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GEOMETRIC ANALYSIS OF THE ACTION FUNCTIONAL OF THE NONLINEAR SUPERSYMMETRIC SIGMA MODEL

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ABSTRACT. The mathematical version of the action functional of the nonlinear supersymmetric model of quantum field theory couples a map from a Riemann surface into a Riemannian manifold with a spinor field along the map. While a simplified version of the model, the so-called Dirac-harmonic map functional, has been extensively studied in the literature in recent years, the full model involves an additional curvature term. Handling the finer analytic aspects caused by this term requires new methods. These are developed in this paper. We analyze the blow-up of solutions. In particular, we show that the energy identities and no neck property hold during the blow-up process. In technical terms, we derive a new exponential decay estimate of some weighted energy on neck domains for the spinor field. This is based on some Hardy-type inequality.

1. INTRODUCTION

The action functionals of quantum field theory provide a rich source for mathematical inspiration. They possess rather intricate and subtle formal properties which are difficult to analyze and to utilize, but which ultimately reward us with rich and beautiful structures that often have profound applications in geometry, like Yang-Mills theory or the Seiberg-Witten model. One of the simplest such functionals is the action functional of the nonlinear sigma model [8, 10]. In its simplest form, it leads to harmonic maps from Riemann surfaces into Riemannian manifolds. The fundamental analysis of the convergence or blow-up of solutions started with the work of Sacks-Uhlenbeck [16], and it has been an active field of research ever since. The key to the behavior of the solutions is the conformal invariance of the action functional. Since the conformal group is not compact, solutions can concentrate at points, leading to the so-called bubbling or blow-up phenomenon. This, in fact, is one of the fundamental properties of basically all these variational problems from quantum field theory, and therefore the work of Sacks-Uhlenbeck turned out to be a model for many other such problems.

When one moves from the basic sigma model to the supersymmetric one, one encounters an additional field, a spinor field along the map. The map and the spinor field are coupled by the

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Euler-Lagrange equations. While in the physical model, the spinor field is anticommuting, there also exists a mathematical version with commuting fields. The solutions of the Euler-Lagrange equations of such a coupled model have been called Dirac-harmonic maps, and they were introduced and studied in Chen-Jost-Li-Wang [5, 6]. In particular, although the analysis turned out to be substantially more difficult and subtle than for standard harmonic maps, in the end a Sacks-Uhlenbeck type blow-up analysis could be achieved, and the key technical points like energy identities and the no-neck property could be achieved, see [5, 6, 7, 13, 17, 18, 19, 21, 22]. Nevertheless, the model of [5, 6] contained an important simplification of the physical action functional, insofar as it did not include an additional curvature term arising from the geometry of the target manifold.

It is the purpose of the present paper to carry out the geometric analysis of the full model, including the curvature term, and to provide a state-of-the-art treatment of the solutions. For simplicity, we call the solutions Dirac-harmonic maps with curvature term. The reason for this name will become apparent below.

As already mentioned, the full supersymmetric nonlinear sigma model includes a curvature term in addition to the map and the spinor (see for instance [10], p.162). That curvature term is needed for supersymmetry. It is of lower order than the other terms and therefore analytically dominated by those. Therefore, it does not affect the basic properties of the Dirac-harmonic map theory. Nevertheless, since the blow-up analysis of Dirac-harmonic maps is very subtle, depending on deep identities and symmetries, for some of the finer aspects, this curvature term needs to be accounted for. In fact, the methods developed so far do not all generalize to include such a curvature term. Therefore, in this paper, we present a new scheme for obtaining the energy identities and no neck property, that is, the finer aspects of the blow-up behavior. This scheme is able to handle the curvature term. To achieve this, we apply a Hardy-type inequality to derive the exponential decay of some weighted energy of a spinor on the neck region. This is the key analytical step of this paper. Some geometric formulae for Dirac-harmonic maps with curvature term have been derived by Branding [2], but our formulae are different from his. Likewise, in [3], he has already carried out some of the easier steps of the analysis. Our analytical results go substantially beyond those in that paper.

This paper is organized as follows. In section 2, we introduce the model and state the main theorem. In section 3, we derive the Euler-Lagrange equations for Dirac-harmonic maps with curvature term where we embed N into $\mathbb{R}^K$ by the Nash-Moser embedding theorem. In section 4, we prove some geometric properties of Dirac-harmonic maps with curvature term. In section 5, we will first study some analytic properties of Dirac-harmonic maps with curvature term, such as small regularity theorem and gap theorem. Secondly, for reader's convenience, we will recall some lemmas in [6, 13] which will be used in this paper. In section 6, we will prove our main Theorem 2.1. We give a proof of the removable singularity Theorem 6.1 in the appendix 7.

2. Dirac-harmonic maps with curvature term and their blow-up behavior

We now proceed to formally introducing the model. We shall first present the standard Dirac-harmonic model and shall then add the curvature term. Let (M, h) be a compact Riemann surface and ΣM be the spinor bundle over M. Let (N, g) be another compact Riemannian manifold. Let ϕ be a map from M to N, ψ a section of the twisted bundle $\Sigma M \otimes \phi^{-1}TN$ with induced metric $\langle , \rangle$ and induced connection $\widetilde{\nabla}$. We consider the action functional

$$L(\phi, \psi) = \int_M \big(|d\phi|^2 + \langle \psi, \not{D}\psi\rangle_{\Sigma M \otimes \phi^{-1}TN}\big) dvol_h, \tag{2.1}$$

where $\not{D}$ is the Dirac operator along the map ϕ (see Section 3 for details). Critical points (ϕ, ψ) of L are called Dirac-harmonic maps from M to N.

The Euler-Lagrange equations of the functional L are

$$\big(\Delta\phi^i + \Gamma^i_{jk} h^{\alpha\beta} \phi^j_\alpha \phi^k_\beta\big) \frac{\partial}{\partial y^i}(\phi(x)) = R(\phi, \psi), \tag{2.2}$$

$$\not{D}\psi = 0, \tag{2.3}$$

where $R(\phi, \psi)$ is defined by

$$R(\phi, \psi) = \frac{1}{2} R^m_{lij}(\phi(x)) \langle \psi^i, \nabla\phi^l \cdot \psi^j \rangle \frac{\partial}{\partial y^m}(\phi(x)).$$

Here R^m_{lij} stands for the Riemann curvature tensor of the target manifold (N, g). One can refer to [5, 6].

As already mentioned, the supersymmetric nonlinear sigma model of quantum field theory includes an additional curvature term in addition to (2.1). This leads us to consider the following functional

$$L_c(\phi, \psi) = \frac{1}{2} \int_M \big(|d\phi|^2 + \langle \psi, \not{D}\psi\rangle_{\Sigma M \otimes \phi^{-1}TN}\big) - \frac{1}{6} R_{ikjl} \langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle dvol. \tag{2.4}$$

The critical points (ϕ, ψ) are called Dirac-harmonic maps with curvature term from M to N. They were first proposed and studied by Chen-Jost-Wang [4]. They proved a type of Liouville theorem for Dirac-harmonic maps with curvature term.

In this paper, we will study some analytic aspects of Dirac-harmonic maps with curvature term including small energy regularity, removability of singularities, energy identity and no neck property. The main result in this paper is that we prove that the energy identities and the no neck property hold when the target manifold is a general compact Riemannian manifold. Here we use ideas of Ding-Tian [9] and Qing-Tian [15]. For the energy identity, we will choose a special cut-off function and use a Hardy-type inequality on $\mathbb{R}^2$ to estimate the energy of ψ on an annulus first. For the no neck property, more delicate estimates of the energy of the sequence on the necks are required. Similarly to the cases of approximate harmonic maps [15] and Dirac-harmonic maps [13], we shall use a type of three circle lemma to prove the exponential decay of the tangential energy of the map ϕ. Differently from Dirac-harmonic maps [13], if we want to get the exponential decay of the whole energy of ϕ, we need to first prove the exponential decay of some weighted energy of ψ. This is achieved in Lemma 6.2 and Lemma 6.4 which are crucial for the proof of our main theorem.

In order to describe our main theorem, we need some notations first.

Let U be a domain of M, we denote the energy of (ϕ,ψ) on U as

$$E(\phi,\psi;U):=\int_U(|d\phi|^2+|\psi|^4),$$

the energy of ϕ as

$$E(\phi;U):=\int_U|d\phi|^2,$$

the energy of ψ as

$$E(\psi;U):=\int_U|\psi|^4.$$

Our main result is the following:

Theorem 2.1. *For a sequence of smooth Dirac-harmonic maps with curvature term $\{(\phi_k,\psi_k)\}$ with uniformly bounded energy*

$$E(\phi_k,\psi_k)\leq\Lambda<\infty,$$

we define the blow-up set

$$\mathcal{S}:=\cap_{r>0}\big\{x\in M|\liminf_{n\to\infty}\int_{D(x,r)}(|d\phi_n|^2+|\psi_n|^4)\geq\epsilon_0)\big\}. \tag{2.5}$$

Then $\mathcal{S}$ is a finite set $\{p_1,...,p_I\}$, where $\epsilon_0>0$ is as in Theorem 5.1. A subsequence, still denoted by $\{(\phi_k,\psi_k)\}$, converges in $C^\infty_{loc}(M\setminus\mathcal{S})$ to a Dirac-harmonic map with curvature term $(\phi,\psi):M\to N$ and there is a finite set of Dirac-harmonic spheres with curvature term $(\sigma_i^l,\xi_i^l):S^2\to N$, $i=1,...,I$; $l=1,...,L_i$ such that

$$\lim_{k\to\infty}E(\phi_k) = E(\phi)+\sum_{i=1}^I\sum_{l=1}^{L_i}E(\sigma_i^l), \tag{2.6}$$

$$\lim_{k\to\infty}E(\psi_k) = E(\psi)+\sum_{i=1}^I\sum_{l=1}^{L_i}E(\xi_i^l), \tag{2.7}$$

and the image $\phi(M)\cup_{i=1}^I\cup_{l=1}^{L_i}(\sigma_i^l(S^2))$ is a connected set.

3. Euler-Lagrange equations

Let (M,g) be a compact Riemann surface with a fixed spin structure, ΣM the spinor bundle over M and $\langle\cdot,\cdot\rangle_{\Sigma M}$ the metric on ΣM. Choosing a local orthonormal basis $e_\alpha,\alpha=1,2$ on M, the usual Dirac operator is defined as $\partial\!\!\!/:=e_\alpha\cdot\nabla_{e_\alpha}$, where ∇ is the spin connection on ΣM and $\cdot$ is the Clifford multiplication. For more details, one can refer to [11]. Let ϕ be a smooth map from M to another compact Riemannian manifold (N,h) with dimension $n\geq 2$. If $\phi^{-1}TN$ is the pull-back bundle of TN by ϕ, we get the twisted bundle $\Sigma M\otimes\phi^{-1}TN$. Naturally, there is a metric $\langle\cdot,\cdot\rangle_{\Sigma M\otimes\phi^{-1}TN}$ on $\Sigma M\otimes\phi^{-1}TN$ which is induced from the metrics on ΣM and $\phi^{-1}TN$. Also we have a natural connection $\widetilde{\nabla}$ on $\Sigma M\otimes\phi^{-1}TN$ which is induced from the connections on ΣM and $\phi^{-1}TN$. Let ψ be a section of the bundle $\Sigma M\otimes\phi^{-1}TN$. In local coordinates, it can be written as

$$\psi=\psi^i\otimes\partial_{y^i}(\phi),$$

where each ψ^i is a standard spinor on M and ∂_{y^i} is the natural local basis on N. Then $\widetilde{\nabla}$ becomes

$$\widetilde{\nabla}\psi=\nabla\psi^i\otimes\partial_{y^i}(\phi)+(\Gamma^i_{jk}\nabla\phi^j)\psi^k\otimes\partial_{y^i}(\phi), \tag{3.1}$$

where Γ^i_{jk} are the Christoffel symbols of the Levi-Civita connection of N. The Dirac operator along the map ϕ is defined by $\not{D}\psi:=e_\alpha\cdot\widetilde{\nabla}_{e_\alpha}\psi$.

Here, we want to point out that the usual Dirac operator $\not{\partial}$ on a surface can be seen as the Cauchy-Riemann operator. Let $(\mathbb{R}^2,dx^2+dy^2)$ be the standard Euclidean space and $e_1=\frac{\partial}{\partial x}$ and $e_2=\frac{\partial}{\partial y}$ be the standard orthonormal frame. A spinor field is simply a map $\psi:\mathbb{R}^2\to\Delta_2=\mathbb{C}^2$, and the action of e_1 and e_2 on spinors can be identified with multiplication with matrices

$$e_1=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad e_2=\begin{pmatrix}0&i\\i&0\end{pmatrix}.$$

If $\psi:=\begin{pmatrix}\psi_1\\\psi_2\end{pmatrix}:\mathbb{R}^2\to\mathbb{C}^2$ is a spinor field, then the Dirac operator is

$$\not{\partial}\psi=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\frac{\partial\psi_1}{\partial x}\\\frac{\partial\psi_2}{\partial x}\end{pmatrix}+\begin{pmatrix}0&i\\i&0\end{pmatrix}\begin{pmatrix}\frac{\partial\psi_1}{\partial y}\\\frac{\partial\psi_2}{\partial y}\end{pmatrix}=2\begin{pmatrix}\frac{\partial\psi_2}{\partial\bar z}\\-\frac{\partial\psi_1}{\partial z}\end{pmatrix}, \tag{3.2}$$

where

$$\frac{\partial}{\partial z}=\frac{1}{2}(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}),\quad \frac{\partial}{\partial\bar z}=\frac{1}{2}(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}).$$

For more details on spin geometry and Dirac operators, one can refer to [11].

The functional

$$L_c(\phi,\psi)=\frac{1}{2}\int_M\left(|d\phi|^2+\langle\psi,\not{D}\psi\rangle_{\Sigma M\otimes\phi^{-1}TN}-\frac{1}{6}R_{ikjl}\langle\psi^i,\psi^j\rangle\langle\psi^k,\psi^l\rangle\right)dvol$$

is conformally invariant. That is, for any conformal diffeomorphism $f:M\to M$, setting

$$\widetilde{\phi}=\phi\circ f\qquad \textit{and}\qquad \widetilde{\psi}=\lambda^{-1/2}\psi\circ f,$$

we have

$$L_c(\widetilde{\phi},\widetilde{\psi})=L_c(\phi,\psi).$$

(For the proof, one can refer to [6]). Here λ is the conformal factor of the conformal map f, *i.e.* $f^*g=\lambda^2g$.

By [4], the Euler-Lagrange equations of the functional L_c are

$$\tau(\phi) = \frac{1}{2}R^m_{lij}(\phi)\langle\psi^i,\nabla\phi^l\cdot\psi^j\rangle\frac{\partial}{\partial y^m}(\phi)-\frac{1}{12}h^{mp}R_{ikjl;p}\langle\psi^i,\psi^j\rangle\langle\psi^k,\psi^l\rangle\frac{\partial}{\partial y^m}(\phi), \tag{3.3}$$

$$\not{D}\psi = \frac{1}{3}R^m_{jkl}\langle\psi^j,\psi^l\rangle\psi^k\frac{\partial}{\partial y^m}(\phi), \tag{3.4}$$

where $\tau(\phi)=\left(-\Delta\phi^i+\Gamma^i_{jk}g^{\alpha\beta}\phi^j_\alpha\phi^k_\beta\right)\frac{\partial}{\partial y^i}(\phi(x))$ is the tension field of ϕ.

By using the Nash-Moser embedding theorem, we embed N into $\mathbb{R}^N$, denoted by $f: N\to\mathbb{R}^K$. Set

$$\phi'=f\circ\phi \quad and \quad \psi'=f_*\psi.$$

Let A be the second fundamental form of f. It is well-known that the tension fields of ϕ and ϕ' satisfy the following relation

$$\tau'(\phi')=A(d\phi,d\phi)+df(\tau(\phi)). \tag{3.5}$$

For simplicity of notation, we identify ϕ with ϕ' and ψ with ψ'. Using the Gauss equation (cf. [5, 2]), we have

$$\begin{aligned}\frac{1}{2}R^m_{lij}(\phi)\langle\psi^i,\nabla\phi^l\cdot\psi^j\rangle &= \frac{1}{2}h^{mk}(A_{ik}A_{jl}-A_{jk}A_{il})\langle\psi^i,\nabla\phi^l\cdot\psi^j\rangle\\ &= h^{mk}A_{ik}A_{jl}Re\langle\psi^i,\nabla\phi^l\cdot\psi^j\rangle\end{aligned}$$

and

$$\begin{aligned}&-\frac{1}{12}h^{mp}R_{ikjl;p}\langle\psi^i,\psi^j\rangle\langle\psi^k,\psi^l\rangle\\ =\;& -\frac{1}{12}h^{mp}\left(A_{ij;p}A_{kl}+A_{ij}A_{kl;p}-A_{il;p}A_{jk}-A_{il}A_{jk;p}\right)\langle\psi^i,\psi^j\rangle\langle\psi^k,\psi^l\rangle\\ =\;& -\frac{1}{6}h^{mp}A_{ij;p}A_{kl}\langle\psi^i,\psi^j\rangle\langle\psi^k,\psi^l\rangle+\frac{1}{6}h^{mp}A_{il;p}A_{jk}Re(\langle\psi^i,\psi^j\rangle\langle\psi^k,\psi^l\rangle)\\ =\;& \frac{1}{6}h^{mp}\big(-A_{ij;p}A_{kl}+A_{il;p}A_{jk}\big)Re(\langle\psi^i,\psi^j\rangle\langle\psi^k,\psi^l\rangle)\end{aligned}$$

where $Re(z)$ denotes the real part of $z\in\mathbb{C}$ and the last equality holds since one can easily verify that $-\frac{1}{6}h^{mp}A_{ij;p}A_{kl}\langle\psi^i,\psi^j\rangle\langle\psi^k,\psi^l\rangle$ are real numbers.

Also one can check that the Dirac operators $\not{D}$ and $\not{D}'$ corresponding to ϕ and ϕ' satisfy

$$\not{D}'\psi'=f_*(\not{D}\psi)+\mathcal{A}(d\phi(e_\alpha),e_\alpha\cdot\psi), \tag{3.6}$$

where

$$\mathcal{A}(d\phi(e_\alpha),e_\alpha\cdot\psi)=(\nabla\phi^i\cdot\psi^j)\otimes A(\partial_{y^i},\partial_{y^j}).$$

Using the Gauss equation again, we have

$$\begin{aligned}&\frac{1}{3}R^m_{jkl}\langle\psi^j,\psi^l\rangle\psi^k\partial_{y^m}(\phi)\\ =\;& \frac{1}{3}h^{mi}\left(A_{ik}A_{jl}-A_{il}A_{jk}\right)\langle\psi^j,\psi^l\rangle\psi^k\partial_{y^m}(\phi)\\ =\;& \frac{1}{3}\left(P(A(\partial_{y^j},\partial_{y^l});\partial_{y^k})-P(A(\partial_{y^j},\partial_{y^k});\partial_{y^l})\right)\langle\psi^j,\psi^l\rangle\psi^k\end{aligned}$$

where $P(\cdot;\cdot)$ is the shape operator, i.e.

$$\langle P(\xi;X),Y\rangle=\langle A(X,Y),\xi\rangle$$

for any $X,Y\in\Gamma(TN)$,$\xi\in\Gamma(T^\perp N)$.

Therefore, we have [1]

$$-\Delta\phi = A(d\phi, d\phi) + Re\left(P(\mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi); \psi)\right) - G(\psi), \tag{3.7}$$

$$\partial\!\!\!/\psi = \mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi) + F(\psi, \psi)\psi \tag{3.8}$$

where

$$Re\left(P(\mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi); \psi)\right) = P(A(\partial_{y^j}, \partial_{y^l}); \partial_{y^i}) Re\langle \psi^i, \nabla\phi^l \cdot \psi^j \rangle;$$
$$G(\psi) = \frac{1}{6}\big(\langle \nabla A_{ij}, A_{kl} \rangle - \langle \nabla A_{il}, A_{jk} \rangle\big) Re(\langle \psi^i, \psi^j \rangle \langle \psi^k, \psi^l \rangle);$$
$$F(\psi, \psi)\psi = \frac{1}{3}\left(P(A(\partial_{y^j}, \partial_{y^l}); \partial_{y^k}) - P(A(\partial_{y^j}, \partial_{y^k}); \partial_{y^l})\right) \langle \psi^j, \psi^l \rangle \psi^k.$$

From the preceding, we can easily get the following proposition (see also [2, Lemma 3.5.]).

Proposition 3.1. *A pair (ϕ, ψ) is a Dirac-harmonic map with curvature term if and only if (ϕ, ψ) satisfies (3.7) and (3.8). Here $\phi : M \to N$ is a map from M to $\mathbb{R}^K$ with $\phi(x) \in N$ for any $x \in M$. The spinor field ψ along the map ϕ is a K-tuple of usual spinors $(\psi^1, ..., \psi^K)$ satisfying the condition that for any normal vector ν of N at $\phi(x)$, we have $\langle \nu, \psi \rangle_{\mathbb{R}^K} = 0$.*

4. Geometric aspects

Certain quantities defined in this part were first given in [2], such as the two-tensor (4.1) and the quadratic differential (4.9). Because of certain subtleties of the computation (for instance, the factors in (4.2) and (4.9) are different from those in [2]) , we nevertheless need to go through all the details here. In fact, it is well known that there is an energy-momentum tensor associated to a harmonic map. For the energy-momentum tensor of Dirac-harmonic maps, see [5, 6]. For Dirac-harmonic maps with curvature terms, one can define a two-tensor by [2]

$$T_{\alpha\beta} := 2\langle d\phi(e_\alpha), d\phi(e_\beta) \rangle - \delta_{\alpha\beta}|d\phi|^2 + Re\langle \psi, e_\alpha \cdot \widetilde{\nabla}_{e_\beta}\psi \rangle - \frac{1}{6}\delta_{\alpha\beta}\langle R(\psi, \psi)\psi, \psi \rangle, \tag{4.1}$$

where e_α is a local orthonormal basis on M and θ^α the dual basis to e_α. Usually, the tensor $T_{\alpha\beta}\theta^\alpha \otimes \theta^\beta$ is called energy-momentum tensor for the functional L_c.

From its definition, it is easy to see that $T_{\alpha\beta}$ is traceless, when (ϕ, ψ) is a Dirac-harmonic map with curvature term. Secondly, we shall prove that the tensor $T_{\alpha\beta}$ is symmetric. In fact, from equation (3.4), we have

$$e_1 \cdot \widetilde{\nabla}_{e_2}\psi = e_2 \cdot \widetilde{\nabla}_{e_1}\psi + \frac{1}{3} e_2 \cdot e_1 \cdot R(\psi, \psi)\psi.$$

[1] There are similar formulas in Lemma 3.5 in [2]. Since, however, there is a crucial difference between the inner products employed here and in [2] (in this paper we are taking the Hermitian inner product for the spinors), we need to supply the precise formulas and the calculations to identify the terms where we need to take the real parts.

[2] see equation (3.1) in [2] for a similar tensor where, however, the real part of the third term $\langle \psi, e_\alpha \cdot \widetilde{\nabla}_{e_\beta}\psi \rangle$ is not taken.

So, we have

$$
\begin{aligned}
T_{12}-T_{21} &= Re\langle\psi, e_1\cdot\widetilde{\nabla}_{e_2}\psi - e_2\cdot\widetilde{\nabla}_{e_1}\psi\rangle \\
&= \frac{1}{3}Re\left(R_{ijkl}\langle\psi^j,\psi^l\rangle\langle\psi^i, e_2\cdot e_1\cdot\psi^k\rangle\right). \qquad (4.2)
\end{aligned}
$$

But, on the other hand, we have

$$
\begin{aligned}
\overline{R_{ijkl}\langle\psi^j,\psi^l\rangle\langle\psi^i, e_2\cdot e_1\cdot\psi^k\rangle} &= R_{ijkl}\langle\psi^l,\psi^j\rangle\langle e_2\cdot e_1\cdot\psi^k,\psi^i\rangle \\
&= R_{klij}\langle\psi^j,\psi^l\rangle\langle e_2\cdot e_1\cdot\psi^i,\psi^k\rangle \\
&= -R_{ijkl}\langle\psi^j,\psi^l\rangle\langle\psi^i, e_2\cdot e_1\cdot\psi^k\rangle.
\end{aligned}
$$

From the preceding, we know $R_{ijkl}\langle\psi^j,\psi^l\rangle\langle\psi^i, e_2\cdot e_1\cdot\psi^k\rangle$ is purely imaginary and thus $T_{12}-T_{21}=0$.

Next, we show that the tensor given in equation (4.1) is conserved (see [2, Proposition 3.2.]) for a similar result for the tensor given there).

Lemma 4.1. *Let (ϕ,ψ) be a smooth Dirac-harmonic map with curvature term from M to N, then the energy-momentum tensor is conserved, i.e.*

$$
\sum_\alpha \nabla_\alpha T_{\alpha\beta} = 0. \qquad (4.3)
$$

Proof. Several related computations have already been provided in [2], but since for our purposes the algebraic details are very important, we need to supply detailed computations to identify the relevant terms where we need to take the real parts.

Define

$$
C_{\alpha\beta} := 2\langle d\phi(e_\alpha), d\phi(e_\beta)\rangle - \delta_{\alpha\beta}|d\phi|^2, \quad D_{\alpha\beta} := Re\langle\psi, e_\alpha\cdot\widetilde{\nabla}_{e_\beta}\psi\rangle
$$

and

$$
E_{\alpha\beta} := \frac{1}{6}\delta_{\alpha\beta}\langle R(\psi,\psi)\psi,\psi\rangle.
$$

Computing directly, we have

$$
\begin{aligned}
\sum_\alpha \nabla_\alpha C_{\alpha\beta} &= 2\langle\tau(\phi), d\phi(e_\beta)\rangle \\
&= h_{mp}R^m_{lij}(\phi)\langle\psi^i,\nabla\phi^l\cdot\psi^j\rangle\phi^p_\beta - \frac{1}{6}R_{ikjl;m}\langle\psi^i,\psi^j\rangle\langle\psi^k,\psi^l\rangle\phi^m_\beta \qquad (4.4)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_\alpha \nabla_\alpha D_{\alpha\beta} &= \sum_\alpha \nabla_\alpha Re\langle\psi, e_\alpha\cdot\widetilde{\nabla}_{e_\beta}\psi\rangle \\
&= \sum_\alpha Re\langle\nabla_\alpha\psi, e_\alpha\cdot\widetilde{\nabla}_{e_\beta}\psi\rangle + Re\langle\psi, \not{D}\widetilde{\nabla}_{e_\beta}\psi\rangle \\
&= -Re\langle\not{D}\psi, \cdot\widetilde{\nabla}_{e_\beta}\psi\rangle + Re\langle\psi, \not{D}\widetilde{\nabla}_{e_\beta}\psi\rangle. \qquad (4.5)
\end{aligned}
$$

Noting that

$$\begin{aligned}
\not{D}\widetilde{\nabla}_{e_\beta}\psi &= e_\alpha\cdot\widetilde{\nabla}_{e_\alpha}\widetilde{\nabla}_{e_\beta}\psi\\
&= e_\alpha\cdot\Big(\widetilde{\nabla}_{e_\beta}\widetilde{\nabla}_{e_\alpha}\psi + R^{\Sigma M}(e_\alpha,e_\beta)\psi^i\otimes\partial_{y^i} + R^m_{lij}\phi^i_\alpha\phi^j_\beta\psi^l\otimes\partial_{y^m}\Big)\\
&= \widetilde{\nabla}_{e_\beta}\not{D}\psi + e_\alpha\cdot R^{\Sigma M}(e_\alpha,e_\beta)\psi^i\otimes\partial_{y^i} + R^m_{lij}\phi^i_\alpha\phi^j_\beta e_\alpha\cdot\psi^l\otimes\partial_{y^m}\\
&= \widetilde{\nabla}_{e_\beta}\not{D}\psi + R^m_{lij}\phi^i_\alpha\phi^j_\beta e_\alpha\cdot\psi^l\otimes\partial_{y^m},
\end{aligned}$$

we get

$$\nabla_\alpha D_{\alpha\beta} = -Re\langle\not{D}\psi,\widetilde{\nabla}_{e_\beta}\psi\rangle + Re\langle\psi,\widetilde{\nabla}_{e_\beta}\not{D}\psi\rangle + \sum_\alpha h_{mp}R^m_{lij}\phi^i_\alpha\phi^j_\beta\langle\psi^p,e_\alpha\cdot\psi^l\rangle. \tag{4.6}$$

Finally, from the equation (3.4), we have

$$\begin{aligned}
\sum_\alpha\nabla_\alpha E_{\alpha\beta} &= \frac{1}{2}\widetilde{\nabla}_\beta\langle\not{D}\psi,\psi\rangle\\
&= \frac{1}{2}Re\langle\widetilde{\nabla}_\beta\not{D}\psi,\psi\rangle + \frac{1}{2}Re\langle\not{D}\psi,\widetilde{\nabla}_\beta\psi\rangle.
\end{aligned}\tag{4.7}$$

By equations (4.4), (4.6) and (4.7), we get

$$\begin{aligned}
\sum_\alpha\nabla_\alpha T_{\alpha\beta} &= \sum_\alpha\nabla_\alpha(C_{\alpha\beta}+D_{\alpha\beta}-E_{\alpha\beta})\\
&= \frac{1}{2}Re\langle\widetilde{\nabla}_\beta\not{D}\psi,\psi\rangle - \frac{3}{2}Re\langle\not{D}\psi,\widetilde{\nabla}_\beta\psi\rangle - \frac{1}{6}R_{ikjl;m}\langle\psi^i,\psi^j\rangle\langle\psi^k,\psi^l\rangle\phi^m_\beta.
\end{aligned}\tag{4.8}$$

From equation (3.4), we easily know

$$\begin{aligned}
Re\langle\widetilde{\nabla}_\beta\not{D}\psi,\psi\rangle &= \frac{1}{3}h_{im}R^m_{jkl;p}\phi^p_\beta\langle\psi^j,\psi^l\rangle\langle\psi^i,\psi^k\rangle + Re(h_{im}R^m_{jkl}\langle\psi^j,\psi^l\rangle\langle\psi^i,\widetilde{\nabla}_{e_\beta}\psi^k\rangle)\\
&= \frac{1}{3}h_{im}R^m_{jkl;p}\phi^p_\beta\langle\psi^j,\psi^l\rangle\langle\psi^i,\psi^k\rangle + 3Re\langle\not{D}\psi,\widetilde{\nabla}_\beta\psi\rangle.
\end{aligned}$$

Then the lemma follows immediately.

□

As a consequence of the above conservation law, we have the following:

Corollary 4.2. *The quadratic differential Tdz^2 is holomorphic, where*

$$T(z)dz^2 = \{|\phi_x|^2-|\phi_y|^2-2i\langle\phi_x,\phi_y\rangle+Re\langle\psi,\frac{\partial}{\partial x}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial x}}\psi\rangle-iRe\langle\psi,\frac{\partial}{\partial x}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial y}}\psi\rangle-\frac{1}{6}\langle R(\psi,\psi)\psi,\psi\rangle\}dz^2. \tag{4.9}$$

5. Analytic aspects and some Lemmas

In this section, we shall show some analytic aspects of Dirac-harmonic maps with curvature term and recall some lemmas which will be useful in subsequent sections.

We start with an ϵ-regularity theorem for Dirac-harmonic maps with curvature term. This kind of estimate was first introduced by Sacks-Uhlenbeck [16] which has been extended to harmonic maps [9] and Dirac-harmonic maps [6]. We refer to [3, Proposition 3.2.] for a similar result following the method in [6]. Here we provide a different proof.

Theorem 5.1 (ϵ_0-regularity theorem). *There is a small constant $\epsilon_0 > 0$ such that if (ϕ, ψ) is a Dirac-harmonic map with curvature term from the unit disc D in $\mathbb{R}^2$ to a compact Riemannian manifold (N, g) and satisfies*

$$E(\phi, \psi; D) = \int_D (|d\phi|^2 + |\psi|^4) < \epsilon_0, \tag{5.1}$$

then

$$\|d\phi\|_{L^\infty(D_{\frac{1}{2}})} \leq C(\|d\phi\|_{L^2(D)} + \|\psi\|^4_{L^4(D)}), \quad \|\psi\|_{L^\infty(D_{\frac{1}{2}})} \leq C\|\psi\|_{L^4(D)}, \tag{5.2}$$

where $C > 0$ is a constant depending only on N.

Proof. Without loss of generality, we assume $\frac{1}{\pi}\int_D \phi = 0$. Choosing a cut-off function $\eta \in C_0^\infty(D)$ satisfying $0 \leq \eta \leq 1, \eta|_{D_{3/4}} \equiv 1, |\nabla\eta| + |\nabla^2\eta| \leq C$, we have

$$\begin{aligned} |\Delta(\eta\phi)| &= |\eta\Delta\phi + 2\nabla\eta\nabla\phi + \phi\Delta\eta| \\ &\leq C\left(|\phi| + |d\phi| + |d\phi||\eta d\phi| + |\psi|^2|\eta d\phi| + |\psi|^4\right) \\ &\leq C(|d\phi| + |\psi|^2)|d(\eta\phi)| + C\left(|\phi| + |d\phi| + |\psi|^4\right). \end{aligned}$$

By the standard second order elliptic estimates, for any $1 < p < 2$, we have

$$\begin{aligned} \|\eta\phi\|_{W^{2,p}(D)} &\leq C\|(|d\phi| + |\psi|^2)|d(\eta\phi)|\|_{L^p(D)} + C(\|d\phi\|_{L^p(D)} + \||\psi|^4\|_{L^p(D)}) \\ &\leq C\|d(\eta\phi)\|_{L^{\frac{2p}{2-p}}(D)}\||d\phi| + |\psi|^2\|_{L^2(D)} + C(\|d\phi\|_{L^p(D)} + \||\psi|^4\|_{L^p(D)}) \\ &\leq C\sqrt{\epsilon_0}\|d(\eta\phi)\|_{L^{\frac{2p}{2-p}}(D)} + C(\|d\phi\|_{L^p(D)} + \||\psi|^4\|_{L^p(D)}), \end{aligned}$$

where we used the Poincare's inequality $\|\phi\|_{L^p(D)} \leq C(p)\|d\phi\|_{L^P(D)}$.

Taking $p = \frac{4}{3}$ and $\epsilon_0 > 0$ sufficiently small, we have

$$\begin{aligned} \|\eta\phi\|_{W^{1,4}(D)} &\leq \|\eta\phi\|_{W^{2,4/3}(D)} \\ &\leq C(\|d\phi\|_{L^{4/3}(D)} + \||\psi|^4\|_{L^{4/3}(D)}). \end{aligned} \tag{5.3}$$

Again, for any $1 < p < 2$, using the elliptic estimates Lemma 5.4, we have

$$\begin{aligned} \|\eta\psi\|_{W^{1,p}(D)} &\leq C\|\partial\!\!\!/(\eta\psi)\|_{L^p(D)} \\ &\leq C\|\nabla\eta\cdot\psi + \eta\partial\!\!\!/\psi\|_{L^p(D)} \\ &\leq C\left(\|\psi\|_{L^p(D)} + \||d\phi||\eta\psi|\|_{L^p(D)} + \||\psi|^2|\eta\psi|\|_{L^p(D)}\right) \\ &\leq C(\|d\phi\|_{L^2(D)} + \||\psi|^2\|_{L^2(D)})\|\eta\psi\|_{L^{\frac{2p}{2-p}}(D)} + C\|\psi\|_{L^p(D)} \\ &\leq C\sqrt{\epsilon_0}\|\eta\psi\|_{L^{\frac{2p}{2-p}}(D)} + C\|\psi\|_{L^p(D)}. \end{aligned}$$

Taking $p = \frac{3}{2}$ and $\epsilon_0 > 0$ sufficiently small, we have

$$\|\eta\psi\|_{L^6(D)} \leq \|\eta\psi\|_{W^{1,3/2}(D)} \leq C\|\psi\|_{L^4(D)}. \tag{5.4}$$

Thus, combining this with (5.3) and taking a suitable cut-off function η, we get

$$\|d\phi\|_{L^4(D_{2/3})} \leq C(\|d\phi\|_{L^2(D)} + \|\psi\|^4_{L^4(D)}). \tag{5.5}$$

Then it is easy to see that the conclusion (5.2) follows from the standard higher elliptic estimates. □

Next, we show a gap theorem for Dirac-harmonic maps with curvature term, which is a special case of Lemma 4.9 in [2]. The proof we give here is different from that in [2].

Proposition 5.2. *Assume that the pair (ϕ, ψ) is a smooth Dirac-harmonic map with curvature term from a standard sphere S^2 to a compact Riemannian manifold N satisfying*

$$\int_{S^2} (|d\phi|^2 + |\psi|^4) < \epsilon_0$$

with ϵ_0 small enough. Then both ϕ and ψ are trivial.

Proof. Step1. Claim: $\|\psi\|_{L^{4/3}(S^2)} \leq C\|\partial\!\!\!/\psi\|_{L^{4/3}(S^2)}$, where ψ is a spinor on S^2 and $C > 0$ is a universal constant.

In fact, if not, then there exists a sequence of spinors $\{\psi_k\}$ on S^2 such that

$$\|\psi_k\|_{L^{4/3}(S^2)} > k\|\partial\!\!\!/\psi_k\|_{L^{4/3}(S^2)}.$$

Without loss of generality, we assume $\|\psi_k\|_{L^{4/3}(S^2)} = 1$, then we have

$$\|\partial\!\!\!/\psi_k\|_{L^{4/3}(S^2)} < \frac{1}{k}. \tag{5.6}$$

By standard elliptic estimates, we get

$$\|\psi_k\|_{W^{1,4/3}(S^2)} \leq C.$$

Thus, there exists a subsequence of $\{\psi_k\}$(we still denote it by $\{\psi_k\}$) and $\eta \in W^{1,4/3}(S^2)$ satisfying

$$\psi_k \to \eta \quad \textit{weakly} \quad \textit{in} \quad W^{1,4/3}(S^2) \quad \textit{and} \quad \textit{strongly} \quad \textit{in} \quad L^{4/3}(S^2). \tag{5.7}$$

Combining this with $\|\psi_k\|_{L^{4/3}(S^2)} = 1$ and the inequality (5.6), we get $\|\eta\|_{L^{4/3}(S^2)} = 1$ and

$$\|\partial\!\!\!/\eta\|_{L^{4/3}(S^2)} = 0. \tag{5.8}$$

So, $\eta = 0$ since on S^2 there is no nontrivial harmonic spinor. This is a contradiction.

Step2. By the standard elliptic estimates, we have

$$\begin{aligned}
\|\psi\|_{L^4(S^2)} &\leq C\|\psi\|_{W^{1,4/3}(S^2)} \\
&\leq C(\|\partial\!\!\!/\psi\|_{L^{4/3}(S^2)} + \|\psi\|_{L^{4/3}(S^2)}) \\
&\leq C\|\partial\!\!\!/\psi\|_{L^{4/3}(S^2)} \\
&\leq C(\||d\phi||\psi|\|_{L^{4/3}(S^2)} + \||\psi|^3\|_{L^{4/3}(S^2)}) \\
&\leq C(\|d\phi\|_{L^2(S^2)} + \|\psi\|^2_{L^4(S^2)})\|\psi\|_{L^4(S^2)} \\
&\leq C\epsilon_0\|\psi\|_{L^4(S^2)}.
\end{aligned}$$

Since ϵ_0 is sufficiently small, we have $\psi = 0$. So

$$\begin{aligned}\||d\phi|\|_{W^{1,4/3}(S^2)} &\le C\|\Delta\phi\|_{L^{4/3}(S^2)} \\ &\le C\||d\phi|^2\|_{L^{4/3}(S^2)} \\ &\le C\||d\phi|\|_{L^2(S^2)}\||d\phi|\|_{L^4(S^2)} \\ &\le C\||d\phi|\|_{L^2(S^2)}\||d\phi|\|_{W^{1,4/3}(S^2)} \le C\epsilon_0\||d\phi|\|_{W^{1,4/3}(S^2)}.\end{aligned}$$

Thus ϕ has to be a constant map. □

The following lemma is a Pohozaev type identity for Dirac-harmonic maps with curvature terms, which is crucial in the subsequent sections, where we will show the singularity removability theorem and estimate the energy of ϕ on the necks in the bubbling process. (For a similar but different identity see [3, Lemma 3.11]. That identity is based on a quadratic differential that is different from ours, and in fact, not holomorphic.)

Lemma 5.3 (Pohozaev identity). *Let $D \subset \mathbb{R}^2$ be the unit disk and (ϕ, ψ) be a smooth Dirac-harmonic map with curvature term on $D \setminus \{0\}$ satisfying $\|d\phi\|_{L^2(D)} + \|\psi\|_{L^4(D)} \le C$, then for any $0 < r < \frac{1}{2}$, we have*

$$\int_0^{2\pi}(r^2|\frac{\partial\phi}{\partial r}|^2 - |\frac{\partial\phi}{\partial\theta}|^2)d\theta = \int_0^{2\pi} Re(\langle\psi, \partial_\theta\cdot\widetilde{\nabla}_{\partial\theta}\psi\rangle)d\theta - \frac{r^2}{6}\int_0^{2\pi}\langle R(\psi,\psi)\psi,\psi\rangle d\theta, \tag{5.9}$$

where (r, θ) are polar coordinates in D centered at 0.

Proof. We shall follow the approach developed for harmonic maps in [16], which was extended to the case of Dirac-harmonic maps in [6, Lemma 4.5].

First, it is easy to check that (ϕ, ψ) is a weak solution on D and hence, by standard elliptic estimates, we may assume

$$\|\nabla\psi\|_{L^{4/3}(D_{\frac{1}{2}})} \le C$$

since we have $\|d\phi\|_{L^2(D)} + \|\psi\|_{L^4(D)} \le C$.

From Corollary 4.2,

$$T(z) = \{|\phi_x|^2 - |\phi_y|^2 - 2i\langle\phi_x,\phi_y\rangle + Re\langle\psi, \frac{\partial}{\partial x}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial x}}\psi\rangle - iRe\langle\psi, \frac{\partial}{\partial x}\cdot\widetilde{\nabla}_{\frac{\partial}{\partial y}}\psi\rangle - \frac{1}{6}\langle R(\psi,\psi)\psi,\psi\rangle\} \tag{5.10}$$

is holomorphic in $D \setminus \{0\}$. Noting that $|\widetilde{\nabla}\psi| \le C(|\nabla\psi| + |d\phi||\psi|)$, we have

$$\int_{D_{\frac{1}{2}}}|T| \le C\int_{D_{\frac{1}{2}}}(|d\phi|^2 + |\psi|^4 + |\psi||\nabla\psi| + |d\phi||\psi|^2) \le C < \infty.$$

Hence $T(z)$ has a pole of order at most one, which implies that $zT(z)$ is holomorphic on $D_{\frac{1}{2}}$ and

$$0 = Im\int_{|z|=r} zT(z)dz = \int_0^{2\pi} Re\left(z^2T(z)\right)d\theta$$

by the Cauchy theorem. One can check

$$Re(z^2T(z)) = r^2|\frac{\partial\phi}{\partial r}|^2 - |\frac{\partial\phi}{\partial\theta}|^2 - Re(\langle\psi, \partial_\theta\cdot\widetilde{\nabla}_{\partial\theta}\psi\rangle) + \frac{r^2}{6}\langle R(\psi,\psi)\psi,\psi\rangle.$$

Then the lemma follows. □

Finally, for convenience, we present some lemmas which will be used in this paper.

Lemma 5.4 (Lemma 4.7 in [6]). *Let u be a complex function satisfying*

$$\begin{cases} \overline{\partial} u = f & \text{in } D, \\ u|_{\partial D} = \varphi, \end{cases} \tag{5.11}$$

with $\varphi \in W^{1,p}(\partial D)$ and $f \in L^p(D)$ for some $p > 1$, where D is the unit disk in $\mathbb{R}^2$ centered at the origin, then the following estimate holds

$$\|u\|_{W^{1,p}(D)} \leq C(\|f\|_{L^p(D)} + \|\varphi\|_{W^{1,p}(\partial D)}), \tag{5.12}$$

where $C > 0$ is a universal constant.
If instead of (5.11), u satisfies

$$\begin{cases} \partial u = f & \text{in } D, \\ u|_{\partial D} = \varphi, \end{cases} \tag{5.13}$$

then the same estimate as above holds.

Lemma 5.5 (Lemma 3.1 in [13]). *Suppose $u \in C^\infty([-2,2] \times S^1, \mathbb{C}^K)$, $v \in C^\infty([-2,2] \times S^1, \mathbb{C}^K)$ satisfy*

$$\Delta u = A^1 u + A^2 \nabla u + A^3 v + \frac{1}{2\pi}\int_0^{2\pi} A^4 u + A^5 \nabla u + A^6 v d\theta, \tag{5.14}$$

$$\overline{\partial} v = B^1 u + B^2 \nabla u + B^3 v + \frac{1}{2\pi}\int_0^{2\pi} B^4 u + B^5 \nabla u + B^6 v d\theta, \tag{5.15}$$

where $A^i, B^j \in C^\infty([-2,2] \times S^1, \mathbb{C}^K)$,$i = 1, ..., 6$ and $j = 1, ..., 6$. Assume $\sum_{i=1}^6 \|A^i\|_{L^\infty} + \sum_{j=1}^6 \|B^j\|_{L^\infty} \leq C < \infty$, then

$$\|u\|_{W^{2,2}([-1,1]\times S^1)} \leq C(\|u\|_{L^2([-2,2]\times S^1)} + \|v\|_{L^2([-2,2]\times S^1)}), \tag{5.16}$$

$$\|v\|_{W^{1,2}([-1,1]\times S^1)} \leq C(\|u\|_{L^2([-2,2]\times S^1)} + \|v\|_{L^2([-2,2]\times S^1)}). \tag{5.17}$$

Let $\Sigma = [0, K] \times S^1$ for a fixed number K. We assume $K = lL$ for some integer l and a large fixed universal number L. Let us denote $P_i = [(i-1)L, iL] \times S^1$ and

$$\|(u, v)\|^2_{L^2(P_i)} = \int_{[(i-1)L, iL]\times S^1} (|u|^2 + |v|^2) dt d\theta.$$

Proposition 5.6 (Proposition 3.3 in [13]). *Suppose $u \in C^\infty(\Sigma, \mathbb{C}^K)$, $v \in C^\infty(\Sigma, \mathbb{C}^K)$ satisfy the equations (5.14) and (5.15) respectively. Assume l and L are given with L large. Then there exists a positive number δ_0 such that, if $\|A^j\|_\infty \leq \delta_0$, $\|B^j\|_\infty \leq \delta_0$ for $j = 1, ..., 6$ and*

$$|\int_{(i-1)L\times S^1} u d\theta|, |\int_{(i-1)L\times S^1} v d\theta|, |\int_{iL\times S^1} u d\theta|, |\int_{iL\times S^1} v d\theta| \leq \delta_0, \tag{5.18}$$

then, for $2 \leq i \leq l-1$,
(a)$\|(u,v)\|_{L^2(P_{i+1})} \leq e^{-\frac{1}{2}L}\|(u,v)\|_{L^2(P_i)}$ implies $\|(u,v)\|_{L^2(P_i)} \leq e^{-\frac{1}{2}L}\|(u,v)\|_{L^2(P_{i-1})}$;

$(b)\|(u,v)\|_{L^2(P_{i-1})} \le e^{-\frac{1}{2}L}\|(u,v)\|_{L^2(P_i)}$ *implies* $\|(u,v)\|_{L^2(P_i)} \le e^{-\frac{1}{2}L}\|(u,v)\|_{L^2(P_{i+1})}$;
(c)*either* $\|(u,v)\|_{L^2(P_i)} \le e^{-\frac{1}{2}L}\|(u,v)\|_{L^2(P_{i+1})}$ *or* $\|(u,v)\|_{L^2(P_i)} \le e^{-\frac{1}{2}L}\|(u,v)\|_{L^2(P_{i-1})}$.

Remark 5.7. In fact, in Lemma 3.1 and Proposition 3.3 of [13], u, v, A^i, B^i, $i = 1, ..., 6$ are real valued functions. However, one can easily check that the proofs are completely the same for complex valued functions.

6. Removable singularities and blow-up analysis

For the blow-up analysis, we need the following removable singularity theorem:

Theorem 6.1 (Theorem 3.12 in [3]). *A smooth Dirac-harmonic map with curvature term (ϕ, ψ) on $D \setminus \{0\}$ with finite energy can be smoothly extended to D.*

Proof. With the help of the Pohozeve identity for Dirac-harmonic maps with curvature term, one can apply similar arguments as in the case of Dirac-harmonic maps [6, Theorem 4.6] to prove this theorem. We shall provide a detailed proof in the appendix. □

Now, we begin to prove our main Theorem 2.1.

Proof. Following the same procedure as the case of approximate harmonic maps in Ding-Tian's paper [9], it suffices to consider the case that there is only one blow-up point $\mathcal{S} = \{p\}$ and there is only one bubble. For cases of two or more bubbles, where a bubble tree forms, one can refer to [12, 14, 20].

By the ϵ-regularity Theorem 5.1, after taking a subsequence, (ϕ_n, ψ_n) converges strongly to some limit Dirac-harmonic map with curvature term $(\phi, \psi) : M \to N$ on $M \setminus D_\delta(p)$ for any $\delta > 0$. Then what we need to prove is that there exists a Dirac-harmonic sphere with curvature term $(\sigma, \xi) : S^2 \to N$ such that

$$\lim_{\delta\to 0}\lim_{n\to\infty} E(\phi_n; D_\delta) = E(\sigma), \tag{6.1}$$

$$\lim_{\delta\to 0}\lim_{n\to\infty} E(\psi_n; D_\delta) = E(\xi), \tag{6.2}$$

and $\phi(D_\delta) \cup \sigma(S^2)$ is a connected set.

By a standard argument of blow-up analysis, there exist $\lambda_n \to 0$ and $x_n \to p$ as $n \to \infty$ such that, for each (ϕ_n, ψ_n),

$$E(\phi_n, \psi_n; D_{\lambda_n}(x_n)) = \sup_{r\le\lambda_n, D_r(x)\subset D_\delta(p)} E(\phi_n, \psi_n; D_r(x)) = \frac{\epsilon_0}{2} > 0.$$

Denoting

$$\phi_n'(x) := \phi_n(x_n + \lambda_n x), \psi_n'(x) := \lambda_n^{1/2}\psi_n(x_n + \lambda_n x), \tag{6.3}$$

then for any $D_1(y) \subset \mathbb{R}^2$, there holds

$$E(\phi_n', \psi_n'; D_1(y)) \le E(\phi_n, \psi_n; D_{\lambda_n}(x_n)) = \frac{\epsilon_0}{2},$$

when n is big enough.

By the ϵ-regularity Theorem 5.1, we can take a subsequence, still denoted by (ϕ'_n, ψ'_n), that strongly converges to some (σ, ξ) in $W^{1,2}(D_R, N) \times L^4(\Sigma D_R \times \mathbb{R}^K)$ for any $R \geq 1$. Thus we get a nonconstant Dirac-harmonic map with curvature term (σ, ξ) on $\mathbb{R}^2$ with bounded energy. By conformal invariance and removable singularity, we get a nonconstant Dirac-harmonic maps with curvature term (σ, ξ) on the whole S^2. So we get the first bubble at the blow-up point p.

We may assume $x_n = 0$. It is well known that under the one bubble assumption, energy identity and neckless property are equivalent to

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{n\to\infty} E(\phi_n, D_\delta \setminus D_{\lambda_n R}) + \lim_{\delta\to 0}\lim_{R\to\infty}\lim_{n\to\infty} E(\psi_n, D_\delta \setminus D_{\lambda_n R}) = 0 \tag{6.4}$$

and

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{n\to\infty} Osc_{D_\delta \setminus D_{\lambda_n R}} \phi_n = 0. \tag{6.5}$$

To prove (6.4) and (6.5), firstly, we introduce a new coordinate system. Let (r, θ) be polar coordinates centered at 0. Let $f : \mathbb{R}^1 \times \mathbb{S}^1 \to \mathbb{R}^2$, $f(t, \theta) = (e^{-t}, \theta)$ $(t, \theta) \in \mathbb{R}^1 \times \mathbb{S}^1$ where $\mathbb{R}^1 \times \mathbb{S}^1$ is equipped with the metric $g = dt^2 + d\theta^2$, which is conformal to the standard Euclidean metric ds^2 on $\mathbb{R}^2$. In fact,

$$(f^{-1})^* g = \frac{1}{r^2} ds^2.$$

For convenience, we will respectively denote

$$\Phi_n := \phi_n \circ f \quad and \quad \Psi_n := e^{-\frac{t}{2}} \psi_n \circ f. \tag{6.6}$$

Denoting $T_0 := -\log\delta$, $T_1 := -\log(r_n R)$, then $D_\delta \setminus D_{r_n R}$ changes to $\Sigma := [T_0, T_1] \times S^1$.

Without loss of generality, we assume $T_1 = T_0 + l_n L$ for some integer l_n and a fixed number L (given in Proposition 5.6). For $1 \leq i \leq l_n$, we also denote $P_i = [T_0 + (i-1)L, T_0 + iL] \times S^1$ and

$$\|(\Phi, \Psi)\|^2_{L^2(P_i)} = \int_{P_i} (|\Phi|^2 + |\Psi|^2) dt d\theta.$$

It is easy to see that (6.4) and (6.5) are equivalent to

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{n\to\infty} E(\Phi_n, \Sigma) + \lim_{\delta\to 0}\lim_{R\to\infty}\lim_{n\to\infty} E(\Psi_n, \Sigma) = 0 \tag{6.7}$$

and

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{n\to\infty} Osc_\Sigma \Phi_n = 0. \tag{6.8}$$

Secondly, we claim: for any $\epsilon > 0$, we have

$$\|d\Phi_n\|_{L^\infty(\Sigma)} + \|\Psi_n\|_{L^\infty(\Sigma)} \leq C\epsilon, \tag{6.9}$$

when n, R and $\frac{1}{\delta}$ are sufficiently large.

In fact, by [9], for any $\epsilon > 0$, we know when n,R and $\frac{1}{\delta}$ are big enough,

$$\|d\Phi_n\|_{L^2(P_{i-1}\cup P_i \cup P_{i+1})} + \|\Psi_n\|_{L^4(P_{i-1}\cup P_i \cup P_{i+1})} < \epsilon \tag{6.10}$$

for the one bubble case. By the ϵ_0-regularity Theorem 5.1, we obtain

$$\|d\Phi_n\|_{L^\infty(P_i)} + \|\Psi_n\|_{L^\infty(P_i)} \leq C\epsilon \tag{6.11}$$

for $1 \le i \le l_n$. Then, the inequality (6.9) follows immediately.

Thirdly, we extend the second fundamental form A to a small tubular neighborhood of the target manifold N, in which A,∇A and $\nabla^2 A$ are uniform bounded depending only on N. Also, we can do this for $\mathcal{A}$ and P.

For simplicity, we will denote Φ_n, Ψ_n by Φ and Ψ respectively.

We define $\Phi^*(t)$ and $\Psi^*(t)$ as follows:

$$\Phi^*(t) = \frac{1}{2\pi}\int_0^{2\pi} \Phi d\theta \quad and \quad \Psi^*(t) = \frac{1}{2\pi}\int_0^{2\pi} \Psi d\theta. \tag{6.12}$$

By (6.9), the difference between Φ and Φ^* is very small. So, Φ^* must live in a small tubular neighborhood of N. Then the function $A(\Phi^*)(d\Phi^*, d\Phi^*)$ is well-defined.

Next, we use the same method as in [13] to compute the equation for $(\Phi - \Phi^*, \Psi - \Psi^*)$. Here, for reader's convenience, we repeat this process again.

Since the equations (3.7) and (3.8) for Dirac-harmonic maps with curvature term are conformally invariant, by equation (3.7), we have

$$\begin{aligned}
\Delta\Phi^*(t) &= \frac{1}{2\pi}\int_0^{2\pi} -A(\Phi)(d\Phi, d\Phi) - Re(P(\Phi)\left(\mathcal{A}(\Phi)(d\Phi(e_\alpha), e_\alpha\cdot\Psi); \Psi\right)) \\
&\quad +\frac{1}{6}\langle\nabla A_{ij}, A_{kl}\rangle\langle\Psi^i, \Psi^j\rangle\langle\Psi^k, \Psi^l\rangle - \frac{1}{6}\langle\nabla A_{il}, A_{jk}\rangle Re(\langle\Psi^i, \Psi^j\rangle\langle\Psi^k, \Psi^l\rangle)d\theta \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned}$$

Computing directly, we have

$$\begin{aligned}
\mathrm{I} &= \frac{1}{2\pi}\int_0^{2\pi} A(\Phi)(d\Phi, d\Phi) - A(\Phi^*)(d\Phi, d\Phi) + A(\Phi^*)(d\Phi, d\Phi) - A(\Phi^*)(d\Phi^*, d\Phi^*) \\
&\quad +A(\Phi^*)(d\Phi^*, d\Phi^*)d\theta \\
&= A(\Phi^*)(d\Phi^*, d\Phi^*) + \frac{1}{2\pi}\int_0^{2\pi} A^4(\Phi - \Phi^*) + A^5\nabla(\Phi - \Phi^*)d\theta,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{II} &= \frac{1}{2\pi}Re\int_0^{2\pi} P(\Phi)\left(\mathcal{A}(\Phi)(d\Phi(e_\alpha), e_\alpha\cdot\Psi); \Psi\right) - P(\Phi^*)\left(\mathcal{A}(\Phi)(d\Phi(e_\alpha), e_\alpha\cdot\Psi); \Psi\right) \\
&\quad +P(\Phi^*)\left(\mathcal{A}(\Phi)(d\Phi(e_\alpha), e_\alpha\cdot\Psi); \Psi\right) - P(\Phi^*)\left(\mathcal{A}(\Phi^*)(d\Phi(e_\alpha), e_\alpha\cdot\Psi); \Psi\right) \\
&\quad +P(\Phi^*)\left(\mathcal{A}(\Phi^*)(d\Phi(e_\alpha), e_\alpha\cdot\Psi); \Psi\right) - P(\Phi^*)\left(\mathcal{A}(\Phi^*)(d\Phi^*(e_\alpha), e_\alpha\cdot\Psi); \Psi\right) \\
&\quad +P(\Phi^*)\left(\mathcal{A}(\Phi^*)(d\Phi^*(e_\alpha), e_\alpha\cdot\Psi); \Psi\right) - P(\Phi^*)\left(\mathcal{A}(\Phi^*)(d\Phi^*(e_\alpha), e_\alpha\cdot\Psi^*); \Psi\right) \\
&\quad +P(\Phi^*)\left(\mathcal{A}(\Phi^*)(d\Phi^*(e_\alpha), e_\alpha\cdot\Psi^*); \Psi\right) - P(\Phi^*)\left(\mathcal{A}(\Phi^*)(d\Phi^*(e_\alpha), e_\alpha\cdot\Psi^*); \Psi^*\right) \\
&\quad +P(\Phi^*)\left(\mathcal{A}(\Phi^*)(d\Phi^*(e_\alpha), e_\alpha\cdot\Psi^*); \Psi^*\right) d\theta \\
&= Re(P(\Phi^*)\left(\mathcal{A}(\Phi^*)(d\Phi^*(e_\alpha), e_\alpha\cdot\Psi^*); \Psi^*\right)) + \frac{1}{2\pi}\int_0^{2\pi} A^4(\Phi - \Phi^*) + A^5\nabla(\Phi - \Phi^*) \\
&\quad +\frac{1}{2\pi}Re\int_0^{2\pi} A^8(\Psi - \Psi^*)d\theta,
\end{aligned}$$

where A^i may differ from line to line and just stands for a symbol satisfying $\|A^i\|_\infty \le C\epsilon$ for $i = 4, 5, 8$.

III and VI can be dealt with in the same way, and we get

$$\begin{aligned}
\Delta(\Phi - \Phi^*) &= A(\Phi)(d\Phi, d\Phi) - A(\Phi^*)(d\Phi^*, d\Phi^*) \\
&\quad + ReP(\Phi)\left(\mathcal{A}(\Phi)(d\Phi(e_\alpha), e_\alpha \cdot \Psi); \Psi\right) - ReP(\Phi^*)\left(\mathcal{A}(\Phi^*)(d\Phi^*(e_\alpha), e_\alpha \cdot \Psi^*); \Psi^*\right) \\
&\quad + \frac{1}{6}\langle \nabla A_{ij}(\Phi), A_{kl}(\Phi)\rangle \langle \Psi^i, \Psi^j\rangle\langle \Psi^k, \Psi^l\rangle - \frac{1}{6}\langle \nabla A_{ij}(\Phi^*), A_{kl}(\Phi^*)\rangle \langle \Psi^{*i}, \Psi^{*j}\rangle\langle \Psi^{*k}, \Psi^{*l}\rangle \\
&\quad - \frac{1}{6}\langle \nabla A_{il}(\Phi), A_{jk}(\Phi)\rangle Re(\langle \Psi^i, \Psi^j\rangle\langle \Psi^k, \Psi^l\rangle) \\
&\quad + \frac{1}{6}\langle \nabla A_{il}(\Phi^*), A_{jk}(\Phi^*)\rangle Re(\langle \Psi^{*i}, \Psi^{*j}\rangle\langle \Psi^{*k}, \Psi^{*l}\rangle) \\
&\quad - \frac{1}{2\pi}\int_0^{2\pi} A^4(\Phi - \Phi^*) + A^5\nabla(\Phi - \Phi^*) + A^6(\Psi - \Psi^*)d\theta \\
&\quad - \frac{1}{2\pi} Re\int_0^{2\pi} A^8(\Psi - \Psi^*)d\theta.
\end{aligned}$$

Using the same method, we get

$$\begin{aligned}
\Delta(\Phi - \Phi^*) &= A^1(\Phi - \Phi^*) + A^2\nabla(\Phi - \Phi^*) + A^3(\Psi - \Psi^*) + Re(A^7(\Psi - \Psi^*)) \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} A^4(\Phi - \Phi^*) + A^5\nabla(\Phi - \Phi^*) + A^6(\Psi - \Psi^*)d\theta \\
&\quad + \frac{1}{2\pi} Re\int_0^{2\pi} A^8(\Psi - \Psi^*)d\theta,
\end{aligned} \tag{6.13}$$

where $\|A^i\|_\infty \le C\epsilon$ for $i = 1, ..., 8$.

Similarly, we have

$$\begin{aligned}
\partial\!\!\!/(\Psi - \Psi^*) &= B^1(\Phi - \Phi^*) + B^2\nabla(\Phi - \Phi^*) + B^3(\Psi - \Psi^*) \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} B^4(\Phi - \Phi^*) + B^5\nabla(\Phi - \Phi^*) + B^6(\Psi - \Psi^*)d\theta,
\end{aligned} \tag{6.14}$$

where $\|B^i\|_\infty \le C\epsilon$ for $i = 1, ..., 6$.

Noting that $Re(A^7(\Psi - \Psi^*))$ and $\frac{1}{2\pi}Re\int A^8(\Psi - \Psi^*)d\theta$ are linear terms and $\|A^j\|_{L^\infty} \le C\epsilon$, $j = 7, 8$, one can easily find that the proofs of Lemma 5.5 and Proposition 5.6 still hold if we add these terms in the equation (5.14) and (5.15). So, for simplicity, we will put these terms into $A^3(\Psi - \Psi^*)$ and $\frac{1}{2\pi}\int A^6(\Psi - \Psi^*)d\theta$ in the sequel.

By (c) of Proposition 5.6, we obtain

$$\begin{aligned}
\|(\Phi - \Phi^*, \Psi - \Psi^*)\|_{L^2(P_i)} &\le e^{-\frac{1}{2}L}\|(\Phi - \Phi^*, \Psi - \Psi^*)\|_{L^2(P_{i+1})} \quad or \\
\|(\Phi - \Phi^*, \Psi - \Psi^*)\|_{L^2(P_i)} &\le e^{-\frac{1}{2}L}\|(\Phi - \Phi^*, \Psi - \Psi^*)\|_{L^2(P_{i-1})}.
\end{aligned}$$

Then, using (a) and (b) of Proposition 5.6, by iterating, we have

$$\begin{aligned}
\|(\Phi - \Phi^*, \Psi - \Psi^*)\|_{L^2(P_i)} &\le e^{-\frac{i}{2}L}\|(\Phi - \Phi^*, \Psi - \Psi^*)\|_{L^2(P_1)} \quad or \\
\|(\Phi - \Phi^*, \Psi - \Psi^*)\|_{L^2(P_i)} &\le e^{-\frac{l_n - i}{2}L}\|(\Phi - \Phi^*, \Psi - \Psi^*)\|_{L^2(P_{l_n})}.
\end{aligned}$$

So, we can get the decay for $\|\Phi-\Phi^*\|_{L^2(P_i)}$ and $\|\Psi-\Psi^*\|_{L^2(P_i)}$, that is

$$\|\Phi-\Phi^*\|_{L^2(P_i)} \leq (e^{-\frac{i}{2}L}+e^{-\frac{l_n-i}{2}L})(\|\Phi-\Phi^*\|_{L^2(P_1)}+\|\Phi-\Phi^*\|_{L^2(P_{l_n})}), \tag{6.15}$$

$$\|\Psi-\Psi^*\|_{L^2(P_i)} \leq (e^{-\frac{i}{2}L}+e^{-\frac{l_n-i}{2}L})(\|\Psi-\Psi^*\|_{L^2(P_1)}+\|\Psi-\Psi^*\|_{L^2(P_{l_n})}). \tag{6.16}$$

Applying Lemma 5.5 to equation (6.13) and equation (6.14), we obtain the energy decay in the θ-direction,

$$\begin{aligned}
\|\frac{\partial\Phi}{\partial\theta}\|_{L^2(P_i)} &\leq \|\nabla(\Phi-\Phi^*)\|_{L^2(P_i)}\\
&\leq C\left(\|\Phi-\Phi^*\|_{L^2(P_i\cup P_{i-1}\cup P_{i+1})}+\|\Psi-\Psi^*\|_{L^2(P_i\cup P_{i-1}\cup P_{i+1})}\right)\\
&\leq C\left(e^{-\frac{i}{2}L}+e^{-\frac{l_n-i}{2}L}\right)\left(\|\Phi-\Phi^*\|_{L^2(P_1\cup P_{l_n})}+\|\Psi-\Psi^*\|_{L^2(P_1\cup P_{l_n})}\right)\\
&\leq C\left(e^{-\frac{i}{2}L}+e^{-\frac{l_n-i}{2}L}\right)\epsilon
\end{aligned} \tag{6.17}$$

and

$$\begin{aligned}
\|\frac{\partial\Psi}{\partial\theta}\|_{L^2(P_i)} &\leq \|\nabla(\Psi-\Psi^*)\|_{L^2(P_i)}\\
&\leq C\left(\|\Phi-\Phi^*\|_{L^2(P_i\cup P_{i-1}\cup P_{i+1})}+\|\Psi-\Psi^*\|_{L^2(P_i\cup P_{i-1}\cup P_{i+1})}\right)\\
&\leq C\left(e^{-\frac{i}{2}L}+e^{-\frac{l_n-i}{2}L}\right)\left(\|\Phi-\Phi^*\|_{L^2(P_1\cup P_{l_n})}+\|\Psi-\Psi^*\|_{L^2(P_1\cup P_{l_n})}\right)\\
&\leq C\left(e^{-\frac{i}{2}L}+e^{-\frac{l_n-i}{2}L}\right)\epsilon.
\end{aligned} \tag{6.18}$$

From Lemma 5.3, we know

$$\int_{P_i}|\frac{\partial\Phi}{\partial t}|^2-\int_{P_i}|\frac{\partial\Phi}{\partial\theta}|^2=\int_{P_i}Re(\langle\Psi,\partial_\theta\cdot\widetilde{\nabla}_{\partial_\theta}\Psi\rangle)-\int_{P_i}\frac{1}{6}\langle R(\Psi,\Psi)\Psi,\Psi\rangle d\theta dt,$$

where $P_i=[T_0+(i-1)L,T_0+iL]\times S^1$. Since

$$\frac{\partial\Psi}{\partial\theta}=\widetilde{\nabla}_{\partial_\theta}\Psi+\sum_{i=1}^K\Psi^i\otimes A(d\Phi(\frac{\partial}{\partial\theta}),\frac{\partial}{\partial z^i}),$$

where $\frac{\partial\Psi}{\partial\theta}=(\frac{\partial\Psi^1}{\partial\theta},...,\frac{\partial\Psi^K}{\partial\theta})$, we get

$$\int_{P_i}|\frac{\partial\Phi}{\partial t}|^2\leq C\left(e^{-\frac{i}{2}L}+e^{-\frac{l_n-i}{2}L}\right)\epsilon+C\int_{P_i}|\Psi|^4,$$

and then

$$\int_{P_i}|d\Phi|^2\leq C\left(e^{-\frac{i}{2}L}+e^{-\frac{l_n-i}{2}L}\right)\epsilon+C\int_{P_i}|\Psi|^4. \tag{6.19}$$

We will get the conclusion (6.7) and (6.8) from Corollary 6.3 and Corollary 6.5, which will be presented later in this section. Then we will finish our proof of Theorem 2.1. □

Lemma 6.2.

$$\lim_{\delta\to0}\lim_{R\to\infty}\lim_{n\to\infty}\int_{D_\delta\setminus D_{\lambda_nR}}\frac{|\psi_n|^2}{|x|}dx=0. \tag{6.20}$$

Proof. The key of the proof is the Hardy-type inequality on $\mathbb{R}^2$ that for any $f \in C_0^\infty(\mathbb{R}^2 \backslash \{0\})$, there holds

$$\|\frac{f}{|x|}\|_{L^1(\mathbb{R}^2)} \leq \|\nabla f\|_{L^1(\mathbb{R}^2)} \tag{6.21}$$

where the constant 1 is the best possible constant (for a simple proof, see [1]).

We choose the cut-off function $\eta \in C_0^\infty(D_\delta \setminus D_{\lambda_n R})$ such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on $D_{\frac{1}{2}\delta} \setminus D_{2\lambda_n R}$ and

$$\begin{aligned} |\nabla \eta| &\leq \frac{C}{\delta} \quad on \quad D_\delta \setminus D_{\frac{1}{2}\delta} \quad and \\ |\nabla \eta| &\leq \frac{C}{\lambda_n R} \quad on \quad D_{2\lambda_n R} \setminus D_{\lambda_n R}. \end{aligned}$$

Taking $f = \eta|\psi_n|^2$ in the inequality (6.21), we get

$$\begin{aligned} \|\eta \frac{|\psi_n|^2}{|x|}\|_{L^1(\mathbb{R}^2)} &\leq \|\nabla(\eta|\psi_n|^2)\|_{L^1(\mathbb{R}^2)} \\ &\leq \|2\eta\psi_n \nabla\psi_n\|_{L^1(\mathbb{R}^2)} + \|\nabla\eta|\psi_n|^2\|_{L^1(\mathbb{R}^2)} \\ &\leq \|2\eta\psi_n \frac{1}{|x|}\frac{\partial\psi_n}{\partial\theta}\|_{L^1(\mathbb{R}^2)} + \|2\eta\psi_n \frac{\partial\psi_n}{\partial r}\|_{L^1(\mathbb{R}^2)} + \|\nabla\eta|\psi_n|^2\|_{L^1(\mathbb{R}^2)}. \end{aligned}$$

On the one hand, we know

$$\frac{\partial\psi_n}{\partial r} = \frac{\partial}{\partial r} \cdot \frac{1}{|x|}\frac{\partial}{\partial\theta} \cdot \frac{1}{|x|}\frac{\partial\psi_n}{\partial\theta} + \frac{\partial}{\partial r} \cdot (\mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi) + F(\psi,\psi)\psi)$$

from equation (3.8). So, we have

$$|2\eta\psi_n \frac{\partial\psi_n}{\partial r}| \leq |2\eta\psi_n \frac{1}{|x|}\frac{\partial\psi_n}{\partial\theta}| + C|\eta|d\phi_n||\psi_n|^2| + C|\eta|\psi_n|^4|.$$

On the other hand, by inequality (6.9), we have

$$|x||d\phi_n| + \sqrt{|x|}|\psi_n| \leq \epsilon \quad on \quad D_\delta \setminus D_{\lambda_n R}. \tag{6.22}$$

Combining these, we get

$$\begin{aligned} \|\eta \frac{|\psi_n|^2}{|x|}\|_{L^1(\mathbb{R}^2)} &\leq 4\|\eta\psi_n \frac{1}{|x|}\frac{\partial\psi_n}{\partial\theta}\|_{L^1(\mathbb{R}^2)} + C\|\eta|d\phi_n||\psi_n|^2\|_{L^1(\mathbb{R}^2)} + C\|\eta|\psi_n|^4\|_{L^1(\mathbb{R}^2)} + \|\nabla\eta|\psi_n|^2\|_{L^1(\mathbb{R}^2)} \\ &\leq 4\|\eta\psi_n \frac{1}{|x|}\frac{\partial\psi_n}{\partial\theta}\|_{L^1(\mathbb{R}^2)} + C\epsilon\|\eta\frac{|\psi_n|^2}{|x|}\|_{L^1(\mathbb{R}^2)} + \|\nabla\eta|\psi_n|^2\|_{L^1(\mathbb{R}^2)}. \end{aligned}$$

Since we can take ϵ sufficiently small, we then have

$$\begin{aligned}
\|\eta\frac{|\psi_n|^2}{|x|}\|_{L^1(\mathbb{R}^2)} &\leq C\|\psi_n\frac{1}{|x|}\frac{\partial\psi_n}{\partial\theta}\|_{L^1(D_\delta\setminus D_{\lambda_n R})}+C\|\nabla\eta|\psi_n|^2\|_{L^1(D_\delta\setminus D_{\lambda_n R})}\\
&= C\|\Psi_n\frac{\partial\Psi_n}{\partial\theta}\|_{L^1(\Sigma)}+C\|\nabla\eta|\psi_n|^2\|_{L^1(D_\delta\setminus D_{\lambda_n R})}\\
&\leq C\sum_{i=1}^{l_n}\|\Psi_n\frac{\partial\Psi_n}{\partial\theta}\|_{L^1(P_i)}+C\|\nabla\eta|\psi_n|^2\|_{L^1(D_\delta\setminus D_{\lambda_n R})}\\
&\leq C\sum_{i=1}^{l_n}\|\frac{\partial\Psi_n}{\partial\theta}\|_{L^1(P_i)}+C\|\nabla\eta|\psi_n|^2\|_{L^1(D_\delta\setminus D_{\lambda_n R})}\\
&\leq C\sum_{i=1}^{l_n}\|\frac{\partial\Psi_n}{\partial\theta}\|_{L^2(P_i)}+C\|\nabla\eta|\psi_n|^2\|_{L^1(D_\delta\setminus D_{\lambda_n R})}\\
&\leq C\epsilon\sum_{i=1}^{l_n}(e^{-\frac{i}{2}L}+e^{-\frac{l_n-i}{2}L})+C\frac{1}{\delta}\||\psi_n|^2\|_{L^1(D_\delta\setminus D_{\frac{1}{2}\delta})}+C\frac{1}{\lambda_n R}\||\psi_n|^2\|_{L^1(D_{2\lambda_n R}\setminus D_{\lambda_n R})}\\
&\leq C\epsilon+C\|\psi_n\|^2_{L^4(D_\delta\setminus D_{\frac{1}{2}\delta})}+C\|\psi_n\|^2_{L^4(D_{2\lambda_n R}\setminus D_{\lambda_n R})}\\
&\leq C\epsilon
\end{aligned}$$

where the last inequality is from (6.10).

Thus,

$$\int_{D_{\frac{1}{2}\delta}\setminus D_{2\lambda_n R}}\frac{|\psi_n|^2}{|x|}dx\leq C\epsilon.$$

Combining this with the assumption (6.10) again, we have

$$\int_{D_\delta\setminus D_{\lambda_n R}}\frac{|\psi_n|^2}{|x|}dx\leq C\epsilon.$$

□

As a corollary of the above lemma, we will immediately get the energy identities.

Corollary 6.3.

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{n\to\infty}E(\Phi_n,\Sigma)+\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{n\to\infty}E(\Psi_n,\Sigma)=0. \tag{6.23}$$

Proof. Firstly, from Lemma 6.2 and (6.22), we get

$$E(\Psi_n,\Sigma)=\int_{D_\delta\setminus D_{\lambda_n R}}|\psi_n|^4dx\leq\int_{D_\delta\setminus D_{\lambda_n R}}\frac{|\psi_n|^2}{|x|}dx\leq C\epsilon,$$

that is

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{n\to\infty}E(\Psi_n,\Sigma)=0. \tag{6.24}$$

Next, by inequality (6.19), we have

$$E(\Phi_n,\Sigma)=\sum_{i=1}^{l_n}\int_{P_i}|d\Phi_n|^2\le C\epsilon+C\int_\Sigma|\Psi_n|^4\le C\epsilon.$$

Then we can get the conclusion

$$\lim_{\delta\to0}\lim_{R\to\infty}\lim_{n\to\infty}E(\Phi_n,\Sigma)+\lim_{\delta\to0}\lim_{R\to\infty}\lim_{n\to\infty}E(\Psi_n,\Sigma)=0. \tag{6.25}$$

□

So far, we have proved the energy identities. In order to get the no neck property, we need the following exponential decay estimates.

Lemma 6.4. *For any $T_0+1\le t_0\le T_1-1$, we have*

$$\int_{D_{e^{-t_0+1}}\setminus D_{e^{-t_0-1}}}\frac{|\psi_n|^2}{|x|}dx\le C\epsilon\big(e^{-\frac14(t_0-T_0)}+e^{-\frac14(T_1-t_0)}\big) \tag{6.26}$$

where $T_0=-\log\delta$, $T_1=-\log(\lambda_nR)$.

Proof. We define

$$f(t):=\int_{D_{e^{-t_0+t}}\setminus D_{e^{-t_0-t}}}\frac{|\psi_n|^2}{|x|}dx$$

for any $t_0\in(T_0,T_1)$ and $0\le t\le\min\{|t_0-T_0|,|t_0-T_1|\}$.

For any $\sigma>0$, taking the cut-off function $\eta\in C_0^\infty(D_{e^{-t_0+t}+\sigma}\setminus D_{e^{-t_0-t}-\sigma})$ such that $0\le\eta\le1$ and $\eta\equiv1$ on $D_{e^{-t_0+t}}\setminus D_{e^{-t_0-t}}$ and $|\nabla\eta|\le\frac{2}{\sigma}$. Taking $f=\eta|\psi_n|^2$ in the Hardy inequality (6.21) and applying similar arguments as in the proof of Lemma 6.2, we will get

$$\begin{aligned}\|\eta\frac{|\psi_n|^2}{|x|}\|_{L^1(\mathbb{R}^2)}&\le\|\nabla(\eta|\psi_n|^2)\|_{L^1(\mathbb{R}^2)}\\&\le\|2\eta\psi_n\nabla\psi_n\|_{L^1(\mathbb{R}^2)}+\|\nabla\eta|\psi_n|^2\|_{L^1(\mathbb{R}^2)}\\&\le\|2\eta\psi_n\frac{1}{|x|}\frac{\partial\psi_n}{\partial\theta}\|_{L^1(\mathbb{R}^2)}+\|2\eta\psi_n\frac{\partial\psi_n}{\partial r}\|_{L^1(\mathbb{R}^2)}+\|\nabla\eta|\psi_n|^2\|_{L^1(\mathbb{R}^2)}\\&\le4\|\eta\psi_n\frac{1}{|x|}\frac{\partial\psi_n}{\partial\theta}\|_{L^1(\mathbb{R}^2)}+C\|\eta|d\phi_n||\psi_n|^2\|_{L^1(\mathbb{R}^2)}+C\|\eta|\psi_n|^4\|_{L^1(\mathbb{R}^2)}+\|\nabla\eta|\psi_n|^2\|_{L^1(\mathbb{R}^2)}\\&\le4\|\eta\psi_n\frac{1}{|x|}\frac{\partial\psi_n}{\partial\theta}\|_{L^1(\mathbb{R}^2)}+C\epsilon\|\eta\frac{|\psi_n|^2}{|x|}\|_{L^1(\mathbb{R}^2)}+\|\nabla\eta|\psi_n|^2\|_{L^1(\mathbb{R}^2)}.\end{aligned}$$

Taking $\epsilon > 0$ sufficiently small such that $C\epsilon \leq \frac{1}{2}$, we have

$$\begin{aligned}
\|\eta\frac{|\psi_n|^2}{|x|}\|_{L^1(\mathbb{R}^2)} &\leq 8\|\psi_n\frac{1}{|x|}\frac{\partial\psi_n}{\partial\theta}\|_{L^1(D_{e^{-t_0+t_{+\sigma}}}\setminus D_{e^{-t_0-t_{-\sigma}}})} + 2\|\nabla\eta|\psi_n|^2\|_{L^1(D_{e^{-t_0+t_{+\sigma}}}\setminus D_{e^{-t_0-t_{-\sigma}}})} \\
&\leq 8\|\Psi_n\frac{\partial\Psi_n}{\partial\theta}\|_{L^1([t_0-t-1,t_0+t+1]\times S^1))} + 2\|\nabla\eta|\psi_n|^2\|_{L^1(D_{e^{-t_0+t_{+\sigma}}}\setminus D_{e^{-t_0-t_{-\sigma}}})} \\
&\leq C\|\frac{\partial\Psi_n}{\partial\theta}\|_{L^2([t_0-t-1,t_0+t+1]\times S^1)} + \frac{4}{\sigma}\||\psi_n|^2\|_{L^1(D_{e^{-t_0+t_{+\sigma}}}\setminus D_{e^{-t_0+t}})} \\
&\quad + \frac{4}{\sigma}\||\psi_n|^2\|_{L^1(D_{e^{-t_0-t}}\setminus D_{e^{-t_0-t_{-\sigma}}})} \\
&\leq C\epsilon\big(e^{-\frac{1}{2}(t_0-t-T_0)} + e^{-\frac{1}{2}(T_1-t_0-t)}\big) + \frac{4}{\sigma}\||\psi_n|^2\|_{L^1(D_{e^{-t_0+t_{+\sigma}}}\setminus D_{e^{-t_0+t}})} \\
&\quad + \frac{4}{\sigma}\||\psi_n|^2\|_{L^1(D_{e^{-t_0-t}}\setminus D_{e^{-t_0-t_{-\sigma}}})}
\end{aligned}$$

where σ is small and the last inequality is from (6.18).

Letting $\sigma \to 0$, we get

$$\begin{aligned}
f(t) &= \int_{D_{e^{-t_0+t}}\setminus D_{e^{-t_0-t}}} \frac{|\psi_n|^2}{|x|}dx \\
&\leq C\epsilon\big(e^{-\frac{1}{2}(t_0-t-T_0)} + e^{-\frac{1}{2}(T_1-t_0-t)}\big) + 4e^{-t_0+t}\int_{\partial D_{e^{-t_0+t}}} \frac{|\psi_n|^2}{|x|} \\
&\quad + 4e^{-t_0-t}\int_{\partial D_{e^{-t_0-t}}} \frac{|\psi_n|^2}{|x|} \\
&\leq C\epsilon\big(e^{-\frac{1}{2}(t_0-t-T_0)} + e^{-\frac{1}{2}(T_1-t_0-t)}\big) + 4f'(t).
\end{aligned}$$

This is

$$\begin{aligned}
(e^{-\frac{1}{4}t}f(t))' &\geq -C\epsilon e^{-\frac{1}{4}t}(e^{-\frac{1}{2}(t_0-t-T_0)} + e^{-\frac{1}{2}(T_1-t_0-t)}) \\
&= -C\epsilon e^{\frac{1}{4}t}\big(e^{-\frac{1}{2}(t_0-T_0)} + e^{-\frac{1}{2}(T_1-t_0)}\big).
\end{aligned} \tag{6.27}$$

Without loss of generality, we may assume $t_0 - T_0 \leq T_1 - t_0$. Then, integrating the above ODE from 1 to $t_0 - T_0$, we get

$$\begin{aligned}
f(1) &\leq Ce^{-\frac{1}{4}(t_0-T_0)}f(t_0-T_0) + C\epsilon e^{-\frac{1}{2}(t_0-T_0)}\int_1^{t_0-T_0} e^{\frac{1}{4}t}dt \\
&\leq C\epsilon e^{-\frac{1}{4}(t_0-T_0)}
\end{aligned}$$

where the second inequality follows from Lemma 6.2.

In the case of $T_1 - t_0 \leq t_0 - T_0$, we can apply similar arguments to get

$$f(1) \leq C\epsilon e^{-\frac{1}{4}(T_1-t_0)}.$$

Thus, we get the exponential decay estimate (6.26). □

Corollary 6.5.

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{n\to\infty} OSC_{\Sigma}\Phi_n = 0. \tag{6.28}$$

Proof. By Lemma 6.4, we have

$$\begin{aligned}\int_{P_i}|\Psi_n|^4 dtd\theta &= \int_{D_{e^{-(T_0+(i-1)L)}}\setminus D_{e^{-(T_0+iL)}}}|\psi_n|^4 dx\\ &\le C\epsilon\int_{D_{e^{-(T_0+(i-1)L)}}\setminus D_{e^{-(T_0+iL)}}}\frac{|\psi_n|^2}{|x|}dx\\ &\le C\epsilon\left(e^{-\frac{i}{4}L}+e^{-\frac{l_n-i}{4}L}\right).\end{aligned}$$

Combining this with (6.19), we get

$$\begin{aligned}\int_{P_i}|d\Phi_n|^2 &\le C\left(e^{-\frac{i}{2}L}+e^{-\frac{l_n-i}{2}L}\right)\epsilon + C\int_{P_i}|\Psi_n|^4\\ &\le C\left(e^{-\frac{i}{2}L}+e^{-\frac{l_n-i}{2}L}\right)\epsilon + C\left(e^{-\frac{i}{4}L}+e^{-\frac{l_n-i}{4}L}\right)\epsilon\\ &\le C\left(e^{-\frac{i}{4}L}+e^{-\frac{l_n-i}{4}L}\right)\epsilon. \qquad (6.29)\end{aligned}$$

So, we have the energy decay

$$\|\nabla\Phi_n\|_{L^2(P_i)} \le C\left(e^{-\frac{i}{8}L}+e^{-\frac{l_n-i}{8}L}\right)\epsilon^{1/2}. \qquad (6.30)$$

Thus, from Theorem 5.1, we have

$$\begin{aligned}Osc_{\Sigma}\Phi_n &\le C\sum_{i=1}^{l_n}(\|d\Phi_n\|_{L^2(P_i)}+\|\Psi_n\|^4_{L^4(P_i)})\\ &\le C\sum_{i=1}^{l_n}\|d\Phi_n\|_{L^2(P_i)} + C\|\Psi_n\|^4_{L^4(\Sigma)}\\ &\le C\epsilon^{1/2}.\end{aligned}$$

This finishes the proof. □

7. APPENDIX

Now, we prove the removable singularity Theorem 6.1.

Proof. With the help of Lemma 5.3, the proof is similar to Theorem 4.6 in [6] which uses the idea of Sacks-Uhlenbeck [16].

Since (ϕ,ψ) has finite energy, by a rescaling transformation, we may assume

$$\int_D(|d\phi|^2+|\psi|^4)<\epsilon^2.$$

According to Theorem 5.1, we have $|x||d\phi|+|x|^{\frac{1}{2}}|\psi|\le C\epsilon$.

Define

$$\phi^*(r)=\frac{1}{2\pi}\int_0^{2\pi}\phi(r,\theta)d\theta,$$

then we have $|\phi(x)-\phi^*(x)|\leq C\epsilon$ and (cf. [16])

$$\int_{D_r}\nabla\phi\nabla(\phi-\phi^*)=\int_{\partial D_r}(\phi-\phi^*)\frac{\partial\phi}{\partial r}-\int_{D_r}(\phi-\phi^*)\Delta\phi.$$

On one hand, we have

$$\begin{aligned}\int_{\partial D_r}(\phi-\phi^*)\frac{\partial\phi}{\partial r}&\leq(\int_{\partial D_r}|\phi-\phi^*|^2)^{\frac{1}{2}}(\int_{\partial D_r}|\frac{\partial\phi}{\partial r}|^2)^{\frac{1}{2}}\\&\leq(\int_{\partial D_r}|\frac{\partial\phi}{\partial\theta}|^2)^{\frac{1}{2}}(\int_{\partial D_r}|\frac{\partial\phi}{\partial r}|^2)^{\frac{1}{2}}\\&\leq r\int_{\partial D_r}|d\phi|^2\end{aligned}$$

and

$$\begin{aligned}\int_{D_r}(\phi-\phi^*)\Delta\phi&\leq C\epsilon\int_{D_r}|d\phi|^2+C\epsilon\int_{D_r}|d\phi||\psi|^2+C\epsilon\int_{D_r}|\psi|^4\\&\leq C\epsilon\int_{D_r}|d\phi|^2+C\epsilon\int_{D_r}|\psi|^4,\end{aligned}$$

where we used the equation (3.7).

On the other hand, we have

$$\begin{aligned}\int_{D_r}\nabla\phi\nabla(\phi-\phi^*)&=\int_{D_r}|\nabla\phi|^2-\int_{D_r}\frac{\partial\phi}{\partial r}\frac{\partial\phi^*}{\partial r}\\&\geq\int_{D_r}|\nabla\phi|^2-(\int_{D_r}|\frac{\partial\phi}{\partial r}|^2)^{\frac{1}{2}}(\int_{D_r}|\frac{\partial\phi^*}{\partial r}|^2)^{\frac{1}{2}}\\&\geq\int_{D_r}|\nabla\phi|^2-\int_{D_r}|\frac{\partial\phi}{\partial r}|^2\\&=\frac{1}{2}\int_{D_r}|\nabla\phi|^2-\frac{1}{2}\int_{D_r}(|\frac{\partial\phi}{\partial r}|^2-\frac{1}{r^2}|\frac{\partial\phi}{\partial\theta}|^2).\end{aligned}$$

From the above, we get

$$\begin{aligned}\frac{1}{2}\int_{D_r}|\nabla\phi|^2&\leq r\int_{\partial D_r}|d\phi|^2+C\epsilon\int_{D_r}|d\phi|^2+C\epsilon\int_{D_r}|\psi|^4+\frac{1}{2}\int_{D_r}(|\frac{\partial\phi}{\partial r}|^2-\frac{1}{r^2}|\frac{\partial\phi}{\partial\theta}|^2)\\&\leq r\int_{\partial D_r}|d\phi|^2+C\epsilon\int_{D_r}|d\phi|^2+C\epsilon\int_{D_r}|\psi|^4\\&\quad+\int_{D_r}\frac{1}{2r^2}|Re(\langle\psi,\partial_\theta\cdot\widetilde{\nabla}_{\partial\theta}\psi\rangle)|+\frac{1}{12}\int_{D_r}|\langle R(\psi,\psi)\psi,\psi\rangle|\\&\leq r\int_{\partial D_r}|d\phi|^2+C\epsilon\int_{D_r}|d\phi|^2+C\int_{D_r}|\psi|^4+C\int_{D_r}|\psi||\nabla\psi|\\&\leq r\int_{\partial D_r}|d\phi|^2+C\epsilon\int_{D_r}|d\phi|^2+C\int_{D_r}|\psi|^4+C\int_{D_r}|\nabla\psi|^{\frac{4}{3}}.\end{aligned}\tag{7.1}$$

Choosing a cut-off function $\eta_\rho \in C_0^\infty(D_{2\rho})$ such that $\eta_\rho \equiv 1$ in D_ρ and $|d\eta_\rho| \le \frac{C}{\rho}$, by the equation (3.8), we have

$$\partial\!\!\!/((1-\eta_\rho)\psi) = (1-\eta_\rho)\mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi) + (1-\eta_\rho)F(\psi,\psi)\psi - \nabla\eta_\rho \cdot \psi.$$

Then, from Lemma 5.4, we get

$$\begin{aligned}\|(1-\eta_\rho)\psi\|_{W^{1,\frac{4}{3}}(D_1)} \le{}& C\|d\phi\|_{L^2(D_1)}\|\psi\|_{L^4(D_1)} + C\|\psi\|^3_{L^4(D_1)} \\ &+ Cr^{\frac{3}{4}}\|\nabla\psi\|_{L^{\frac{4}{3}}(\partial D_1)} + Cr^{\frac{1}{4}}\|\psi\|_{L^4(\partial D_1)} + C\|\nabla\eta_\rho\cdot\psi\|_{L^{\frac{4}{3}}(D_1)}.\end{aligned}$$

Noting that

$$\lim_{\rho\to 0}\frac{1}{\rho}\int_{D_{2\rho}}|\psi|^{\frac{4}{3}} = 0$$

and the smallness of $\|d\phi\|_{L^2(D_1)} + \|\psi\|_{L^4(D_1)}$ and the Sobolev embedding theorem, we obtain

$$\int_{D_1}|\psi|^4 \le C(\int_{\partial D_1}|\nabla\psi|^{\frac{4}{3}})^3 + C\int_{\partial D_1}|\psi|^4.$$

By rescaling, we have for any $0 \le r \le 1$

$$\begin{aligned}\int_{D_r}|\psi|^4 &\le C(r\int_{\partial D_r}|\nabla\psi|^{\frac{4}{3}})^3 + Cr\int_{\partial D_r}|\psi|^4 \\ &\le Cr\int_{\partial D_r}|\nabla\psi|^{\frac{4}{3}} + Cr\int_{\partial D_r}|\psi|^4. \qquad (7.2)\end{aligned}$$

Noting that $\overline{\psi} := \frac{1}{2\pi}\int_{\partial D_1}\psi$ is a constant, by equation (3.8), we have

$$\partial\!\!\!/(\psi - \overline{\psi}) = \mathcal{A}(d\phi(e_\alpha), e_\alpha\cdot(\psi-\overline{\psi})) + F(\psi,\psi)(\psi-\overline{\psi}) + \mathcal{A}(d\phi(e_\alpha), e_\alpha\cdot\overline{\psi}) + F(\psi,\psi)\overline{\psi}$$

in $D_1 \setminus \{0\}$.

Using a similar argument as above, we have

$$\begin{aligned}\|\psi-\overline{\psi}\|_{W^{1,4/3}(D_1)} \le{}& C(\|d\phi\|_{L^2(D_1)} + \|\psi\|^2_{L^4(D_1)})\|\psi-\overline{\psi}\|_{L^4(D_1)} \\ &+ C\||d\phi||\overline{\psi}| + |\psi|^2|\overline{\psi}|\|_{L^{4/3}(D_1)} + C\|\psi-\overline{\psi}\|_{W^{1,4/3}(\partial D_1)} \\ \le{}& C(\|d\phi\|_{L^2(D_1)} + \|\psi\|^2_{L^4(D_1)})\|\psi-\overline{\psi}\|_{W^{1,4/3}(D_1)} \\ &+ C\||d\phi||\overline{\psi}| + |\psi|^2|\overline{\psi}|\|_{L^{4/3}(D_1)} + C\|\nabla\psi\|_{L^{4/3}(\partial D_1)},\end{aligned}$$

where the second inequality comes from the Sobolev embedding and Poincare inequality.

Also, by the smallness of $\|d\phi\|_{L^2(D_1)} + \|\psi\|_{L^4(D_1)}$, we have

$$\begin{aligned}\|\nabla\psi\|_{L^{4/3}(D_1)} &\le C\||d\phi||\overline{\psi}| + |\psi|^2|\overline{\psi}|\|_{L^{4/3}(D_1)} + C\|\nabla\psi\|_{L^{4/3}(\partial D_1)} \\ &\le C|\overline{\psi}|\|d\phi\|_{L^2(D_1)} + C|\overline{\psi}|\|\psi\|^2_{L^4(D_1)} + C\|\nabla\psi\|_{L^{4/3}(\partial D_1)} \\ &\le C\|\psi\|_{L^4(\partial D_1)}\|d\phi\|_{L^2(D_1)} + C\|\psi\|_{L^4(\partial D_1)}\|\psi\|^2_{L^4(D_1)} + C\|\nabla\psi\|_{L^{4/3}(\partial D_1)}.\end{aligned}$$

So we get

$$\int_{D_1} |\nabla\psi|^{\frac{4}{3}} \le C\int_{\partial D_1} |\nabla\psi|^{\frac{4}{3}} + C(\int_{\partial D_1} |\psi|^4)^{\frac{1}{3}}(\int_{D_1} |d\phi|^2)^{\frac{2}{3}} + C(\int_{\partial D_1} |\psi|^4)^{\frac{1}{3}}(\int_{D_1} |\psi|^4)^{\frac{2}{3}}$$
$$\le C\int_{\partial D_1} |\nabla\psi|^{\frac{4}{3}} + \epsilon_1(\int_{D_1} |d\phi|^2 + \int_{D_1} |\psi|^4) + \frac{C}{\epsilon_1}(\int_{\partial D_1} |\psi|^4)$$

where ϵ_1 is a small constant. Hence, for $0 \le r \le 1$,

$$\int_{D_r} |\nabla\psi|^{\frac{4}{3}} \le Cr\int_{\partial D_r} |\nabla\psi|^{\frac{4}{3}} + \epsilon_1(\int_{D_r} |d\phi|^2 + \int_{D_r} |\psi|^4) + \frac{Cr}{\epsilon_1}(\int_{\partial D_r} |\psi|^4). \tag{7.3}$$

Combining (7.1),(7.2) with (7.3), we have for any $0 \le r \le 1$

$$\begin{aligned} &\int_{D_r} |d\phi|^2 + \int_{D_r} |\psi|^4 + \int_{D_r} |\nabla\psi|^{\frac{4}{3}} \\ &\le Cr(\int_{\partial D_r} |d\phi|^2 + \int_{\partial D_r} |\psi|^4 + \int_{\partial D_r} |\nabla\psi|^{\frac{4}{3}}). \end{aligned} \tag{7.4}$$

Denoting $F(r) := \int_{D_r} |d\phi|^2 + \int_{D_r} |\psi|^4 + \int_{D_r} |\nabla\psi|^{\frac{4}{3}}$, then this implies

$$F(r) \le CrF'(r).$$

Integrating this inequality yields

$$F(r) \le F(1)r^{\frac{1}{C}}. \tag{7.5}$$

By Theorem 5.1, we can easily conclude

$$\phi \in W^{1,2p}(D_1), \quad \psi \in W^{1,q}(D_1) \tag{7.6}$$

for some $p > 1$ and $q > \frac{4}{3}$. Higher regularity follows by the standard bootstrap method. One can refer to [6]. This completes the proof of Theorem 6.1. ∎

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