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by

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We study the geometric global quantum discord (GGQD) of two-qubit systems. We give an approach for deriving analytical formulae of GGQD for arbitrary two-qubit states. Detailed examples are presented.

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I. INTRODUCTION

The correlations between the subsystems A and B of a bipartite system play significant roles in many information processing tasks [1]. Such correlations can classified according to the probability distributions of the measurement outcomes from measuring the subsystems A and B. For any quantum entangled states, the probability distributions of the measurement outcomes from measuring the subsystem A will depend on the probability distributions of the measurement outcomes from measuring the subsystem B. Nevertheless, it is still possible that the correlations between the measurement outcomes from measuring the subsystem A and from measuring the subsystem B can be described by classical probability distributions. A quantum state is said to admit a local hidden variable model if all the measurement outcomes can be modeled as a classical random distribution over a probability space. The states admitting LHV models do not violate any Bell inequalities. While the states that do not admit any LHV models violate at least one Bell inequality.

For any separable states, the probability distributions of the measurement outcomes from measuring the subsystem A are independent of the probability distributions of the measurement outcomes from measuring the subsystem B. However, these separable states may be further classified as classically correlated states and quantum correlated ones, depending on the possibility to learn all the mutual information by measuring one of the subsystems. Such property is characterized by so called quantum discord [2–5]. It has been shown that the quantum discord is required for some information processing like assisted optimal state discrimination [7].

In recent years more relevant quantities such as geometric quantum discord (GQD) [6, 8, 9] have been proposed. However, in the original definitions both the quantum discord and the geometric quantum discord are not symmetric with respect to the subsystems. From a symmetric extension of the quantum discord the global quantum discord has been presented [10]. Furthermore, a geometric quantum discord for multipartite states, called geometric global quantum discord (GGQD), has been proposed [11]. Nevertheless, similar to the original discord, it is extremely difficult to compute the GGQD for generally given quantum states. In this article, we study the GGQD for arbitrary two-qubit systems. We derive explicit expressions of GGQD for arbitrary two-qubit states.

The paper is organized as follows. In section II we review the GQD and GGQD. We derive the analytical formula of GGQD for arbitrary two-qubit states. In section III, as examples we present the GGQD for X-states. Conclusions and discussions are given in section IV.

II. GEOMETRIC GLOBAL QUANTUM DISCORD OF TWO-QUBIT STATES

For a bipartite state ρ_{AB} in a composite system AB, the total correlation between A and B is measured by the quantum mutual information

$$I(\rho_{AB}) = S(\rho_A) - S(\rho_A|\rho_B),$$

where ρ_A, ρ_B are the reduced density matrices associated with the subsystems A and B, $S(\rho_A|\rho_B)$ is conditional entropy, $S(\rho) = -Tr(\rho \log_2 \rho)$ is the Von Neuman entropy. One may also introduce the following quantity to characterize the quantum mutual information,

$$J(\rho_{AB}) = S(\rho_A) - S(\rho_{AB}|\{\Pi_B^j\}),$$

where $S(\rho_{AB}|\{\prod_B^j\}) = \sum_j p_j S(\rho_{A|j})$, $\rho_{A|j} = \frac{1}{p_j}\langle b_j|\rho_{AB}|b_j\rangle$, $\prod_B^j = |b_j\rangle\langle b_j|$ is a set of projectors, p_j denotes the probability of obtaining the jth measurement outcome.

The quantities $I(\rho_{AB})$ and $J(\rho_{AB})$ are equivalent in the classical case. but distinct in the quantum case. The difference defined by $D(\rho_{AB}) = I(\rho_{AB}) - J(\rho_{AB})$ is called the discord of the ρ_{AB}. As the measurement is signal measurement of bipartite system, the global quantum discord $D(\rho_{A_1A_2\cdots A_N})$ for an arbitrary multipartite state $\rho_{A_1A_2\cdots A_N}$ is defined by,

$$D(\rho_{A_1A_2\cdots A_N}) = \min_{\{\Pi_k\}}[S(\rho_{A_1A_2\cdots A_N})\|\Phi(\rho_{A_1A_2\cdots A_N}) - \sum_{j=1}^{N} S(\rho_{A_j}\|\Phi_j(\rho_{A_j}))],$$

under all local measurements $\{\Pi_{A_1}^{j_1} \bigotimes \cdots \bigotimes \Pi_{A_N}^{j_N}\}$, where $\Phi_j(\rho_{A_j}) = \sum\limits_i \Pi_{A_i}^i \rho_{A_j} \Pi_{A_i}^i$ and $\Phi(\rho_{A_1A_2\cdots A_N}) = \sum\limits_k \Pi_k \rho_{A_1A_2\cdots A_N} \Pi_k$, with $\Pi_k = \Pi_{A_1}^{j_1} \bigotimes \cdots \bigotimes \Pi_{A_N}^{j_N}$ and k denoting the index string $(j_1 \cdots j_N)$.

Following the concept of global quantum discord, the geometric global quantum discord (GGQD) is defined by

$$D^{GG}(\rho_{A_1A_2\cdots A_N}) = \min_{\sigma_{A_1A_2\cdots A_N}} \{Tr[\rho_{A_1A_2\cdots A_N} - \sigma_{A_1A_2\cdots A_N}]^2 | D(\sigma_{A_1A_2\cdots A_N}) = 0\},$$

which is equivalent to [11],

$$D^{GG}(\rho_{A_1A_2\cdots A_N}) = \sum_{\alpha_1,\alpha_2,\cdots,\alpha_N} C^2_{\alpha_1\alpha_2\cdots\alpha_N} - \max_{\Pi} \sum_{i_1 i_2 \cdots i_N} (\sum_{\alpha_1,\alpha_2,\cdots,\alpha_N} A_{\alpha_1 i_1} A_{\alpha_2 i_2} \cdots A_{\alpha_N i_N} C_{\alpha_1\alpha_2\cdots\alpha_N})^2, \qquad (1)$$

where $C_{\alpha_1\alpha_2\cdots\alpha_N}$ and $A_{\alpha_k i_k}$ are determined as follows. For any k, $1 \leq k \leq N$, let $L(H_k)$ be the real Hilbert space consisting of all Hermitian operators on H_k, with the inner product $\langle X|X^T\rangle = Tr(XX^T)$ for X, $X^T \in L(H_k)$, for all k, and for given orthonormal basis $\{X_{\alpha_k}\}_{\alpha_k=1}^{n_k^2}$ of $L(H_k)$ and orthonormal basis $\{|i_k\rangle\}_{i_k=1}^{n_k}$ of H_k. $C_{\alpha_1\alpha_2\cdots\alpha_N}$ and $A_{\alpha_k i_k}$ are given by the following equations,

$$\rho_{A_1A_2\cdots A_N} = \sum_{\alpha_1,\alpha_2,\cdots,\alpha_N} C_{\alpha_1\alpha_2\cdots\alpha_N} X_{\alpha_1} \bigotimes X_{\alpha_2} \bigotimes \cdots \bigotimes X_{\alpha_N}$$

and

$$A_{\alpha_k i_k} = \langle i_k | X_{\alpha_k} | i_k \rangle.$$

Consider now the GGQD of two-qubit states. For bipartite qubit states ρ_{AB}, Eq. (1) can be simplified,

$$D^{GG}(\rho_{AB}) = \sum_{\alpha_1,\alpha_2} C^2_{\alpha_1\alpha_2} - \max_{\Pi} \sum_{i_1 i_2} (\sum_{\alpha_1,\alpha_2} A_{\alpha_1 i_1} A_{\alpha_2 i_2} C_{\alpha_1\alpha_2})^2.$$

Moreover, $\{X_m = \frac{\sigma_m^A}{\sqrt{2}}\}$, $\{Y_n = \frac{\sigma_n^B}{\sqrt{2}}\}$ are the orthonormal bases, with σ_m^A, σ_n^B, $m,n = 0,1,2,3$, the Pauli matrices associated with the subsystems A and B respectively. Therefore,

$$D^{GG}(\rho_{AB}) = Tr(CC^T) - \max_{AB} tr(ACB^T BC^T A^T),$$

with $A = (A_{im})$, $B = (B_{jn})$, $A_{im} = Tr(|i\rangle\langle i|X_m)$, $B_{jn} = Tr(|j\rangle\langle j|Y_n)$, where $\{|i\rangle\}$ and $\{|j\rangle\}$ are any orthonormal bases. $C = (C_{mn})$ is given by $C_{mn} = tr\rho_{AB} X_m \bigotimes Y_n$. From a similar approach in [8], the matrices C, A and B can be written in the following forms,

$$C = (C_{mn}) = \frac{1}{2}\begin{pmatrix} 1 & y^T \\ x & T \end{pmatrix}, \tag{2}$$

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & a \\ 1 & -a \end{pmatrix}, \qquad a = (a_1, a_2, a_3) = \sqrt{2}(A_{11}, A_{12}, A_{13}),$$

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & b \\ 1 & -b \end{pmatrix}, \qquad b = (b_1, b_2, b_3) = \sqrt{2}(B_{11}, B_{12}, B_{13})$$

and

$$Tr(ACB^T BC^T A^T) = \frac{1}{4}[1 + y^T b^T b y + a(xx^T + Tb^T bT^T)a^T]. \tag{3}$$

Note that under local unitary transformations, any two-qubit state can write as

$$\rho_{AB} = \begin{pmatrix} \rho_{00} & \rho_{01} & \rho_{02} & \rho_{03} \\ \rho_{01}^* & \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{02}^* & \rho_{12}^* & \rho_{22} & \rho_{23} \\ \rho_{03}^* & \rho_{13}^* & \rho_{23}^* & \rho_{33} \end{pmatrix}.$$

Therefore

$$C = \frac{1}{2}\begin{pmatrix} \rho_{00}+\rho_{11}+\rho_{22}+\rho_{33} & 2(\rho_{01}+\rho_{23}) & 0 & \rho_{00}-\rho_{11}+\rho_{22}-\rho_{33} \\ 2(\rho_{02}+\rho_{13}) & 2(\rho_{12}+\rho_{03}) & 0 & 2(\rho_{02}-\rho_{13}) \\ 0 & 0 & 2(\rho_{12}-\rho_{03}) & 0 \\ \rho_{00}+\rho_{11}-\rho_{22}-\rho_{33} & 2(\rho_{01}-\rho_{23}) & 0 & \rho_{00}-\rho_{11}-\rho_{22}+\rho_{33} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} c_{00} & c_{01} & 0 & c_{03} \\ c_{10} & c_{11} & 0 & c_{13} \\ 0 & 0 & c_{22} & 0 \\ c_{30} & c_{31} & 0 & c_{33} \end{pmatrix}. \tag{4}$$

Then from Eq.(2) we have

$$x = \begin{pmatrix} 2(\rho_{02}+\rho_{13}) \\ 0 \\ \rho_{00}+\rho_{11}-\rho_{22}-\rho_{33} \end{pmatrix}, \tag{5}$$

$$y^T = \begin{pmatrix} 2(\rho_{01}+\rho_{23}) & 0 & \rho_{00}-\rho_{11}+\rho_{22}-\rho_{33} \end{pmatrix}, \tag{6}$$

$$T = \begin{pmatrix} 2(\rho_{12}+\rho_{03}) & 0 & 2(\rho_{02}-\rho_{13}) \\ 0 & 2(\rho_{12}-\rho_{03}) & 0 \\ 2(\rho_{01}-\rho_{23}) & 0 & \rho_{00}-\rho_{11}-\rho_{22}+\rho_{33} \end{pmatrix}. \tag{7}$$

Substituting Eq.(5)-(7) into Eq.(3), we obtain

$$\begin{aligned} Tr(ACB^TBC^TA^T) =& \frac{1}{4}[(c_{00}^2+c_{01}+c_{03}^2)+(c_{10}^2+c_{11}+c_{13}^2)a_1^2+(c_{30}^2+c_{31}+c_{33}^2)a_3^2+2(c_{10}c_{30}+c_{11}c_{31}+c_{13}c_{33})a_1a_3 \\ &+2c_{01}c_{03}b_1b_3+2c_{01}c_{12}a_2b_1b_2+2c_{03}c_{22}a_2b_2b_3+c_{22}^2a_2^2b_2^2 \\ &+2c_{11}c_{13}a_1^2b_1b_3+2c_{31}c_{33}a_3^2b_1b_3+2(c_{13}c_{33}+c_{13}c_{31})a_1a_3b_1b_3]. \end{aligned}$$

The key point to compute GGQD is to obtain the maximal value of $Tr(ACB^TBC^TA^T)$. Let

$$\begin{aligned} f =& (c_{00}^2+c_{01}+c_{03}^2)+(c_{10}^2+c_{11}+c_{13}^2)a_1^2+(c_{30}^2+c_{31}+c_{33}^2)a_3^2+2(c_{10}c_{30}+c_{11}c_{31}+c_{13}c_{33})a_1a_3 \\ &+2c_{01}c_{03}b_1b_3+2c_{01}c_{12}a_2b_1b_2+2c_{03}c_{22}a_2b_2b_3+c_{22}^2a_2^2b_2^2 \\ &+2c_{11}c_{13}a_1^2b_1b_3+2c_{31}c_{33}a_3^2b_1b_3+2(c_{13}c_{33}+c_{13}c_{31})a_1a_3b_1b_3. \end{aligned} \tag{8}$$

Set $M_0 = (c_{00}^2+c_{01}+c_{03}^2)+(c_{10}^2+c_{11}+c_{13}^2)a_1^2+(c_{30}^2+c_{31}+c_{33}^2)a_3^2+2(c_{10}c_{30}+c_{11}c_{31}+c_{13}c_{33})a_1a_3$, $M_{13} = 2c_{01}c_{03}+2c_{11}c_{13}a_1^2+2c_{31}c_{33}a_3^2+2(c_{11}c_{33}+c_{13}c_{31})a_1a_3$, $M_{12}=2c_{01}c_{22}a_2$, $M_{23}=2c_{03}c_{22}a_2$ and $M_{22}=c_{22^2}a_2^2$. Then $f=M_0+M_{13}b_1b_3+M_{12}b_1b_2+M_{23}b_2b_3+M_{22}b_2^2$. To obtain the maximal value of $Tr(ACB^TBC^TA^T)$ is just to obtain the maximal value of $\frac{1}{4}f$.

By taking a coordinate transformation $b_1=\cos\theta_1\sin\theta_2$, $b_2=\sin\theta_1\sin\theta_2$ and $b_3=\cos\theta_2$, we have

$$\begin{cases} \dfrac{\partial f}{\partial\theta_1} = -M_{13}\sin\theta_2\cos\theta_2\sin\theta_1+M_{23}\sin\theta_2\cos\theta_2\cos\theta_1-M_{12}\sin\theta_2\cos\theta_2\sin\theta_1+M_{22}\sin^2\theta_2\sin\theta_1\cos\theta_1=0, \\ \dfrac{\partial f}{\partial\theta_2} = M_{13}\cos\theta_1\cos^2\theta_2-M_{13}\cos\theta_1\sin^2\theta_2+M_{23}\sin\theta_1\cos^2\theta_2-M_{23}\sin\theta_1\sin^2\theta_2 \\ \quad +M_{12}\cos\theta_1\cos^2\theta_2-M_{12}\cos\theta_1\sin^2\theta_2+2M_{22}\sin^2\theta_1\sin\theta_2\cos\theta_2=0. \end{cases}$$

The solutions of the above two equations can be classified by the following twelve cases:

1. $\theta_2=0$, $\cos^2\theta_1=\frac{M_{23}^2}{(M_{12}+M_{13})^2+M_{23}^2}$, $\sin^2\theta_1=\frac{(M_{12}+M_{23})^2}{(M_{12}+M_{13})^2+M_{23}^2}$;
2. $\theta_2=\Pi$, $\cos^2\theta_1=\frac{M_{23}^2}{(M_{12}+M_{13})^2+M_{23}^2}$, $\sin^2\theta_1=\frac{(M_{12}+M_{23})^2}{(M_{12}+M_{13})^2+M_{23}^2}$;
3. $\theta_1=0$, $\theta_2=0$, $M_{13}+M_{12}=0$;
4. $\theta_1=0$, $\theta_2=\Pi$, $M_{13}+M_{12}=0$;
5. $\theta_1=0$, $\theta_2=\frac{\Pi}{4}$, $M_{23}=0$;
6. $\theta_1=0$, $\theta_2=\frac{3\Pi}{4}$, $M_{23}=0$;
7. $\theta_1=\Pi$, $\theta_2=0$, $M_{13}+M_{12}=0$;
8. $\theta_1=\Pi$, $\theta_2=\Pi$, $M_{13}+M_{12}=0$;
9. $\theta_1=\Pi$, $\theta_2=\frac{\Pi}{4}$, $M_{23}=0$;
10. $\theta_1=\Pi$, $\theta_2=\frac{3\Pi}{4}$, $M_{23}=0$;

11. $\cos^2\theta_1 = (M_{13} - M_{23} + M_{12})^2$,
$$\cos^2\theta_2 = \frac{M_{22}\sin^2\theta_1 + \sqrt{(M_{13}\cos\theta_1 + M_{23}\sin\theta_1 + M_{12}\cos\theta_1)^2 + M_{22}^2\sin^4\theta_1}}{2\sqrt{(M_{13}\cos\theta_1 + M_{23}\sin\theta_1 + M_{12}\cos\theta_1)^2 + M_{22}^2\sin^4\theta_1}};$$

12. $4M_{22}^2(M_{13} - M_{23} + M_{12})\cos\theta_1 - 4M_{22}^2(M_{13} - M_{23} + M_{12})\cos^3\theta_1 - 4M_{22}^2(M_{13} + M_{23})\cos^3\theta_1 + (M_{13} - M_{23} + M_{12})^2(M_{13} + M_{12})\cos\theta_1 - 4M_{22}^2 M_{23}\cos^2\theta_1\sin\theta_1 + (M_{13} - M_{23} + M_{12})^2 M_{23}\sin\theta_1 = 0$.

Substituting the above solutions of $\frac{\partial f}{\partial\theta_1} = \frac{\partial f}{\partial\theta_2} = 0$ into Eq. (8), one gets that f becomes a function of the parameters a_1, a_2 and a_3. Set further $a_1 = \cos\theta_3\sin\theta_4$, $a_2 = \sin\theta_3\sin\theta_4$, $a_3 = \cos\theta_4$ in $\max\limits_{\theta_1,\theta_2} f$. One can repeat the above procedure to find $\max\limits_{A,B} Tr(ACB^TBC^TA^T) = \frac{1}{4}\max\limits_{\theta_1,\theta_2,\theta_3,\theta_4} f = \frac{1}{4}\max\limits_{\theta_3,\theta_4}\max\limits_{\theta_1,\theta_2} f$. Here the value of $\max\limits_{\theta_1,\theta_2} f$ depends on M_{ij} which is a function of θ_3 and θ_4.

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III. GEOMETRIC GLOBAL QUANTUM DISCORD FOR A CLASS OF TWO-QUBIT STATES

We apply now our approach to compute some two-qubit states. Let us first consider the X-state, which, under local unitary transformations, has a form

$$\rho_{AB} = \begin{pmatrix} \rho_{00} & \rho_{01} & \rho_{02} & \rho_{03} \\ \rho_{01}^* & \rho_{11} & -\rho_{03} & \rho_{13} \\ \rho_{02}^* & -\rho_{03}^* & \rho_{22} & \rho_{23} \\ \rho_{03}^* & \rho_{13}^* & \rho_{23}^* & \rho_{33} \end{pmatrix}. \tag{9}$$

We have

$$f = (c_{00}^2 + c_{01}^2) + (c_{10}^2 + c_{13}^2)a_1^2 + (c_{30}^2 + c_{33}^2)a_3^2 + 2(c_{10}c_{30} + c_{13}c_{33})a_1a_3 + 2c_{01}c_{22}a_2b_1b_2 + c_{22}^2a_2^2b_2^21b_3. \tag{10}$$

Denote f_i to be f under the ith solution of the twelve solutions of $\frac{\partial f}{\partial\theta_1} = \frac{\partial f}{\partial\theta_2} = 0$ in the last section. Under the third solution $\theta_1 = 0$, $\theta_2 = 0$, $M_{13} + M_{12} = 0$, i.e., $b_1 = 0$, $b_2 = 0$, $b_3 = 1$, we get

$$f_3 = (c_{00}^2 + c_{01}^2) + (c_{10}^2 + c_{13}^2)a_1^2 + (c_{30}^2 + c_{33}^2)a_3^2 + 2(c_{10}c_{30} + c_{13}c_{33})a_1a_3.$$

From the forth solution $\theta_1 = 0$, $\theta_2 = \Pi$, $M_{13} + M_{12} = 0$, i.e., $b_1 = 0$, $b_2 = 0$, $b_3 = -1$, we obtain $f_3 = f_4$. Similarly, from the fifth to tenth solutions, we have

$$f_5 = (c_{00}^2 + c_{01}^2) + (c_{10}^2 + c_{13}^2)a_1^2 + (c_{30}^2 + c_{33}^2)a_3^2 + 2(c_{10}c_{30} + c_{13}c_{33})a_1a_3 = f_3,$$

$f_6 = f_3$, $f_7 = f_3$, $f_8 = f_3$, $f_9 = f_3$, $f_{10} = f_3$ respectively. Hence we can conclude that $\max_{\theta_1,\theta_2} f = f_3$, $\max_{AB} f = \max_{\theta_1,\theta_2,\theta_3,\theta_4} f = \max_{\theta_3,\theta_4} f_3$. Therefore

$$\begin{aligned}\max_{AB} f &= \max_{\theta_3,\theta_4}[(c_{00}^2 + c_{01}^2) + (c_{10}^2 + c_{13}^2)a_1^2 + (c_{30}^2 + c_{33}^2)a_3^2 + 2(c_{10}c_{30} + c_{13}c_{33})a_1a_3] \\ &= \max_{a_1,a_2,a_3}[(c_{00}^2 + c_{01}^2) + (c_{10}^2 + c_{13}^2)a_1^2 + (c_{30}^2 + c_{33}^2)a_3^2 + 2(c_{10}c_{30} + c_{13}c_{33})a_1a_3].\end{aligned} \tag{11}$$

Accounting to that $a_1^2 + a_2^2 + a_3^2 = 1$ and a_2 does not appear in f_3, we set $a_2 = 0$ and $a_1 = \cos\theta_3$, $a_1 = \sin\theta_3$. Then

$$f_3 = (c_{00}^2 + c_{01}^2 + c_{10}^2 + c_{13}^2) + (c_{30}^2 + c_{33}^2 - c_{10}^2 - c_{13}^2)\sin^2\theta_3 + 2(c_{10}c_{30} + c_{13}c_{33})\sin\theta_3\cos\theta_3$$

and

$$\frac{\partial f_3}{\partial \theta_3} = (c_{30}^2 + c_{33}^2 - c_{10}^2 - c_{13}^2)\sin 2\theta_3 + 2(c_{10}c_{30} + c_{13}c_{33})\cos 2\theta_3 = 0,$$

which give rise to that either $\theta_3 = \frac{\Pi}{4}, \frac{3\Pi}{4}$ if $c_{30}^2 + c_{33}^2 - c_{10}^2 - c_{13}^2 = 0$, or

$$\theta_3 = \frac{1}{2}\arctan\frac{2(c_{10}c_{30} + c_{13}c_{33})}{c_{30}^2 + c_{33}^2 - c_{10}^2 - c_{13}^2}$$

if $c_{30}^2 + c_{33}^2 - c_{10}^2 - c_{13}^2 \neq 0$. Substituting the results to (11), we have the GGQD for the state (9).

As an detailed example, let us consider

$$\rho = \frac{1}{4}(I \bigotimes I - \sigma_y \bigotimes \sigma_y + C_3\sigma_z \bigotimes \sigma_z) = \begin{pmatrix} 1+C_3 & 0 & 0 & 1 \\ 0 & 1-C_3 & -1 & 0 \\ 0 & -1 & 1-C_3 & 0 \\ 1 & 0 & 0 & 1+C_3 \end{pmatrix},$$

which is a state of the form (9). From (4) we have for this state,

$$C = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & C_3 \end{pmatrix}.$$

We have $f = 1 + C_3^2 a_3^2 + a_2^2 b_2^2$. $f = 2 + (C_3^2 - 1)a_3^2$ if $C_3^2 - 1 \geq 0$, and $\max f = C_3^2 + 1$. Hence

$$max\, Tr(ACB^T BC^T A^T) = \frac{1}{4}(C_3^2 + 1), \qquad Tr(CC^T) = \frac{1}{4}(C_3^2 + 2).$$

We have

$$D^{GG}(\rho) = Tr(CC^T) - \max_{AB} Tr(ACB^T BC^T A^T) = \frac{1}{4}.$$

If $C_3^2 - 1 < 0$, then $\max f = 2$,

$$max\, Tr(ACB^T BC^T A^T) = \frac{1}{2}, \qquad Tr(CC^T) = \frac{1}{4}(C_3^2 + 2).$$

We have

$$D^{GG}(\rho) = Tr(CC^T) - \max_{AB} Tr(ACB^T BC^T A^T) = \frac{1}{4}C_3^2.$$

In conclusion we have

$$D^{GG}(\rho) = Tr(CC^T) - \max_{AB} Tr(ACB^T BC^T A^T) = \frac{1 + C_3^2 - \max\{1, C_3^2\}}{4}.$$

This result coincides with the one for $N = 2$, $C_1 = 0$ and $C_2 = -1$ in [11].

IV. CONCLUSIONS AND DISCUSSIONS

We have computed the geometric global quantum discord for arbitrarily two-qubit states. ...

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