A refinement of Franks’ theorem

by

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A REFINEMENT OF FRANKS’ THEOREM

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Abstract. In this paper, we give a refinement of Franks’ theorem [Fr92], which answers two questions raised by Kang [Kang14].

Résumé. Dans cet article, nous donnons un raffinement du théorème de Franks [Fr92], il répond aux questions proposées par Kang [Kang14].

1. INTRODUCTION

Let $A = \mathbb{R}/\mathbb{Z} \times (0, 1)$ (resp. $\tilde{A} = \mathbb{R}/\mathbb{Z} \times [0, 1]$) be the open annulus (resp. the closed annulus). In 1992, Franks [Fr92] (and [Fr96]) proved the following celebrated theorem:

Theorem 1 (Franks). Suppose $F$ be an area preserving homeomorphism of the open or closed annulus which is isotopic to the identity. If $F$ has at least one fixed or periodic point then $F$ must have infinitely many interior periodic points.

Kang [Kang14, Section 1.2] raised the following questions when he studied the reversible maps on planar domains:

Suppose that $F$ is an area preserving homeomorphism of the open or closed annulus which is isotopic to the identity. If $\text{Per}_{\text{odd}}(F) \neq \emptyset$, does it imply that $\sharp \text{Per}_{\text{odd}}(F) = +\infty$ where $\text{Per}_{\text{odd}}(F)$ is the set of odd periodic points of $F$? Furthermore, let $k \in \mathbb{N}$ and $n \in \mathbb{N}$ be two numbers such that $(n, k) = 1$. If $\text{Per}_k(F) \neq \emptyset$, does it imply that $\sharp \{ \bigcup_{(k', n) = 1} \text{Per}_{k'}(F) \} = +\infty$ where $\text{Per}_k(F)$ is the set of $k$-prime-periodic points of $F$? Here $z$ is a $k$-prime-periodic of $F$ means that $z$ is not a $l$-periodic point of $F$ if $l < k$.

In this paper, we answer his questions. We have the following theorems

Theorem 2. If $\text{Per}_{\text{odd}}(F) \neq \emptyset$, then $\sharp \text{Per}_{\text{odd}}(F) = +\infty$.

Theorem 3. Assume that $k, n_0 \in \mathbb{N}$ which satisfy that $(k, n_0) = 1$. If $\text{Per}_k(F) \neq \emptyset$, then

$$\sharp \left\{ \bigcup_{(k', n_0) = 1} \text{Per}_{k'}(F) \right\} = +\infty.$$
Remark 1. Theorem 3 implies Theorem 2. Indeed, if $\text{Per}_{\text{odd}}(F) \neq \emptyset$, there is $k \in 2\mathbb{Z} + 1$ such that $\text{Per}_k(F) \neq \emptyset$. Taken $n_0 = 2$, then Theorem 2 follows from Theorem 3.

Hence, we only need to prove Theorem 3. We will introduce some mathematical objects and recall some well-known facts in Section 2. We will prove Theorem 3 in Section 3.

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2. Preliminaries

2.1. Rotation vector. Let us introduce the classical notion of rotation vector which was defined originally in [Sch57]. Let $M$ be a smooth manifold. Suppose that $F$ is the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$ on $M$. Let $\text{Rec}^+(F)$ be the set of positively recurrent points of $F$. If $z \in \text{Rec}^+(F)$, we fix an open disk $U \subset M$ containing $z$, and write $\{F^{n_k}(z)\}_{k \geq 1}$ for the subsequence of the positive orbit of $z$ obtained by keeping the points that are in $U$. For any $k \geq 0$, choose a simple path $\gamma_{F^{n_k}(z),z}$ in $U$ joining $F^{n_k}(z)$ to $z$. The homology class $[\Gamma_k]_M \in H_1(M,\mathbb{Z})$ of the loop $\Gamma_k = I^{n_k}(z)\gamma_{F^{n_k}(z),z}$ does not depend on the choice of $\gamma_{F^{n_k}(z),z}$. Say that $z$ has a rotation vector $\rho_{M,I}(z) \in H_1(M,\mathbb{R})$ if

$$\lim_{l \to +\infty} \frac{1}{n_{k_l}}[\Gamma_{k_l}]_M = \rho_{M,I}(z)$$

for any subsequence $\{F^{n_{k_l}}(z)\}_{l \geq 1}$ which converges to $z$. Neither the existence nor the value of the rotation vector depends on the choice of $U$. Let $\mathcal{M}(F)$ be the set of Borel finite measures on $M$ whose elements are invariant by $F$. If $\mu \in \mathcal{M}(F)$ and $M$ is compact, we can define the rotation vector $\rho_{M,I}(z)$ for $\mu$-almost every positively recurrent point [Lec06] (see also Section 1.3 in [Wang11]). If we suppose that the rotation vector $\rho_{M,I}(z)$ is $\mu$-integrable, we define the rotation vector of the measure

$$\rho_{M,I}(\mu) = \int_M \rho_{M,I} \, d\mu \in H_1(M,\mathbb{R}).$$

Remark that the definition of rotation vector here is the same as the homological rotation vector that was defined by Franks in [Fr92] on the positively recurrent points set.

The following theorem is due to Franks ([Fr92, Fr96]):

Theorem 4. Let $M$ be an oriented surface of genus $0$ with $\chi(M) \leq 0$, that is, $M = S^2 \setminus \{x_1, x_2, \ldots, x_n\}$ where $n \geq 2$. Suppose that $F$ is the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$ on $M$ and preserves a finite measure $\mu$ of $M$ with total support. If $\rho_{M,I}(\mu) = 0$, then $F$ has a fixed point in the interior of $M$.

Denote by $\text{Fix}_{\text{Cont},I}(F)$ the set of contractible fixed points of $F$, that is, $x \in \text{Fix}_{\text{Cont},I}(F)$ if and only if $x$ is a fixed point of $F$ and the oriented loop $I(x) : t \to F_t(x)$ defined on $[0,1]$ is contractible on $M$. In [Lec06, Theorem 8.1], Le Calvez proved the following deep result
Theorem 5. Suppose that $M$ is a surface (without boundary) and $I = (F_t)_{t \in [0,1]}$ is an isotopy on $M$ from $\text{Id}_M$ to $F$. We suppose that $F$ has no contractible fixed points. Then there exists an oriented topological foliation $\mathcal{F}$ on $M$ such that, for all $z \in M$, the trajectory $I(z)$ is homotopic to an arc $\gamma$ joining $z$ and $F(z)$ in $M$ which is positively transverse to $\mathcal{F}$. That means that for every $t_0 \in [0,1]$ there exists an open neighborhood $V \subset M$ of $\gamma(t_0)$ and an orientation preserving homeomorphism $h : V \to (−1,1)^2$ which sends the foliation $\mathcal{F}$ on the horizontal foliation (oriented with $x$ positive) to the universal cover of $\mathcal{F}$ on the horizontal foliation (oriented with $x_1$ increasing) such that the map $t \mapsto p_2(h(\gamma(t)))$ defined in a neighborhood of $t_0$ is strictly increasing where $p_2(x_1, x_2) = x_2$.

We say that $X \subseteq \text{Fix}_{\text{Cont},I}(F)$ is unlinked if there exists an isotopy $I' = (F'_t)_{t \in [0,1]}$ homotopic to $I$ which fixes every point of $X$, that is, $F'_t(x) = x$ for all $t \in [0,1]$ and $x \in X$. Moreover, we say that $X$ is a maximal unlinked set, if any set $X' \subseteq \text{Fix}_{\text{Cont},I}(F)$ which strictly contains $X$ is not unlinked. If $\#\text{Fix}_{\text{Cont},I}(F) < \infty$, there must be a set $X \subseteq \text{Fix}_{\text{Cont},I}(F)$ which is a maximal unlinked set. By Theorem 5, there exists an oriented topological foliation $\mathcal{F}$ on $M \setminus X$ (or, equivalently, a singular oriented foliation $\mathcal{F}$ on $M$ with $X$ equal to the singular set) such that, for all $z \in M \setminus X$, the trajectory $I(z)$ is homotopic to an arc $\gamma$ joining $z$ and $F(z)$ in $M \setminus X$ which is positively transverse to $\mathcal{F}$.

2.2. Rotation number. We denote by $\pi$ the covering map of the open annulus (resp. the closed annulus) $\pi : \mathbb{R} \times (0,1) \ (\text{resp. } \mathbb{R} \times [0,1]) \to \mathbb{A} \ (\text{resp. } \bar{\mathbb{A}})$

$$(x,y) \mapsto (x+\mathbb{Z}, y),$$

and by $T$ the generator of the covering transformation group

$$T : \mathbb{R} \times (0,1) \ (\text{resp. } \mathbb{R} \times [0,1]) \to \mathbb{R} \times (0,1) \ (\text{resp. } \mathbb{R} \times [0,1])$$

$$(x,y) \mapsto (x+1,y).$$

Write respectively $S$ and $N$ for the lower and the upper end of $\mathbb{A}$.

In the following, we denote by $\mathbb{A}$ the open or closed annulus unless an explicit mention. We call essential circle in $\mathbb{A}$ every simple closed curve which is not null-homotopic. Let $F$ be a homeomorphism of $\mathbb{A}$. We say that $F$ satisfies the intersection property if any essential circle in $\mathbb{A}$ meets its image by $F$. We denote the space of all homeomorphisms of $\mathbb{A}$ which are isotopic to the identity as Homeo$^o_*(\mathbb{A})$ and its subspace whose elements additionally have the intersection property as Homeo$^o_\text{int}(*)(\mathbb{A})$. It is easy to see that a homeomorphism of $\mathbb{A}$ that preserves a finite measure with total support satisfies the intersection property.

When $F \in \text{Homeo}_*(\mathbb{A})$, we define the rotation number of a positively recurrent point as follows. We say that a positively recurrent point $z$ has a rotation number $\rho(f; z) \in \mathbb{R}$ for a lift $f$ of $F$ to the universal cover of $\mathbb{A}$, if for every subsequence $\{F^{n_k}(z)\}_{k \geq 0}$ of $\{F^n(z)\}_{n \geq 0}$ which converges to $z$, we have

$$\lim_{k \to +\infty} \frac{p_1 \circ f^{n_k}(\tilde{z}) - p_1(\tilde{z})}{n_k} = \rho(f; z)$$

where $\tilde{z} \in \pi^{-1}(z)$ and $p_1$ is the first projection $p_1(x,y) = x$. In particular, the rotation number $\rho(f; z)$ always exists and is rational when $z$ is a fixed or periodic point of $F$. Let
Rec\(^+\)(F) be the set of positively recurrent points of \(F\). We denote the set of rotation numbers of positively recurrent points of \(F\) as Rot(\(f\)).

It is well known that a positively recurrent point of \(F\) is also a positively recurrent point of \(F^q\) for all \(q \in \mathbb{N}\) (see the appendix of [Wang14]). By the definition of rotation number, we easily get that the following elementary properties.

1. \(\rho(T^k \circ f; z) = \rho(f; z) + k\), and hence Rot\((T^k \circ f) = \text{Rot}(f) + k\) for every \(k \in \mathbb{Z}\);
2. \(\rho(f^q; z) = q\rho(f; z)\), and hence Rot\((f^q) = q\text{Rot}(f)\) for every \(q \in \mathbb{N}\).

Suppose that \(z \in \text{Rec}^+(F)\) and \(\bar{z} \in \pi^{-1}(z)\). We define \(\mathcal{E}(z) \subset \mathbb{R} \cup \{-\infty, +\infty\}\) by saying that \(\rho \in \mathcal{E}(z)\) if there exists a sequence \(\{n_k\}_{k=1}^{+\infty} \subset \mathbb{N}\) such that

- \(\lim_{k \to +\infty} F^{n_k}(z) = z\);
- \(\lim_{k \to +\infty} \frac{p_1(f^{n_k}(\bar{z})) - p_1(\bar{z})}{n_k} = \rho\).

Define \(\rho^- (f; z) = \inf \mathcal{E}(z)\) and \(\rho^+ (f; z) = \sup \mathcal{E}(z)\). Obviously, we have that \(\rho(f; z)\) exists if and only if \(\rho^- (f; z) = \rho^+ (f; z) \in \mathbb{R}\). Note that the set \(\mathcal{E}(z)\) is a bounded set when \(\mathcal{A}\) is a closed annulus (by compactness) and might be an unbounded set when \(\mathcal{A}\) is an open annulus. However, the set \(\mathcal{E}(z)\) is still a bounded set if \(\mathcal{T}\text{Fix}(F) < +\infty\) (see [Lec01]). By the definitions, it is easy to see that \(\rho^- (f; z)\) and \(\rho^+ (f; z)\) satisfy the same properties as \(\rho(f; z)\).

**Remark 2.** A lift \(f\) of \(F\) is one-to-one corresponding to an identity isotopy \(I\) (mod homotopy). Observing that \(H_1(\mathcal{A}, \mathbb{R}) \simeq \mathbb{R}\), the rotation number \(\rho(f; z)\) is nothing else but the rotation vector \(\rho_{\mathcal{A}; I}(z)\) where the time-one map of the lift identity isotopy of \(I\) to the universal cover is \(f\).

The following Theorem is due to Franks [Fr88] when \(\mathcal{A}\) is closed annulus and \(F\) has no wandering point, and it was improved by Le Calvez [Lec06] (see also [Wang14]) when \(\mathcal{A}\) is open annulus and \(F\) satisfies the intersection property:

**Theorem 6.** Let \(F \in \text{Homeo}^c(\mathcal{A})\) and \(f\) be a lift of \(F\) to the universal cover of \(\mathcal{A}\). Suppose that there exist two recurrent points \(z_1\) and \(z_2\) such that \(-\infty \leq \rho^- (f; z_1) < \rho^+ (f; z_2) \leq +\infty\). Then for any rational number \(p/q \in [\rho^- (f; z_1), \rho^+ (f; z_2)]\) written in an irreducible way, there exists a periodic point of period \(q\) whose rotation number is \(p/q\).

### 3. Proof of the theorems

**Proof of the Theorem 3.** If \(\sharp\text{Per}_k(F) = +\infty\), we have nothing to do. Hence we assume that \(\sharp\text{Per}_k(F) < +\infty\). Let \(\text{Per}_k(F) = \{x_1, \ldots, x_m\}\) and \(G = F^k\). If \(\sharp\text{Fix}(G) = +\infty\), then there exists \(t|k\) with \((t, n_0) = 1\) such that \(\sharp\text{Per}_t(F) = +\infty\). We have done. Therefore, we can assume that \(1 \leq \sharp\text{Fix}(G) < +\infty\). Assume that \(\text{Fix}(G) = \{y_1, \ldots, y_m\}\). By Theorem 6, we can choose a lift \(f\) of \(F\) to the universal cover of \(\mathcal{A}\) or \(\mathcal{A}'\) such that \(\rho(f; x_i) = \frac{k'}{q'} \in [0, 1)\) for \(i = 1, \ldots, m\) where \(k' \in \mathbb{Z}\). The proof will be divided into two cases: \(k' = 0\) and \(k' \neq 0\).
In the case of \( k' \neq 0 \), by Theorem 1, we know that \( 2 \text{Per}(G) = +\infty \). Hence \( \text{Per}(G) \setminus \text{Fix}(G) \neq \emptyset \). Note that \( \rho(f^k; y_i) = k' \) for all \( i \). For any \( z \in \text{Per}(G) \setminus \text{Fix}(G) \), we assume that \( \rho(f^k; z) = \frac{p}{q} \) with \( q \geq 1 \) and \((p, q) = 1\). By Theorem 4, there is a fixed point \( \rho(f^k; z') = \frac{p}{q} \). Observing that the point \( z' \) is a \( ks \)-periodic point of \( F \), the conclusion follows in this case.

We now consider the case of \( k' = 0 \). If the annulus is a closed annulus \( \tilde{A} \), we consider its interior \( \tilde{A} \). Let \( S^2 = \tilde{A} \cup \{N, S\} \). Note that \( \chi(S^2 \setminus \{N, S, y_1, \cdots, y_m\}) \leq 0 \) (=0 when the annulus is closed and \( \text{Fix}(G) \subset \partial \tilde{A} \)). Write \( M = S^2 \setminus \{N, S, y_1, \cdots, y_m\} \).

When \( \chi(M) = 0 \), we work on the closed annulus \( \tilde{A} \). We have that \( \rho(f^k, y_i) = 0 \) and \( \rho(f^k, z) = 0 \) for any \( z \in \text{Rec}^+(G) \) (by Theorem 6). By Theorem 4, there is a fixed point of \( G \) in the interior of \( \tilde{A} \) which contradicts the fact that \( \text{Fix}(G) \subset \partial \tilde{A} \).

In the case of \( \chi(M) < 0 \), we follow the idea of Le Calvez [Lec06, Theorem 9.3]. We choose an identity isotopy \( I_0 = (F_t)_{t \in [0, 1]} \) on \( S^2 \) such that \( F_t \) fixes \( N \) and \( S \) for every \( t \), and the time-one map of the lift identity isotopy of \( I_0 |_\tilde{A} \) to \( \mathbb{R} \times (0, 1) \) is \( f^k \). Furthermore, as \( \rho(f^k, y_i) = 0 \) for all \( i \), we can suppose that \( I_0 \) fixes \( N \) and \( S \) and one point of \( \{y_1, \cdots, y_m\} \) (e.g., we can modify \( I_0 \) though the technique in the proof of Lemma 1.2 in [Wang11, Section 1.4] without changing the homotopic class of \( I_0 |_\tilde{A} \), say \( y_1 \). Identify \( G \) as a map of \( S^2 \). As \( 2 \text{Fix}(G) < +\infty \), there is a maximal unlink set \( \{N, S, y_1\} \subset X \subset \text{Fix}(G) = \{N, S, y_1, \cdots, y_m\} \) and an identity isotopy \( I_1 \) which is homotopic to \( I_0 \) with fixed endpoints such that \( I_1 \) fixes every point of \( X \). It is clear that \( 2X \geq 3 \) in this case. By Theorem 5, there exists an oriented topological foliation \( F \) on \( S^2 \setminus X \) such that, for all \( z \in \tilde{A} \), \( I_1(z) \) is homotopic to an arc \( \gamma \) joining \( z \) and \( G(z) \) in \( S^2 \setminus X \) which is positively transverse to \( F \). For any leaf \( \lambda \in F \), the \( \alpha \)-limit set and \( \omega \)-limit set of \( \lambda \) must belong to two distinct points of \( X \) respectively since \( X \) is finite and \( G \) is symplectic. Choose a leaf \( \lambda \in F \) which connects two different points \( z_1 \) and \( z_2 \) of \( X \). We consider the following open annulus \( A_{z_1, z_2} = S^2 \setminus \{z_1, z_2\} \).

We choose a small open disk \( U \) near \( \lambda \) such that, \( U \cap \lambda = \emptyset \) and for any \( z \in U \), \( \lambda \wedge I_1(z) \geq 1 \) where \( I_1(z) \) is the trajectory of \( z \) under the isotopy \( I_1 \). We define the first return map

\[
\Phi : \text{Rec}^+(G) \cap U \rightarrow \text{Rec}^+(G) \cap U,
\]

where \( \tau(z) \) is the first return time, that is, the least number \( n \geq 1 \) such that \( G^n(z) \in U \). By Poincaré Recurrence Theorem, this map is defined \( \mu \)-a.e. on \( U \). For every couple \( (z', z'') \in U^2 \), choose a simple path \( \gamma_{z', z''} \) in \( U \) joining \( z' \) to \( z'' \). For every \( z \in \text{Rec}^+(G) \cap U \) and \( n \geq 1 \), define

\[
\tau_n(z) = \sum_{i=0}^{n-1} \tau(\Phi^i(z)), \quad \Gamma^n_z = I_1^{\tau_n(z)}(z) \gamma_{\Phi^\tau_n(z), z}, \quad m(z) = \Gamma_1^n \wedge \lambda, \quad m_n(z) = \sum_{i=0}^{n-1} m(\Phi^i(z)).
\]
It is well known that $\tau \in L^1(U, \mathbb{R})$ (see, e.g., [Wang11, Section 1.3]). Hence $\tau_n/n$ converges $\mu$-a.e. on $\text{Rec}^+(G) \cap U$. It is clear that $m_n/n \geq 1$ for all $n \geq 1$ and $z \in \text{Rec}^+(G) \cap U$. This implies that $m_n/\tau_n > 0$ for $\mu$-a.e. on $\text{Rec}^+(G) \cap U$. Observe that

$$E(z) \subset \left[ \inf_n \left\{ \frac{m_n(z)}{\tau_n(z)} \right\}, \sup_n \left\{ \frac{m_n(z)}{\tau_n(z)} \right\} \right] \subset \mathbb{R}$$

when the limit of $\tau_n(z)/n$ exists, where the definition $E(z)$ one can refer to Remark 2. We get that $\rho_{A_{z_1,z_2,I_1}}(z) > 0$ for $\mu$-a.e. on $\text{Rec}^+(G) \cap U$. We also have that $\rho_{A_{z_1,z_2,I_1}}(y) = 0$ for all $y \in X \setminus \{z_1, z_2\}$. Then Theorem 3 follows by Theorem 6. □

**References**


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