 0$. Moreover, since Ψ is harmonic, we have

$$f^\alpha_{\bar z} = 0 = g^\alpha_z.$$

Therefore

$$D\Theta = dz \wedge \nabla_{\partial_z}\Theta + d\bar z \wedge \nabla_{\partial_{\bar z}}\Theta = (f^\alpha \bar g^\beta)_{\bar z}(d\bar z \wedge dz) \otimes (\theta_\alpha \wedge \theta_\beta) = 0,$$

and

$$D^*\Theta = -\frac{2}{\lambda}\left(\iota_{\partial_{\bar z}}\nabla_{\partial_z}\Theta + \iota_{\partial_z}\nabla_{\partial_{\bar z}}\Theta\right) = -\frac{2}{\lambda}(f^\alpha \bar g^\beta)_{\bar z} \otimes (\theta_\alpha \wedge \theta_\beta) = 0.$$

□

¿From now on, we assume that N is also a Riemann surface and choose a local conformal parameter $\phi = u + \sqrt{-1}v$ of N and the metric of N is given by $\rho(\phi)\,|d\phi|^2$. Decompose $d\Phi$ as follows:

$$d\Phi = \partial\Phi + \bar\partial\Phi + \partial\bar\Phi + \bar\partial\bar\Phi,$$

where

$$\partial\Phi = \phi_z \mathrm{d}z \otimes \partial_\phi, \quad \bar{\partial}\Phi = \phi_{\bar{z}} \mathrm{d}\bar{z} \otimes \partial_\phi,$$

and

$$\partial\bar{\Phi} = \bar{\phi}_z \mathrm{d}z \otimes \partial_{\bar{\phi}}, \quad \bar{\partial}\bar{\Phi} = \bar{\phi}_{\bar{z}} \mathrm{d}\bar{z} \otimes \partial_{\bar{\phi}}.$$

It is clear that $\overline{\partial\Phi} = \bar{\partial}\bar{\Phi}$, $\overline{\bar{\partial}\Phi} = \partial\bar{\Phi}$. Moreover

$$\|\mathrm{d}\Phi\|^2 = 2\|\partial\Phi\|^2 + 2\left\|\bar{\partial}\Phi\right\|^2, \quad J(\Phi) = \|\partial\Phi\|^2 - \left\|\bar{\partial}\Phi\right\|^2.$$

Here $J(\Phi)$ is the Jacobian of Φ. If e is a local unit tangent vector field of M, then

$$\partial\Phi = \frac{1}{4}\left(\mathrm{Id} - \sqrt{-1}J^N\right) \circ \mathrm{d}\Phi \circ \left(\mathrm{Id} - \sqrt{-1}J^M\right), \quad \bar{\partial}\Phi = \frac{1}{4}\left(\mathrm{Id} - \sqrt{-1}J^N\right) \circ \mathrm{d}\Phi \circ \left(\mathrm{Id} + \sqrt{-1}J^M\right),$$

and

$$\partial\bar{\Phi} = \frac{1}{4}\left(\mathrm{Id} + \sqrt{-1}J^N\right) \circ \mathrm{d}\Phi \circ \left(\mathrm{Id} - \sqrt{-1}J^M\right), \quad \bar{\partial}\bar{\Phi} = \frac{1}{4}\left(\mathrm{Id} + \sqrt{-1}J^N\right) \circ \mathrm{d}\Phi \circ \left(\mathrm{Id} + \sqrt{-1}J^M\right).$$

We get that following formulae

$$\|\partial\Phi\|^2 = \frac{1}{4}\|\mathrm{d}\Phi\|^2 + \frac{1}{2}J(\Phi), \quad \left\|\bar{\partial}\Phi\right\|^2 = \frac{1}{4}\|\mathrm{d}\Phi\|^2 - \frac{1}{2}J(\Phi), \quad J(\Phi) = \left\langle J^N(\mathrm{d}\Phi(e)), \mathrm{d}\Phi(J^M(e))\right\rangle.$$

In this special case, split the Dirac bundle $\Sigma M \otimes \Phi^{-1}TN$ as follows:

$$\Sigma M \otimes \Phi^{-1}TN = \left(L \otimes \Phi^{-1}K_N^{-1}\right) \oplus \left(\Lambda^{0,1}L \otimes \Phi^{-1}K_N^{-1}\right) \oplus \left(L \otimes \Phi^{-1}\bar{K}_N^{-1}\right) \oplus \left(\Lambda^{0,1}L \otimes \Phi^{-1}\bar{K}_N^{-1}\right)$$

and rewrite the spinor Ψ as follows:

$$\Psi = f\partial_\phi + \mathrm{d}\bar{z} \otimes g\partial_\phi + \bar{p}\partial_{\bar{\phi}} + \mathrm{d}\bar{z} \otimes \bar{q}\partial_{\bar{\phi}}.$$

Lemma 2.3. *Ψ is harmonic if and only if $f\partial_\phi, q\partial_\phi$ are holomorphic and $g\partial_\phi, p\partial_\phi$ are anti-holomorphic, i.e.,*

$$f_{\bar{z}} + (\log\rho)_\phi \phi_{\bar{z}} f = 0, \quad q_{\bar{z}} + (\log\rho)_\phi \phi_{\bar{z}} q = 0,$$

and

$$g_z + (\log\rho)_\phi \phi_z g = 0, \quad p_z + (\log\rho)_\phi \phi_z p = 0.$$

Proof. A direct computation gives that

$$\begin{aligned} \not{D}\Psi = \sqrt{2}\mathrm{d}\bar{z} \otimes &\left\{\left(f_{\bar{z}} + (\log\rho)_\phi \phi_{\bar{z}} f\right)\partial_\phi + \left(\bar{p}_{\bar{z}} + (\log\rho)_{\bar{\phi}} \bar{\phi}_{\bar{z}} \bar{p}\right)\partial_{\bar{\phi}}\right\} \\ &- \frac{2\sqrt{2}}{\lambda}\left\{\left(g_z + (\log\rho)_\phi \phi_z g\right)\partial_\phi + \left(\bar{q}_z + (\log\rho)_{\bar{\phi}} \bar{\phi}_z \bar{q}\right)\partial_{\bar{\phi}}\right\}. \end{aligned}$$

□

Set

$$\Theta := (f\bar{g} - \bar{p}q)\,\rho(\phi)\mathrm{d}z, \tag{2.1}$$

then Θ is holomorphic. In fact,

Lemma 2.4. *If Ψ is harmonic, then $f\bar{g}\rho\mathrm{d}z$ and $\bar{p}q\rho\mathrm{d}z$ are both holomorphic.*

Proof. It is a consequence of Lemma 2.2. Here we give another direct proof. We only prove that $f\bar{g}\rho \mathrm{d}z$ is a holomorphic $(1,0)$-form. Applying Lemma 2.3

$$(\log(f\bar{g}\rho))_{\bar{z}} = (\log f)_{\bar{z}} + (\log \bar{g})_{\bar{z}} + (\log\rho)_\phi\phi_{\bar{z}} + (\log\rho)_{\bar{\phi}}\bar{\phi}_{\bar{z}} = 0.$$

This equality implies that $f\bar{g}\rho$ is a local holomorphic function. Hence $f\bar{g}\rho\mathrm{d}z$ is holomorphic. □

As a direct application, we give a new proof of the following result of L.Yang[19].

Theorem 2.5. *There is no coupled Dirac-harmonic map from the 2-sphere equipped with an arbitrary metric to any Riemann surface.*

Proof. Since there is no nontrivial holomorphic 1-form on the 2-sphere equipped with any metric, we can apply Lemma 2.4 together with Lemma 2.1 to complete the proof of this theorem. □

Now we can state the following

Proposition 2.6. *In our complex notation, the Euler-Lagrange equations become*

$$\begin{cases} \phi_{z\bar{z}} + (\log\rho)_\phi\phi_z\phi_{\bar{z}} + \dfrac{\sqrt{2}}{4}\kappa^N(\phi)\left(\phi_{\bar{z}}(f\bar{g} - \bar{p}q) - \phi_z\left(\bar{f}g - p\bar{q}\right)\right)\rho = 0 \\ f_{\bar{z}} + (\log\rho)_\phi\phi_{\bar{z}}f = 0, \\ q_{\bar{z}} + (\log\rho)_\phi\phi_{\bar{z}}q = 0, \\ g_z + (\log\rho)_\phi\phi_z g = 0, \\ p_z + (\log\rho)_\phi\phi_z p = 0. \end{cases}$$

Proof. We rewrite the functional L as

$$L(\Phi,\Psi) = \frac{1}{2}\int_M \|\mathrm{d}\Phi\|^2 + \langle \not{D}\Psi,\Psi\rangle = 2\int_M \left\|\bar{\partial}\Phi\right\|^2 + \frac{1}{2}\int_M \langle \not{D}\Psi,\Psi\rangle + \int_M J(\Phi).$$

Let $\Phi = \phi + t\eta$ and fix the coefficients of Ψ, i.e.,

$$\Psi = f\partial_\Phi + \bar{p}\partial_{\bar{\Phi}} + \mathrm{d}\bar{z}\otimes g\partial_\Phi + \mathrm{d}\bar{z}\otimes\bar{q}\partial_{\bar{\Phi}}.$$

Moreover, suppose Ψ is harmonic along the map ϕ. Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} L(\Phi,\Psi) =& 2\,\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\int_M \left\|\bar{\partial}\phi + t\bar{\partial}\eta\right\|^2 + \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\int_M \langle \not{D}\Psi,\Psi\rangle \\ =& \frac{2}{i}\int_M \mathrm{Re}\left\{\phi_{\bar{z}}\left(\bar{\eta}_z + (\log\rho)_{\bar{\phi}}\bar{\phi}_z\bar{\eta}\right)\right\}\rho\mathrm{d}z\wedge\mathrm{d}\bar{z} + \frac{1}{2}\int_M \left\langle \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\not{D}\Psi,\Psi\right\rangle \\ =& -\frac{2}{i}\int_M \mathrm{Re}\left\{\left(\phi_{z\bar{z}} + (\log\rho)_\phi\phi_z\phi_{\bar{z}}\right)\bar{\eta}\right\}\rho\mathrm{d}z\wedge\mathrm{d}\bar{z} + \frac{1}{2}\int_M \left\langle \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\not{D}\Psi,\Psi\right\rangle \end{aligned}$$

Since Ψ is harmonic, then $\rho f\bar{g}$ and $\rho\bar{p}q$ are holomorphic. By Lemma 2.3

$$\begin{aligned} \int_M \langle \not{D}\Psi,\Psi\rangle =& \frac{\sqrt{2}}{i}\int_M \rho\left(\left(f_{\bar{z}} + (\log\rho)_\phi\phi_{\bar{z}}f\right)\bar{g} + \left(\bar{p}_{\bar{z}} + (\log\rho)_{\bar{\phi}}\bar{\phi}_{\bar{z}}\bar{p}\right)q\right)\mathrm{d}z\wedge\mathrm{d}\bar{z} \\ &- \frac{\sqrt{2}}{i}\int_M \rho\left(\left(g_z + (\log\rho)_\phi\phi_z g\right)\bar{f} + \left(\bar{q}_z + (\log\rho)_{\bar{\phi}}\bar{\phi}_z\bar{q}\right)p\right)\mathrm{d}z\wedge\mathrm{d}\bar{z} \\ =& \frac{2\sqrt{2}}{i}\int_M \mathrm{Re}\left\{\left(\left(f_{\bar{z}} + (\log\rho)_\phi\phi_{\bar{z}}f\right)\bar{g} + \left(\bar{p}_{\bar{z}} + (\log\rho)_{\bar{\phi}}\bar{\phi}_{\bar{z}}\bar{p}\right)q\right)\right\}\rho\mathrm{d}z\wedge\mathrm{d}\bar{z}. \end{aligned}$$

Therefore,

$$
\begin{aligned}
\frac{1}{2}\int_M \left\langle \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\not{D}\Psi, \Psi\right\rangle =& \frac{\sqrt{2}}{i}\int_M \mathrm{Re}\left\{\left(\left((\log\rho)_{\phi\phi}\eta + (\log\rho)_{\phi\bar\phi}\bar\eta\right)\phi_{\bar z} + (\log\rho)_{\phi}\eta_{\bar z}\right) f\bar g\right\}\rho\mathrm{d}z\wedge\mathrm{d}\bar z\\
&+\frac{\sqrt{2}}{i}\int_M \mathrm{Re}\left\{\left(\left((\log\rho)_{\bar\phi\phi}\eta + (\log\rho)_{\bar\phi\bar\phi}\bar\eta\right)\bar\phi_{\bar z} + (\log\rho)_{\bar\phi}\bar\eta_{\bar z}\right)\bar p q\right\}\rho\mathrm{d}z\wedge\mathrm{d}\bar z\\
=& \frac{\sqrt{2}}{i}\int_M \mathrm{Re}\left\{(\log\rho)_{\phi\bar\phi}\left(\phi_{\bar z}\bar\eta - \bar\phi_{\bar z}\eta\right) f\bar g\right\}\rho\mathrm{d}z\wedge\mathrm{d}\bar z\\
&+\frac{\sqrt{2}}{i}\int_M \mathrm{Re}\left\{(\log\rho)_{\phi\bar\phi}\left(\bar\phi_{\bar z}\eta - \phi_{\bar z}\bar\eta\right)\bar p q\right\}\rho\mathrm{d}z\wedge\mathrm{d}\bar z\\
=& -\frac{\sqrt{2}}{2i}\int_M \rho\kappa^N\,\mathrm{Re}\left\{\left(\phi_{\bar z} f\bar g - \phi_z\bar f g\right)\bar\eta\right\}\rho\mathrm{d}z\wedge\mathrm{d}\bar z\\
&-\frac{\sqrt{2}}{2i}\int_M \rho\kappa^N\,\mathrm{Re}\left\{\left(\phi_z p\bar q - \phi_{\bar z}\bar p q\right)\bar\eta\right\}\rho\mathrm{d}z\wedge\mathrm{d}\bar z\\
=& -\frac{\sqrt{2}}{2i}\int_M \rho\kappa^N\,\mathrm{Re}\left\{\phi_{\bar z}\left(f\bar g - \bar p q\right)\bar\eta\right\}\rho\mathrm{d}z\wedge\mathrm{d}\bar z\\
&+\frac{\sqrt{2}}{2i}\int_M \rho\kappa^N\,\mathrm{Re}\left\{\phi_z\left(\bar f g - p\bar q\right)\bar\eta\right\}\rho\mathrm{d}z\wedge\mathrm{d}\bar z.
\end{aligned}
$$

The rest of the proof is obvious. □

3. Dirac-harmonic map between closed Riemann surfaces

In this section, we let M, N be closed Riemann surfaces. Denote

$$
\begin{aligned}
h(L\otimes\Phi^{-1}K_N^{-1}) &:= \dim_{\mathbb{C}}\left\{f\partial_\phi \in \Gamma(L\otimes\Phi^{-1}K_N^{-1}) \text{ is harmonic}\right\},\\
h(\Lambda^{0,1}L\otimes\Phi^{-1}K_N^{-1}) &:= \dim_{\mathbb{C}}\left\{\mathrm{d}\bar z\otimes g\partial_\phi \in \Gamma(\Lambda^{0,1}L\otimes\Phi^{-1}K_N^{-1}) \text{ is harmonic}\right\},\\
h(L\otimes\Phi^{-1}\bar K_N^{-1}) &:= \dim_{\mathbb{C}}\left\{\bar p\partial_{\bar\phi} \in \Gamma(L\otimes\Phi^{-1}\bar K_N^{-1}) \text{ is harmonic}\right\},\\
h(\Lambda^{0,1}L\otimes\Phi^{-1}\bar K_N^{-1}) &:= \dim_{\mathbb{C}}\left\{\mathrm{d}\bar z\otimes \bar q\partial_{\bar\phi} \in \Gamma(\Lambda^{0,1}L\otimes\Phi^{-1}\bar K_N^{-1}) \text{ is harmonic}\right\}.
\end{aligned}
$$

Then we have

Lemma 3.1.

$$
\begin{aligned}
\Lambda^{0,1}L\otimes\Phi^{-1}\bar K_N^{-1} &\cong (L\otimes\Phi^{-1}K_N^{-1})^*,\\
\Lambda^{0,1}L\otimes\Phi^{-1}K_N^{-1} &\cong (L\otimes\Phi^{-1}K_N)^*,\\
L\otimes\Phi^{-1}\bar K_N^{-1} &\cong L\otimes\Phi^{-1}K_N,
\end{aligned}
$$

and hence

$$
\begin{aligned}
h(\Lambda^{0,1}L\otimes\Phi^{-1}\bar K_N^{-1}) = h(L\otimes\Phi^{-1}K_N^{-1}) = l(L\otimes\Phi^{-1}K_N^{-1}),\\
h(\Lambda^{0,1}L\otimes\Phi^{-1}K_N^{-1}) = h(L\otimes\Phi^{-1}\bar K_N^{-1}) = l(L\otimes\Phi^{-1}K_N),
\end{aligned}
$$

where

$$l(D) := \dim_{\mathbb{C}} \{ f \text{ is meromorphic on } M : (f) + D \geq 0 \} .$$

Proof. We note that

$$\begin{aligned}
\Lambda^{0,1} L \otimes \Phi^{-1} \bar{K}_N^{-1} &\cong \bar{K}_M \otimes L \otimes \Phi^{-1} (K_N^*)^{-1} \\
&\cong K_M^* \otimes L \otimes \Phi^{-1} (K_N^{-1})^* \\
&\cong L^* \otimes L^* \otimes L \otimes \Phi^{-1} (K_N^{-1})^* \\
&\cong L^* \otimes \Phi^{-1} (K_N^{-1})^* \\
&\cong (L \otimes \Phi^{-1} K_N^{-1})^* .
\end{aligned}$$

The other two isomorphisims can be obtained in a similar way and the dimension formulae follow from divisor and line bundle theory. □

Lemma 3.2.

$$\begin{aligned}
\deg(L \otimes \Phi^{-1} K_N^{-1}) &= g_M - 1 - \deg(\Phi)(2g_N - 2), \\
\deg(L \otimes \Phi^{-1} K_N) &= g_M - 1 + \deg(\Phi)(2g_N - 2),
\end{aligned}$$

where g_M, g_N is the genus of M, N respectively.

Proof. The first formula follows from

$$\deg(L \otimes \Phi^{-1} K_N^{-1}) = \deg(L) - \deg(\Phi) \deg(K_N) = g_M - 1 - \deg(\Phi)(2g_N - 2).$$

The second formula can be obtained similarly. See[19] for similar results.

□

By using Lemma 3.1 and Lemma 3.2, we can reprove a result of Yang[19]:

Theorem 3.3. *Suppose* (1.2) *holds, then every Dirac-harmonic map* (Φ, Ψ) *is uncoupled.*

Proof. Under the assumption (1.2), we get either $g_M - 1 - \deg(\Phi)(2g_N - 2) < 0$ and hence

$$h(L \otimes \Phi^{-1} K_N^{-1}) = h(\Lambda^{0,1} L \otimes \Phi^{-1} \bar{K}_N^{-1}) = 0,$$

or $g_M - 1 + \deg(\Phi)(2g_N - 2) < 0$ and hence

$$h(\Lambda^{0,1} L \otimes \Phi^{-1} K_N^{-1}) = h(L \otimes \Phi^{-1} \bar{K}_N^{-1}) = 0.$$

In either case, by the construction of the holomorphic $(1,0)$-form Θ (see (2.1)), we know that Θ must be trivial and Φ then is harmonic. □

Corollary 3.4. *Suppose* (1.3) *holds, then every Dirac-harmonic map must be a holomorphic or anti-holomorphic map coupled with a harmonic spinor along this map.*

Proof. Note that (1.2) holds if (1.3) is valid. Then using the theory for harmonic map[9;13;18] or Theorem A.1, we get this corollary. □

Using the Riemann-Roch formula (c.f.[14]), we have the following

Proposition 3.5.

$$\begin{aligned}
h(L \otimes \Phi^{-1} K_N^{-1}) - h(\Lambda^{0,1} L \otimes \Phi^{-1} K_N^{-1}) &= -2 \deg(\Phi)(g_N - 1), \\
h(L \otimes \Phi^{-1} \bar{K}_N^{-1}) - h(\Lambda^{0,1} L \otimes \Phi^{-1} \bar{K}_N^{-1}) &= 2 \deg(\Phi)(g_N - 1).
\end{aligned}$$

Proof. The Riemann Roch formula says that for every divisor D,

$$l(D) = \deg(D) - g_M + 1 + l(K_M \otimes D^{-1}).$$

Applying Lemma 3.1 and Lemma 3.2, we know that

$$\begin{aligned} h(L \otimes \Phi^{-1} K_N^{-1}) =& l(L \otimes \Phi^{-1} K_N^{-1}) \\ =& \deg(L \otimes \Phi^{-1} K_N^{-1}) - g_M + 1 + l(L \otimes \Phi^{-1} K_N) \\ =& -2\deg(\Phi)(g_N - 1) + l(L \otimes \Phi^{-1} K_N) \\ =& -2\deg(\Phi)(g_N - 1) + h(\Lambda^{0,1} L \otimes \Phi^{-1} K_N^{-1}). \end{aligned}$$

The second identity can be proved similarly. □

Now we can prove the existence Theorem 1.1 for Dirac-harmonic maps.

Proof of Theorem 1.1. First, we choose metrics on M and N such that there is a harmonic map $\Phi : M \longrightarrow N$ homotopic to ϕ. Eells and Lemaire[8] proved that such metrics always exist except the case when $g_M = 1, g_N = 0, \left|\deg(\phi)\right| = 1$. Second, by using Proposition 3.5, we know that the space of harmonic spinors along the map Φ with the associated form $\Theta = 0$ is a complex linear space with complex dimension at least $4\left|\deg(\Phi)(g_N - 1)\right|$. To see this, if $\deg(\Phi)(g_N - 1) \leq 0$, then by Proposition 3.5, we have $h(L \otimes \Phi^{-1} K_N^{-1}) \geq 2|\deg(\Phi)(g_N - 1)|$ and $h(\Lambda^{0,1} L \otimes \Phi^{-1} \bar{K}_N^{-1}) \geq 2|\deg(\Phi)(g_N - 1)|$, if we choose harmonic spinors Ψ as the following form

$$\Psi = f\partial_\phi + \mathrm{d}\bar{z} \otimes \bar{q}\partial_{\bar{\phi}},$$

then we see that such Ψ's form a complex vector space with dimension at least $4\left|\deg(\Phi)(g_N - 1)\right|$.

Similarly, if $\deg(\Phi)(g_N - 1) > 0$, then we can choose

$$\Psi = \bar{p}\partial_{\bar{\phi}} + \mathrm{d}\bar{z} \otimes g\partial_\phi,$$

and such harmonic spinors also form a complex vector space with dimension at least $4\left|\deg(\Phi)(g_N - 1)\right|$. According to the definition of the associated form Θ (c.f. (2.1))

$$\Theta = (f\bar{g} - \bar{p}q)\rho\mathrm{d}z,$$

we know that $\Theta = 0$. In particular, the harmonic map Φ couples such kind of harmonic spinors must be Dirac-harmonic. □

Proof of Theorem 1.2. We firstly consider the case that

$$\deg(\Phi)(g_N - 1) \geq 0.$$

Then, according to Proposition 3.5, we know that

$$h(L \otimes \Phi^{-1} \bar{K}_N^{-1}) = 2\deg(\Phi)(g_N - 1) + h(\Lambda^{0,1} L \otimes \Phi^{-1} \bar{K}_N^{-1}).$$

Lemma 3.2 together with (1.2) implies that

$$\deg(L \otimes \Phi^{-1} K_N^{-1}) = g_M - 1 - 2\deg(\Phi)(g_N - 1) < 0.$$

Therefore, Lemma 3.1 implies that

$$h(L \otimes \Phi^{-1} K_N^{-1}) = h(\Lambda^{0,1} L \otimes \Phi^{-1} \bar{K}_N^{-1}) = l(L \otimes \Phi^{-1} K_N^{-1}) = 0.$$

Thus,

$$h(L \otimes \Phi^{-1} \bar{K}_N^{-1}) = h(\Lambda^{0,1} L \otimes \Phi^{-1} K_N^{-1}) = 2\deg(\Phi)(g_N - 1).$$

Hence the space of harmonic spinors along the map Φ is a complex linear space with dimension $4|\deg(\Phi)(g_N-1)|$.

The case of $\deg(\Phi)(g_N-1)<0$ can be handled in a similar way. To see this, we note that

$$\deg(L\otimes\Phi^{-1}K_N)=g_M-1+2\deg(\Phi)(g_N-1)<0.$$

and hence the following hold

$$h(L\otimes\Phi^{-1}\bar{K}_N^{-1})=h(\Lambda^{0,1}L\otimes\Phi^{-1}K_N^{-1})=l(L\otimes\Phi^{-1}K_N)=0.$$

$$h(L\otimes\Phi^{-1}K_N^{-1})=h(\Lambda^{0,1}L\otimes\Phi^{-1}\bar{K}_N^{-1})=-2\deg(\Phi)(g_N-1)>0.$$

Again, we have that the space of harmonic spinors along the map Φ is a complex linear space with dimension $4|\deg(\Phi)(g_N-1)|$.

□

Proposition 3.6. *There is no nontrivial Dirac-harmonic map from the* 2*-sphere to the* 2*-torus.*

Remark 3.1. Branding[3] proved that there is no nontrivial Dirac-harmonic map from S^2 to T^n (both equipped with standard metrics) by using the Bochner method.

Proof of Proposition 3.6. Suppose (Φ,Ψ) is a Dirac-harmonic map from the 2-sphere to the 2-torus, then we can apply Theorem 1.2 since (1.2) holds ($g_M=0,g_N=1$). In particular, Ψ must be trivial. Then applying the theory of harmonic maps[9;13;18] or Theorem A.1, we know that Φ is holomorphic or anti-holomorphic. The Riemann-Hurwitz formula (c.f.[6;10;14]) says that if Φ is a non-constant (anti-)holomorphic map, then

$$\left|\deg(\Phi)\right|\chi(N)=\chi(M)+r,\quad r\geq 0,$$

which contradicts the assumption $\chi(M)=2$ and $\chi(N)=0$. As a consequence, Φ must be a constant. □

The following Proposition is a consequence of Theorem 3.3 and a result of Schoen and Yau[17] and Sampson[16].

Proposition 3.7. *Let M,N be two closed Riemann surfaces of the same genus and assume that the metric of N has negative Gauss curvature. Suppose* (Φ,Ψ) *is a Dirac-harmonic map from M to N and* $\deg(\Phi)=1$. *Then Φ is a diffeomorphism.*

Proof. By known results about harmonic map[9;13;18] or Theorem A.2, we need only to prove that Φ is harmonic. That is a direct consequence of Theorem 3.3 since $g_M=g_N\geq 2$. □

Finally, we give a

Proof of Theorem 1.3. According to Corollary 3.4, we know that Φ is (anti-)holomorphic. Without loss of generality, assume Φ is holomorphic, then the spinor Ψ along the map Φ can be written as

$$\Psi=f\partial_\phi+\mathrm{d}\bar{z}\otimes\bar{q}\partial_{\bar{\phi}}.$$

By assumption (1.3), Φ is not constant and hence the zeros of $\partial\Phi$ are isolated.

$$\begin{aligned}\Psi=&-\frac{\sqrt{2}f}{2\phi_z}\partial_{\bar{z}}\cdot\mathrm{d}\bar{z}\otimes\partial\Phi(\partial_z)+\frac{\sqrt{2}\bar{q}}{\lambda\bar{\phi}_{\bar{z}}}\partial_z\cdot\bar{\partial}\bar{\Phi}(\partial_{\bar{z}})\\=&\frac{2}{\lambda}\left(\partial_{\bar{z}}\cdot\eta\otimes\partial\Phi(\partial_z)+\partial_z\cdot\eta\otimes\bar{\partial}\bar{\Phi}(\partial_{\bar{z}})\right),\end{aligned}$$

where

$$\eta = \frac{\sqrt{2}\bar{q}}{2\bar{\phi}_{\bar{z}}} - \frac{\sqrt{2}\lambda f}{4\phi_z}\mathrm{d}\bar{z}$$

is a spinor on M with possibly isolated singularities.

Notice that

$$f_{\bar{z}} = 0 = q_{\bar{z}}$$

since $\phi_{\bar{z}} = 0$ and Ψ is harmonic.

$$\nabla_{\partial_z}\eta = -\left(\frac{\sqrt{2}\lambda f}{4\phi_z}\right)_z \mathrm{d}\bar{z}, \quad \nabla_{\partial_{\bar{z}}}\eta = \left(\frac{\sqrt{2}\bar{q}}{2\bar{\phi}_{\bar{z}}}\right)_{\bar{z}},$$

and

$$\partial\!\!\!/\eta = \frac{2}{\lambda}\left(\left(\frac{\lambda f}{2\phi_z}\right)_z + \frac{\lambda}{2}\left(\frac{\bar{q}}{\bar{\phi}_{\bar{z}}}\right)_{\bar{z}}\mathrm{d}\bar{z}\right).$$

Here we have used the relationship

$$\partial_z \cdot 1 = \frac{\sqrt{2}\lambda}{2}\mathrm{d}\bar{z}, \quad \partial_{\bar{z}} \cdot \mathrm{d}\bar{z} = -\sqrt{2}.$$

Noting that $\partial_z \cdot \mathrm{d}\bar{z} = 0 = \partial_{\bar{z}} \cdot 1$, we get

$$\nabla_{\partial_z}\eta + \frac{1}{2}\partial_z \cdot \partial\!\!\!/\eta = 0, \quad \nabla_{\partial_{\bar{z}}}\eta + \frac{1}{2}\partial_{\bar{z}} \cdot \partial\!\!\!/\eta = 0$$

which means that η is a twistor spinor. □

Appendix A. Some known results about harmonic maps

For readers' convenience, here we list some known results about harmonic maps between closed Riemann surfaces. All results can be found in[9;13;18]. Let M and N be two closed Riemann surfaces with local conformal parameter $z = x + \sqrt{-1}y$ and $\phi = u + \sqrt{-1}v$ respectively. Denote the metric of M and N locally by $\lambda(z)\,|\mathrm{d}z|^2$ and $\rho(\phi)\,|\mathrm{d}\phi|^2$ respectively. Then a smooth map $\Phi : M \longrightarrow N$ is harmonic if and only if

$$\phi_{z\bar{z}} + (\log\rho)_\phi \phi_z \phi_{\bar{z}} = 0.$$

Based on this equation, we get the following Bochner foumulae

$$\Delta_M \log\|\partial\Phi\| = \kappa^M - \kappa^N J(\phi),$$
$$\Delta_M \log\left\|\bar{\partial}\Phi\right\| = \kappa^M + \kappa^N J(\phi).$$

Therefore, if Φ is a harmonic map, then $\partial\Phi$ has an isolated zero set if $\partial\Phi$ is not identically zero, while $\bar{\partial}\Phi$ has an isolated zero set if $\bar{\partial}\Phi$ is not identically zero. Hence, according to these Bochner formulae, we get

Theorem A.1 ([9]). *Suppose Φ is harmonic and*

$$g_M - 1 < \left|\deg(\Phi)(g_N - 1)\right|,$$

then Φ is either holomorphic or anti-holomorphic.

Theorem A.2 ([17]). *Suppose Φ is harmonic, $g_M = g_N, \deg(\Phi) = 1$ and $\kappa^N \leq 0$, then Φ is a diffeomorphism.*

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