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A REFINEMENT OF FRANKS' THEOREM

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Abstract. In this paper, we give a refinement of Franks' theorem [Fr92], which answers two questions raised by Kang [Kang14].

Résumé. Dans cet article, nous donnons un raffinement du théorème de Franks [Fr92], il répond aux questions proposés par Kang [Kang14].

1. introduction

Let $\mathbb{A} = \mathbb{R}/\mathbb{Z} \times (0,1)$ (resp. $\bar{\mathbb{A}} = \mathbb{R}/\mathbb{Z} \times [0,1]$) be the open annulus (resp. the closed annulus). In 1992, Franks [Fr92] (and [Fr96]) proved the following celebrated theorem:

Theorem 1 (Franks)**.** *Suppose F be an area preserving homeomorphism of the open or closed annulus which is isotopic to the identity. If F has at least one fixed or periodic point then F must have infinitely many interior periodic points.*

Kang [Kang14, Section 1.2] raised the following questions when he studied the reversible maps on planar domains:

Suppose that F is an area preserving homeomorphism of the open or closed annulus which is isotopic to the identity. If $\mathrm{Per}_{\mathrm{odd}}(F) \neq \emptyset$, does it imply that $\sharp\mathrm{Per}_{\mathrm{odd}}(F) = +\infty$ where $\mathrm{Per}_{\mathrm{odd}}(F)$ is the set of odd periodic points of F? Furthermore, let $k \in \mathbb{N}$ and $n \in \mathbb{N}$ be two numbers such that $(n,k) = 1$. If $\mathrm{Per}_k(F) \neq \emptyset$, does it imply that $\sharp\{\bigcup_{(k',n)=1} \mathrm{Per}_{k'}(F)\} = +\infty$ where $\mathrm{Per}_k(F)$ is the set of k-prime-periodic points of F? Here z is a k-prime-periodic of F means that z is not a l-periodic point of F if $l < k$.

In this paper, we answer his questions. We have the following theorems

Theorem 2. *If $\mathrm{Per}_{\mathrm{odd}}(F) \neq \emptyset$, then $\sharp\mathrm{Per}_{\mathrm{odd}}(F) = +\infty$.*

Theorem 3. *Assume that $k, n_0 \in \mathbf{N}$ which satisfy that $(k, n_0) = 1$. If $\mathrm{Per}_k(F) \neq \emptyset$, then*

$$\sharp\left\{\bigcup_{(k',n_0)=1} \mathrm{Per}_{k'}(F)\right\} = +\infty.$$

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Remark 1. Theorem 3 implies Theorem 2. Indeed, if $\mathrm{Per}_{\mathrm{odd}}(F) \neq \emptyset$, there is $k \in 2\mathbf{Z}+1$ such that $\mathrm{Per}_k(F) \neq \emptyset$. Taken $n_0 = 2$, then Theorem 2 follows from Theorem 3.

Hence, we only need to prove Theorem 3. We will introduce some mathematical objects and recall some well-known facts in Section 2. We will prove Theorem 3 in Section 3.

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2. Preliminaries

2.1. **Rotation vector.** Let us introduce the classical notion of rotation vector which was defined originally in [Sch57]. Let M be a smooth manifold. Suppose that F is the time-one map of an identity isotopy $I = (F_t)_{t\in[0,1]}$ on M. Let $\mathrm{Rec}^+(F)$ be the set of positively recurrent points of F. If $z \in \mathrm{Rec}^+(F)$, we fix an open disk $U \subset M$ containing z, and write $\{F^{n_k}(z)\}_{k\geq 1}$ for the subsequence of the positive orbit of z obtained by keeping the points that are in U. For any $k \geq 0$, choose a simple path $\gamma_{F^{n_k}(z),z}$ in U joining $F^{n_k}(z)$ to z. The homology class $[\Gamma_k]_M \in H_1(M,\mathbb{Z})$ of the loop $\Gamma_k = I^{n_k}(z)\gamma_{F^{n_k}(z),z}$ does not depend on the choice of $\gamma_{F^{n_k}(z),z}$. Say that z has a *rotation vector* $\rho_{M,I}(z) \in H_1(M,\mathbb{R})$ if

$$\lim_{l\to+\infty} \frac{1}{n_{k_l}}[\Gamma_{k_l}]_M = \rho_{M,I}(z)$$

for any subsequence $\{F^{n_{k_l}}(z)\}_{l\geq 1}$ which converges to z. Neither the existence nor the value of the rotation vector depends on the choice of U. Let $\mathcal{M}(F)$ be the set of Borel finite measures on M whose elements are invariant by F. If $\mu \in \mathcal{M}(F)$ and M is compact, we can define the rotation vector $\rho_{M,I}(z)$ for μ-almost every positively recurrent point [Lec06] (see also Section 1.3 in [Wang11]). If we suppose that the rotation vector $\rho_{M,I}(z)$ is μ-integrable, we define the *rotation vector of the measure*

$$\rho_{M,I}(\mu) = \int_M \rho_{M,I}\,\mathrm{d}\mu \in H_1(M,\mathbb{R}).$$

Remark that the definition of rotation vector here is the same as the homological rotation vector that was defined by Franks in [Fr92] on the positively recurrent points set.

The following theorem is due to Franks ([Fr92, Fr96]):

Theorem 4. *Let M be an oriented surface of genus 0 with $\chi(M) \leq 0$, that is, $M = \mathbf{S}^2 \setminus \{x_1, x_2, \cdots, x_n\}$ where $n \geq 2$. Suppose that F is the time-one map of an identity isotopy $I = (F_t)_{t\in[0,1]}$ on M and preserves a finite measure μ of M with total support. If $\rho_{M,I}(\mu) = 0$, then F has a fixed point in the interior of M.*

Denote by $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ the set of contractible fixed points of F, that is, $x \in \mathrm{Fix}_{\mathrm{Cont},I}(F)$ if and only if x is a fixed point of F and the oriented loop $I(x) : t \mapsto F_t(x)$ defined on $[0,1]$ is contractible on M. In [Lec06, Theorem 8.1], Le Calvez proved the following deep result

Theorem 5. *Suppose that M is a surface (without boundary) and $I = (F_t)_{t\in[0,1]}$ is an isotopy on M from Id_M to F. We suppose that F has no contractible fixed points. Then there exists an oriented topological foliation $\mathcal{F}$ on M such that, for all $z \in M$, the trajectory $I(z)$ is homotopic to an arc γ joining z and $F(z)$ in M which is positively transverse to $\mathcal{F}$. That means that for every $t_0 \in [0,1]$ there exists an open neighborhood $V \subset M$ of $\gamma(t_0)$ and an orientation preserving homeomorphism $h : V \to (-1,1)^2$ which sends the foliation $\mathcal{F}$ on the horizontal foliation (oriented with x_1 increasing) such that the map $t \mapsto p_2(h(\gamma(t)))$ defined in a neighborhood of t_0 is strictly increasing where $p_2(x_1, x_2) = x_2$.*

We say that $X \subseteq \mathrm{Fix}_{\mathrm{Cont},I}(F)$ is *unlinked* if there exists an isotopy $I' = (F'_t)_{t\in[0,1]}$ homotopic to I which fixes every point of X, that is, $F'_t(x) = x$ for all $t \in [0,1]$ and $x \in X$. Moreover, we say that X is a *maximal unlinked set*, if any set $X' \subseteq \mathrm{Fix}_{\mathrm{Cont},I}(F)$ which strictly contains X is not unlinked. If $\sharp\mathrm{Fix}_{\mathrm{Cont},I}(F) < \infty$, there must be a set $X \subseteq \mathrm{Fix}_{\mathrm{Cont},I}(F)$ which is a maximal unlinked set. By Theorem 5, there exists an oriented topological foliation $\mathcal{F}$ on $M \setminus X$ (or, equivalently, a singular oriented foliation $\mathcal{F}$ on M with X equal to the singular set) such that, for all $z \in M \setminus X$, the trajectory $I(z)$ is homotopic to an arc γ joining z and $F(z)$ in $M \setminus X$ which is positively transverse to $\mathcal{F}$.

2.2. **Rotation number.** We denote by π the covering map of the open annulus (resp. the closed annulus)

$$\begin{array}{rcl} \pi : \mathbb{R}\times(0,1) \quad (\text{resp.} \quad \mathbb{R}\times[0,1]) & \to & \mathbb{A} \quad (\text{resp.} \quad \bar{\mathbb{A}}) \\ (x,y) & \mapsto & (x+\mathbb{Z}, y), \end{array}$$

and by T the generator of the covering transformation group

$$\begin{array}{rcl} T : \mathbb{R}\times(0,1) \quad (\text{resp.} \quad \mathbb{R}\times[0,1]) & \to & \mathbb{R}\times(0,1) \quad (\text{resp.} \quad \mathbb{R}\times[0,1]) \\ (x,y) & \mapsto & (x+1, y). \end{array}$$

Write respectively S and N for the lower and the upper end of $\mathbb{A}$.

In the following, we denote by $\mathbb{A}$ the open or closed annulus unless an explicit mention. We call *essential circle* in $\mathbb{A}$ every simple closed curve which is not null-homotopic. Let F be a homeomorphism of $\mathbb{A}$. We say that F satisfies the *intersection property* if any essential circle in $\mathbb{A}$ meets its image by F. We denote the space of all homeomorphisms of $\mathbb{A}$ which are isotopic to the identity as $\mathrm{Homeo}_*(\mathbb{A})$ and its subspace whose elements additionally have the intersection property as $\mathrm{Homeo}_*^{\wedge}(\mathbb{A})$. It is easy to see that a homeomorphism of $\mathbb{A}$ that preserves a finite measure with total support satisfies the intersection property.

When $F \in \mathrm{Homeo}_*(\mathbb{A})$, we define the rotation number of a positively recurrent point as follows. We say that a positively recurrent point z has a *rotation number* $\rho(f;z) \in \mathbb{R}$ for a lift f of F to the universal cover of $\mathbb{A}$, if for every subsequence $\{F^{n_k}(z)\}_{k\geq 0}$ of $\{F^n(z)\}_{n\geq 0}$ which converges to z, we have

$$\lim_{k\to+\infty} \frac{p_1 \circ f^{n_k}(\widetilde{z}) - p_1(\widetilde{z})}{n_k} = \rho(f;z)$$

where $\widetilde{z} \in \pi^{-1}(z)$ and p_1 is the first projection $p_1(x,y) = x$. In particular, the rotation number $\rho(f;z)$ always exists and is rational when z is a fixed or periodic point of F. Let

$\mathrm{Rec}^+(F)$ be the set of positively recurrent points of F. We denote the set of rotation numbers of positively recurrent points of F as $\mathrm{Rot}(f)$.

It is well known that a positively recurrent point of F is also a positively recurrent point of F^q for all $q \in \mathbb{N}$ (see the appendix of [Wang14]). By the definition of rotation number, we easily get that the following elementary properties.

1. $\rho(T^k \circ f; z) = \rho(f; z) + k$, and hence $\mathrm{Rot}(T^k \circ f) = \mathrm{Rot}(f) + k$ for every $k \in \mathbb{Z}$;
2. $\rho(f^q; z) = q\rho(f; z)$, and hence $\mathrm{Rot}(f^q) = q\mathrm{Rot}(f)$ for every $q \in \mathbb{N}$.

Suppose that $z \in \mathrm{Rec}^+(F)$ and $\widetilde{z} \in \pi^{-1}(z)$. We define $\mathcal{E}(z) \subset \mathbb{R} \cup \{-\infty, +\infty\}$ by saying that $\rho \in \mathcal{E}(z)$ if there exists a sequence $\{n_k\}_{k=1}^{+\infty} \subset \mathbb{N}$ such that

- $\lim\limits_{k \to +\infty} F^{n_k}(z) = z$;
- $\lim\limits_{k \to +\infty} \dfrac{p_1(f^{n_k}(\widetilde{z})) - p_1(\widetilde{z})}{n_k} = \rho$.

Define $\rho^-(f; z) = \inf \mathcal{E}(z)$ and $\rho^+(f; z) = \sup \mathcal{E}(z)$. Obviously, we have that $\rho(f; z)$ exists if and only if $\rho^-(f; z) = \rho^+(f; z) \in \mathbb{R}$. Note that the set $\mathcal{E}(z)$ is a bounded set when $\mathbb{A}$ is a closed annulus (by compactness) and might be an unbounded set when $\mathbb{A}$ is an open annulus. However, the set $\mathcal{E}(z)$ is still a bounded set if $\sharp\mathrm{Fix}(\mathrm{F}) < +\infty$ (see [Lec01]). By the definitions, it is easy to see that $\rho^-(f; z)$ and $\rho^+(f; z)$ satisfy the same properties as $\rho(f; z)$.

Remark 2. A lift f of F is one-to-one corresponding to an identity isotopy I (mod homptopy). Observing that $H_1(\mathbb{A}, \mathbb{R}) \simeq \mathbb{R}$, the rotation number $\rho(f; z)$ is nothing else but the rotation vector $\rho_{\mathbb{A},I}(z)$ where the time-one map of the lift identity isotopy of I to the universal cover is f.

The following Theorem is due to Franks [Fr88] when $\mathbb{A}$ is closed annulus and F has no wandering point, and it was improved by Le Calevez [Lec06] (see also [Wang14]) when $\mathbb{A}$ is open annulus and F satisfies the intersection property:

Theorem 6. *Let $F \in \mathrm{Homeo}_*^\wedge(\mathbb{A})$ and f be a lift of F to the universal cover of $\mathbb{A}$. Suppose that there exist two recurrent points z_1 and z_2 such that $-\infty \le \rho^-(f; z_1) < \rho^+(f; z_2) \le +\infty$. Then for any rational number $p/q \in]\rho^-(f; z_1), \rho^+(f; z_2)[$ written in an irreducible way, there exists a periodic point of period q whose rotation number is p/q.*

3. Proof of the theorems

Proof of the Theorem 3. If $\sharp\mathrm{Per}_k(F) = +\infty$, we have nothing to do. Hence we assume that $\sharp\mathrm{Per}_k(F) < +\infty$. Let $\mathrm{Per}_k(F) = \{x_1, \cdots, x_m\}$ and $G = F^k$. If $\sharp\mathrm{Fix}(G) = +\infty$, then there exists $t \,|\, k$ with $(t, n_0) = 1$ such that $\sharp\mathrm{Per}_t(F) = +\infty$. We have done. Therefore, we can assume that $1 \le \sharp\mathrm{Fix}(G) < +\infty$. Assume that $\mathrm{Fix}(G) = \{y_1, \cdots, y_{m'}\}$. By Theorem 6, we can choose a lift f of F to the universal cover of $\mathbb{A}$ or $\bar{\mathbb{A}}$ such that $\rho(f; x_i) = \frac{k'}{k} \in [0, 1)$ for $i = 1, \cdots, m$ where $k' \in \mathbb{Z}$. The proof will be divided into two cases: $k' = 0$ and $k' \neq 0$.

In the case of $k' \neq 0$, by Theorem 1, we know that $\sharp\mathrm{Per}(G) = +\infty$. Hence $\mathrm{Per}(G) \setminus \mathrm{Fix}(G) \neq \emptyset$. Note that $\rho(f^k; y_i) = k'$ for all i. For any $z \in \mathrm{Per}(G) \setminus \mathrm{Fix}(G)$, we assume that $\rho(f^k; z) = \frac{p}{q}$ with $q \geq 1$ and $(p, q) = 1$ if $p \neq 0$. W.l.o.g., we assume that $k' < \frac{p}{q}$. Then by Theorem 6, for every $\frac{r}{s} \in (k', \frac{p}{q})$ with $(r, s) = 1$ and $(s, n_0) = 1$, there exists $z' \in \mathrm{Per}(G)$ such that $\rho(f^k; z') = \frac{r}{s}$. Observing that the point z' is a ks-periodic point of F, the conclusion follows in this case.

We now consider the case of $k' = 0$. If the annulus is a closed annulus $\bar{\mathbb{A}}$, we consider its interior $\mathbb{A}$. Let $\mathbf{S}^2 = \mathbb{A} \sqcup \{N, S\}$. Note that $\chi(\mathbf{S}^2 \setminus \{N, S, y_1, \cdots, y_{m'}\}) \leq 0$ (=0 when the annulus is closed and $\mathrm{Fix}(G) \subset \partial\bar{\mathbb{A}}$). Write $M = \mathbf{S}^2 \setminus \{N, S, y_1, \cdots, y_{m'}\}$.

When $\chi(M) = 0$, we work on the closed annulus $\bar{\mathbb{A}}$. We have that $\rho(f^k, y_i) = 0$ and $\rho(f^k, z) = 0$ for any $z \in \mathrm{Rec}^+(\mathrm{G})$ (by Theorem 6). By Theorem 4, there is a fixed point of G in the interior of $\bar{\mathbb{A}}$ which contradicts the fact that $\mathrm{Fix}(G) \subset \partial\bar{\mathbb{A}}$.

In the case of $\chi(M) < 0$, we follow the idea of Le Calvez [Lec06, Theorem 9.3]. We choose an identity isotopy $I_0 = (F_t)_{t\in[0,1]}$ on $\mathbf{S}^2$ such that F_t fixes N and S for every t, and the time-one map of the lift identity isotopy of $I_0|_{\mathbb{A}}$ to $\mathbb{R} \times (0, 1)$ is f^k. Furthermore, as $\rho(f^k, y_i) = 0$ for all i, we can suppose that I_0 fixes N, S and one point of $\{y_1, \cdots, y_{m'}\}$ (e.g., we can modify I_0 though the technique in the proof of Lemma 1.2 in [Wang11, Section 1.4] without changing the homotopic class of $I_0|_{\mathbb{A}}$), say y_1. Identify G as a map of $\mathbf{S}^2$. As $\sharp\mathrm{Fix}(G) < +\infty$, there is a maximal unlink set $\{N, S, y_1\} \subseteq X \subseteq \mathrm{Fix}(G) = \{N, S, y_1, \cdots, y_{m'}\}$ and an identity isotopy I_1 which is homotopic to I_0 with fixed endpoints such that I_1 fixes every point of X. It is clear that $\sharp X \geq 3$ in this case. By Theorem 5, there exists an oriented topological foliation $\mathcal{F}$ on $\mathbf{S}^2 \setminus X$ such that, for all $z \in \mathbf{S}^2 \setminus X$, the trajectory $I_1(z)$ is homotopic to an arc γ joining z and $G(z)$ in $\mathbf{S}^2 \setminus X$ which is positively transverse to $\mathcal{F}$. For any leaf $\lambda \in \mathcal{F}$, the α-limit set and ω-limit set of λ must belong to two distinct points of X respectively since X is finite and G is symplectic. Choose a leaf $\lambda \in \mathcal{F}$ which connects two different points z_1 and z_2 of X. We consider the following open annulus $A_{z_1,z_2} = \mathbf{S}^2 \setminus \{z_1, z_2\}$.

We choose a small open disk U near λ such that, $U \cap \lambda = \emptyset$ and for any $z \in U$, $\lambda \wedge I_1(z) \geq 1$ where $I_1(z)$ is the trajectory of z under the isotopy I_1. We define the first return map

$$\begin{array}{rcl} \Phi : \mathrm{Rec}^+(G) \cap U & \to & \mathrm{Rec}^+(G) \cap U, \\ z & \mapsto & G^{\tau(z)}(z), \end{array}$$

where $\tau(z)$ is the first return time, that is, the least number $n \geq 1$ such that $G^n(z) \in U$. By Poincaré Recurrence Theorem, this map is defined μ-a.e. on U. For every couple $(z', z'') \in U^2$, choose a simple path $\gamma_{z',z''}$ in U joining z' to z''. For every $z \in \mathrm{Rec}^+(G) \cap U$ and $n \geq 1$, define

$$\tau_n(z) = \sum_{i=0}^{n-1} \tau(\Phi^i(z)), \quad \Gamma_z^n = I_1^{\tau_n(z)}(z)\gamma_{\Phi^n(z),z}, \quad m(z) = \Gamma_z^1 \wedge \lambda, \quad m_n(z) = \sum_{i=0}^{n-1} m(\Phi^i(z)).$$

It is well known that $\tau \in L^1(U, \mathbb{R})$ (see, e.g., [Wang11, Section 1.3]). Hence τ_n/n converges μ-a.e. on $\mathrm{Rec}^+(G) \cap U$. It is clear that $m_n/n \geq 1$ for all $n \geq 1$ and $z \in \mathrm{Rec}^+(G) \cap U$. This implies that $m_n/\tau_n > 0$ for μ-a.e. on $\mathrm{Rec}^+(G) \cap U$. Observe that

$$\mathcal{E}(z) \subset [\inf_n \left\{ \frac{m_n(z)}{\tau_n(z)} \right\}, \sup_n \left\{ \frac{m_n(z)}{\tau_n(z)} \right\}] \subset \mathbb{R}$$

when the limit of $\tau_n(z)/n$ exists, where the definition $\mathcal{E}(z)$ one can refer to Remark 2. We get that $\rho^-_{A_{z_1,z_2},I_1}(z) > 0$ for μ-a.e. on $\mathrm{Rec}^+(G) \cap U$. We also have that $\rho_{A_{z_1,z_2},I_1}(y) = 0$ for all $y \in X \setminus \{z_1, z_2\}$. Then Theorem 3 follows by Theorem 6. □

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