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A GLOBAL WEAK SOLUTION OF THE DIRAC-HARMONIC MAP FLOW

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Abstract. We show the existence of a global weak solution of the heat flow for Dirac-harmonic maps from compact Riemann surfaces with boundary when the energy of the initial map and the L^2-norm of the boundary values of the spinor are sufficiently small. The solution is unique and regular with the exception of at most finitely many singular times. We also discuss the behavior at the singularities of the flow. As an application, we deduce some existence results for Dirac-harmonic maps.

1. introduction

Motivated by the nonlinear supersymmetric sigma model from quantum field theory, Dirac-harmonic maps are critical points of an energy functional that couples maps with spinor fields. They were introduced in Chen-Jost-Li-Wang [8, 9]. This subject generalizes the theory of harmonic maps and harmonic spinors. The particular structure of the coupling which comes from the nonlinear supersymmetric sigma model is crucial for their subtle geometric and analytical properties. This structure needs to be very carefully exploited and combined with some of the most powerful and advanced techniques and results in geometric analysis in order to derive regularity, existence and uniqueness results. This is the context of the present paper. We shall discuss and analyze a parabolic version of the model and show the existence of a unique global weak solution under some smallness assumptions on the initial data. As is to be expected for such problems, we encounter the possibility of finite time blow-up, and therefore the weak solution in general will not be strong. But at least, it can be continued across such a singularity as a weak solution.

1.1. **The Dirac-harmonic variational problem.** In order to discuss our results in more detail, we now need to become more technical. Let us first present the Dirac-harmonic model, which this paper is about. Let (M, g) be a Riemann surface with a fixed spin structure, ΣM the spin bundle over M and $\langle \cdot, \cdot \rangle_{\Sigma M}$ the metric on ΣM. Choosing a local orthonormal basis $e_\alpha, \alpha = 1, 2$ on M, the usual Dirac operator is defined as $\partial\!\!\!/ := e_\alpha \cdot \nabla_{e_\alpha}$, where ∇ is the spin connection on ΣM and $\cdot$ is the Clifford multiplication. This multiplication is skew-adjoint:

$$\langle X \cdot \psi, \varphi \rangle_{\Sigma M} = -\langle \psi, X \cdot \varphi \rangle_{\Sigma M}$$

for any $X \in \Gamma(TM)$, $\psi,\ \varphi \in \Gamma(\Sigma M)$.

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The usual Dirac operator $\partial\!\!\!/$ on a surface can be seen as the (doubled) Cauchy-Riemann operator. Consider $\mathbb{R}^2$ with the Euclidean metric $dx^2 + dy^2$. Let $e_1 = \frac{\partial}{\partial x}$ and $e_2 = \frac{\partial}{\partial y}$ be the standard orthonormal frame. A spinor field on $\mathbb{R}^2$ is simply a map $\psi : \mathbb{R}^2 \to \Sigma\mathbb{R}^2 = \mathbb{C}^2$, and the action of e_1 and e_2 on spinors can be identified with multiplication with matrices

$$e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

If $\psi := \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{C}^2$ is a spin field, then the Dirac operator is

$$\partial\!\!\!/\psi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \psi_1}{\partial x} \\ \frac{\partial \psi_2}{\partial x} \end{pmatrix} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \psi_1}{\partial y} \\ \frac{\partial \psi_2}{\partial y} \end{pmatrix} = 2 \begin{pmatrix} \frac{\partial \psi_2}{\partial \bar{z}} \\ -\frac{\partial \psi_1}{\partial z} \end{pmatrix}, \tag{1.1}$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}).$$

For more details on spin geometry and Dirac operators, we can refer to [21].

Let ϕ be a smooth map from M to some compact Riemannian manifold (N, h) with dimension $n \geq 2$. If $\phi^{-1}TN$ is the pull-back bundle of TN by ϕ, we get the twisted bundle $\Sigma M \otimes \phi^{-1}TN$. Naturally, there is a metric $\langle\cdot,\cdot\rangle_{\Sigma M\otimes\phi^{-1}TN}$ on $\Sigma M \otimes \phi^{-1}TN$ which is induced from the metrics on ΣM and $\phi^{-1}TN$. Also we have a natural connection $\widetilde{\nabla}$ on $\Sigma M \otimes \phi^{-1}TN$ which is induced from the connections on ΣM and $\phi^{-1}TN$. Let ψ be a section of the bundle $\Sigma M\otimes\phi^{-1}TN$. In local coordinates, it can be written as

$$\psi = \psi^i \otimes \partial_{y^i}(\phi),$$

where each ψ^i is a standard spinor on M and ∂_{y^i} is the natural local basis on N. Then $\widetilde{\nabla}$ becomes

$$\widetilde{\nabla}\psi = \nabla\psi^i \otimes \partial_{y^i}(\phi) + (\Gamma^i_{jk}\nabla\phi^j)\psi^k \otimes \partial_{y^i}(\phi), \tag{1.2}$$

where Γ^i_{jk} are the Christoffel symbols of the Levi-Civita connection of N. The Dirac operator along the map ϕ is defined by $D\!\!\!\!/\,\psi := e_\alpha \cdot \widetilde{\nabla}_{e_\alpha}\psi$.

We consider the action functional

$$L(\phi, \psi) = \int_M \Big(|d\phi|^2 + \langle\psi, D\!\!\!\!/\,\psi\rangle_{\Sigma M\otimes\phi^{-1}TN}\Big) dvol_g. \tag{1.3}$$

Critical points (ϕ, ψ) of L are called Dirac-harmonic maps from M to N.

By the Nash embedding theorem, we embed N into some $\mathbb{R}^N$. The Euler-Lagrange equations of the functional L are

$$\tau(\phi) = \mathcal{P}(\mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi); \psi), \tag{1.4}$$

$$\partial\!\!\!/\psi = \mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi), \tag{1.5}$$

where $\tau(\phi) = \Delta\phi - A(\phi)(d\phi, d\phi)$ is the tension field of the map ϕ, A is the second fundamental form of N in $\mathbb{R}^N$, $\mathcal{A}$ and $\mathcal{P}$ are defined as follows

$$\mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi) := (\nabla\phi^i \cdot \psi^j) \otimes A(\partial_{y^i}, \partial_{y^j}),$$

$$\mathcal{P}(\mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi); \psi) := P(A(\partial_{y^l}, \partial_{y^j}); \partial_{y^i}) Re(\langle\psi^i, d\phi^l \cdot \psi^j\rangle).$$

Here $P(\xi;\cdot)$ denotes the shape operator satisfying $\langle P(\xi;X),Y\rangle = \langle A(X,Y),\xi\rangle$ for any $X,Y\in\Gamma(TN)$ and $Re(z)$ denotes the real part of $z\in\mathbb{C}$. We refer to [8, 9, 31, 30, 12, 26, 19] for more details.

1.2. **The heat flow approach.** A key difficulty arises from the fact that the action functional L is not bounded from below. Therefore, classical variational approaches developed for harmonic maps cannot be applied to study the existence of Dirac-harmonic maps. There have been other approaches, such as [18, 7, 2, 4]. Here, we shall pursue that approach that seems most promising to us for addressing the existence question in general terms. This is a heat flow that couples a parabolic second order equation for the map with a first order elliptic equation for the spinor. That is, the solution of the first order Dirac type equation is carried along a harmonic map type heat flow. That harmonic map heat flow is the prototype, and when the spinor vanishes, this is what we get. However, the case of interest for us is of course when the spinor is not trivial. The Dirac equation for the spinor might then be considered as a side condition or constraint that depends nonlinearly on the flow. This approach was introduced in [10], and their heat flow for Dirac-harmonic maps looks as follows. For $\Phi\in C^{2,1,\alpha}(M\times(0,T];N)$ and $\Psi\in C^{1,0,\alpha}(M\times(0,T];\Sigma M\otimes\Phi^{-1}TN)$

$$\begin{cases}\partial_t\Phi = \tau(\Phi) - \mathcal{P}(\mathcal{A}(d\Phi(e_\alpha),e_\alpha\cdot\Psi);\Psi), & in \quad M\times[0,T);\\ \partial\!\!\!/\Psi = \mathcal{A}(d\Phi(e_\alpha),e_\alpha\cdot\Psi), & in \quad M\times[0,T).\end{cases}\tag{1.6}$$

We impose the boundary-initial data

$$\begin{cases}\Phi(x,t) = \phi(x,t), & on \quad M\times\{0\}\cup\partial M\times[0,T];\\ \mathcal{B}\Psi(x,t) = \mathcal{B}\psi(x,t), & on \quad \partial M\times[0,T],\end{cases}\tag{1.7}$$

where $\phi\in C^{2,1,\alpha}(M\times\{0\}\cup\partial M\times[0,T];N)$, $\psi\in C^{1,0,\alpha}(\partial M\times[0,T];\Sigma M\otimes\Phi^{-1}TN)$ and $\mathcal{B}=\mathcal{B}^{\pm}$ is the Chiral boundary operator defined as follows:

$$\begin{aligned}\mathcal{B}^{\pm}: L^2(\Sigma M\otimes\Phi^{-1}TN|_{\partial M}) &\to L^2(\Sigma M\otimes\Phi^{-1}TN|_{\partial M})\\ \psi &\mapsto \frac{1}{2}\left(Id\pm\overrightarrow{n}\cdot G\right)\cdot\psi,\end{aligned}\tag{1.8}$$

where $\overrightarrow{n}$ is the outward unite normal vector field on ∂M, $G = ie_1\cdot e_2$ is the Chiral operator defined using a local orthonormal frame $\{e_\alpha\}_{\alpha=1}^2$ on M and satisfying:

$$G^2 = \mathrm{Id},\quad G^* = G,\quad \nabla G = 0,\quad GX\cdot = -X\cdot G,\tag{1.9}$$

for any $X\in\Gamma(TM)$. One can also take $\mathcal{B}$ to be the MIT bag boundary operator $\mathcal{B}^{\pm}_{MIT}$ or the J-boundary operator $\mathcal{B}^{\pm}_J$ as considered in [10]. For convenience, in the sequel, we shall only consider the case of chiral boundary conditions and omit the other two cases of boundary conditions, as the arguments for them are the same.

In [10], a short-time existence and uniqueness result for the flow (1.6) and (1.7) was obtained:

Theorem 1.1 (Theorem 1.3, [10]). *Let $M^m (m\geq 2)$ be a compact spin Riemannian manifold with smooth boundary ∂M, N be a compact Riemannian manifold. Suppose that*

$$\phi\in\cap_{T>0}C^{2,1,\alpha}(M\times[0,T];N)$$

and

$$\psi\in\cap_{T>0}C^{1,0,\alpha}(\partial M\times[0,T];\Sigma M\otimes\Phi^{-1}TN)$$

for some $0 < \alpha < 1$*, then the problem consisting of* (1.6) *and* (1.7) *admits a unique solution*

$$\Phi \in \cap_{0<t<s<T_1} C^{2,1,\alpha}(M \times [t,s]) \cap C^0(M \times [0,T_1], N)$$

and

$$\Psi \in \cap_{0<t<s<T_1} C^{1,0,\alpha}(M \times [t,s]) \cap C^{1,0,0}(M \times [0,T_1]; \Sigma M \otimes \Phi^{-1}TN)$$

for some time $T_1 > 0$ *which is characterized by*

$$\limsup_{t \nearrow T_1} \|\nabla \Phi(\cdot, t)\|_{C^0(M)} = \infty.$$

For Dirac-geodesics and their heat flows, we refer to [11]. For the evolution problem of regularized Dirac-geodesics, see [3].

1.3. **Global existence and main results.** In this paper, we shall study the *global* existence of the flow (1.6) in dimension $\dim M = 2$ with the following boundary-initial data:

$$\begin{cases} \Phi(x,t) = \varphi(x), & on \quad \partial M \times [0,T]; \\ \Phi(x,0) = \phi_0(x), & in \quad M; \\ \mathcal{B}\Psi(x,t) = \mathcal{B}\psi_0(x), & on \quad \partial M \times [0,T]; \\ \phi_0(x) = \varphi(x), & on \quad \partial M. \end{cases} \tag{1.10}$$

Set

$$W^{1,2}(M,N) := \left\{ \phi \in W^{1,2}(M,\mathbb{R}^N) \ with \ \phi(x) \in N \ for \ a.e. \ x \in M \right\},$$

$$W^{1,4/3}(M, \Sigma M \otimes \Phi^{-1}TN) := \left\{ \psi \in W^{1,4/3}(M, \Sigma M \otimes \mathbb{R}^N) \ with \ \psi(x) \in \Sigma M \otimes \Phi^{-1}TN \ for \ a.e. \ x \in M \right\}.$$

Let N_{δ_0} be the δ_0−tubular neighborhood of N in $\mathbb{R}^N$. Then there exists $\delta_0 > 0$ small enough, such that the nearest point projection map $\Pi_N : N_{\delta_0} \to N$ is smooth, *i.e.* $|x - \Pi_N(x)| = d(x,N)$. Given $\Phi \in W^{1,2}(M,N)$, we define

$$\begin{aligned} \not{D}_\Phi : W^{1,4/3}(\Sigma M \otimes \mathbb{R}^N) &\to L^{4/3}(\Sigma M \otimes \mathbb{R}^N) \\ \Psi &\mapsto \not{\partial}\Psi - \mathcal{A}(d\Phi(e_\alpha), e_\alpha \cdot (D\Pi_N|_\Phi \circ \Psi)), \end{aligned}$$

where $D\Pi_N|_\Phi : \mathbb{R}^N \to T_\Phi N$ is the projection. It is easy to see that $\not{D}_\Phi \Psi = \not{D}\Psi$ for $\Psi \in W^{1,4/3}(\Sigma M \otimes \Phi^{-1}TN)$.

Denote the energy of Φ on $\Omega \subset M$ by

$$E(\Phi, \Omega) = \frac{1}{2} \int_\Omega |\nabla \Phi|^2 dM$$

and denote the the energy of Ψ on $\Omega \subset M$ by

$$E(\Psi, \Omega) = \int_\Omega |\Psi|^4 dM.$$

For simplicity, $E(\Phi) = E(\Phi, M)$ and $E(\Psi) = E(\Psi, M)$.

When we have a non-vanishing spinor field Ψ, the total energy of the map $E(\Phi(t))$ is not necessarily non-increasing in t, in contrast to what one knows for the ordinary harmonic map heat flow. However, by exploring the hidden structure of our elliptic-parabolic system (1.6) with boundary-initial data (1.10), we can still show that $E(\Phi(t))$ is uniformly bounded in t - a key property for our flow, allowing for seeking a global weak solution with at most finitely many singularities, in the same spirit as is demonstrated by Struwe in [27]. The remaining difficulty then is that in general we do not have good control of the energy of the spinor field $E(\Psi(t))$ as t approaches the first singular time T_1, when the map blows up.

To overcome this, we shall impose some boundary-initial constraint on ϕ_0 and ψ_0. To be more precise, we shall first define a constant $\Lambda = \Lambda(M, N)$.

$$\Lambda := \sup\{\, \widetilde{\Lambda} \in [0, \infty] : For\ any\ (\phi, \psi) \in W^{1,2}(M, N) \times W^{1,4/3}(\Sigma M \otimes \mathbb{R}^N),\ if\ E(\phi) \le \widetilde{\Lambda}^2,$$

$$then\ \|\psi\|_{W^{1,4/3}(M)} \le C(M, N, \widetilde{\Lambda})(\|\not{D}_\phi \psi\|_{L^{4/3}(M)} + \|\mathcal{B}\psi\|_{W^{1/4,4/3}(\partial M)})\,\}. \tag{1.11}$$

In the above definition (1.11), if we consider $\phi \in W^{1,p}(M, N)$ with $p > 2$ and replace $E(\phi)$ with $\|\phi\|_{W^{1,p}}$, then the corresponding $\Lambda = \infty$ (see Lemma 2.6 or Theorem 1.1 in [10]). However, in the critical case of $\phi \in W^{1,2}(M, N)$, we do not know whether Λ is ∞ or not.

In fact, the constant Λ defined above has a positive lower bound (see Lemma 2.9).

More precisely, we have

$$\Lambda \ge \frac{1}{\sqrt{2}\,\Lambda_1 \cdot \Lambda_2 \cdot \Lambda_3} > 0, \tag{1.12}$$

where $\Lambda_1 = \Lambda_1(M, N) > 0$ (see Lemma 2.7) is the elliptic estimate constant for the usual Dirac operator $\not{\partial}$:

$$\|\psi\|_{W^{1,4/3}(M)} \le \Lambda_1 \left(\|\not{\partial}\psi\|_{L^{4/3}(M)} + \|\mathcal{B}\psi\|_{W^{1/4,4/3}(\partial M)}\right), \quad \forall\, \psi \in W^{1,4/3}(\Sigma M \otimes \mathbb{R}^N). \tag{1.13}$$

$\Lambda_2 = \Lambda_2(M, N) > 0$ is the following Sobolev embedding constant:

$$\|f\|_{L^4(M)} \le \Lambda_2 \|f\|_{W^{1,4/3}(M)}, \quad \forall\, f \in W^{1,4/3}(M, \mathbb{R}^N) \tag{1.14}$$

and $\Lambda_3 > 0$ denotes any upper bound of the L^∞−norm $\|A\|_{L^\infty(N)}$ of the spinorial extension of the second fundamental form $\mathcal{A}$:

$$|\mathcal{A}(d\Phi(e_\alpha), e_\alpha \cdot \Psi)| \le \|A\|_{L^\infty(N)} |d\Phi||\Psi|^2, \tag{1.15}$$

for any $(\Phi, \Psi) \in W^{1,2}(M, N) \times W^{1,4/3}(\Sigma M \otimes \Phi^{-1}TN)$.

Now we are able to state our first main result:

Theorem 1.2. *Let M be a compact Riemann spin surface with smooth boundary ∂M and let $N \subset \mathbb{R}^N$ be a compact Riemannian manifold. Suppose $\phi_0 \in H^1(M, N)$, $\varphi \in C^{2+\alpha}(\partial M, N)$, $\psi_0 \in C^{1+\alpha}(\partial M, \Sigma M \otimes \varphi^{-1}TN)$ and satisfy the following boundary-initial constraint:*

$$E(\phi_0) + \sqrt{2}\|\mathcal{B}\psi_0\|^2_{L^2(\partial M)} < \Lambda^2, \tag{1.16}$$

where $\Lambda = \Lambda(M, N) > 0$ is the constant defined in (1.11)*. Then there exists a global weak solution of* (1.6) *with the boundary-initial data* (1.10)*, which is defined in $M \times [0, \infty)$ and satisfies*

$$E(\Phi(t)) + \int_{M^t} |\partial_t \Phi|^2 dM dt \le E(\phi_0) + \sqrt{2}\|\mathcal{B}\psi_0\|^2_{L^2(\partial M)}, \quad \forall\, t \ge 0,$$

$$E(\Phi(t)) + \frac{1}{2}\int_{\partial M} \langle \vec{n} \cdot \mathcal{B}\psi_0, \Psi\rangle(t) \le E(\Phi(s)) + \frac{1}{2}\int_{\partial M} \langle \vec{n} \cdot \mathcal{B}\psi_0, \Psi\rangle(s), \quad \forall\, 0 \le s \le t < \infty.$$

Moreover, there exists an integer $K > 0$ depending only on M, N, $E(\phi_0)$, $\|\varphi\|_{C^{2+\alpha}(\partial M)}$ and $\|\mathcal{B}\psi_0\|_{C^{1+\alpha}(\partial M)}$ and there exist finitely many singular times $\{T_k\}$, $1 \le k \le K$, characterized by the condition

$$\limsup_{\substack{x\in M\\ t\nearrow T_k}} E(\Phi(t); B_R^M(x)) > \overline{\epsilon} \quad \text{for all} \quad R > 0, \tag{1.17}$$

where $\overline{\epsilon} > 0$ is the constant defined in Theorem 4.1 and $B_R^M(x)$ is the geodesic ball in M, satisfying

$$\Phi \in C_{loc}^{2,1,\alpha}(M \times ((0,\infty) \setminus \{T_k\}_{k=1}^K)) \text{ and } \Psi \in C_{loc}^{1,0,\alpha}(M \times ((0,\infty) \setminus \{T_k\}_{k=1}^K)). \tag{1.18}$$

Moreover, we show that at each singular time $\{T_k\}$, when energy of the map concentrates, after suitable space-time rescaling, a bubble, namely, a nontrivial Dirac-harmonic sphere splits off.

Theorem 1.3. *Let (Φ, Ψ) be a solution to* (1.6) *with the boundary-initial data* (1.10) *from Theorem 1.2. Suppose T_1 is a singular time, i.e.*

$$\limsup_{\substack{x\in M\\ t\nearrow T_1}} E(\Phi(t); B_R^M(x)) > \overline{\epsilon} \quad \text{for all} \quad R > 0. \tag{1.19}$$

There exist sequences $t_i \nearrow T_1$, $x_i \to x_0 \in M$, $r_i \to 0$ and a nontrivial Dirac-harmonic map $(\widetilde{\Phi}, \widetilde{\Psi}) : \mathbb{R}^2 \to N \times (\Sigma\mathbb{R}^2 \otimes \widetilde{\Phi}^{-1}TN)$, such that

(1) *if $x_0 \in M \setminus \partial M$, then as $i \to \infty$,*

$$\Phi_i(x) := \Phi(x_i + r_i x, t_i) \to \widetilde{\Phi}(x) \quad \text{in} \quad C^1_{loc}(\mathbb{R}^2) \quad \text{and}$$
$$\Psi_i(x) := \sqrt{r_i}\Psi(x_i + r_i x, t_i) \to \widetilde{\Psi}(x) \quad \text{in} \quad C^1_{loc}(\mathbb{R}^2).$$

$(\widetilde{\Phi}, \widetilde{\Psi})$ has finite energy and conformally extends to a smooth Dirac-harmonic sphere.

(2) *if $x_0 \in \partial M$, then $\frac{dist(x_i,\partial M)}{r_i} \to \infty$ and the same bubbling statement as in* (1) *holds.*

In Theorem 1.3, for a boundary blow-up point, the case that $\frac{dist(x_i,\partial M)}{r_i}$ is uniformly bounded cannot occur. Otherwise, one obtains a bubbling solution with certain boundary constraints. However this cannot happen, due to the following result, which can be reduced to the harmonic map case considered by Lemaire [22].

Theorem 1.4. *Let $(\Phi, \Psi) : \mathbb{R}^2_+ \to N$ be a smooth Dirac-harmonic map with boundary data $\Phi|_{\partial\mathbb{R}^2_+} = const.$ and $\mathcal{B}\Psi|_{\partial\mathbb{R}^2_+} = 0$ and satisfying*

$$\int_{\mathbb{R}^2_+} |\nabla\Phi|^2 dx + \int_{\mathbb{R}^2_+} |\Psi|^4 dx < \infty,$$

where $\mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2 | x_2 \ge 0\}$ and $\partial\mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2 | x_2 = 0\}$. Then Φ must be a constant map and $\Psi \equiv 0$.

As an important application of the heat flow, we can obtain some existence results of Dirac-harmonic maps.

Theorem 1.5. *Let* (Φ, Ψ) *be a solution to* (1.6) *with the boundary-initial data* (1.10) *as obtained in Theorem 1.2 and defined in* $[0, \infty)$*. Then there exists* $t_i \nearrow \infty$ *such that* $(\Phi(\cdot, t_i), \Psi(\cdot, t_i))$ *converges weakly in* $W^{1,2}(M) \times W^{1,4/3}(M)$ *to a Dirac-harmonic map* $(\Phi_\infty, \Psi_\infty) \in C^{2+\alpha}(M, N) \times C^{1+\alpha}(M, \Sigma M \otimes \Phi_\infty^{-1} TN)$ *with boundary data* $\Phi_\infty|_{\partial M} = \varphi$ *and* $\mathcal{B}\Psi_\infty|_{\partial M} = \mathcal{B}\psi_0$*.*

If the boundary-initial data are small enough, the map part of the limiting Dirac-harmonic map $(\Phi_\infty, \Psi_\infty)$ obtained in the above theorem has to be homotopic to the initial map ϕ_0,

Corollary 1.6. *We define a constant* $\epsilon_0 = \epsilon_0(N) > 0$*:*

$$\epsilon_0 := \inf \Big\{ E(\phi) \mid (\phi, \psi) : S^2 \to N \text{ is a nontrivial smooth Dirac-harmonic map} \Big\}.$$

For any $\phi_0 \in H^1(M, N) \cap C^0(M, N)$*,* $\varphi \in C^{2+\alpha}(\partial M, N)$*,* $\psi_0 \in C^{1+\alpha}(\partial M, \Sigma M \otimes \varphi^{-1} TN)$*, if*

$$E(\phi_0) + \sqrt{2} \|\mathcal{B}\psi_0\|^2_{L^2(\partial M)} < \min \{\Lambda^2, \epsilon_0\}, \tag{1.20}$$

where $\Lambda > 0$ *is defined in* (1.11)*, there exists a Dirac-harmonic map* $(\phi, \psi) : M \to N$ *with* ϕ *lying in the same homotopy class of* ϕ_0*.*

Remark 1.7. In the case of $\mathcal{B}\psi_0 \equiv 0$, by triviality of $\ker(\not{D}_\phi; \mathcal{B})$ for a regular map $\Phi \in W^{1,p}(M, N)$ with $p > 2$ (see Theorem 1.1 in [10] or Lemma 2.6), Ψ has to vanish and hence our problem (1.6) and (1.10) reduces to the classical harmonic map flow with Dirichlet boundary condition. In this case, there is no constraint on the initial energy $E(\phi_0)$ in order to obtain a global weak solution. See e.g. [17, 6, 13] for related works. The finer qualitative behavior at the singularities of our flow will be addressed in a subsequent work.

This paper is organized as follows. In Section 2, we recall some lemmas which will be used in this paper, such as a covering lemma, an interpolation inequality and some elliptic estimates for the first order equation. In Section 3, we derive some a priori estimates which ensure the local existence for initial data with lower regularity. Also, we prove a small energy regularity theorem and the uniqueness of solution in this section. In Section 4, some existence results including local existence and global existence (Theorem 1.2) are proved and the characterization of the singularities is derived. In Section 5, we prove Theorem 1.3, Theorem 1.4, Theorem 1.5 and Corollary 1.6.

Notations: Denote $\Omega_s^t = \Omega \times [s, t]$, $M_s^t = M \times [s, t]$, $M^T = M \times [0, T]$ and denote the standard Sobolev and Hölder spaces by $W_p^{m,n}(M^T)$, $C^{m+\alpha, n+\beta}(M^T)$ and $C^{m,n,\alpha}(M^T) = C^{m+\alpha, n+\alpha/2}(M^T)$. Finally,

$$\begin{aligned} V(M_s^t) := \{(\Phi, \Psi) : M \times [s, t] \to N \times (\Sigma M \otimes \Phi^{-1} TN) | \sup_{s \le \sigma \le t} \|\nabla \Phi\|_{L^2(M)} + \sup_{s \le \sigma \le t} \|\Psi\|_{W^{1,4/3}(M)} \\ + \sup_{s \le \sigma \le t} \|\Psi\|_{L^8(M)} + \int_{M_s^t} (|\partial_t \Phi|^2 + |\nabla^2 \Phi|^2) dM dt < \infty\}. \end{aligned}$$

2. Preliminaries and some lemmas

In this section, we first recall some lemmas which will be used in this paper and then derive the properties of the constant Λ defined in (1.11).

Lemma 2.1 (II, Theorem 2.2 and Remark 2.1, P. 62, P. 63 in [20] or Lemma 4.1 in [14])**.** *For any smooth bounded domain $\Omega \subset \mathbb{R}^2$ and any function $u \in H^1(\Omega)$, there exists a constant $C > 0$ depending on the shape of Ω such that*

$$\int_\Omega |u|^4 dx \leq C \int_\Omega |u|^2 dx \Big(\int_\Omega |\nabla u|^2 dx + \frac{1}{|\Omega|} \int_\Omega |u|^2 dx \Big), \tag{2.1}$$

where $|\Omega|$ is the volume of Ω.

Lemma 2.2 (Lemma 3.3 in [27])**.** *There exist constants $K > 0, R_0 > 0$ depending only on M such that for any $R \in (0, R_0]$, there exists a cover of M by balls $B_R^M(x_i)$ with the property that at any point $x \in M$ at most K of the balls $B_{2R}^M(x_i)$ meet.*

Lemma 2.3. *There exist constants $C > 0, R_0 > 0$ depending only on M, such that for any $T \leq \infty$, any $u \in C^\infty(M^T)$, any $R \in (0, R_0]$ and any function $\eta \in C_0^\infty(B_R(x_0))$, $x_0 \in M$ depending only on the distance $|x - x_0|$ and non-increasing as a function of this distance, there holds*

$$\int_M |\nabla u|^4 \eta dMdt \leq C \sup_{0 \leq t \leq T} \int_{B_R^M(x_0)} |\nabla u|^2(x,t) dM \cdot \Big(\int_{M^T} |\nabla^2 u|^2 \eta dMdt + R^{-2} \int_{M^T} |\nabla u|^2 \eta dMdt \Big) \tag{2.2}$$

where $B_R^M(x_0)$ is the geodesic ball on the M. Moreover, we have

$$\int_M |\nabla u|^4 dMdt \leq C \sup_{(x_0,t) \in M^T} \int_{B_R^M(x_0)} |\nabla u|^2(x,t) dM \cdot \Big(\int_{M^T} |\nabla^2 u|^2 dMdt + R^{-2} \int_{M^T} |\nabla u|^2 dMdt \Big). \tag{2.3}$$

Proof. The idea is the same as in Struwe's paper [27], using the density of step functions in L^∞ space and the covering argument in Lemma 2.2. One can refer to Lemma 3.1, Lemma 3.2 in [27] or Lemma 4.1, Lemma 4.2 in [14] for a detailed proof. □

The next lemma provides a Green formula for the Dirac operator $\not{D}$ along a map ϕ.

Lemma 2.4 (Proposition 3.2 in [12])**.** *For any $\psi, \omega \in W^{1,3/4}(\Sigma M \otimes \phi^{-1}TN)$, we have*

$$\int_M \langle \psi, \not{D}\omega \rangle = \int_M \langle \not{D}\psi, \omega \rangle - \int_{\partial M} \langle \overrightarrow{n} \cdot \psi, \omega \rangle \tag{2.4}$$

where $\langle \psi, \omega \rangle := h_{ij} \langle \psi^i, \omega^j \rangle$.

We next present a modified version of Proposition 3.1 in [10], which will play a crucial role in controlling the total energy of the map along our flow (see Section 3).

Proposition 2.5. *Suppose that $\phi \in W^{1,2}(M, N)$ and $\psi \in W^{1,4/3}(\Sigma M \otimes \phi^{-1}TN)$, then*

$$\Big| \int_{\partial M} (||\psi||^2 - 2||\mathcal{B}\psi||^2) \Big| \leq 2||\psi||_{L^4(M)} ||\not{D}\psi||_{L^{4/3}(M)}. \tag{2.5}$$

Proof. We use the observation of Proposition 3.1 in [10]. Denoting

$$X := \frac{1}{2} \langle \psi, e_\alpha \cdot G \cdot \psi \rangle e_\alpha,$$

then

$$\langle X, \overrightarrow{n} \rangle = \frac{1}{2}\langle \psi, \overrightarrow{n} \cdot G \cdot \psi \rangle,$$

$$\|\mathcal{B}\psi\|^2 = \frac{1}{2}\|\psi\|^2 - \langle X, \overrightarrow{n} \rangle,$$

and

$$\operatorname{div} X = -Re\langle \not{D}\psi, G \cdot \psi \rangle.$$

By the boundary trace embedding theorem $W^{1,4/3}(M) \to L^2(\partial M)$ (see Lemma 5.19 in [1]), we have $\psi \in L^2(\partial M)$ and hence $X \in L^1(\partial M)$. Combined with the fact that div $X \in L^1(M)$, by the divergence theorem, we have

$$|\int_{\partial M} \langle X, \overrightarrow{n} \rangle| = |\int_M \operatorname{div} X \, dx| = |\int_M Re\langle \not{D}\psi, G \cdot \psi \rangle dx| \le \|\psi\|_{L^4(M)} \|\not{D}\psi\|_{L^{4/3}(M)}.$$

Thus,

$$|\int_{\partial M} (\|\psi\|^2 - 2\|\mathcal{B}\psi\|^2)| \le 2\|\psi\|_{L^4(M)} \|\not{D}\psi\|_{L^{4/3}(M)}. \tag{2.6}$$

□

Next, we recall some elliptic estimates from [10].

Lemma 2.6 (Theorem 1.2 in [10])**.** *Suppose $\phi \in W^{1,2p^*}(M, N)$, $p^* > 1$ and $\psi \in W^{1,p}(\Sigma M \otimes \mathbb{R}^N)$, $1 < p < p^*$ satisfy*

$$\begin{cases} \not{D}_\phi \psi = \xi, & in \quad M; \\ \mathcal{B}\psi = \mathcal{B}\psi_0, & on \quad \partial M, \end{cases} \tag{2.7}$$

then there exists a constant $C = C(p, M, N, \|\phi\|_{W^{1,2p^*}(M)}) > 0$ such that

$$\|\psi\|_{W^{1,p}(M)} \le C \left(\|\xi\|_{L^p(M)} + \|\mathcal{B}\psi\|_{W^{1-1/p,p}(\partial M)} \right).$$

As a special case of Lemma 2.6, when $\phi \equiv const.$, we have

Lemma 2.7. *For any $1 < p < \infty$, there exists a constant $C = C(p, M, N) > 0$ such that for any $\psi \in W^{1,p}(\Sigma M \otimes \mathbb{R}^N)$ there holds*

$$\|\psi\|_{W^{1,p}(M)} \le C(p, M, N) \left(\|\not{\partial}\psi\|_{L^p(M)} + \|\mathcal{B}\psi\|_{W^{1-1/p,p}(\partial M)} \right).$$

Here, $C(\frac{4}{3}, M, N) = \Lambda_1(M, N)$ defined in (1.13).

Taking $\not{D}$ to be the usual Dirac operator $\not{\partial}$ in Theorem 4.4 of [10], we get

Lemma 2.8 (Theorem 4.4 in [10])**.** *Suppose $\psi \in W^{1,p}(\Sigma M \otimes \mathbb{R}^N)$, $1 < p < \infty$ and $f \in C^\alpha(\Sigma M \otimes \mathbb{R}^N)$, $0 < \alpha < 1$ satisfy*

$$\not{\partial}\psi = f \quad in \quad M,$$

then there exists a constant $C = C(\alpha, M, N) > 0$ such that $\psi \in C^{1+\alpha}(\Sigma M \otimes \mathbb{R}^N)$ and

$$\|\psi\|_{C^{1+\alpha}(M)} \le C(\alpha, M, N) \left(\|\not{\partial}\psi\|_{C^\alpha(M)} + \|\mathcal{B}\psi\|_{C^{1+\alpha}(\partial M)} \right).$$

Lemma 2.6 provides the elliptic estimate and the uniqueness result for ϕ regular enough, namely $\phi \in W^{1,2p^*}(M,N)$ for some $p^* > 1$. However, if ϕ is only in $W^{1,2}(M,N)$, then the corresponding estimate may not hold. In this critical case, we need to use the constant Λ defined in (1.11) in order to obtain the elliptic estimate and the uniqueness result. Now we show that there is a positive lower bound of the constant Λ.

Lemma 2.9. *The constant* Λ *defined in* (1.11) *satisfies* (1.12).

Proof. Let $\Lambda_3 > \|A\|_{L^\infty(N)}$ be a constant. For any $0 < \widetilde{\Lambda} < \frac{1}{\sqrt{2}\Lambda_1\cdot\Lambda_2\cdot\Lambda_3}$, it is sufficient to prove that if $E(\phi) \le \widetilde{\Lambda}^2$, then Ker $(\not{D}_\phi; \mathcal{B}) = 0$ and

$$\|\psi\|_{W^{1,4/3}(M)} \le C(M,N,\widetilde{\Lambda})(\|\not{D}_\phi\psi\|_{L^{4/3}(M)} + \|\mathcal{B}\psi\|_{W^{1/4,4/3}(\partial M)} + \|\psi\|_{L^{4/3}(M)}).$$

In fact, suppose

$$\begin{cases} \not{D}_\phi\psi = \xi, & in \quad M; \\ \mathcal{B}\psi = \mathcal{B}\psi_0, & on \quad \partial M, \end{cases}$$

by Lemma 2.7, we have

$$\begin{aligned}
\|\psi\|_{W^{1,4/3}(M)} &\le \Lambda_1 \left(\|\mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot D\pi \circ \psi)\|_{L^{4/3}(M)} + \|\not{D}_\phi\psi\|_{L^{4/3}(M)} + \|\mathcal{B}\psi_0\|_{W^{1/4,4/3}(\partial M)}\right) \\
&\le \Lambda_1 \left(\|A\|_{L^\infty(N)}\||d\phi||D\pi \circ \psi|\|_{L^{4/3}(M)} + \|\not{D}_\phi\psi\|_{L^{4/3}(M)} + \|\mathcal{B}\psi_0\|_{W^{1/4,4/3}(\partial M)}\right) \\
&\le \Lambda_1\|A\|_{L^\infty(N)}\|d\phi\|_{L^2(M)}\|\psi\|_{L^4(M)} + \Lambda_1\|\not{D}_\phi\psi\|_{L^{4/3}(M)} + \Lambda_1\|\mathcal{B}\psi_0\|_{W^{1/4,4/3}(\partial M)} \\
&\le \Lambda_1\Lambda_2\|A\|_{L^\infty(N)}\|d\phi\|_{L^2(M)}\|\psi\|_{W^{1,4/3}(M)} + \Lambda_1\|\not{D}_\phi\psi\|_{L^{4/3}(M)} + \Lambda_1\|\mathcal{B}\psi_0\|_{W^{1/4,4/3}(\partial M)} \\
&\le \sqrt{2}\,\widetilde{\Lambda}\Lambda_1\Lambda_2\|A\|_{L^\infty(N)}\|\psi\|_{W^{1,4/3}(M)} + \Lambda_1\|\not{D}_\phi\psi\|_{L^{4/3}(M)} + \Lambda_1\|\mathcal{B}\psi_0\|_{W^{1/4,4/3}(\partial M)}.
\end{aligned}$$

Since $\sqrt{2}\,\widetilde{\Lambda}\Lambda_1\Lambda_2\|A\|_{L^\infty(N)} < 1$, we get

$$\|\psi\|_{W^{1,4/3}(M)} \le C(M,N,\widetilde{\Lambda})(\|\not{D}_\phi\psi\|_{L^{4/3}(M)} + \|\mathcal{B}\psi_0\|_{W^{1/4,4/3}(\partial M)}).$$

□

In fact, we can show that the constant Λ in (1.11) has the following equivalent definition:

$$\begin{aligned}
\Lambda := \sup\{\widetilde{\Lambda} \in [0,\infty] : &\ For\ any\ \phi \in W^{1,2}(M,N),\ if\ E(\phi) \le \widetilde{\Lambda}^2,\ then\ \mathrm{Ker}(\not{D}_\phi;\mathcal{B}) = 0\ and \\
&\ for\ any\ \psi \in W^{1,4/3}(\Sigma M \otimes \mathbb{R}^N),\ there\ holds \\
&\ \|\psi\|_{W^{1,4/3}(M)} \le C(M,N,\widetilde{\Lambda})(\|\not{D}_\phi\psi\|_{L^{4/3}(M)} + \|\mathcal{B}\psi\|_{W^{1/4,4/3}(\partial M)} + \|\psi\|_{L^{4/3}(M)})\}.
\end{aligned} \tag{2.8}$$

To see this, we first show that

Lemma 2.10. *Suppose* $\phi \in W^{1,2}(M,N)$ *and* $\psi \in W^{1,4/3}(\Sigma M \otimes \mathbb{R}^N)$ *satisfies*

$$\begin{cases} \not{D}_\phi\psi = \xi, & in \quad M; \\ \mathcal{B}\psi = \mathcal{B}\psi_0, & on \quad \partial M. \end{cases}$$

If

$$E(\phi) \le \widetilde{\Lambda}^2 < \Lambda^2,$$

where Λ *is defined as in* (2.8)*, then there exists a constant* $C(M,N,\widetilde{\Lambda}) > 0$ *such that*

$$\|\psi\|_{W^{1,4/3}(M)} \le C(M,N,\widetilde{\Lambda})(\|\not{D}_\phi\psi\|_{L^{4/3}(M)} + \|\mathcal{B}\psi\|_{W^{1/4,4/3}(\partial M)}). \tag{2.9}$$

Proof. In fact, by the definition of Λ in (2.8), we have $\mathrm{Ker}(\not{D}_\phi;\mathcal{B})=0$ and

$$\|\psi\|_{W^{1,4/3}(M)}\le C(M,N,\widetilde{\Lambda})(\|\not{D}_\phi\psi\|_{L^{4/3}(M)}+\|\mathcal{B}\psi\|_{W^{1/4,4/3}(\partial M)}+\|\psi\|_{L^{4/3}(M)}). \tag{2.10}$$

If the conclusion (2.9) does not hold, there exists a sequence $\phi_n\in W^{1,2}(M,N)$, $\psi_n\in W^{1,4/3}(M,\Sigma M\otimes\mathbb{R}^N)$, satisfying

$$E(\phi_n)\le\widetilde{\Lambda}^2<\Lambda^2,$$

but,

$$\|\psi_n\|_{W^{1,4/3}(M)}\ge n\,(\|\not{D}_{\phi_n}\psi_n\|_{L^{4/3}(M)}+\|\mathcal{B}\psi_n\|_{W^{1/4,4/3}(\partial M)}).$$

Without loss of generality, we may assume $\|\psi_n\|_{L^{4/3}(M)}=1$. By (2.10), we have

$$(1-\frac{C}{n})(\|\not{D}_{\phi_n}\psi_n\|_{L^{4/3}(M)}+\|\mathcal{B}\psi_n\|_{W^{1/4,4/3}(\partial M)})\le\frac{C}{n}\,\|\psi_n\|_{L^{4/3}(M)}.$$

Using (2.10) again, we get $\|\psi_n\|_{W^{1,4/3}(M)}\le C$ when n is big enough. Then there exists a subsequence of $\{(\phi_n,\psi_n)\}$, still denoted by $\{(\phi_n,\psi_n)\}$, and $\phi\in W^{1,2}(M,N)$, $\psi\in W^{1,4/3}(\Sigma M\otimes\mathbb{R}^N)$ such that

$$\begin{aligned}&\psi_n\rightharpoonup\psi \text{ weakly in } W^{1,4/3}(M),\quad d\phi_n\rightharpoonup d\phi \text{ weakly in } L^2(M),\\ &\psi_n\to\psi \text{ strongly in } L^2(M).\end{aligned}$$

So, we have $\|\psi\|_{L^{4/3}(M)}=1$, $E(\phi)\le E(\phi_n)\le\widetilde{\Lambda}^2<\Lambda^2$ and

$$\begin{cases}\not{D}_\phi\psi=0, & in \quad M;\\ \mathcal{B}\psi=0, & on \quad \partial M,\end{cases}$$

Since $\mathrm{Ker}(\not{D}_\phi;\mathcal{B})=0$, we get $\psi\equiv0$. This is a contradiction to $\|\psi\|_{L^{4/3}(M)}=1$. Thus, the estimate (2.9) follows. □

From Lemma 2.10, it is easy to get the following:

Corollary 2.11. *The definition of* (1.11) *is equivalent to* (2.8).

Finally, we provide the ϵ-regularity estimate for the Dirac equation. We remark that the interior regularity for weak solutions was proved in [26] (Theorem 3.4) and the boundary regularity for weak solutions in the homogeneous boundary value case was shown in [26] (Theorem 3.5).

Lemma 2.12. *Let $B_1\subset\mathbb{R}^2$ and $\phi\in W^{1,2}(B_1,N)$, $\psi\in W^{1,q}(B_1,\mathbb{C}^2\otimes\mathbb{R}^N)$ satisfy*

$$\not{D}_\phi\psi=0\quad on\quad B_1.$$

Then for any $2\le q<\infty$, there exist $\epsilon=\epsilon(q,N)>0$ and $C=C(q,N)>0$ such that whenever

$$\|d\phi\|_{L^2(B_1)}\le\epsilon,$$

then

$$\|\psi\|_{L^q(B_{1/2})}+\|\nabla\psi\|_{W^{1,\frac{2q}{2+q}}(B_{1/2})}\le C(q,N)\|\psi\|_{L^4(B_1)}. \tag{2.11}$$

Moreover, let $B_1^+ := \{(x_1, x_2) \in B_1; x_2 \geq 0\} \subset \mathbb{R}^2$ *and* $\partial^0 B_1^+ := \partial B_1^+ \cap \{(x_1, x_2) \in B_1; x_2 = 0\}$. *If* $\psi \in W^{1,q}(B_1^+, \mathbb{C}^2 \otimes \mathbb{R}^N)$, $\mathcal{B}\psi_0 \in W^{1-1/q,q}(\partial^0 B_1^+, \mathbb{C}^2 \otimes \mathbb{R}^N)$, $\phi \in W^{1,2}(B_1^+, N)$ *satisfy*

$$\begin{cases} \not{D}_\phi \psi = 0 & in \quad B_1^+; \\ \mathcal{B}\psi = \mathcal{B}\psi_0 & on \quad \partial^0 B_1^+. \end{cases}$$

Then there exist $\epsilon = \epsilon(q, N) > 0$ *and* $C = C(q, N) > 0$ *such that whenever*

$$\|d\phi\|_{L^2(B_1^+)} \leq \epsilon,$$

then

$$\|\psi\|_{L^q(B_{1/2}^+)} + \|\nabla\psi\|_{W^{1,\frac{2q}{2+q}}(B_{1/2}^+)} \leq C(q, N)(\|\psi\|_{L^4(B_1^+)} + \|\mathcal{B}\psi_0\|_{W^{1-1/q,q}(\partial^0 B_1^+)}). \tag{2.12}$$

Proof. Taking a cut-off function $\eta \in C_0^\infty(B_1)$ such that $\eta|_{B_{3/4}} \equiv 1$ and $|\nabla\eta| \leq C$, by the standard elliptic estimates, for any $1 < p < 2$, we have

$$\begin{aligned} \|\eta\psi\|_{W^{1,p}(B_1)} &\leq C(\|\not{\partial}(\eta\psi)\|_{L^p(B_1)} + \|\psi\|_{L^4(B_1)}) \\ &\leq C(\|d\phi\|_{L^2(B_1)} \|\eta\psi\|_{L^{\frac{2p}{2-p}}(B_1)} + \|\psi\|_{L^4(B_1)}) \\ &\leq C\epsilon\, \|\eta\psi\|_{W^{1,p}(B_1)} + C\|\psi\|_{L^4(B_1)}. \end{aligned}$$

Then we can get the interior estimate (2.11) by the Sobolev embedding theorem.

For the boundary estimates, we also need to choose a cut-off function $\eta \in C_0^\infty(B_1^+)$ such that $\eta|_{B_{3/4}^+} \equiv 1$ and $|\nabla\eta| \leq C$, by Lemma 2.7, we get

$$\begin{aligned} \|\eta\psi\|_{W^{1,p}(B_1^+)} &\leq C(\|\not{\partial}(\eta\psi)\|_{L^p(B_1^+)} + \|\eta\mathcal{B}\psi_0\|_{W^{1-1/p,p}(\partial B_1^+)}) \\ &\leq C(\|d\phi\|_{L^2(B_1^+)} \|\eta\psi\|_{L^{\frac{2p}{2-p}}(B_1^+)} + \|\psi\|_{L^4(B_1^+)} + \|\eta\mathcal{B}\psi_0\|_{W^{1-1/p,p}(\partial B_1^+)}) \\ &\leq C\epsilon\, \|\eta\psi\|_{W^{1,p}(B_1^+)} + C(\|\psi\|_{L^4(B_1^+)} + \|\mathcal{B}\psi_0\|_{W^{1-1/p,p}(\partial^0 B_1^+)}). \end{aligned}$$

Then we can get the boundary estimate (2.12) by Sobolev embedding again. □

3. A priori estimates

In this section, we shall first show some elementary properties of the flow, in particular we show that the energy of the map $E(\Phi(t))$ is uniformly bounded from above. Then, we impose the boundary-initial constraint (1.16) and prove some a-priori estimates, an ϵ-regularity and a uniqueness result, which will be used in the next section to get the existence results.

First, we need the following proposition.

Proposition 3.1. *Suppose* $(\Phi, \Psi) \in V(M^T)$ *is a solution of* (1.6) *with the boundary-initial data* (1.10)*, then we have*

$$\begin{aligned} \int_{M^T} \langle \mathcal{P}(\mathcal{A}(d\Phi(e_\alpha), e_\alpha \cdot \Psi); \Psi), \frac{\partial\Phi}{\partial t} \rangle &= -\frac{1}{2} \int_0^T \frac{d}{dt} \int_{\partial M} \langle \mathcal{B}\psi_0, \overrightarrow{n} \cdot \Psi \rangle(t) dt \\ &= -\frac{1}{2} \int_{\partial M} \langle \mathcal{B}\psi_0, \overrightarrow{n} \cdot \Psi \rangle(T) + \frac{1}{2} \int_{\partial M} \langle \mathcal{B}\psi_0, \overrightarrow{n} \cdot \Psi \rangle(0). \end{aligned} \tag{3.1}$$

Proof. Computing directly, we have

$$
\begin{aligned}
\frac{d}{dt}\not{D}\Psi &= \frac{d}{dt}(e_\alpha \cdot \nabla_{e_\alpha}(\Psi^i \otimes \partial_{y^i})) \\
&= \frac{d}{dt}(e_\alpha \cdot \nabla_{e_\alpha}\Psi^i \otimes \partial_{y^i}) + \frac{d}{dt}(e_\alpha \cdot \Psi^i \otimes \nabla_{e_\alpha}\partial_{y^i}) \\
&= e_\alpha \cdot \nabla_{e_\alpha}(\frac{d}{dt}\Psi^i) \otimes \partial_{y^i} + e_\alpha \cdot \nabla_{e_\alpha}\Psi^i \otimes \nabla_{\frac{d}{dt}}\partial_{y^i} + e_\alpha \cdot \nabla_{\frac{d}{dt}}\Psi^i \otimes \nabla_{e_\alpha}\partial_{y^i} + e_\alpha \cdot \Psi^i \otimes \nabla_{\frac{d}{dt}}\nabla_{e_\alpha}\partial_{y^i}.
\end{aligned}
$$

Noting that

$$\nabla_{\frac{d}{dt}}\nabla_{e_\alpha}\partial_{y^i} = \nabla_{e_\alpha}\nabla_{\frac{d}{dt}}\partial_{y^i} + R(d\Phi(\partial_t), d\Phi(e_\alpha))\partial_{y^i},$$

we get

$$\frac{d}{dt}\not{D}\Psi = \not{D}(\frac{d}{dt}\Psi) + e_\alpha \cdot \Psi^i \otimes R(d\Phi(\partial_t), d\Phi(e_\alpha))\partial_{y^i}. \tag{3.2}$$

Since $\not{D}\Psi = 0$, we have

$$
\begin{aligned}
0 &= \int_{M^T} \langle \Psi, \frac{d}{dt}\not{D}\Psi\rangle dMdt \\
&= \int_{M^T} \langle \Psi, \not{D}(\frac{d}{dt}\Psi)\rangle dMdt + \int_{M^T} \langle \Psi, e_\alpha \cdot \Psi^i \otimes R(d\Phi(\partial_t), d\Phi(e_\alpha))\partial_{y^i}\rangle dMdt \\
&= \int_0^T \mathbf{I}dt + \int_0^T \mathbf{II}dt.
\end{aligned}
$$

On the one hand, by the definition of $\mathcal{B}$ (see (1.8)), we have

$$2\mathcal{B}\Psi = \Psi \pm \overrightarrow{n} \cdot G \cdot \Psi \quad \textit{and} \quad 0 = \frac{d}{dt}(2\mathcal{B}\Psi) = 2\mathcal{B}\dot{\Psi} = \dot{\Psi} \pm \overrightarrow{n} \cdot G \cdot \dot{\Psi}$$

where $\dot{\Psi} := \frac{d}{dt}\Psi$. Combining this with Lemma 2.4 and (1.9), we can get

$$
\begin{aligned}
\mathbf{I} &= \int_M \langle \not{D}\Psi, \dot{\Psi}\rangle dM - \int_{\partial M} \langle \overrightarrow{n} \cdot \Psi, \dot{\Psi}\rangle \\
&= -\int_{\partial M} \langle \overrightarrow{n} \cdot \Psi, \dot{\Psi}\rangle \\
&= -\int_{\partial M} \langle \overrightarrow{n} \cdot \Psi, \mp\overrightarrow{n} \cdot G \cdot \dot{\Psi}\rangle \\
&= -\int_{\partial M} \langle \mp\overrightarrow{n} \cdot G \cdot \Psi, \overrightarrow{n} \cdot \dot{\Psi}\rangle \\
&= -\int_{\partial M} \langle \Psi, \overrightarrow{n} \cdot \dot{\Psi}\rangle + \int_{\partial M} \langle 2\mathcal{B}\Psi, \overrightarrow{n} \cdot \dot{\Psi}\rangle,
\end{aligned}
$$

then we have

$$\mathbf{I} = -\int_{\partial M} \langle \overrightarrow{n} \cdot \Psi, \dot{\Psi}\rangle = \int_{\partial M} \langle \mathcal{B}\Psi, \overrightarrow{n} \cdot \dot{\Psi}\rangle = \frac{d}{dt}\int_{\partial M} \langle \mathcal{B}\psi_0, \overrightarrow{n} \cdot \Psi\rangle.$$

On the other hand, using the equation of Gauss, we get

$$
\begin{aligned}
\mathbf{II} &= \int_M \langle \Psi, e_\alpha \cdot \Psi^i \otimes R^m_{ijk} \partial_t \Phi^j d\Phi^k(e_\alpha) \partial_{y^m} \rangle dM \\
&= \int_M R_{mijk} \langle \Psi^m, \nabla \Phi^k \cdot \Psi^i \rangle \partial_t \Phi^j dM \\
&= \int_M [\langle A(\partial_{y^m}, \partial_{y^j}), A(\partial_{y^i}, \partial_{y^k}) \rangle_{\mathbf{R}^N} - A(\partial_{y^m}, \partial_{y^k}), A(\partial_{y^i}, \partial_{y^j}) \rangle_{\mathbf{R}^N}] \\
&\quad \cdot \langle \Psi^m, \nabla \Phi^k \cdot \Psi^i \rangle \partial_t \Phi^j dM \\
&= 2 \int_M \langle A(\partial_{y^m}, \partial_{y^j}), A(\partial_{y^i}, \partial_{y^k}) \rangle_{\mathbf{R}^N} Re(\langle \Psi^m, \nabla \Phi^k \cdot \Psi^i \rangle) \partial_t \Phi^j dM \\
&= 2 \int_M \langle \mathcal{P}(\mathcal{A}(d\Phi(e_\alpha), e_\alpha \cdot \Psi); \Psi), \partial_t \Phi \rangle. \qquad (3.3)
\end{aligned}
$$

Then the equality (3.1) follows immediately. □

Lemma 3.2. *Suppose* $(\Phi, \Psi) \in V(M^T)$ *is a solution of* (1.6) *with the boundary-initial data* (1.10)*, then there holds*

$$E(\Phi(t)) + \int_{M^t} |\partial_t \Phi|^2 dM dt \leq E(\phi_0) + \sqrt{2} \|\mathcal{B}\psi_0\|^2_{L^2(\partial M)}.$$

Moreover, $E(\Phi(t)) + \frac{1}{2} \int_{\partial M} \langle \overrightarrow{n} \cdot \mathcal{B}\psi_0, \Psi \rangle$ *is absolutely continuous on* $[0, T]$ *and non-increasing.*

Proof. Multiplying the equation (1.6) by $\partial_t \Phi$ and using the Lemma 2.4, we have

$$
\begin{aligned}
\int_{M^t_s} |\partial_t \Phi|^2 dM - \int_{M^t_s} \Delta \Phi \partial_t \Phi dM &= -\int_{M^t_s} \langle \mathcal{P}(\mathcal{A}(d\Phi(e_\alpha), e_\alpha \cdot \Psi); \Psi), \partial_t \Phi \rangle \\
&= \frac{1}{2} \int_s^t \frac{d}{dt} \int_{\partial M} \langle \mathcal{B}\psi_0, \overrightarrow{n} \cdot \Psi \rangle dt,
\end{aligned}
$$

for any $0 \leq s \leq t \leq T$. Integrating by parts, we get

$$\frac{1}{2} \int_s^t \frac{d}{dt} \int_M |\nabla \Phi|^2 dM dt + \int_{M^t_s} |\partial_t \Phi|^2 dM dt = \frac{1}{2} \int_s^t \frac{d}{dt} \int_{\partial M} \langle \mathcal{B}\psi_0, \overrightarrow{n} \cdot \Psi \rangle dt. \qquad (3.4)$$

So, we have

$$
\begin{aligned}
E(\Phi(t)) + \int_{M^t} |\partial_t \Phi|^2 dM dt &\leq E(\phi_0) + \frac{1}{2} \left| \int_{\{0\} \times \partial M} \langle \mathcal{B}\psi_0, \overrightarrow{n} \cdot \Psi \rangle \right| + \frac{1}{2} \left| \int_{\{t\} \times \partial M} \langle \mathcal{B}\psi_0, \overrightarrow{n} \cdot \Psi \rangle \right| \\
&\leq E(\phi_0) + \sqrt{2} \|\mathcal{B}\psi_0\|^2_{L^2(\partial M)},
\end{aligned}
$$

where the last inequality follows from Proposition 2.5 since $\not{D}\psi \equiv 0$. Also, we have

$$\int_s^t \frac{d}{dt} \left(\frac{1}{2} \int_M |\nabla \Phi|^2 dM + \frac{1}{2} \int_{\partial M} \langle \overrightarrow{n} \cdot \mathcal{B}\psi_0, \Psi \rangle \right) dt = -\int_{M^t_s} |\partial_t \Phi|^2 dM dt, \qquad (3.5)$$

and the claims follow. □

Next, we shall study the flow with the boundary-initial constraint (1.16), namely

$$E(\phi_0) + \sqrt{2}\|\mathcal{B}\psi_0\|^2_{L^2(\partial M)} < \Lambda^2,$$

where Λ is the constant in Theorem 1.2.

Lemma 3.3. *Suppose* $(\Phi, \Psi) \in V(M^T)$ *is a solution of* (1.6) *with the boundary-initial data* (1.10) *that satisfies the boundary-initial constraint* (1.16). *Then*

$$\|\Psi(\cdot, t)\|_{W^{1,4/3}(M)} \leq C(M, E(\phi_0) + \sqrt{2}\|\mathcal{B}\psi_0\|^2_{L^2(\partial M)})\|\mathcal{B}\psi_0\|_{W^{1/4,4/3}(\partial M)}, \quad \forall\ 0 \leq t \leq T. \tag{3.6}$$

Proof. By Lemma 3.2, we know

$$E(\Phi(t)) \leq E(\phi_0) + \sqrt{2}\|\mathcal{B}\psi_0\|^2_{L^2(\partial M)} < \Lambda^2. \tag{3.7}$$

Since Ψ satisfies the first order elliptic equation

$$\begin{cases} \not{D}_\Phi \Psi = 0, & in \quad M; \\ \mathcal{B}\Psi = \mathcal{B}\psi_0, & on \quad \partial M, \end{cases}$$

along the flow, by Lemma 2.10, we have

$$\|\Psi(\cdot, t)\|_{W^{1,4/3}(M)} \leq C(M, E(\phi_0) + \sqrt{2}\|\mathcal{B}\psi_0\|^2_{L^2(\partial M)})\|\mathcal{B}\psi_0\|_{W^{1/4,4/3}(\partial M)}. \tag{3.8}$$

□

Lemma 3.4. *Suppose* $\phi_0 \in H^1(M, N)$, $\varphi \in H^{3/2}(\partial M, N)$, $\phi_0|_{\partial M} = \varphi$, $\psi_0 \in W^{3/8,8/5}(\partial M, \Sigma M \otimes \varphi^{-1}TN)$ *and satisfy the boundary-initial constraint* (1.16). *Then there exists constants* $\epsilon_1 = \epsilon_1(M, N) > 0$ *and* $C = C(M, N) > 0$, *such that if* $(\Phi, \Psi) \in V(M^T)$ *is a solution of* (1.6) *with the boundary-initial data* (1.10) *and satisfies*

$$\epsilon(R) := \sup_{(x,t)\in M^T} E(\Phi(t); B^M_R(x)) \leq \epsilon_1 \quad \text{for all} \quad R \in (0, R_0],$$

then there hold the estimates

$$\sup_{0\leq t\leq T} \|\Psi(\cdot, t)\|_{L^8(M)} \leq \frac{C}{R^{1/4}}\|\mathcal{B}\psi_0\|_{W^{3/8,8/5}(\partial M)} \tag{3.9}$$

and

$$\int_{M^T} |\nabla^2\Phi|^2 dMdt \leq C(1 + \frac{T}{R^2})E(\Phi(0)) + \frac{CT}{R^2}(1 + \|\varphi\|^2_{H^{3/2}(\partial M)} + \|\mathcal{B}\psi_0\|^8_{W^{3/8,8/5}(\partial M)}). \tag{3.10}$$

Proof. By Lemma 2.2, we know there exists a cover of M by balls $B^M_R(x_i)$ with the property that at any point $x \in M$ at most K of the balls $B^M_{2R}(x_i)$ meet. By Lemma 2.12, if $B^M_{2R}(x_i) \cap \partial M = \emptyset$, then

$$\|\psi\|_{L^8(B_R)} \leq \frac{C}{R^{1/4}}\|\psi\|_{L^4(B_{2R})};$$

If $B^M_{2R}(x_i) \cap \partial M \neq \emptyset$, then

$$\|\psi\|_{L^8(B^M_R)} \leq \frac{C}{R^{1/4}}(\|\psi\|_{L^4(B^M_{2R})} + \|\mathcal{B}\psi_0\|_{W^{3/8,8/5}(\partial B^M_{2R} \cap \partial M)});$$

Combining these, we have

$$\begin{aligned}
\|\psi\|_{L^8(M)} &\leq \sum_i \|\psi\|_{L^8(B_R^M(x_i))} \\
&\leq \frac{C}{R^{1/4}}\big(\|\psi\|_{L^4(M)} + \|\mathcal{B}\psi_0\|_{W^{3/8,8/5}(\partial M)}\big) \\
&\leq \frac{C}{R^{1/4}}\big(\|\mathcal{B}\psi_0\|_{W^{1/4,4/3}(\partial M)} + \|\mathcal{B}\psi_0\|_{W^{3/8,8/5}(\partial M)}\big) \\
&\leq \frac{C}{R^{1/4}}\|\mathcal{B}\psi_0\|_{W^{3/8,8/5}(\partial M)},
\end{aligned}$$

where the third inequality follows from Lemma 3.3.

Multiplying the first equation of (1.6) by $-\Delta\Phi$ and integrating over M^T, we obtain

$$\begin{aligned}
&E(\Phi(T)) - E(\Phi(0)) + \int_{M^T} |\Delta\Phi|^2 dMdt \\
&= -\int_{M^T} \partial_t\Phi \cdot \Delta\Phi dMdt + \int_{M^T} |\Delta\Phi|^2 dMdt \\
&= -\int_{M^T} A(\Phi)(d\Phi, d\Phi)\Delta\Phi dMdt + \int_{M^T} \mathcal{P}(\mathcal{A}(d\Phi(e_\alpha), e_\alpha \cdot \Psi); \Psi)\Delta\Phi dMdt \\
&\leq \frac{1}{2}\int_{M^T} |\Delta\Phi|^2 dMdt + C\int_{M^T} |\nabla\Phi|^4 dMdt + \int_{M^T} |\Psi|^8 dMdt.
\end{aligned}$$

So,

$$\begin{aligned}
&E(\Phi(T)) + \frac{1}{2}\int_{M^T} |\Delta\Phi|^2 dMdt \\
&\leq E(\Phi(0)) + C\int_{M^T} |\nabla\Phi|^4 dMdt + \int_{M^T} |\Psi|^8 dMdt \\
&\leq E(\Phi(0)) + \int_{M^T} |\Psi|^8 dMdt + C \sup_{(x_0,t)\in M^T} \int_{B_R^M(x_0)} |\nabla\Phi|^2(x,t) dM \\
&\quad \cdot \left(\int_{M^T} |\nabla^2\Phi|^2 dMdt + R^{-2}\int_{M^T} |\nabla\Phi|^2 dMdt\right),
\end{aligned} \tag{3.11}$$

where the last inequality follows from (2.3).

By the theory of elliptic equations, there exists a unique solution $g \in H^2(M, \mathbb{R}^N)$ for

$$\begin{cases} \Delta g = 0 & in \quad M, \\ \quad g = \varphi & on \quad \partial M, \end{cases} \tag{3.12}$$

such that

$$\|g\|_{H^2(M)} \leq C(M, N)\|\varphi\|_{H^{3/2}(\partial M)}. \tag{3.13}$$

Since $\Phi - g \in H^1_0(M)$, then by standard elliptic theory, we have

$$
\begin{aligned}
\int_{M^T} |\nabla^2\Phi|^2 dMdt &\le \int_{M^T} |\nabla^2(\Phi - g)|^2 dMdt + \int_{M^T} |\nabla^2 g|^2 dMdt \\
&\le C\int_{M^T} |\Delta(\Phi - g)|^2 dMdt + C(M)T\|\varphi\|^2_{H^{3/2}(\partial M)} \\
&= C\int_{M^T} |\Delta\Phi|^2 dMdt + C(M)T\|\varphi\|^2_{H^{3/2}(\partial M)}. \qquad (3.14)
\end{aligned}
$$

Combining this with (3.11), (3.9) and Lemma 3.2, we get

$$
\begin{aligned}
&E(\Phi(t)) + \frac{1}{2}\int_{M^T} |\Delta\Phi|^2 dMdt \\
&\le E(\Phi(0)) + \int_{M^T} |\Psi|^8 dMdt + C\epsilon_1\left(\int_{M^T} |\nabla^2\Phi|^2 dMdt + R^{-2}\int_{M^T} |\nabla\Phi|^2 dMdt\right) \\
&\le C\epsilon_1 \int_{M^T} |\Delta\Phi|^2 dMdt + C(1 + \frac{T}{R^2})E(\Phi(0)) + \frac{CT}{R^2}(1 + \|\varphi\|^2_{H^{3/2}(\partial M)} + \|\mathcal{B}\psi_0\|^8_{W^{3/8,8/5}(\partial M)}).
\end{aligned}
$$

Taking ϵ_1 small enough, we obtain

$$
\int_{M^T} |\Delta\Phi|^2 dMdt \le C(1 + \frac{T}{R^2})E(\Phi(0)) + \frac{CT}{R^2}(1 + \|\varphi\|^2_{H^{3/2}(\partial M)} + \|\mathcal{B}\psi_0\|^8_{W^{3/8,8/5}(\partial M)}).
$$

Then the estimate (3.10) follows from (3.14) immediately. □

By taking a similar choice of testing function as in Lemma 3.8 of [28] or Lemma 4.5 in [14], we obtain

Lemma 3.5. *Suppose* $\phi_0 \in H^1(M, N)$, $\varphi \in H^{3/2}(\partial M, N)$, $\phi_0|_{\partial M} = \varphi$, $\psi_0 \in W^{3/8,8/5}(\partial M, \Sigma M \otimes \varphi^{-1}TN)$ *and satisfy* (1.16). *Then there exist constants* $\epsilon_2 = \epsilon_2(M, N) > 0$ *and* $C = C(M, N) > 0$, *such that if* $(\Phi, \Psi) \in V(M^T)$ *is a solution of* (1.6) *with the boundary-initial data* (1.10) *that satisfies*

$$
\epsilon(R) := \sup_{(x,t)\in M^T} E(\Phi(t); B^M_R(x)) \le \epsilon_2 \quad \text{for all} \quad R \in (0, R_0],
$$

then there holds the estimate

$$
\begin{aligned}
&E(\Phi(T); B^M_R(x_0)) + \int_{(B^M_R(x_0))^T} |\nabla^2\Phi|^2 dMdt \\
&\le E(\Phi(0); B^M_{2R}(x_0)) + C\frac{T}{R^2}(1 + E(\phi_0) + \|\varphi\|^2_{H^{3/2}(M)} + \|\mathcal{B}\psi_0\|^8_{W^{3/8,8/5}(\partial M)}).
\end{aligned}
$$

Proof. Fixing $x_0 \in M$, taking a cut-off function $\eta \in C^\infty_0(B^M_{2R}(x_0))$ such that $\eta|_{B_R} \equiv 1$, $|\nabla\eta| \le \frac{C}{R}$ and $|\nabla^2\eta| \le \frac{C}{R^2}$, then multiplying the first equation of (1.6) by $-\Delta\Phi\eta^2$ and integrating over M^T, we obtain

$$
\begin{aligned}
&-\int_{M^T} \partial_t\Phi\Delta\Phi\eta^2 dMdt + \int_{M^T} |\Delta\Phi|^2\eta^2 dMdt \\
&= -\int_{M^T} A(\Phi)(d\Phi, d\Phi)\Delta\Phi\eta^2 dMdt + \int_{M^T} \mathcal{P}(\mathcal{A}(d\Phi(e_\alpha), e_\alpha \cdot \Psi); \Psi)\Delta\Phi\eta^2 dMdt,
\end{aligned}
$$

then integrating by parts, we have

$$
\begin{aligned}
&\frac{1}{2}\int_{M^T}\partial_t|\nabla\Phi|^2\eta^2 dMdt+\int_{M^T}|\Delta\Phi|^2\eta^2 dMdt\\
&\leq \int_{M^T}|\partial_t\Phi||\nabla\Phi||2\eta\nabla\eta|dMdt+\frac{1}{2}\int_{M^T}|\Delta\Phi|^2\eta^2 dMdt+C\int_{M^T}|\nabla\Phi|^4\eta^2 dMdt+\int_{M^T}|\Psi|^8\eta^2 dMdt\\
&\leq \delta\int_{M^T}|\partial_t\Phi|^2\eta^2 dMdt+C(\delta)\int_{M^T}|\nabla\Phi|^2|\nabla\eta|^2 dMdt+\frac{1}{2}\int_{M^T}|\Delta\Phi|^2\eta^2 dMdt\\
&\quad +C\int_{M^T}|\nabla\Phi|^4\eta^2 dMdt+\int_{M^T}|\Psi|^8\eta^2 dMdt.
\end{aligned}
$$

Noting that

$$
|\partial_t\Phi|\leq|\nabla^2\Phi|+C|\nabla\Phi|^2+C|\Psi|^2|\nabla\Phi|,
$$

we get

$$
\begin{aligned}
&\frac{1}{2}\int_{M^T}\partial_t|\nabla\Phi|^2\eta^2 dMdt+\frac{1}{2}\int_{M^T}|\Delta\Phi|^2\eta^2 dMdt\\
&\leq \delta\int_{M^T}|\nabla^2\Phi|^2\eta^2 dMdt+C(\delta)\int_{M^T}|\nabla\Phi|^2|\nabla\eta|^2 dMdt+C\int_{M^T}|\nabla\Phi|^4\eta^2 dMdt\\
&\quad +\int_{M^T}|\Psi|^8\eta^2 dMdt\\
&\leq (\delta+C\epsilon_2)\int_{M^T}|\nabla^2\Phi|^2\eta^2 dMdt+\frac{C(\delta)}{R^2}\int_{(B^M_{2R}(x_0))^T}|\nabla\Phi|^2 dMdt+\int_{M^T}|\Psi|^8\eta^2 dMdt, \qquad (3.15)
\end{aligned}
$$

where the last inequality follows from the same argument as (3.11).

Since $\Phi\eta-g\eta\in H^1_0(M)$ (see (3.12)), then by standard elliptic theory, we have

$$
\begin{aligned}
\int_{M^T}|\nabla^2(\Phi\eta)|^2 dMdt &\leq \int_{M^T}|\nabla^2(\Phi\eta-g\eta)|^2 dMdt+\int_{M^T}|\nabla^2(g\eta)|^2 dMdt\\
&\leq C\int_{M^T}|\Delta(\Phi\eta-g\eta)|^2 dMdt+\int_{M^T}|\nabla^2(g\eta)|^2 dMdt\\
&\leq C\int_{M^T}|\Delta\Phi|^2\eta^2 dMdt+C(M)\int_{M^T}(|\nabla\Phi|^2|\nabla\eta|^2+|\Phi|^2|\nabla^2\eta|^2\\
&\quad +|\nabla^2 g|^2\eta^2+|\nabla g|^2|\nabla\eta|^2+|g|^2|\nabla^2\eta|^2)dMdt. \qquad (3.16)
\end{aligned}
$$

By (3.15) and (3.16), we get

$$
\begin{aligned}
&\int_{M^T}\partial_t|\nabla\Phi|^2\eta^2 dMdt+\int_{M^T}|\nabla^2\Phi|^2\eta^2 dMdt\\
&\leq C(\delta+C\epsilon_2)\int_{M^T}|\nabla^2\Phi|^2\eta^2 dMdt+\frac{C(\delta)}{R^2}\int_{(B^M_{2R(x_0)})^T}|\nabla\Phi|^2 dMdt+\int_{M^T}|\Psi|^8\eta^2 dMdt\\
&\quad +C\int_{M^T}(|\nabla\Phi|^2|\nabla\eta|^2+|\Phi|^2|\nabla^2\eta|^2+|\nabla^2 g|^2\eta^2+|\nabla g|^2|\nabla\eta|^2+|g|^2|\nabla^2\eta|^2)dMdt.
\end{aligned}
$$

Taking $\delta > 0$ and $\epsilon_2 > 0$ sufficiently small such that $C(\delta + C\epsilon_2) \le 1/2$, then with (3.9), (3.13) and Lemma 3.2 we have

$$\begin{aligned}
&\int_{M^T} \partial_t |\nabla\Phi|^2 \eta^2 dMdt + \int_{M^T} |\nabla^2\Phi|^2\eta^2 dMdt \\
&\le \frac{C(\delta)}{R^2} \int_{(B^M_{2R}(x_0))^T} |\nabla\Phi|^2 dMdt + \int_{M^T} |\Psi|^8 \eta^2 dMdt + C\frac{T}{R^2}(1 + \|\varphi\|^2_{H^{3/2}(M)}) \\
&\le C\frac{T}{R^2}(1 + E(\phi_0) + \|\varphi\|^2_{H^{3/2}(M)} + \|\mathcal{B}\psi_0\|^8_{W^{3/8,8/5}(\partial M)}).
\end{aligned}$$

Thus, we get the estimate

$$\begin{aligned}
&E(\Phi(T); B^M_R(x_0)) + \int_{(B^M_R(x_0))^T} |\nabla^2\Phi|^2 dMdt \\
&\le E(\Phi(0); B^M_{2R}(x_0)) + C\frac{T}{R^2}(1 + E(\phi_0) + \|\varphi\|^2_{H^{3/2}(M)} + \|\mathcal{B}\psi_0\|^8_{W^{3/8,8/5}(\partial M)}).
\end{aligned}$$

□

Next, we obtain the $\epsilon-$ regularity

Lemma 3.6. *Suppose that $\phi_0 \in H^1(M,N)$, $\varphi \in C^{2+\alpha}(\partial M, N)$ and $\psi_0 \in C^{1+\alpha}(\partial M, \Sigma M \otimes \varphi^{-1}TN)$ satisfy the boundary-initial constraint (1.16). Let $(\Phi,\Psi) \in V(M^T)$ be a solution of (1.6) with boundary-initial data (1.10). Given $z_0 = (x_0, t_0) \in M \times (0, T]$, denote $P^M_R(z_0) := B^M_R(x_0) \times [t_0 - R^2, t_0]$. Assume that $\Phi \in C^{2+\alpha,1+\frac{\alpha}{2}}(P^M_R(z_0), N)$ and $\Psi \in C^{1+\alpha}(P^M_R(z_0), \Sigma M \otimes \Phi^{-1}TN)$. Then there exist two positive constants $\epsilon_3 = \epsilon_3(M, N, E(\phi_0), \|\varphi\|_{C^{2+\alpha}(\partial M)}, \|\mathcal{B}\psi_0\|_{C^{1+\alpha}(\partial M)}) > 0$ and $C = C(\alpha, R, M, N, E(\phi_0), \|\varphi\|_{C^{2+\alpha}(\partial M)}, \|\mathcal{B}\psi_0\|_{C^{1+\alpha}(\partial M)}) > 0$ such that if*

$$\sup_{[t_0-R^2,t_0]} E(\Phi(t), B^M_R(x_0)) \le \epsilon_3,$$

then

$$\sqrt{R}\|\Psi\|_{L^\infty(P^M_{R/2}(z_0))} + R\|\nabla\Phi\|_{L^\infty(P^M_{R/2}(z_0))} \le C \tag{3.17}$$

and for any $0 < \beta < 1$,

$$\sup_{t_0-\frac{R^2}{4}\le t\le t_0} \|\Psi(t)\|_{C^{1+\alpha}(B^M_{R/2}(z_0))} + \|\Phi\|_{C^{1,0,\beta}(P^M_{R/2}(z_0))} \le C(\beta), \tag{3.18}$$

Moreover, if

$$\sup_{x_0\in M} \sup_{[t_0-R^2,t_0]} E(\Phi(t), B^M_R(x_0)) \le \epsilon_1,$$

then

$$\|\Psi\|_{C^{1,0,\alpha}(M\times[t_0-\frac{R^2}{8},t_0])} + \|\Phi\|_{C^{2,1,\alpha}(M\times[t_0-\frac{R^2}{8},t_0])} \le C. \tag{3.19}$$

Proof.

Step 1: We derive (3.18) and (3.19) from (3.17).

Taking the cut-off function $\eta \in C_0^\infty(P_R^M(z_0))$ such that $0 \leq \eta \leq 1$, $\eta|_{P^M_{3R/4}(z_0)} \equiv 1$, $|\nabla^j \eta| \leq \frac{C}{R^j}$, $j = 1, 2$ and $|\partial_t \eta| \leq \frac{C}{R^2}$, set $U = \eta\Phi$, then

$$\begin{cases} U_t - \Delta U = f, & in \quad P_R^M(z_0); \\ U(x,t) = 0, & on \quad B_R^M(z_0) \times \{t = t_0 - R^2\}; \\ U(x,t) = \eta\varphi, & on \quad \partial M \times (t_0 - R^2, t_0), \end{cases}$$

where

$$f := \eta(\partial_t - \Delta)\Phi + \partial_t \eta \Phi - 2\nabla\eta\nabla\Phi - \Phi\Delta\eta.$$

By standard parabolic theory, for any $1 < p < \infty$, we have

$$\|U\|_{W_p^{2,1}(P_R^M(z_0))} \leq C(\|f\|_{L^p(P_R^M(z_0))} + \|\eta\varphi\|_{W_p^{2,1}(\partial P_R^M(z_0))}) \leq C(1 + \|\varphi\|_{C^2(\partial M)})$$

since $f \in L^\infty$ under the equation (1.6) and assumption (3.17). Then for any $0 < \beta = 1 - 4/p < 1$, we obtain

$$\|\nabla\Phi\|_{C^{\beta,\beta/2}(P^M_{3R/4}(z_0))} \leq \|\nabla U\|_{C^{\beta,\beta/2}(P_R^M(z_0))} \leq C\|U\|_{W_p^{2,1}(P_R^M(z_0))} \leq C(\beta)(1 + \|\varphi\|_{C^2(\partial M)}). \tag{3.20}$$

Taking the cut-off function $\chi \in C_0^\infty(B_R^M(z_0))$ such that $0 \leq \chi \leq 1$, $\chi|_{B^M_{3R/4}(z_0)} \equiv 1$ and $|\nabla^j \chi| \leq \frac{C}{R^j}$, $j = 1, 2$, set $V = \chi\Psi$, then

$$\begin{cases} \partial\!\!\!/ V = h, & in \quad B_R^M(z_0); \\ \mathcal{B}V(x) = \chi\mathcal{B}\psi_0, & on \quad \partial B_R^M(z_0), \end{cases}$$

where $h = \chi\partial\!\!\!/\Psi + \nabla\chi \cdot \Psi \in L^\infty$. By Lemma 2.7 and Sobolev embedding, we have

$$\|\Psi\|_{C^{1-n/p}(B^M_{3R/4}(z_0))} \leq C\|V\|_{W^{1,p}(B_R^M(z_0))} \leq C(1 + \|\mathcal{B}\psi_0\|_{C^1(\partial M)}) \tag{3.21}$$

for any $2 < p < \infty$. Combining (3.20) with (3.21), we know $\partial\!\!\!/\Psi \in C^\alpha(B^M_{R/2}(z_0))$ and by the Schauder estimates Lemma 2.8 and taking some suitable cut-off function as before, we have

$$\|\Psi(t)\|_{C^{1+\alpha}(B^M_{R/2}(z_0))} \leq C(1 + \|\mathcal{B}\psi_0\|_{C^{1+\alpha}(\partial M)})(1 + \|\varphi\|_{C^2(\partial M)}) \tag{3.22}$$

for any $t_0 - \frac{R^2}{4} \leq t \leq t_0$. Then the inequality (3.18) follows from (3.20), (3.22) immediately.

In order to prove (3.19), noting that we can rewrite the equation $\partial\!\!\!/\Psi = \mathcal{A}(d\Phi(e_\alpha), e_\alpha \cdot \Psi)$ as

$$\partial\!\!\!/\Psi + \Omega\Psi = 0$$

where $\Omega = [\nu(\Phi), d\nu(\Phi)]$ and $\{\nu^i\}_{i=n+1}^N$ is an orthonormal basis of normal bundle $T^\perp N$ (see Remark 2.1 in [10]), then for any $t_0 - \frac{R^2}{4} < t, s < t_0$, we have

$$\begin{cases} \partial\!\!\!/(\Psi(\cdot,t) - \Psi(\cdot,s)) = -\Omega(\cdot,t)(\Psi(\cdot,t) - \Psi(\cdot,s)) + (\Omega(\cdot,s) - \Omega(\cdot,t))\Psi(\cdot,s) & in \quad M; \\ \mathcal{B}(\Psi(\cdot,t) - \Psi(\cdot,s)) = 0 & on \quad \partial M. \end{cases}$$

Since $d\Omega = [d\nu(\Phi), d\nu(\Phi)]$, with (3.20) and (3.22), according to Theorem 4.1 in [10], for any $0 < \beta < 1$, by Sobolev embedding, we have

$$\|\Psi(\cdot,t) - \Psi(\cdot,s)\|_{C^\beta(M)} \leq C(\|\Omega(\cdot,t) - \Omega(\cdot,s)\|_{L^\infty(M)}) \leq C|s-t|^\beta.$$

So, we get $\|\Psi\|_{C^{1,0,\alpha}(M\times[t_0-\frac{R^2}{4},t_0])} \le C$ and

$$\begin{cases} \partial_t\Phi - \Delta\Phi & \in C^{\beta,\beta/2}(M\times[t_0-\frac{R^2}{4},t_0]) \quad for \quad any \quad 0<\beta<1; \\ \Phi|_{\partial M} = \varphi & \in C^{2+\alpha}(\partial M). \end{cases}$$

Taking some suitable cut-off function and by standard Schauder estimates of parabolic equation, we have $\Phi \in C^{2,1,\alpha}(M\times[t_0-\frac{R^2}{8},t_0])$ and

$$\|\Phi\|_{C^{2,1,\alpha}(M\times[t_0-\frac{R^2}{8},t_0])} \le C\big(\|\partial_t\Phi-\Delta\Phi\|_{C^{\alpha,\alpha/2}(M\times[t_0-\frac{R^2}{4},t_0])} + \|\Phi\|_{C^0(M\times[t_0-\frac{R^2}{4},t_0])} + \|\varphi\|_{C^{2+\alpha}(\partial M)}\big) \le C.$$

So we have proved (3.19).

Step 2: We prove (3.17).

We follow as similar idea as in [25, 23]. Without loss of generality, we may assume $R = 1$. Choose $0 \le \rho < 1$ such that

$$(1-\rho)^2 \sup_{P^M_\rho(z_0)} |\nabla\Phi|^2 = \max_{0\le\sigma\le 1}\{(1-\sigma)^2 \sup_{P^M_\sigma(z_0)} |\nabla\Phi|^2\}$$

and then choose $z_1 = (x_1,t_1) \in P^M_\rho(z_0)$ such that

$$|\nabla\Phi|^2(z_1) = \sup_{P^M_\rho(z_0)} |\nabla\Phi|^2 := e.$$

We claim:

$$(1-\rho)^2 e \le 4.$$

We proceed by contradiction. If $(1-\rho)^2 e > 4$, we set

$$u(x,t) := \Phi(x_1 + e^{-\frac{1}{2}}x, t_1 + e^{-1}t) \quad and \quad v(x) := e^{-\frac{1}{4}}\Psi(x_1 + e^{-\frac{1}{2}}x).$$

Denoting $P_r(0) = B_r(0)\times[-r^2,0] \subset \mathbb{R}^2$ and

$$S_r := P_r(0) \cap \{(x,t)|(x_1 + e^{-\frac{1}{2}}x, t_1 + e^{-1}t) \in P^M_1(0)\},$$

then $u \in C^{2,1,\alpha}(S_1)$, $v \in C^{1,0,\alpha}(S_1)$, and they satisfy

$$\begin{cases} \partial_t u = \tau(u) - \mathcal{P}(\mathcal{A}(du(e_\alpha), e_\alpha\cdot v); v), & in \quad S_1; \\ \partial\!\!\!/ v = \mathcal{A}(du(e_\alpha), e_\alpha \cdot v), & in \quad S_1, \end{cases} \tag{3.23}$$

with the boundary data

$$\begin{cases} u(x,t) = \varphi(x_1 + e^{-\frac{1}{2}}x), & if \quad x_1 + e^{-\frac{1}{2}}x \in \partial M; \\ \mathcal{B}v(x,t) = e^{-\frac{1}{4}}\mathcal{B}\psi_0(x_1 + e^{-\frac{1}{2}}x), & if \quad x_1 + e^{-\frac{1}{2}}x \in \partial M. \end{cases} \tag{3.24}$$

Moreover, we have

$$\sup_{S_1}|\nabla u|^2 = e^{-1}\sup_{P^M_{e^{-1/2}}(z_1)} |\nabla\Phi|^2 \le e^{-1}\sup_{P^M_{\rho+e^{-1/2}}(z_0)} |\nabla\Phi|^2 \le e^{-1}\sup_{P^M_{\frac{1+\rho}{2}}(z_0)} |\nabla\Phi|^2 \le 4$$

and

$$|\nabla u|^2(0) = e^{-1}|\nabla\Phi|^2(z_1) = 1.$$

Since v satisfies the equation $\partial\!\!\!/ v = \mathcal{A}(du(e_\alpha), e_\alpha \cdot v)$ and there holds

$$|du| \leq 2, \qquad \sup_{-1\leq t\leq 0} \|v\|_{L^4(B_1)} \leq \|\Psi\|_{L^4(M)} \leq C,$$

where in the last step we have used Lemma 3.3. By elliptic theory, we have

$$\sup_{-1\leq t\leq 0} \|v\|_{L^\infty(B_{3/4})} \leq C \sup_{-1\leq t\leq 0} \|v\|_{W^{1,4}(B_{3/4})} \leq C(\|\mathcal{B}\psi_0\|_{C^1(\partial M)}).$$

Next, we want to show that there exists a constant $C > 0$ such that

$$1 \leq C \int_{S_{3/4}} |\nabla u|^2 dxdt. \tag{3.25}$$

If C does not exist, then we can find a sequence $\{(u_i, v_i)\}$ satisfying

$$\begin{cases} \partial_t u_i = \tau(u_i) - \mathcal{P}(\mathcal{A}(du_i(e_\alpha), e_\alpha \cdot v_i); v_i), & in \quad S_{3/4}; \\ \partial\!\!\!/ v_i = \mathcal{A}(du_i(e_\alpha), e_\alpha \cdot v_i), & in \quad S_{3/4}, \end{cases} \tag{3.26}$$

with the boundary data

$$\begin{cases} u_i(x,t) = \varphi(x_1 + e^{-\frac{1}{2}} x), & if \quad x_1 + e^{-\frac{1}{2}} x \in \partial M; \\ \mathcal{B}v_i(x,t) = e^{-\frac{1}{4}} \mathcal{B}\psi_0(x_1 + e^{-\frac{1}{2}} x), & if \quad x_1 + e^{-\frac{1}{2}} x \in \partial M \end{cases} \tag{3.27}$$

and

$$\sup_{S_{3/4}} (|\nabla u_i| + |v_i|) \leq C, \tag{3.28}$$

$$|\nabla u_i|^2(0) = 1, \tag{3.29}$$

$$\int_{S_{3/4}} |\nabla u_i|^2 dxdt \leq \frac{1}{i}. \tag{3.30}$$

By **Step 1** (since (u_i, v_i) satisfy (3.26), (3.27) and (3.28)), we have

$$\|\nabla u_i\|_{C^{\beta,\beta/2}(S_{1/2})} \leq C(\beta)$$

for any $0 < \beta < 1$.

Therefore, there exist a subsequence of $\{u_i\}$ (we still denote it by $\{u_i\}$) and a function $\overline{u} \in C^{1,0,\gamma}(S_{1/2})$ such that

$$\nabla u_i \to \nabla \overline{u} \quad in \quad C^{\gamma,\gamma/2}(S_{1/2})$$

where $0 < \gamma < \beta$. Then by (3.30), we know

$$\int_{S_{1/2}} |\nabla \overline{u}|^2 dxdt = 0 \tag{3.31}$$

which implies $\nabla \overline{u} \equiv 0$ in $S_{1/2}$. But, (3.29) tells us $|\nabla \overline{u}|(0) = 1$. This is impossible and then (3.25) must be true. Thus, we have

$$1 \leq C \int_{S_{3/4}} |\nabla u|^2 dxdt \leq C \sup_{-1<t<0} \int_{B^M_{e^{\frac{1}{2}}}(x_1)} |\nabla \Phi|^2 (t_1 + e^{-1} t) dx \leq C \sup_{-1<t<0} \int_{B^M_1(z_0)} |\nabla \Phi|^2(t) dx \leq C\epsilon_3.$$

Choosing $\epsilon_3 > 0$ sufficiently small leads to a contradiction, so we must have $(1-\rho)^2 e \le 4$ and then

$$(1-3/4)^2 \sup_{P^M_{3/4}(z_0)} |\nabla\Phi|^2 \le (1-\rho)^2 e \le 4.$$

Since Ψ satisfies the equation $\partial\!\!\!/\Psi = \mathcal{A}(d\Phi(e_\alpha), e_\alpha\cdot\Psi)$ and $\|d\Phi\|_{L^\infty(P^M_{3/4}(z_0))} \le 8$, $\|\Psi\|_{L^4(M)} \le C$, by the elliptic theory of first order equations and Sobolev embedding again, we shall easily obtain

$$\|\Psi\|_{L^\infty(P^M_{1/2}(z_0))} \le C.$$

Thus we get the inequality (3.17). This finishes the proof of the lemma. □

Finally, we show the uniqueness result.

Theorem 3.7. *Let $\phi_0 \in H^1(M,N)$, $\phi_0|_{\partial M} = \varphi \in H^{3/2}(\partial M, N)$ and $\psi_0 \in W^{3/8,8/5}(\partial M, \Sigma M\otimes\varphi^{-1}TN)$ satisfy the boundary-initial constraint* (1.16). *Furthermore, suppose that $(\Phi_i,\Psi_i) \in V(M^T)$, $i = 1,2$ are weak solutions of* (1.6) *with the same boundary-initial data* (1.10). *Then $(\Phi_1,\Psi_1) \equiv (\Phi_2,\Psi_2)$ in M^T.*

Proof. Let $W := \Phi_1 - \Phi_2$, $\Omega := \Psi_1 - \Psi_2$ and denote $|\nabla U| := |\nabla\Phi_1| + |\nabla\Phi_2|$, $|V| := |\Psi_1| + |\Psi_2|$. Since $(\Phi_i,\Psi_i) \in V(M^T)$, $i = 1,2$ are weak solutions to (1.6), we have

$$\begin{aligned}|\partial_t W - \Delta W| &\le |A(\Phi_1)(d\Phi_1, d\Phi_1) - A(\Phi_2)(d\Phi_2, d\Phi_2)| \\ &\quad + |\mathcal{P}(\mathcal{A}(d\Phi_1(e_\alpha), e_\alpha\cdot\Psi_1);\Psi_1) - \mathcal{P}(\mathcal{A}(d\Phi_2(e_\alpha), e_\alpha\cdot\Psi_2);\Psi_2)| \\ &\le C(|W|(|\nabla U|^2 + |\nabla U||V|^2) + |\nabla W|(|\nabla U| + |V|^2) + |\Omega||\nabla U||V|).\end{aligned}$$

Multiplying the above inequality by W and integrating over M^t, we obtain

$$\begin{aligned}&\frac{1}{2}\int_{M^t} \partial_t |W|^2 dMdt - \int_{M^t} \Delta W\cdot W dMdt \\ &= \frac{1}{2}\int_M |W|^2 dM + \int_{M^t} |\nabla W|^2 dMdt \\ &\le C\int_{M^t} (|W|^2(|\nabla U|^2 + |\nabla U||V|^2) + |W||\nabla W|(|\nabla U| + |V|^2) + |W||\Omega||\nabla U||V|) \\ &\le C(\int_{M^t} |W|^4 dMdt)^{1/2}((\int_{M^t} |\nabla U|^4 dMdt)^{1/2} + (\int_{M^t} |V|^8 dMdt)^{1/2}) \\ &\quad + C(\int_{M^t} |W|^4 dMdt)^{1/4}(\int_{M^t} |\nabla W|^2 dMdt)^{1/2}((\int_{M^t} |\nabla U|^4 dMdt)^{1/4} + (\int_{M^t} |V|^8 dMdt)^{1/4}) \\ &\quad + C(\int_{M^t} |W|^4 dMdt)^{1/4}(\int_{M^t} |\Omega|^2 dMdt)^{1/2}(\int_{M^t} |\nabla U|^8 dMdt)^{1/8}(\int_{M^t} |V|^8 dMdt)^{1/8} \\ &\le C\varepsilon(t)(\int_{M^t} |W|^4 dMdt)^{1/2} + C\varepsilon(t)(\int_{M^t} |W|^4 dMdt)^{1/4}(\int_{M^t} |\nabla W|^2 dMdt)^{1/2} \\ &\quad + C\varepsilon(t)(\int_{M^t} |W|^4 dMdt)^{1/4}(\int_{M^t} |\Omega|^2 dMdt)^{1/2}\end{aligned}$$

for any $t \in (0, T]$ and $\varepsilon(t) \to 0$ as $t \to 0$.

Noticing that $\not{D}_{\Phi_l}\Psi_l = 0$, $l = 1, 2$, we have

$$
\begin{aligned}
|\not{D}_{\Phi_2}\Omega| &= |\not{\partial}\Omega - \mathcal{A}(\Phi_2)(d\Phi_2(e_\alpha), e_\alpha \cdot D\Pi_N|_{\Phi_2} \circ \Omega)| \\
&= |\mathcal{A}(\Phi_1)(d\Phi_1(e_\alpha), e_\alpha \cdot \Psi_1) - \mathcal{A}(\Phi_2)(d\Phi_2(e_\alpha), e_\alpha \cdot D\Pi_N|_{\Phi_2} \circ \Psi_1)| \\
&= |(\nabla\Phi_1^i \cdot \Psi_1^j) \otimes A(D\Pi_N|_{\Phi_1} \circ \partial_{y^i}, D\Pi_N|_{\Phi_1} \circ \partial_{y^j}) - (\nabla\Phi_2^i \cdot \Psi_1^j) \\
&\quad \otimes A(D\Pi_N|_{\Phi_2} \circ \partial_{y^i}, D\Pi_N|_{\Phi_2} \circ \partial_{y^j})| \\
&\le |(\nabla\Phi_1^i \cdot \Psi_1^j) \otimes (A(D\Pi_N|_{\Phi_1} \circ \partial_{y^i}, D\Pi_N|_{\Phi_1} \circ \partial_{y^j}) - A(D\Pi_N|_{\Phi_2} \circ \partial_{y^i}, D\Pi_N|_{\Phi_2} \circ \partial_{y^j}))| \\
&\quad + |(\nabla\Phi_1^i \cdot \Psi_1^j - \nabla\Phi_2^i \cdot \Psi_1^j) \otimes A(D\Pi_N|_{\Phi_2} \circ \partial_{y^i}, D\Pi_N|_{\Phi_2} \circ \partial_{y^j})| \\
&\le C|W||\nabla U||V| + C|\nabla W||V|,
\end{aligned}
$$

where $1 \le i, j \le N$ and $\{\partial_{y^i}\}_{i=1}^N$ is the standard basis of $\mathbb{R}^N$.

Since $E(\Phi_2(t)) \le E(\phi_0) + \sqrt{2}\|\mathcal{B}\psi_0\|^2_{L^2(\partial M)} < \Lambda^2$ and $\mathcal{B}\Omega = 0$ on ∂M, by definition of Λ in (1.11), we have

$$
\begin{aligned}
\|\Omega\|_{W^{1,4/3}(M)} &\le C\left(\||W||\nabla U||V|\|_{L^{4/3}(M)} + \||\nabla W||V|\|_{L^{4/3}(M)}\right) \\
&\le C\||W|\|_{L^4(M)}\||\nabla U|\|_{L^4(M)}\||V|\|_{L^4(M)} + C\||\nabla W|\|_{L^2(M)}\||V|\|_{L^4(M)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|\Omega\|_{L^4(M)} &\le C\|\Omega\|_{W^{1,4/3}(M)} \\
&\le C\||W|\|_{L^4(M)}\||\nabla U|\|_{L^4(M)}\||V|\|_{L^4(M)} + C\||\nabla W|\|_{L^2(M)}\||V|\|_{L^4(M)}
\end{aligned}
$$

and

$$
\begin{aligned}
(\int_{M^t} |\Omega|^2 dMdt)^{1/2} &\le C(\int_0^t (\int_M |\Omega|^4 dM)^{1/2} dt)^{1/2} \\
&\le C(\int_0^t (\int_M |W|^4 dM \cdot \int_M |\nabla U|^4 dM \cdot \int_M |V|^4 dM)^{1/2} dt)^{1/2} \\
&\quad + C(\int_0^t \int_M |\nabla W|^2 dM \cdot (\int_M |V|^4 dM)^{1/2} dt)^{1/2} \\
&\le C(\int_0^t (\int_M |W|^4 dM)^{1/2} \cdot (\int_M |\nabla U|^4 dM)^{1/2} dt)^{1/2} \\
&\quad + C(\int_0^t \int_M |\nabla W|^2 dMdt)^{1/2} \\
&\le C(\int_0^t \int_M |W|^4 dMdt)^{1/4} \cdot (\int_0^t \int_M |\nabla U|^4 dMdt)^{1/4} \\
&\quad + C(\int_0^t \int_M |\nabla W|^2 dMdt)^{1/2} \\
&\le \varepsilon(t)(\int_0^t \int_M |W|^4 dMdt)^{1/4} + C(\int_0^t \int_M |\nabla W|^2 dMdt)^{1/2}.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
&\frac{1}{2}\int_M |W|^2(\cdot,t)dM + \int_{M^t} |\nabla W|^2 dMdt\\
&\leq C\varepsilon(t)(\int_{M^t} |W|^4 dMdt)^{1/2} + C\varepsilon(t)(\int_{M^t} |W|^4 dMdt)^{1/4}(\int_{M^t} |\nabla W|^2 dMdt)^{1/2}\\
&\leq C\varepsilon(t)(\int_{M^t} |W|^4 dMdt)^{1/2} + \frac{1}{2}\int_{M^t} |\nabla W|^2 dMdt
\end{aligned}
\tag{3.32}
$$

By the covering Lemma 2.2 and inequality (2.1), we have

$$
\begin{aligned}
\int_{M^t} |W|^4 dMdt &\leq C\int_0^t \int_M |W|^2 dM(\int_M |\nabla W|^2 dM + \int_M |W|^2 dM)dt\\
&\leq C\sup_{0\leq s\leq t}\int_M |W|^2 dM(\int_{M^t} |\nabla W|^2 dMdt + \int_{M^t} |W|^2 dMdt)\\
&\leq C(\sup_{0\leq s\leq t}\int_M |W|^2 dM + \int_{M^t} |\nabla W|^2 dM)^2.
\end{aligned}
\tag{3.33}
$$

Combing (3.32) with (3.33), we have

$$
\frac{1}{2}\int_M |W|^2(\cdot,t)dM + \frac{1}{2}\int_{M^t} |\nabla W|^2 dMdt \leq \varepsilon(t)(\sup_{0\leq s\leq t}\int_M |W|^2(\cdot,t)dM + \int_{M^t} |\nabla W|^2 dM).
$$

Without loss of generality, we may assume

$$
\int_M |W|^2(\cdot,t)dM = \sup_{0\leq s\leq t}\int_M |W|^2(\cdot,t)dM.
$$

Since $\varepsilon(t) \to 0$ as $t \to 0$, then there exists $S \in (0,T]$ such that

$$
\int_M |W|^2(\cdot,S)dM + \int_{M^S} |\nabla W|^2 dMdt = 0
$$

and $W \equiv 0$ in M^S. Thus, $\Omega \equiv 0$ in M^S by the fact $\mathrm{Ker}(\not{D}_\Phi;\mathcal{B}) = 0$. Iterating we obtain the lemma. □

4. Local and global existence results

In this section, under the boundary-initial constraint (1.16), we show the local existence of our flow for some initial map $\phi_0 \in H^1(M,N)$ and then show the existence of a global weak solution, completing the proof of Theorem 1.2.

Theorem 4.1 (Local existence)**.** *Suppose $\phi_0 \in H^1(M,N)$, $\varphi \in C^{2+\alpha}(\partial M,N)$, $\psi_0 \in C^{1+\alpha}(\partial M,\Sigma M\otimes \varphi^{-1}TN)$ and satisfy the boundary-initial constraint* (1.16). *Then there exists a unique solution $(\Phi,\Psi) \in \cup_{T'<T_1} V(M^{T'})$ of* (1.6) *with boundary-initial data* (1.10) *which is defined in $M\times[0,T_1)$, satisfying*

$$
\Phi \in C^{2,1,\alpha}_{loc}(M\times(0,T_1),N) \text{ and } \Psi \in C^{1,0,\alpha}_{loc}(M\times(0,T_1),\Sigma M\otimes\Phi^{-1}TN)
$$

where T_1 is characterized by the condition

$$
\lim_{t\nearrow T_1}\sup_{(x,t)\in M^{T_1}} E(\Phi(t);B^M_R(x)) > \overline{\epsilon} \quad \textit{for all} \quad R>0
\tag{4.1}
$$

and $\overline{\epsilon} = \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$ *is a constant.*

Moreover, $E(\Phi(t)) + \frac{1}{2}\int_{\partial M}\langle \overrightarrow{n} \cdot \mathcal{B}\psi_0, \Psi\rangle$ *is absolutely continuous and non-increasing in* $[0, T_1)$.

Proof.

Step 1: There exists a sequence $\phi_{0m} \in C^{2+\alpha}(M)$ such that

$$\phi_{0m} \to \phi_0 \quad strongly \quad in \quad H^1(M);$$
$$\varphi_m := \phi_{0m}|_{\partial M} \to \varphi \quad strongly \quad in \quad C^{2+\alpha}(\partial M).$$

In fact, let $g \in C^{2+\alpha}(M)$ be a harmonic function satisfying (3.12). Since $\phi_0 - g \in H_0^1(M)$, choosing $u_{0m} \in C_0^\infty(M)$ such that $u_{0m} \to \phi_0 - g$ in $H^1(M)$, then $\phi_{0m} = u_{0m} + g$ is the desired sequence.

Step 2: Local existence.

By Theorem 1.1, there exist $T_m > 0$ and $\Phi_m \in C^{2,1,\alpha}_{loc}(M \times (0, T_m))$, $\Psi_m \in C^{1,0,\alpha}_{loc}(M \times (0, T_m))$ which solve (1.6) with the boundary-initial data $\phi_{0m}, \varphi_m, \psi_0$.

Since $\phi_{0m} \to \phi_0$ strongly in $H^1(M)$, there exists some $R > 0$ such that for all $x \in M$,

$$E(\phi_{0m}; B^M_{2R}(x)) < \frac{\overline{\epsilon}}{4}.$$

Then by Lemma 3.5, if $T = O(R^2\overline{\epsilon})$, we have

$$\sup_{(x,t)\in M^T} E(\Phi_m(\cdot, t); B^M_R(x)) < \overline{\epsilon}. \tag{4.2}$$

So, combining (4.2) with Lemma 3.5, Lemma 3.6 and Theorem 1.1, we may assume $T_m \geq T = O(R^2\overline{\epsilon})$. Using Lemma 3.2, Lemma 3.3 and Lemma 3.4, we have

$$\|(\Phi_m, \Psi_m)\|^2_{V(M^T)} \leq C.$$

Furthermore, by Lemma 3.6, we have

$$\|\Psi_m\|_{C^{1,0,\alpha}(M\times[\delta,T])} + \|\Phi_m\|_{C^{2,1,\alpha}(M\times[\delta,T])} \leq C(\alpha, R, \delta, T). \tag{4.3}$$

According to the weak compactness, there exists a subsequence of $\{(\Phi_m, \Psi_m)\}$ which for convenience we still denote by $\{(\Phi_m, \Psi_m)\}$, and a function $(\Phi, \Psi) \in V(M^T)$ such that as $m \to \infty$,

$$\begin{aligned}
\partial_t\Phi_m &\rightharpoonup \partial_t\Phi \quad weakly \quad in \quad L^2(M^T),\\
\nabla^2\Phi_m &\rightharpoonup \nabla^2\Phi \quad weakly \quad in \quad L^2(M^T),\\
\nabla\Phi_m &\rightharpoonup \nabla\Phi \quad weakly \quad in \quad L^\infty(0, T; L^2(M)),\\
\Psi_m &\rightharpoonup \Psi \quad weakly \quad in \quad L^\infty(0, T; W^{1,4/3}(M)),
\end{aligned}$$

where $L^\infty(0, T; \|\cdot\|) := \sup_{0\leq t\leq T}\|\cdot\|$. In addition, by the Sobolev embedding theory, we get

$$\begin{aligned}
\nabla\Phi_m &\to \nabla\Phi \quad strongly \quad in \quad L^2(M^T),\\
\nabla\Phi_m &\rightharpoonup \nabla\Phi \quad weakly \quad in \quad L^4(M^T),\\
\Psi_m &\rightharpoonup \Psi \quad weakly \quad in \quad L^2(M^T).
\end{aligned}$$

Then it is easy to check that $(\Phi,\Psi)\in V(M^T)$ is a weak solution of (1.6) with (1.10) in the sense of distributions. Moreover, from (4.3), we know $\Phi\in\cap_{0<s<T}C^{2,1,\alpha}(M\times[s,T])$, $\Psi\in\cap_{0<s<T}C^{1,0,\alpha}(M\times[s,T])$ and then (Φ,Ψ) is a classical solution of (1.6). The short-time existence Theorem 1.1 guarantees the existence of a solution to (1.6) using $\Phi(T)$ as the new initial data and the solution can be continued to a larger time interval. Repeating this argument, the solution can be continued until the first time of energy concentration, that is, when $t=T_1$, the condition (4.1) is satisfied.

Finally, from Lemma 3.2 and Lemma 3.6, we have

$$E(\Phi(t))+\frac{1}{2}\int_{\partial M}\langle\overrightarrow{n}\cdot\mathcal{B}\psi_0,\Psi\rangle(t)$$

is absolutely continuous and non-increasing in $[0,T_1)$ and

$$\Phi\in C^{2,1,\alpha}_{loc}(M\times(0,T_1),N)\ and\ \Psi\in C^{1,0,\alpha}_{loc}(M\times(0,T_1),\Sigma M\otimes\Phi^{-1}TN).$$

□

Remark 4.2. If $\varphi\in C^\infty(\partial M,N)$, $\psi_0\in C^\infty(\partial M,\Sigma M\otimes\varphi^{-1}TN)$, then the solution will be regular in $M\times(0,T_1)$.

Next, we prove our main Theorem 1.2.

Proof of Theorem 1.2. By Theorem 4.1, there exists an unique local solution (Φ,Ψ) on M^{T_1} satisfying

$$\Phi\in C^{2,1,\alpha}_{loc}(M\times(0,T_1))\ and\ \Psi\in C^{1,0,\alpha}_{loc}(M\times(0,T_1)),$$

where T_1 is the first singular time. Next, we claim: there exist $\Phi(\cdot,T_1)\in H^1(M,N)$ and $\Psi(\cdot,T_1)\in W^{1,4/3}(M,\Sigma M\otimes\Phi(\cdot,T_1)^{-1}TN)$ such that

$$\begin{aligned}\Phi(\cdot,t)&\rightharpoonup\Phi(\cdot,T_1)\quad weakly\quad in\quad H^1(M),\\ \Psi(\cdot,t)&\rightharpoonup\Psi(\cdot,T_1)\quad weakly\quad in\quad W^{1,4/3}(M)\end{aligned}$$

as $t\to T_1$.

In fact, by Lemma 3.2 and Lemma 3.3, for any sequence $t_i\to T_1$, there exists a subsequence (also denoted by t_i) such that $\Phi(\cdot,t_i)\to\Phi(\cdot,T_1)$ weakly in $H^1(M)$ and $\Psi(\cdot,t_i)\to\Psi(\cdot,T_1)$ weakly in $W^{1,4/3}(M)$ as $i\to\infty$. So, we just need to show the weak limits $\Phi(\cdot,T_1)$ and $\Psi(\cdot,T_1)$ are independent of the choice of the time sequence. Let $s_i\to T_1$ be another time sequence and the corresponding weak limit $\widehat{\Phi}(\cdot,T_1)$, then

$$\begin{aligned}&\int_M|\Phi(\cdot,T_1)-\widehat{\Phi}(\cdot,T_1)|^2dx\\ &=\int_M\langle\Phi(\cdot,T_1)-\widehat{\Phi}(\cdot,T_1),\Phi(\cdot,T_1)-\Phi(\cdot,t_i)\rangle dx+\int_M\langle\Phi(\cdot,T_1)-\widehat{\Phi}(\cdot,T_1),\Phi(\cdot,t_i)\\ &\quad-\Phi(\cdot,s_i)\rangle dx+\int_M\langle\Phi(\cdot,T_1)-\widehat{\Phi}(\cdot,T_1),\Phi(\cdot,s_i)-\widehat{\Phi}(\cdot,T_1)\rangle dx\end{aligned}\tag{4.4}$$

for any $i\geq1$. Noting that

$$\int_M|\Phi(\cdot,t_i)-\Phi(\cdot,s_i)|^2dx=\int_M|\int_{s_i}^{t_i}\frac{\partial\Phi}{\partial t}dt|^2dx\leq|s_i-t_i||\int_{M^{t_i}_{s_i}}|\frac{\partial\Phi}{\partial t}|^2dxdt|$$

and $\int_{M^{T_1}} |\frac{\partial\Phi}{\partial t}|^2 dxdt \leq C$ (see Lemma 3.2), letting $i \to \infty$ in (4.4), by Hölder's inequality and the fact $\Phi(\cdot, t_i) \rightharpoonup \Phi(\cdot, T_1)$ weakly in $H^1(M)$, we obtain

$$\int_M |\Phi(\cdot, T_1) - \widehat{\Phi}(\cdot, T_1)|^2 dx = 0.$$

Thus, $\Phi(\cdot, T_1) = \widehat{\Phi}(\cdot, T_1)$, and with Lemma 2.10, the uniqueness of the weak limit $\Psi(\cdot, T_1)$ follows immediately.

Since T_1 is a singular time, there exists at least one singular point $\{(x^1, T_1)\}$ satisfying

$$\limsup_{t\nearrow T_1} E(\Phi(t); B_R^M(x^1)) > \overline{\epsilon} \text{ for all } R > 0. \tag{4.5}$$

Then, we have

$$\begin{aligned}
E(\Phi(T_1)) &= \lim_{R\to 0} E(\Phi(T_1), M \setminus B_R^M(x^1)) \\
&\leq \lim_{R\to 0} \liminf_{t\nearrow T_1} E(\Phi(t), M \setminus B_R^M(x^1)) \\
&= \lim_{R\to 0} \liminf_{t\nearrow T_1} \left(E(\Phi(t)) - E(\Phi(t), B_R^M(x^1))\right) \\
&\leq \liminf_{t\nearrow T_1} E(\Phi(t)) - \lim_{R\to 0} \limsup_{t\nearrow T_1} E(\Phi(t), B_R^M(x^1)) \\
&\leq \liminf_{t\nearrow T_1} E(\Phi(t)) - \overline{\epsilon}
\end{aligned}$$

and

$$\frac{1}{2}\int_{\partial M} \langle \vec{n} \cdot \mathcal{B}\psi_0, \Psi\rangle(T_1) = \frac{1}{2} \liminf_{t\nearrow T_1} \int_{\partial M} \langle \vec{n} \cdot \mathcal{B}\psi_0, \Psi\rangle(t)$$

where equality follows from the trace theory. Thus

$$E(\Phi(T_1)) + \frac{1}{2}\int_{\partial M} \langle \vec{n} \cdot \mathcal{B}\psi_0, \Psi\rangle(T_1) \leq \liminf_{t\nearrow T_1} \left(E(\Phi(t)) + \frac{1}{2}\int_{\partial M} \langle \vec{n} \cdot \Psi\mathcal{B}\psi_0, \rangle(t)\right) - \overline{\epsilon}. \tag{4.6}$$

By Theorem 4.1, we can continue (Φ, Ψ) to some larger time interval $[0, T_2]$ by solving (1.6) with the new initial data $\Phi(T_1)$ on $[T_1, T_2]$ and piecing together the solutions at T_1. It is easy to see that (Φ, Ψ) is a distribution solution to (1.6) on all of M^{T_2} and satisfies

$$E(\Phi(t)) + \frac{1}{2}\int_{\partial M} \langle \vec{n} \cdot \mathcal{B}\psi_0, \Psi\rangle(t) \leq E(\Phi(s)) + \frac{1}{2}\int_{\partial M} \langle \vec{n} \cdot \mathcal{B}\psi_0, \Psi\rangle(s)$$

for any $0 \leq s \leq t < T_2$. Iterating this process, we obtain a global solution defined on $M \times [0, \infty)$. Let $\{T_k\}_{k=1}^K$ be the singular times at which (Φ, Ψ) can attain singularities. According to (4.6), we have

$$\begin{aligned}
E(\Phi(T_K)) + \frac{1}{2}\int_{\partial M} \langle \vec{n} \cdot \mathcal{B}\psi_0, \Psi\rangle(T_K) &\leq \liminf_{t\nearrow T_K} \left(E(\Phi(t)) + \frac{1}{2}\int_{\partial M} \langle \vec{n} \cdot \mathcal{B}\psi_0, \Psi\rangle(t)\right) - \epsilon_1 \\
&\leq E(\Phi(0)) + \frac{1}{2}\int_{\partial M} \langle \vec{n} \cdot \mathcal{B}\psi_0, \Psi\rangle(0) - \sum_{k=1}^K \overline{\epsilon}.
\end{aligned}$$

Then

$$E(\Phi(T_K)) \leq E(\Phi(0)) + |\frac{1}{2}\int_{\partial M}\langle\overrightarrow{n}\cdot\mathcal{B}\psi_0,\Psi\rangle(T_K)| + \frac{1}{2}|\int_{\partial M}\langle\overrightarrow{n}\cdot\mathcal{B}\psi_0,\Psi\rangle(0)| - K\overline{\epsilon}$$
$$\leq E(\phi_0) + \sqrt{2}||\mathcal{B}\psi_0||^2_{L^2(\partial M)} - K\overline{\epsilon}.$$

This implies

$$K \leq \frac{E(\phi_0) + \sqrt{2}||\mathcal{B}\psi_0||^2_{L^2(\partial M)}}{\overline{\epsilon}}.$$

Hence there are at most finitely many singular times. □

5. Behavior of singularities

In this section, we shall study the behavior of singularities of the global weak solution derived in the previous section by using blow-up analysis. Theorem 1.3, Theorem 1.4, Theorem 1.5 and Corollary 1.6 will be proved in this section.

Proof of Theorem 1.3. Let T_1 be a singular time, i.e.

$$\limsup_{\substack{x\in M\\ t\nearrow T_1}} E(\Phi(t); B^M_R(x)) > \overline{\epsilon} \text{ for all } R > 0.$$

From Lemma 3.6, we know

$$\Phi \in C^{2,1,\alpha}_{loc}(M\times[T_1-\delta^2,T_1))$$

for some small $\delta > 0$. Then by the standard blowup argument, there exist sequences $t_i \nearrow T_1$, $x_i \to x_0 \in M$, $r_i \to 0$ such that

$$E(\Phi(t_i), B^M_{r_i}(x_i)) = \sup_{\substack{(x,t)\in M\times[T_1-\delta^2,t_i]\\ B^M_r(x)\subset M,\ r\leq r_i}} E(\Phi(t), B^M_r(x)) = \frac{\overline{\epsilon}}{2}. \tag{5.1}$$

By Lemma 3.5, for any $T_1-\delta^2 \leq s \leq t_i < T_1$, we have

$$E(\Phi(t_i); B^M_{r_i}(x_i)) \leq E(\Phi(s); B^M_{2r_i}(x_i)) + \widehat{C}\,\frac{t_i - s}{r_i^2},$$

where $\widehat{C} := C(1 + E(\phi_0) + ||\varphi||^2_{H^{3/2}(M)} + ||\mathcal{B}\psi_0||^8_{W^{3/8,8/5}(\partial M)}) > 0$ is a constant. Denoting $T = \frac{\overline{\epsilon}}{4\widehat{C}}$, then we have

$$E(\Phi(s); B^M_{2r_i}(x_i)) \geq \frac{\overline{\epsilon}}{4} \tag{5.2}$$

for any $s \in [t_i - Tr_i^2, t_i]$.

We first prove the second statement (2).

Step 1: Let $x_0 \in \partial M$ and we prove the statement (1) under the assumption that

$$\limsup_{i\to\infty} \frac{dist(x_i,\partial M)}{r_i} \to \infty.$$

By taking subsequences, we may assume $\lim_{i\to\infty}\frac{dist(x_i,\partial M)}{r_i}\to\infty$. Assume $t_i-\frac{\delta^2}{4}>T_1-\delta^2$, define

$$B_i:=\{x\in\mathbb{R}^2|x_i+r_ix\in B^M_\delta(x_0)\}$$

and

$$\begin{aligned}u_i(x,t)&:=\Phi(x_i+r_ix,t_i+r_i^2t)\\ v_i(x,t)&:=\sqrt{r_i}\,\Psi(x_i+r_ix,t_i+r_i^2t).\end{aligned}$$

Then (u_i,v_i) lives in $B_i\times[-\frac{\delta^2}{4r_i^2},0]$ which tends to $\mathbb{R}^2\times\mathbb{R}_-$ as $i\to\infty$ and satisfies

$$\begin{cases}\partial_tu_i=\tau(u_i)-\mathcal{P}(\mathcal{A}(du_i(e_\alpha),e_\alpha\cdot v_i);v_i), & in\quad B_i\times[-\frac{\delta^2}{4r_i^2},0];\\ \not{\partial}v_i=\mathcal{A}(du_i(e_\alpha),e_\alpha\cdot v_i), & in\quad B_i\times[-\frac{\delta^2}{4r_i^2},0],\end{cases}\tag{5.3}$$

with the boundary data

$$\begin{cases}u_i(x,t)=\varphi(x_1+r_ix), & if\quad x_i+r_ix\in\partial M;\\ \mathcal{B}v_i(x,t)=\sqrt{r_i}\mathcal{B}\psi_0(x_i+r_ix), & if\quad x_i+r_ix\in\partial M.\end{cases}\tag{5.4}$$

By Lemma 3.2 and Lemma 3.3, we have

$$\int_{-T}^0\int_{B_i}|\partial_tu_i|^2dxdt\le\int_{t_i-r_i^2T}^{t_i}\int_M|\partial_t\Phi|^2dMdt\to0,\quad as\ i\to\infty,\tag{5.5}$$

and

$$\sup_{\frac{\delta^2}{4r_i^2}\le t\le0}\|v_i\|_{L^4(B_i)}\le\sup_{T_1-\delta^2\le t\le T_1}\|\Psi\|_{L^4(M)}\le C,\tag{5.6}$$

$$\sup_{\frac{\delta^2}{4r_i^2}\le t\le0}\|du_i\|_{L^2(B_i)}\le\sup_{T_1-\delta^2\le t\le T_1}\|d\Phi\|_{L^2(M)}\le C.\tag{5.7}$$

By (5.1), we can see that

$$\sup_{-T\le t\le0}\sup_{x\in B_i}\int_{B_1(x)\cap B_i}|\nabla u_i|^2(y,t)dy\le\sup_{\substack{(x,t)\in B^M_\delta(x_0)\times[T_1-\delta^2,t_i]\\ B^M_r(x)\subset B^M_\delta(x_0),\ r\le r_i}}E(\Phi(t),B^M_r(x))=\frac{\overline{\epsilon}}{2}.$$

So, for any $x\in\mathbb{R}^2$, when i is sufficiently large, we have

$$\sup_{-T\le t\le0}\int_{B_1(x)}|\nabla u_i|^2(y,t)dy\le\frac{\overline{\epsilon}}{2}.\tag{5.8}$$

Combining (5.6), (5.8) with Lemma 3.6, we have

$$\sup_{-\frac{T}{2}\le t\le0}\|v_i(\cdot,t)\|_{C^{1+\alpha}(B_{1/2}(x))}+\sup_{-\frac{T}{2}\le t\le0}\|u_i(\cdot,t)\|_{C^{1+\alpha}(B_{1/2}(x))}\le C\tag{5.9}$$

which tells us

$$\sup_{-\frac{T}{2}\le t\le0}\|v_i(\cdot,t)\|_{C^{1+\alpha}_{loc}(\mathbb{R}^2)}+\sup_{-\frac{T}{2}\le t\le0}\|u_i(\cdot,t)\|_{C^{1+\alpha}_{loc}(\mathbb{R}^2)}\le C.\tag{5.10}$$

From (5.10) and (5.5), we can find $\sigma_i \in [-\frac{T}{2}, 0]$ such that as $i \to \infty$, there holds

$$\int_{B_i} |\partial_t u_i|^2(x, \sigma_i) dx \to 0 \tag{5.11}$$

and

$$\|v_i(\cdot, \sigma_i)\|_{C^{1+\alpha}_{loc}(\mathbb{R}^2)} + \|u_i(\cdot, \sigma_i)\|_{C^{1+\alpha}_{loc}(\mathbb{R}^2)} \leq C. \tag{5.12}$$

Therefore, there exists a subsequence of $(u_i(\cdot, \sigma_i), v_i(\cdot, \sigma_i))$ and a limit field $(\widetilde{\Phi}, \widetilde{\Psi}) \in C^1_{loc}(\mathbb{R}^2)$ such that

$$u_i(\cdot, \sigma_i) \to \widetilde{\Phi} \quad in \quad C^1_{loc}(\mathbb{R}^2) \quad and$$
$$v_i(\cdot, \sigma_i) \to \widetilde{\Psi} \quad in \quad C^1_{loc}(\mathbb{R}^2).$$

Setting $t = \sigma_i$ in the equation (5.3) and letting $i \to \infty$, it is easy to see that $(\widetilde{\Phi}, \widetilde{\Psi})$ is a Dirac-harmonic map with

$$\frac{\bar{\epsilon}}{4} \leq \|\nabla \widetilde{\Phi}\|_{L^2(\mathbb{R}^2)} + \|\widetilde{\Psi}\|_{L^4(\mathbb{R}^2)} \leq C,$$

where the above inequality follows from (5.6), (5.7) and (5.2). Taking $t_i + r_i^2 \sigma_i$ as the new time sequence, then we get that

$$\Phi_i(x) = u_i(x, \sigma_i) = \Phi(x_i + r_i x, t_i + r_i^2 \sigma_i)$$
$$\Psi_i(x) = v_i(x, \sigma_i) = \sqrt{r_i} \Psi(x_i + r_i x, t_i + r_i^2 \sigma_i)$$

is the desired sequence in the theorem.

Step 2: If $x_0 \in \partial M$, then $\limsup_{i \to \infty} \frac{dist(x_i, \partial M)}{r_i} \to \infty$.

If not, there exists a converging subsequence of $\frac{dist(x_i, \partial M)}{r_i}$. Without loss of generality, we may assume $\frac{dist(x_i, \partial M)}{r_i} \to a$ as $i \to \infty$. Then

$$B_i \to \mathbb{R}^2_a := \{(x^1, x^2) | x^2 \geq -a\}.$$

Noting that for any $x \in \{x^2 = -a\}$ on the boundary, $x_i + r_i x \to x_0$ and

$$u_i(x, t) = \varphi(x_i + r_i x) \quad if \quad x_i + r_i x \in \partial M;$$
$$\mathcal{B} v_i(x, t) = \sqrt{r_i} \mathcal{B} \psi_0(x_i + r_i x) \quad if \quad x_i + r_i x \in \partial M;$$

By Lemma 3.6 and (5.1), for any $B_R(0) \subset \mathbb{R}^2, R > 0$, we have

$$\sup_{-\frac{T}{2} \leq t \leq 0} \|v_i(\cdot, t)\|_{C^{1+\alpha}(B_R(0) \cap B_i)} + \sup_{-\frac{T}{2} \leq t \leq 0} \|u_i(\cdot, t)\|_{C^{1+\alpha}_{loc}(B_R(0) \cap B_i)} \leq C. \tag{5.13}$$

Using a similar argument as in **Step 1**, we can obtain a C^1 field $(\widetilde{\Phi}, \widetilde{\Psi})$ satisfying

$$\frac{\bar{\epsilon}}{4} \leq \|\nabla \widetilde{\Phi}\|_{L^2(\mathbb{R}^2_a)} + \|\widetilde{\Psi}\|_{L^4(\mathbb{R}^2_a)} \leq C, \tag{5.14}$$

and a sequence $\sigma_i \in [-\frac{T}{2}, 0]$ such that as $i \to \infty$, there hold

$$\|u_i(\cdot,\sigma_i) - \widetilde{\Phi}\|_{C^1(B_i \cap B_R(0))} \to 0$$
$$\|v_i(\cdot,\sigma_i) - \widetilde{\Psi}\|_{C^1(B_i \cap B_R(0))} \to 0,$$

for any $R > 0$ where $B_R(0) \subset \mathbb{R}^2$ is the standard ball with radius R and centered at 0. Moreover, $(\widetilde{\Phi}, \widetilde{\Psi})$ is a Dirac-harmonic map satisfying

$$\begin{cases} \tau(\widetilde{\Phi}) = \mathcal{P}(\mathcal{A}(d\widetilde{\Phi}(e_\alpha), e_\alpha \cdot \widetilde{\Psi}); \widetilde{\Psi}), & in \quad \mathbb{R}^2_a; \\ \partial\!\!\!/ \widetilde{\Psi} = \mathcal{A}(d\widetilde{\Phi}(e_\alpha), e_\alpha \cdot \widetilde{\Psi}), & in \quad \mathbb{R}^2_a, \end{cases} \tag{5.15}$$

with the boundary data

$$\begin{cases} \widetilde{\Phi}(x,t) = \varphi(x_0), & on \quad \partial\mathbb{R}^2_a; \\ \mathcal{B}\widetilde{\Psi}(x,t) = 0, & on \quad \partial\mathbb{R}^2_a. \end{cases} \tag{5.16}$$

Then, by Theorem 1.4, we get $\widetilde{\Phi} \equiv \varphi(x_0)$ and $\widetilde{\Psi} \equiv 0$. This contradicts (5.14). The second statement (2) is proved.

For the first statement (1), the argument is almost the same as in **Step 1**, so we omit it. The proof of theorem is finished. □

Now, we begin to prove Theorem 1.4.

Proof of Theorem 1.4. Denoting

$$f(z) := i\,\frac{z-i}{z+i} : \mathbb{R}^2_+ \to B_1(0)$$

where $B_1(0) = \{u + iv | u^2 + v^2 \le 1\} \subset \mathbb{R}^2$ is the unit ball, it is well known that f is conformal satisfying

$$(f^{-1})^*(dzd\overline{z}) = \frac{4}{(u^2 + (v-1)^2)^2}(du^2 + dv^2)$$

and $f(i) = 0$, $\{f(x_1, x_2) | x_1 \in \mathbb{R}, x_2 = 0\} = \partial B_1 \setminus \{i\}$. Defining

$$\Phi' = \Phi \circ f^{-1} \quad and \quad \Psi' = \frac{u^2 + (v-1)^2}{2}\Psi \circ f^{-1},$$

then $(\Phi', \Psi') : B_1 \setminus \{i\} \to N \times \Phi^{-1}TN$ is a smooth Dirac-harmonic map with the boundary data $\Phi'|_{\partial B_1 \setminus \{i\}} = const.$ and $\mathcal{B}\Psi'|_{\partial B_1 \setminus \{i\}} = 0$ satisfying

$$\int_{B_1} |\nabla\Phi'|^2 dx + \int_{B_1} |\Psi'|^4 dx < \infty.$$

It is known that the equation of Φ' can be written as an elliptic system with an anti-symmetric potential [30, 12, 26]:

$$\Delta\Phi' = \Omega \cdot \nabla\Phi',$$

with $\Omega \in L^2(B_1, so(N) \otimes \mathbb{R}^2)$ satisfying $|\Omega| \le C(|\nabla\Phi'| + |\Psi'|^2)$. Then by taking pure Dirichlet conditions in the boundary regularity Theorem 1.2 in [26] (or see Remark 1.3 in [24]) and bootstrapping, we get $\Phi' \in W^{2,p}(B_1)$ for any $1 < p < \infty$. By the boundary elliptic estimates of first order equations of Ψ', we shall get $\Psi' \in W^{1,p}(B_1)$ for any $1 < p < \infty$. Furthermore, by the standard bootstrap method, we can get higher regularity, *i.e*. (Φ', Ψ') can be smoothly extended to B_1. By Lemma 2.6, we get $\Psi' = 0$ in B_1. Thus, Φ' is a harmonic map from B_1 to N with constant boundary data. By the result of Lemaire [22], Φ' is a constant map. Then Φ must be a constant map, $\Psi \equiv 0$ and we finished the proof. □

Without the continuity of local energy near the singular time (see Lemma 3.5), we don't know whether the singular set at time infinity (if $T = \infty$ is a singular time) is a finite set or not (see [16, 29] for a similar phenomenon in the cases of higher order heat flows). However, thanks to the weak compactness Theorem 1.9 in [30], we can still prove the existence Theorem 1.5.

Proof of Theorem 1.5. By Theorem 1.2 and Lemma 3.3, we know

$$\int_0^\infty \int_M |\partial_t \Phi|^2 dMdt + \sup_{0\le t<\infty} E(\Phi(\cdot,t)) + \sup_{0\le t<\infty} \|\Psi(\cdot,t)\|_{W^{1,\frac{4}{3}}(M)} \le C < \infty.$$

Thus, there exists a time sequence $t_i \nearrow \infty$ and $(\Phi_\infty, \Psi_\infty) \in W^{1,2}(M) \times W^{1,4/3}(M)$ with boundary data $\Phi_\infty|_{\partial M} = \varphi \in C^{2+\alpha}(\partial M)$ and $\mathcal{B}\Psi_\infty|_{\partial M} = \mathcal{B}\psi_0 \in C^{1+\alpha}(\partial M)$, such that

$$\|\partial_t \Phi(\cdot, t_i)\|_{L^2(M)} \to 0$$

and

$$(\Phi(\cdot, t_i), \Psi(\cdot, t_i)) \rightharpoonup (\Phi_\infty, \Psi_\infty)$$

weakly in $W^{1,2}(M) \times W^{1,4/3}(M)$.

By weak compactness Theorem 1.9 in [30], we know $(\Phi_\infty, \Psi_\infty)$ is a weakly Dirac-harmonic map from M with boundary data $\Phi_\infty|_{\partial M} = \varphi \in C^{2+\alpha}(\partial M)$ and $\mathcal{B}\Psi_\infty|_{\partial M} = \mathcal{B}\psi_0 \in C^{1+\alpha}(\partial M)$. Then, using the same argument as in the proof of Theorem 1.4, we get $\Phi_\infty \in C^{2+\alpha}(M)$ and $\Psi_\infty \in C^{1+\alpha}(M)$. This finishes the proof. □

Proof of Corollary 1.6. We shall first show that the constant $\epsilon_0 = \epsilon_0(N) > 0$ is well-defined. We claim: there exists a constant $\epsilon(N) > 0$ such that, for any smooth Dirac-harmonic map sphere $(\phi, \psi) : S^2 \to N$, if $E(\phi) \le \epsilon(N)$, then both ϕ and ψ are trivial.

In fact, by Lemma 4.9 in [5] or Proposition 5.2 in [19], we have

$$\|\psi\|_{L^{4/3}(S^2)} \le C\|\partial\!\!\!/\psi\|_{L^{4/3}(S^2)},$$

where $C > 0$ is a universal constant. By standard elliptic estimates and Sobolev embedding, we have

$$\begin{aligned}
\|\psi\|_{L^4(S^2)} &\le C\|\psi\|_{W^{1,4/3}(S^2)} \\
&\le C(\|\partial\!\!\!/\psi\|_{L^{4/3}(S^2)} + \|\psi\|_{L^{4/3}(S^2)}) \\
&\le C\|\partial\!\!\!/\psi\|_{L^{4/3}(S^2)} \\
&\le C\||d\phi||\psi|\|_{L^{4/3}(S^2)} \\
&\le C\|d\phi\|_{L^2(S^2)}\|\psi\|_{L^4(S^2)} \le C\epsilon(N)\|\psi\|_{L^4(S^2)}.
\end{aligned}$$

Choosing $\epsilon(N) > 0$ sufficiently small, we have $\psi = 0$. So

$$\begin{aligned}
\||d\phi|\|_{W^{1,4/3}(S^2)} &\leq C\|\Delta\phi\|_{L^{4/3}(S^2)} \\
&\leq C\||d\phi|^2\|_{L^{4/3}(S^2)} \\
&\leq C\||d\phi|\|_{L^2(S^2)}\||d\phi|\|_{L^4(S^2)} \\
&\leq C\||d\phi|\|_{L^2(S^2)}\||d\phi|\|_{W^{1,4/3}(S^2)} \leq C\epsilon(N)\||d\phi|\|_{W^{1,4/3}(S^2)}.
\end{aligned}$$

Again, taking $\epsilon(N) > 0$ sufficiently small, ϕ has to be a constant map.

Next, it is sufficient to prove that no blow-up will occur along the flow. In fact, if the flow blows up at some singular time $T \leq \infty$, then by Theorem 1.5, some nontrivial Dirac-harmonic spheres appear. Assume $(\widetilde{\Phi}, \widetilde{\Psi})$ is one, then by Theorem 1.5, it is easy to see that

$$E(\widetilde{\Phi}) \leq \limsup_{t\to T} E(\Phi).$$

However, by Lemma 3.2, we have

$$\epsilon_0 \leq E(\widetilde{\Phi}) \leq \limsup_{t\to T} E(\Phi) \leq E(\phi_0) + \sqrt{2}\|\mathcal{B}\psi_0\|^2_{L^2(\partial M)} < \min\{\Lambda^2, \epsilon_0\}.$$

This is a contradiction which finishes the proof. □

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