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Hamiltonian diffeomorphisms to Hamiltonian
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A GENERALIZATION OF CLASSICAL ACTION OF HAMILTONIAN DIFFEOMORPHISMS TO HAMILTONIAN HOMEOMORPHISMS

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ABSTRACT. In symplectic geometry, a classical object is the notion of action function, defined on the set of contractible fixed points of the time-one map of a Hamiltonian isotopy. On closed surfaces, we give a dynamical interpretation of this function that permits us to generalize it in the case of a diffeomorphism isotopic to identity that preserves a Borel finite measure of rotation vector zero. We define a boundedness property on the contractible fixed points set of the time-one map of an identity isotopy, which includes the case where the time-one map is a diffeomorphism and the simple case where the set of contractible fixed points of the time-one map is finite. We generalize the classical function to any homeomorphism, provided that the boundedness condition is satisfied. Finally, we define the action spectrum which is invariant under conjugation by an orientation and measure preserving homeomorphism.

0. INTRODUCTION

Suppose that (M, ω) is a symplectic manifold with $\pi_2(M) = 0$. Let $I = (F_t)_{t \in \mathbb{R}}$ be a Hamiltonian flow on M with $F_0 = \text{Id}_M$ and $F_1 = F$. Suppose that $H : \mathbb{R} \times M \rightarrow \mathbb{R}$, one-periodic in time, is the Hamiltonian function generating the flow I . Denote by $\text{Fix}_{\text{Cont}, I}(F)$ the set of contractible fixed points of F , that is, $x \in \text{Fix}_{\text{Cont}, I}(F)$ if and only if x is a fixed point of F and the oriented loop $I(x) : t \mapsto F_t(x)$ defined on $[0, 1]$ is contractible on M . The classical action function is defined, up to an additive constant, on $\text{Fix}_{\text{Cont}, I}(F)$ as follows (see Section 3.1 for the details)

$$\mathcal{A}_H(x) = \int_{D_x} \omega - \int_0^1 H(t, F_t(x)) dt,$$

where $x \in \text{Fix}_{\text{Cont}, I}(F)$ and $D_x \subset M$ is any 2-simplex with $\partial D_x = I(x)$. Since $\pi_2(M) = 0$ the integral $\int_{D_x} \omega$ does not depend on the choice of the disc D_x .

When M is compact, among the properties of F , one may notice the fact that it preserves the volume form $\omega^n = \omega \wedge \cdots \wedge \omega$ and that the “rotation vector” $\rho_{M, I}(\mu) \in H_1(M, \mathbb{R})$ (see Section 1.3) of the finite measure μ induced by ω^n vanishes. In the case of a closed symplectic surface, the fact that a diffeomorphism isotopic to identity preserves an area form ω whose rotation vector is zero characterizes the fact that it is the time-one map of a 1-periodic Hamiltonian isotopy (see Section 3.1).

Let M be a closed oriented surface with positive genus. In this case, M is an aspherical closed surface with the property $\pi_2(M) = 0$. We say that an isotopy $I = (F_t)_{t \in [0, 1]}$ on M is an *identity isotopy* if $F_0 = \text{Id}_M$. We extend the identity isotopy $I = (F_t)_{t \in [0, 1]}$ to \mathbb{R} by writing $F_{t+1} = F_t \circ F_1$. If we replace the area form ω by a finite Borel measure μ which is invariant by F , and the Hamiltonian flow $I = (F_t)_{t \in \mathbb{R}}$ by an extended identity isotopy

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$I' = (F'_t)_{t \in \mathbb{R}}$ with $F'_1 = F$ (the isotopy I' is not necessary smooth and preserving the measure for every t), which satisfies that $\rho_{M, I'}(\mu) = 0$, can we define an “action function” which generalizes the classical one? In this article we will give a positive answer.

Furthermore, since the rotation vector is defined in the C^0 -case, one may naturally ask whether there are similar results when F is only a homeomorphism. In this case, we define a weak boundedness property, written WB-property, which is a certain boundedness condition about linking numbers of contractible fixed points, and which includes the case where F is a diffeomorphism and the simple case where the set $\text{Fix}_{\text{Cont}, I}(F)$ is finite. Roughly speaking, this property prevents the dynamics to be too wild in a neighborhood of every contractible fixed point of F (see Section 1.5 for more details). Through the WB-property, we define a new action function with the following desired properties and prove that it is a generalization of the classical function. It can be naturally generalized for

- Any homeomorphism isotopic to the identity that preserves a finite Borel measure of rotation vector zero with full support and with no atoms on the contractible fixed points set, provided that the WB-property is satisfied;
- Any homeomorphism isotopic to the identity that preserves a finite ergodic Borel measure μ of rotation vector zero with no atoms on the contractible fixed points set, provided that the WB-property is satisfied.

The goal of this article is to give a precise dynamical explanation of the classical action function that can be extended to more general cases. In addition, we investigate some elementary properties of the new action function. In further articles, we will give more properties and give some applications (see, e.g., [Wang11, Chapter 6 and 7]).

The main results of this article are summarized as follows.

Let M be a closed oriented surface with genus $g \geq 1$ and F be a homeomorphism on M . Denote by $\text{Homeo}(M)$ (resp. $\text{Diff}(M)$, $\text{Diff}^1(M)$) the group of all homeomorphisms (resp. diffeomorphisms, C^1 -diffeomorphisms) on M and by $\mathcal{M}(F)$ the set of Borel finite measures on M that are invariant by F . In this paper, we always assume that a measure $\mu \in \mathcal{M}(F)$ has no atoms on $\text{Fix}_{\text{Cont}, I}(F)$. Denote by $\text{Homeo}_*(M)$ the subgroup of $\text{Homeo}(M)$ whose elements are isotopic to Id_M .

Theorem 0.1. *Let $F \in \text{Homeo}_*(M)$ be the time-one map of an identity isotopy I on M . Suppose that $\mu \in \mathcal{M}(F)$ and $\rho_{M, I}(\mu) = 0$. In each of the following cases*

- $F \in \text{Diff}(M)$ (not necessarily C^1);
- I satisfies the WB-property and the measure μ has full support;
- I satisfies the WB-property and the measure μ is ergodic,

an action function L_μ can be defined, which generalizes the classical case.

For any $q \geq 1$, we define an identity isotopy I^q on M : $I^q(z) = \prod_{k=0}^{q-1} I(F^k(z))$ for $z \in M$. We get the following iteration formula:

Proposition 0.2. *Under the hypotheses of Theorem 0.1, for every two distinct contractible fixed points a and b of F , we have $I_\mu(I^q; a, b) = qI_\mu(I; a, b)$ for all $q \geq 1$, where $I_\mu(I; a, b) = L_\mu(I; b) - L_\mu(I; a)$.*

Under the same hypotheses as Theorem 0.1, we define the *action spectrum* of I as follows (up to an additive constant):

$$\sigma(I) = \{L_\mu(z) \mid z \in \text{Fix}_{\text{Cont}, I}(F)\} \subset \mathbb{R}.$$

Fix a measure $\nu \in \mathcal{M}(F)$. Denote by $\text{Homeo}^+(M, \nu)$ the subgroup of $\text{Homeo}(M)$ whose elements preserve the measure ν and the orientation. We have the following conjugation invariance property:

Proposition 0.3. *The action spectrum is invariant by conjugation in $\text{Homeo}^+(M, \mu)$.*

The article is organized as follows. In Section 1, we first introduce some notations and recall the precise definitions of some important mathematical objects. In particular, we define the linking number on contractible fixed points and the boundedness properties. In Section 2, we recall some well known results of plane or annulus homeomorphism, and extend some results of Franks to serve as the technical preliminaries of this article. In Section 3, we recall the definition of the classical action function in symplectic geometry and analyze how to generalize it to a more general situation on closed oriented surfaces. In the end of this section, our main theorem is stated. In Section 4, we extend the definition of the linking number defined in Section 1 to positively recurrent points, which is one of the main ingredients of this article. In Section 5, we first study the boundedness of the extended linking number when it exists, and then study the existence and the boundedness of the linking number in the conservative case. In Section 6, based on the extended linking number and its properties studied in Section 4 and Section 5, we define a new action function and prove that it is a generalization of the classical one, which is our main theorem. In Appendix, we construct two examples to further complete our results.

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1. NOTATION AND DEFINITIONS

We denote by $|\cdot|$ the usual Euclidean metric on \mathbb{R}^k or \mathbb{C}^k and by $\mathbf{S}^{k-1} = \{x \in \mathbb{R}^k \mid |x| = 1\}$ the unit sphere.

If A is a set, we write $\sharp A$ for the cardinality of A . If G is a group and e is its unit element, we write $G^* = G \setminus \{e\}$. If (S, σ, μ) is a measure space and V is any finite dimensional linear space, denote by $L^1(S, V, \mu)$ the set of μ -integrable functions from S to V . If X is a topological space and A is a subset of X , denote by $\text{Int}_X(A)$ and $\text{Cl}_X(A)$ respectively the interior and the closure of A . We will omit the subscript X if there is no any confusion.

1.1. Identity isotopies. An *identity isotopy* $I = (F_t)_{t \in [0,1]}$ on M is a continuous path

$$\begin{aligned} [0, 1] &\rightarrow \text{Homeo}(M) \\ t &\mapsto F_t \end{aligned}$$

such that $F_0 = \text{Id}_M$, the last set being endowed with the compact-open topology. We naturally extend this map to \mathbb{R} by writing $F_{t+1} = F_t \circ F_1$. We can also define the inverse isotopy of I as $I^{-1} = (F_{-t})_{t \in [0,1]} = (F_{1-t} \circ F_1^{-1})_{t \in [0,1]}$.

A *path* on a manifold M is a continuous map $\gamma : J \rightarrow M$ defined on a nontrivial interval J (up to an increasing reparametrization). We can talk of a proper path (i.e. $\gamma^{-1}(K)$ is compact for any compact set K) or a compact path (i.e. J is compact). When γ is a compact path, $\gamma(\inf J)$ and $\gamma(\sup J)$ are the *ends* of γ . We say that a compact

path γ is a *loop* if the two ends of γ coincide. The inverse of the path γ is defined by $\gamma^{-1} : t \mapsto \gamma(-t)$, $t \in -J$. If $\gamma_1 : J_1 \rightarrow M$ and $\gamma_2 : J_2 \rightarrow M$ are two paths such that

$$b_1 = \sup J_1 \in J_1, \quad a_2 = \inf J_2 \in J_2 \quad \text{and} \quad \gamma_1(b_1) = \gamma_2(a_2),$$

then the *concatenation* γ_1 and γ_2 is defined on $J = J_1 \cup (J_2 + (b_1 - a_2))$ in the classical way, where $(J_2 + (b_1 - a_2))$ represents the translation of J_2 by $(b_1 - a_2)$:

$$\gamma_1\gamma_2(t) = \begin{cases} \gamma_1(t) & \text{if } t \in J_1; \\ \gamma_2(t + a_2 - b_1) & \text{if } t \in J_2 + (b_1 - a_2). \end{cases}$$

Let \mathcal{I} be an interval (maybe infinite) of \mathbb{Z} . If $\{\gamma_i : J_i \rightarrow M\}_{i \in \mathcal{I}}$ is a family of compact paths satisfying that $\gamma_i(\sup(J_i)) = \gamma_{i+1}(\inf(J_{i+1}))$ for every $i \in \mathcal{I}$, then we can define their concatenation $\prod_{i \in \mathcal{I}} \gamma_i$.

If $\{\gamma_i\}_{i \in \mathcal{I}}$ is a family of compact paths where $\mathcal{I} = \bigsqcup_{j \in \mathcal{J}} \mathcal{I}_j$ and \mathcal{I}_j is an interval of \mathbb{Z} such that $\prod_{i \in \mathcal{I}_j} \gamma_i$ is well defined (in the concatenation sense) for every $j \in \mathcal{J}$, we define their *product* by abusing notations:

$$\prod_{i \in \mathcal{I}} \gamma_i = \prod_{j \in \mathcal{J}} \prod_{i \in \mathcal{I}_j} \gamma_i.$$

The *trajectory* of a point z for the isotopy $I = (F_t)_{t \in [0,1]}$ is the oriented path $I(z) : t \mapsto F_t(z)$ defined on $[0, 1]$. Suppose that $\{I_k\}_{1 \leq k \leq k_0}$ is a family of identity isotopies on M . Write $I_k = (F_{k,t})_{t \in [0,1]}$. We can define a new identity isotopy $I_{k_0} \cdots I_2 I_1 = (F_t)_{t \in [0,1]}$ by concatenation as follows

$$(1.1) \quad F_t(z) = F_{k, k_0 t - (k-1)}(F_{k-1,1} \circ F_{k-2,1} \circ \cdots \circ F_{1,1}(z)) \quad \text{if} \quad \frac{k-1}{k_0} \leq t \leq \frac{k}{k_0}.$$

In particular, $I^{k_0}(z) = \prod_{k=0}^{k_0-1} I(F^k(z))$ when $I_k = I$ for all $1 \leq k \leq k_0$.

We write $\text{Fix}(F)$ for the set of fixed points of F . A fixed point z of $F = F_1$ is *contractible* if $I(z)$ is homotopic to zero. We write $\text{Fix}_{\text{Cont}, I}(F)$ for the set of contractible fixed points of F , which obviously depends on I .

1.2. The algebraic intersection number. The choice of an orientation on M permits us to define the algebraic intersection number $\Gamma \wedge \Gamma'$ between two loops. We keep the same notation $\Gamma \wedge \gamma$ for the algebraic intersection number between a loop and a path γ when it is defined, for example, when γ is proper or when γ is compact path whose extremities are not in Γ . Similarly, we write $\gamma \wedge \gamma'$ for the algebraic intersection number of two path γ and γ' when it is defined, for example, when γ and γ' are compact paths and the ends of γ (resp. γ') are not on γ' (resp. γ). If Γ is a loop on a smooth manifold M , write $[\Gamma]_M \in H_1(M, \mathbb{Z})$ for the homology class of Γ . It is clear that the value $\Gamma \wedge \gamma$ does not depends on the choice of the path γ that fixes its endpoints when $[\Gamma]_M = 0$.

1.3. Rotation vector.

1.3.1. The definition of rotation vector. Let us introduce the classical notion of rotation vector which was defined originally in [St57]. Suppose that $F \in \text{Homeo}_*(M)$ is the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$. Let $\text{Rec}^+(F)$ be the set of positively recurrent points of F . If $z \in \text{Rec}^+(F)$, we fix an open disk $U \subset M$ containing z , and write $\{F^{n_k}(z)\}_{k \geq 1}$ for the subsequence of the positive orbit of z obtained by keeping the points that are in U . For any $k \geq 0$, choose a simple path $\gamma_{F^{n_k}(z), z}$ in U joining $F^{n_k}(z)$ to z .

The homology class $[\Gamma_k]_M \in H_1(M, \mathbb{Z})$ of the loop $\Gamma_k = I^{n_k}(z)\gamma_{F^{n_k}(z), z}$ does not depend on the choice of $\gamma_{F^{n_k}(z), z}$. Say that z has a *rotation vector* $\rho_{M, I}(z) \in H_1(M, \mathbb{R})$ if

$$\lim_{l \rightarrow +\infty} \frac{1}{n_{k_l}} [\Gamma_{k_l}]_M = \rho_{M, I}(z)$$

for any subsequence $\{F^{n_{k_l}}(z)\}_{l \geq 1}$ which converges to z . Neither the existence nor the value of the rotation vector depends on the choice of U .

1.3.2. The existence of rotation number in the compact case. Suppose that M is compact and that F is the time-one map of an identity isotopy $I = (F_t)_{t \in [0, 1]}$ on M . Recall that $\mathcal{M}(F)$ is the set of Borel finite measures on M whose elements are invariant by F . If $\mu \in \mathcal{M}(F)$, we can define the rotation vector $\rho_{M, I}(z)$ for μ -almost every positively recurrent point [Lec05]. Let us explain why.

Let U be an open disk of M that is the interior of a closed topological disk. For every couple $(z', z'') \in U^2$, choose a simple path $\gamma_{z', z''}$ in U joining z' to z'' . We can define the first return map $\Phi : \text{Rec}^+(F) \cap U \rightarrow \text{Rec}^+(F) \cap U$ and write $\Phi(z) = F^{\tau(z)}(z)$, where $\tau(z)$ is the first return time, that is, the least number $n \geq 1$ such that $F^n(z) \in U$. By Poincaré Recurrence Theorem, this map is defined μ -almost everywhere on U . For every $z \in \text{Rec}^+(F) \cap U$ and $n \geq 1$, define

$$\tau_n(z) = \sum_{i=0}^{n-1} \tau(\Phi^i(z)), \quad \Gamma_z^n = I^{\tau_n(z)}(z)\gamma_{\Phi^n(z), z}.$$

Observe now that

$$[\Gamma_z^n]_M = \sum_{i=0}^{n-1} [\Gamma_{\Phi^i(z)}^1]_M.$$

By the classical Kac's lemma (see [Kac47]), we have

$$\int_U \tau \, d\mu = \mu \left(\bigcup_{k \geq 0} F^k(U) \right) = \mu \left(\bigcup_{k \in \mathbb{Z}} F^k(U) \right).$$

Indeed, we have the following measurable partitions (modulo sets of measure zero):

$$U = \bigsqcup_{i \geq 1} U_i \quad \text{and} \quad \bigcup_{k \geq 0} F^k(U) = \bigsqcup_{i \geq 1} \bigsqcup_{0 \leq j \leq i-1} F^j(U_i),$$

where $U_i = \tau^{-1}(\{i\})$, therefore

$$\mu \left(\bigcup_{k \geq 0} F^k(U) \right) = \sum_{i \geq 1} \sum_{0 \leq j \leq i-1} \mu(U_i) = \sum_{i \geq 1} i \mu(U_i) = \int_U \tau \, d\mu.$$

Hence, we get $\tau \in L^1(U, \mathbb{R}, \mu)$. In the case where M is compact, let us prove that the function $z \mapsto [\Gamma_z^1]_M / \tau(z)$ is bounded on $\text{Rec}^+(F) \cap U$ and hence that the map $z \mapsto [\Gamma_z^1]_M$ belongs to $L^1(U, H_1(M, \mathbb{R}), \mu)$.

Indeed, it is sufficient to prove that for every cohomology class $\kappa \in H^1(M, \mathbb{R})$, there exists a constant K_κ such that $|\langle \kappa, [\Gamma_z^1]_M \rangle| \leq K_\kappa \tau(z)$. Let λ be a closed form that represents κ . The function $g_\lambda : z \mapsto \int_{I(z)} \lambda$ is well defined, since λ is closed, and continuous. It is bounded since M is compact. As $\text{Cl}(U)$ is a closed disk, we can find an open disk U' containing $\text{Cl}(U)$ and a primitive h_λ of λ on U' . This primitive is bounded on $\text{Cl}(U)$. This implies that for every $z \in \text{Rec}^+(F) \cap U$, we have

$$\begin{aligned}
|\langle [\lambda], [\Gamma_z^1]_M \rangle| &= \left| \int_{\Gamma_z^1} \lambda \right| = \left| \sum_{i=0}^{\tau(z)-1} \int_{I(F^i(z))} \lambda + \int_{\lambda_{\Phi(z),z}} \lambda \right| \\
&\leq \tau(z) \max_{z \in M} |g_\lambda(z)| + 2 \sup_{z \in U} |h_\lambda(z)| \\
&\leq \tau(z) (\max_{z \in M} |g_\lambda(z)| + 2 \sup_{z \in U} |h_\lambda(z)|).
\end{aligned}$$

By Birkhoff Ergodic Theorem, for μ -almost every point on $\text{Rec}^+(F) \cap U$, the sequence $\{\tau_n(z)/n\}_{n \geq 1}$ converges to a real number $\tau^*(z) \geq 1$, and the sequence $\{[\Gamma_z^n]_M/n\}_{n \geq 1}$ converges to $[\Gamma_z^*]_M \in H_1(M, \mathbb{R})$. The positively recurrent points of F in U are exactly the positively recurrent points of Φ because U is open. We deduce that μ -almost every point $z \in \text{Rec}^+(F) \cap U$ has a rotation vector $\rho_{M,I}(z) = [\Gamma_z^*]_M / \tau^*(z)$. Since U is arbitrarily chosen, we deduce that μ -almost every point $z \in \text{Rec}^+(F)$ has a rotation vector. The function $z \mapsto [\Gamma_z^1]_M / \tau(z)$ is bounded on $\text{Rec}^+(F) \cap U$, so is the function

$$\rho_{M,I} : z \mapsto \lim_{n \rightarrow +\infty} \frac{\sum_{i=0}^{n-1} [\Gamma_{\Phi^i(z)}^1]_M}{\sum_{i=0}^{n-1} \tau(\Phi^i(z))}$$

on $\text{Rec}^+(F) \cap U$. As M can be covered by finitely many such open disks, we deduce that $\rho_{M,I}$ is uniformly bounded on $\text{Rec}^+(F)$. Therefore, we can define the *rotation vector of the measure*

$$\rho_{M,I}(\mu) = \int_M \rho_{M,I} d\mu \in H_1(M, \mathbb{R}).$$

1.3.3. The rotation number of an open annulus. Let $\mathbb{A} = \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ be the open annulus. Let us denote the covering map

$$\begin{aligned}
\pi : \mathbb{R}^2 &\rightarrow \mathbb{A} \\
(x, y) &\mapsto (x + \mathbb{Z}, y),
\end{aligned}$$

and T the generator of the covering transformation group

$$\begin{aligned}
T : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\
(x, y) &\mapsto (x + 1, y).
\end{aligned}$$

When $F \in \text{Homeo}_*(\mathbb{A})$, we have a simple way to define the “rotation vector” given in Section 1.3.1 if we observe that $H_1(\mathbb{A}, \mathbb{R}) = \mathbb{R}$. We will say that a positively recurrent point z has a *rotation number* $\rho_{\mathbb{A}, \tilde{F}}(z)$ for a lift \tilde{F} of F to the universal cover \mathbb{R}^2 of \mathbb{A} , if for every subsequence $\{F^{n_k}(z)\}_{k \geq 1}$ of $\{F^n(z)\}_{n \geq 1}$ which converges to z , we have

$$\lim_{k \rightarrow +\infty} \frac{p_1 \circ \tilde{F}^{n_k}(\tilde{z}) - p_1(\tilde{z})}{n_k} = \rho_{\mathbb{A}, \tilde{F}}(z)$$

for every $\tilde{z} \in \pi^{-1}(z)$, where $p_1 : (x, y) \mapsto x$ is the first projection. We denote the set of rotation numbers of positively recurrent points of F for \tilde{F} as $\text{Rot}(\tilde{F})$. In particular, the rotation number $\rho_{\mathbb{A}, \tilde{F}}(z)$ always exists when z is a fixed point of F . We denote the set of rotation numbers of fixed points of F as $\text{Rot}_{\text{Fix}(F)}(\tilde{F})$.

It is well known that a positively recurrent point of F is also a positively recurrent point of F^q for all $q \in \mathbb{N}$ (see the appendix of [Wang14]). By the definition of rotation number, we easily get that $\text{Rot}(\tilde{F})$ satisfies the following elementary properties.

1. $\text{Rot}(T^k \circ \tilde{F}) = \text{Rot}(\tilde{F}) + k$ for every $k \in \mathbb{Z}$;
2. $\text{Rot}(\tilde{F}^q) = q\text{Rot}(\tilde{F})$ for every $q \geq 1$.

1.4. Linking number of contractible fixed points.

1.4.1. We begin by recalling some results about identity isotopies, which will be often used in the literature.

Remark 1.1. Suppose that M is an oriented compact surface and that F is the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$ on M . When $z \in \text{Fix}_{\text{Cont}, I}(F)$, there is another identity isotopy $I' = (F'_t)_{t \in [0,1]}$ homotopic to I with fixed endpoints such that I' fixes z (see, for example, [Jau14, Proposition 2.15]), that is, there is a continuous map $H : [0, 1] \times [0, 1] \times M \rightarrow M$ such that

- $H(0, t, z) = F_t(z)$ and $H(1, t, z) = F'_t(z)$ for all $t \in [0, 1]$;
- $H(s, 0, z) = \text{Id}_M(z)$ and $H(s, 1, z) = F(z)$ for all $s \in [0, 1]$;
- $F'_t(z) = z$ for all $t \in [0, 1]$.

Lemma 1.2. Let \mathbf{S}^2 be the 2-sphere and $I = (F_t)_{t \in [0,1]}$ be an identity isotopy on \mathbf{S}^2 . For every three different fixed points z_i ($i = 1, 2, 3$) of F_1 , there exists another identity isotopy $I' = (F'_t)_{t \in [0,1]}$ from $\text{Id}_{\mathbf{S}^2}$ to F_1 such that I' fixes z_i ($i = 1, 2, 3$).

Proof. We identify the sphere \mathbf{S}^2 to the Riemann sphere $\mathbb{C} \cup \{\infty\}$. The Möbius transformation $\mathcal{M}(z) = \frac{az+b}{cz+d}$ maps the triple (v_1, v_2, v_3) to the triple $(\omega_1, \omega_2, \omega_3)$ (see Chapter 3 of [Nee97] for a beautifully illustrated introduction to Möbius transformations) where

$$(1.2) \quad \begin{aligned} a &= \det \begin{pmatrix} v_1\omega_1 & \omega_1 & 1 \\ v_2\omega_2 & \omega_2 & 1 \\ v_3\omega_3 & \omega_3 & 1 \end{pmatrix} & b &= \det \begin{pmatrix} v_1\omega_1 & v_1 & \omega_1 \\ v_2\omega_2 & v_2 & \omega_2 \\ v_3\omega_3 & v_3 & \omega_3 \end{pmatrix} \\ c &= \det \begin{pmatrix} v_1 & \omega_1 & 1 \\ v_2 & \omega_2 & 1 \\ v_3 & \omega_3 & 1 \end{pmatrix} & d &= \det \begin{pmatrix} v_1\omega_1 & v_1 & 1 \\ v_2\omega_2 & v_2 & 1 \\ v_3\omega_3 & v_3 & 1 \end{pmatrix}. \end{aligned}$$

If one of the points v_i or w_i in Formula 1.2 is ∞ , then we first divide all four determinants by this variable and then take the limit as the variable approaches ∞ . Replacing v_i, w_i by $v_i(t) = F_t(z_i)$ and $w_i(t) = z_i$ ($i = 1, 2, 3$ and $t \in [0, 1]$) in the matrices above, we get the *matrix functions* a_t, b_t, c_t and d_t .

Let

$$\mathcal{M}(t, z) = \frac{a_t z + b_t}{c_t z + d_t}$$

and

$$I'(z)(t) = F'_t(z) = \mathcal{M}(t, F_t(z)).$$

By the construction, I' is an isotopy of \mathbf{S}^2 from $\text{Id}_{\mathbf{S}^2}$ to F_1 that fixes z_i ($i = 1, 2, 3$). \square

As a consequence, we have the following corollary.

Corollary 1.3. Let $I = (F_t)_{t \in [0,1]}$ be an identity isotopy on \mathbb{C} . For any two different fixed points z_1 and z_2 of F_1 , there exists another identity isotopy I' from $\text{Id}_{\mathbb{C}}$ to F_1 such that I' fixes z_1 and z_2 .

Remark 1.4. Let $z_i \in \mathbf{S}^2$ ($i = 1, 2, 3$) and $\text{Homeo}_*(\mathbf{S}^2, z_1, z_2, z_3)$ be the identity component of the space of all homeomorphisms of \mathbf{S}^2 leaving z_i ($i = 1, 2, 3$) pointwise fixed (for the compact-open topology). It is well known that $\pi_1(\text{Homeo}_*(\mathbf{S}^2, z_1, z_2, z_3)) = 0$ (see [Ham66, Han92]). It implies that any two identity isotopies $I, I' \subset \text{Homeo}_*(\mathbf{S}^2, z_1, z_2, z_3)$ with fixed endpoints are homotopic. As a consequence, let $\text{Homeo}_*(\mathbb{C}, z_1, z_2)$ be the identity component of the space of all homeomorphisms of \mathbb{C} leaving two different points z_1 and z_2 of \mathbb{C} pointwise fixed, we have $\pi_1(\text{Homeo}_*(\mathbb{C}, z_1, z_2)) = 0$.

1.4.2. Let M be a surface that is homeomorphic to the complex plane \mathbb{C} and $I = (F_t)_{t \in [0,1]}$ be an identity isotopy on M . Let us define the *linking number* $i_I(z, z') \in \mathbb{Z}$ for every two different fixed points z and z' of F_1 . It is the degree of the map $\xi : \mathbf{S}^1 \rightarrow \mathbf{S}^1$ defined by

$$\xi(e^{2i\pi t}) = \frac{h \circ F_t(z') - h \circ F_t(z)}{|h \circ F_t(z') - h \circ F_t(z)|},$$

where $h : M \rightarrow \mathbb{C}$ is a homeomorphism. The linking number does not depend on the chosen h .

It is well known that $U(1)$ is a strong deformation retract of $\text{Homeo}_*(\mathbb{C})$ (see [Kne26] or [Ler01, Theorem 2.9]). Consider the isotopy $R = (r_t)_{t \in [0,1]}$ where $r_t = e^{2i\pi t}$. If $I = (F_t)_{t \in [0,1]}$ is an identity isotopy and $k \in \mathbb{Z}$, we can define the identity isotopy $R^k I$ by concatenation. If $I' = (F'_t)_{t \in [0,1]}$ is another identity isotopy with $F'_1 = F_1$, then there exists a unique integer k such that I' is homotopic to $R^k I$.

Therefore, if $I = (F_t)_{t \in [0,1]}$ and $I' = (F'_t)_{t \in [0,1]}$ are two identity isotopies on M with $F'_1 = F_1$, then there exist $k \in \mathbb{Z}$ such that $i_{I'}(z, z') = i_I(z, z') + k$ for any distinct fixed points z' and z' of F_1 .

1.4.3. Let F be the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$ on a closed oriented surface M of genus $g \geq 1$ and \tilde{F} be the time-one map of the lifted identity isotopy $\tilde{I} = (\tilde{F}_t)_{t \in [0,1]}$ on the universal cover \tilde{M} of M . When $g > 1$, it is well known that $\pi_1(\text{Homeo}_*(M)) \simeq 0$ (see [Ham66]). It implies that any two identity isotopies $I, I' \subset \text{Homeo}_*(M)$ with fixed endpoints are homotopic. Hence, I is unique up to homotopy, it implies that \tilde{F} is uniquely defined and does not depend on the choice of the isotopy from Id_M to F . When $g = 1$, $\pi_1(\text{Homeo}_*(M)) \simeq \mathbb{Z}^2$ (see [Ham65]), \tilde{F} depends on the isotopy I . The universal cover \tilde{M} is homeomorphic to \mathbb{C} .

Let $\pi : \tilde{M} \rightarrow M$ be the covering map and G be the covering transformation group. Denote respectively by Δ and $\tilde{\Delta}$ the diagonal of $\text{Fix}_{\text{Cont}, I}(F) \times \text{Fix}_{\text{Cont}, I}(F)$ and the diagonal of $\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})$. Endow the surface M with a Riemannian metric and denote by d the distance induced by the metric. Lift the Riemannian metric to \tilde{M} and write \tilde{d} for the distance induced by the metric.

We define the *linking number* $i(\tilde{F}; \tilde{z}, \tilde{z}')$ for every pair $(\tilde{z}, \tilde{z}') \in (\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta}$ as

$$(1.3) \quad i(\tilde{F}; \tilde{z}, \tilde{z}') = i_{\tilde{F}}(\tilde{z}, \tilde{z}').$$

This is a special case of the linking number that we have defined in Section 1.4.2.

We give some properties of $i(\tilde{F}; \tilde{z}, \tilde{z}')$ as follows.

(P1): $i(\tilde{F}; \tilde{z}, \tilde{z}')$ is locally constant on $(\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta}$;

(P2): $i(\tilde{F}; \tilde{z}, \tilde{z}')$ is invariant by covering transformation, that is,

$$i(\tilde{F}; \alpha(\tilde{z}), \alpha(\tilde{z}')) = i(\tilde{F}; \tilde{z}, \tilde{z}') \quad \text{for every } \alpha \in G;$$

(P3): $i(\tilde{F}; \tilde{z}, \tilde{z}') = 0$ if $\pi(\tilde{z}) = \pi(\tilde{z}')$;

(P4): there exists K such that $i(\tilde{F}; \tilde{z}, \tilde{z}') = 0$ if $\tilde{d}(\tilde{z}, \tilde{z}') \geq K$.

Indeed, P1 is true by continuity. P2 is true because the linking number does not change when you replace h by $h \circ \alpha$ (see Section 1.4.2). By Remark 1.1, we can choose an isotopy I' that is homotopic to I and fixes $\pi(\tilde{z})$, then the lift \tilde{I}' of I' fixes \tilde{z} and \tilde{z}' . Thus P3 holds. Finally, let

$$K = \sup\{\tilde{d}(\tilde{F}_t(\tilde{z}), \tilde{F}_{t'}(\tilde{z})) \mid (t, t', \tilde{z}) \in [0, 1]^2 \times \text{Fix}(\tilde{F})\}.$$

The value K is well defined because $\text{Fix}_{\text{Cont}, I}(F) = \pi(\text{Fix}(\tilde{F}))$ is compact and $\tilde{F}_t \circ \alpha = \alpha \circ \tilde{F}_t$ for all $t \in [0, 1]$ and $\alpha \in G$. Obviously, when $\tilde{d}(\tilde{z}, \tilde{z}') \geq 3K$, $i(\tilde{F}; \tilde{z}, \tilde{z}') = 0$. We get P4.

1.4.4. In the rest of the paper, when we take two distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} , it does not mean that $\pi(\tilde{a})$ and $\pi(\tilde{b})$ are distinct.

Fix two distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} . For any $z \in \text{Fix}_{\text{Cont}, I}(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$, we define *the linking number of z for \tilde{a} and \tilde{b}* as

$$(1.4) \quad i(\tilde{F}; \tilde{a}, \tilde{b}, z) = \sum_{\pi(\tilde{z})=z} \left(i(\tilde{F}; \tilde{a}, \tilde{z}) - i(\tilde{F}; \tilde{b}, \tilde{z}) \right).$$

We will extend it to the case where $z \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$ in Section 4. Note here that the linking number only depends on $\pi(\tilde{a})$ and $\pi(\tilde{b})$ in the case where z is a contractible fixed point of F , but the extension of $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ for $z \in \text{Rec}^+(F) \setminus \text{Fix}_{\text{Cont}, I}(F)$ in Section 4 depends on the choices of \tilde{a} and \tilde{b} .

1.5. The weak boundedness property and the boundedness property. We can compactify \tilde{M} into a sphere by adding a point ∞ at infinity and the lift \tilde{F} may be extended by fixing this point. In all the text, we write $\mathbf{S} = \tilde{M} \sqcup \{\infty\}$. If \tilde{a} and \tilde{b} are distinct fixed points of \tilde{F} , the restriction of \tilde{F} to the annulus $A_{\tilde{a}, \tilde{b}} = \mathbf{S} \setminus \{\tilde{a}, \tilde{b}\}$ denoted by $\tilde{F}_{\tilde{a}, \tilde{b}}$, has a natural lift $\widehat{F}_{\tilde{a}, \tilde{b}}$ to the universal cover $\widehat{A}_{\tilde{a}, \tilde{b}}$ of $A_{\tilde{a}, \tilde{b}}$ that fixes the preimages of ∞ by the covering projection $\widehat{\pi}_{\tilde{a}, \tilde{b}} : \widehat{A}_{\tilde{a}, \tilde{b}} \rightarrow A_{\tilde{a}, \tilde{b}}$. Denote by $T_{\tilde{a}, \tilde{b}}$ the generator of $H_1(A_{\tilde{a}, \tilde{b}}, \mathbb{R})$ defined by the oriented boundary of a small disk centered at \tilde{a} .

If $\pi(\tilde{a}) \neq \pi(\tilde{b})$, by Remark 1.1, there exist two identity isotopies I' and I'' homotopic to I with fixed endpoints such that I' fixes $\pi(\tilde{a})$ and I'' fixes $\pi(\tilde{b})$. However, in general, there does not exist an identity isotopy I''' homotopic to I with fixed endpoints such that I''' fixes both $\pi(\tilde{a})$ and $\pi(\tilde{b})$, which is an obstacle that prevents us to generalize the action function to a more general cases (see Section 3.3). That is a reason that we introduce the following lemma.

Lemma 1.5. *If \tilde{z} is another fixed point of \tilde{F} which is different from \tilde{a} , \tilde{b} and ∞ , then the rotation number of $\tilde{z} \in A_{\tilde{a}, \tilde{b}}$ for the natural lift $\widehat{F}_{\tilde{a}, \tilde{b}}$ is equal to $i(\tilde{F}; \tilde{a}, \tilde{z}) - i(\tilde{F}; \tilde{b}, \tilde{z})$, that is,*

$$\rho_{A_{\tilde{a}, \tilde{b}}, \widehat{F}_{\tilde{a}, \tilde{b}}}(\tilde{z}) = i(\tilde{F}; \tilde{a}, \tilde{z}) - i(\tilde{F}; \tilde{b}, \tilde{z}).$$

Proof. If J and J' are two isotopies of \tilde{M} from $\text{Id}_{\tilde{M}}$ to \tilde{F} , then there exists $k \in \mathbb{Z}$ such that $i_J = i_{J'} + k$ (see Section 1.4.2). Therefore, if \tilde{a}, \tilde{b} and \tilde{z} are distinct fixed points of \tilde{F} , the quantity $i_J(\tilde{a}, \tilde{z}) - i_J(\tilde{b}, \tilde{z})$ is independent of J and hence equals to $i(\tilde{F}; \tilde{a}, \tilde{z}) - i(\tilde{F}; \tilde{b}, \tilde{z})$ if we choose $J = \tilde{I}$ where \tilde{I} is the identity isotopy in Section 1.4.3. Suppose now that J is an isotopy that fixes \tilde{a} and \tilde{b} . The trajectory $J(\tilde{z})$ defines a loop in the sphere \mathbf{S} . If $\gamma_{\tilde{a}, \infty}$ and $\gamma_{\tilde{b}, \infty}$ are two paths in \mathbf{S} that join respectively \tilde{a} and \tilde{b} to ∞ , we have $i_J(\tilde{a}, \tilde{z}) = \gamma_{\tilde{a}, \infty} \wedge J(\tilde{z})$ and $i_J(\tilde{b}, \tilde{z}) = \gamma_{\tilde{b}, \infty} \wedge J(\tilde{z})$. The loop $J(\tilde{z})$ being homologous to zero in \mathbf{S} , we deduce that $i(\tilde{F}; \tilde{a}, \tilde{z}) - i(\tilde{F}; \tilde{b}, \tilde{z}) = i_J(\tilde{a}, \tilde{z}) - i_J(\tilde{b}, \tilde{z}) = \gamma_{\tilde{a}, \tilde{b}} \wedge J(\tilde{z})$, where $\gamma_{\tilde{a}, \tilde{b}}$ is a path in \mathbf{S} that joins \tilde{a} to \tilde{b} . Note that this integer is nothing else but the rotation number of \tilde{z} for the lift $\hat{F}_{\tilde{a}, \tilde{b}}$ defined by $T_{\tilde{a}, \tilde{b}}$. \square

Remark here that, by the definition $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ of Section 1.4.4, we have

$$i(\tilde{F}; \tilde{a}, \tilde{b}, z) = \sum_{\pi(\tilde{z})=z} i(\tilde{F}; \tilde{a}, \tilde{z}) - i(\tilde{F}; \tilde{b}, \tilde{z}) = \sum_{\pi(\tilde{z})=z} \rho_{A_{\tilde{a}, \tilde{b}}, \hat{F}_{\tilde{a}, \tilde{b}}}(\tilde{z}).$$

Definition 1.6. We say that I satisfies *the weak boundedness property at $\tilde{a} \in \text{Fix}(\tilde{F})$ (WB-property at \tilde{a})* if $i(\tilde{F}; \tilde{a}, \tilde{b})$ is uniformly bounded for all fixed point $\tilde{b} \in \text{Fix}(\tilde{F}) \setminus \{\tilde{a}\}$. We say that I satisfies *the weak boundedness property (WB-property)* if it satisfies the weak boundedness property at every $\tilde{a} \in \text{Fix}(\tilde{F})$. We say that I satisfies *the boundedness property (B-property)* if the set of $i(\tilde{F}; \tilde{a}, \tilde{b})$ where $(\tilde{a}, \tilde{b}) \in (\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta}$ is bounded.

Let us now study the WB-property and B-property. First, we note that the set of all WB-property points of I is dense in $\text{Fix}(\tilde{F})$ ([Ler14]).

Lemma 1.7. *Let \tilde{a} and \tilde{b} be two distinct fixed points of \tilde{F} . The following statements are equivalent*

- (1) *I satisfies the WB-property at \tilde{a} and \tilde{b} ;*
- (2) *there exists $K \geq 0$ such that $\left| \rho_{A_{\tilde{a}, \tilde{b}}, \hat{F}_{\tilde{a}, \tilde{b}}}(\tilde{c}) \right| \leq K$ for all fixed point $\tilde{c} \in \text{Fix}(\tilde{F}) \setminus \{\tilde{a}, \tilde{b}\}$.*

Proof. From Lemma 1.5, we have (1) \Rightarrow (2) immediately. Next we prove (2) \Rightarrow (1) by contradiction. Without loss of generality, we suppose that there exists a sequence $\{\tilde{c}_n\}_{n \geq 1} \subset \text{Fix}(\tilde{F}) \setminus \{\tilde{a}, \tilde{b}\}$ such that $\lim_{n \rightarrow +\infty} i(\tilde{F}; \tilde{a}, \tilde{c}_n) = +\infty$ (the case $\lim_{n \rightarrow +\infty} i(\tilde{F}; \tilde{a}, \tilde{c}_n) = -\infty$ is similar). Lemma 1.5 and the hypothesis (2) imply that $\lim_{n \rightarrow +\infty} i(\tilde{F}; \tilde{b}, \tilde{c}_n) = +\infty$. The property P4 implies that the sequence $\{\tilde{c}_n\}_{n \geq 1}$ is bounded. The property P1 implies that $\lim_{n \rightarrow +\infty} \tilde{c}_n = \tilde{a}$ and $\lim_{n \rightarrow +\infty} \tilde{c}_n = \tilde{b}$, which gives a contradiction. \square

Lemma 1.8. *For any two distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} , if F and F^{-1} are differentiable at $\pi(\tilde{a})$ and $\pi(\tilde{b})$, then $\rho_{A_{\tilde{a}, \tilde{b}}, \hat{F}_{\tilde{a}, \tilde{b}}}(\tilde{z})$ is uniformly bounded for any $\tilde{z} \in \text{Rec}^+(\tilde{F}) \setminus \{\tilde{a}, \tilde{b}\}$ if it exists. In particular, $\rho_{A_{\tilde{a}, \tilde{b}}, \hat{F}_{\tilde{a}, \tilde{b}}}(\tilde{c})$ is uniformly bounded for any fixed point $\tilde{c} \in \text{Fix}(\tilde{F}) \setminus \{\tilde{a}, \tilde{b}\}$.*

Proof. Let $\bar{A}_{\tilde{a}, \tilde{b}} = S_{\tilde{a}} \sqcup A_{\tilde{a}, \tilde{b}} \sqcup S_{\tilde{b}}$ where $S_{\tilde{a}}$ and $S_{\tilde{b}}$ are the tangent unit circles at \tilde{a} and \tilde{b} such that $\bar{A}_{\tilde{a}, \tilde{b}}$ is the natural compactification of $A_{\tilde{a}, \tilde{b}}$. The maps F and F^{-1} are differentiable

at $\pi(\tilde{a})$ and $\pi(\tilde{b})$. Hence the lift \tilde{F} (resp. \tilde{F}^{-1}) of F (resp. F^{-1}) to \tilde{M} is differentiable at \tilde{a} and \tilde{b} . By the method of *blowing-up*, it induces a homeomorphism $f : \bar{A}_{\tilde{a},\tilde{b}} \rightarrow \bar{A}_{\tilde{a},\tilde{b}}$,

$$f(u) = \begin{cases} \tilde{F}_{\tilde{a},\tilde{b}}(u) & \text{when } u \in A_{\tilde{a},\tilde{b}} \\ \frac{D\tilde{F}(\tilde{a}).u}{|D\tilde{F}(\tilde{a}).u|} & \text{when } u \in S_{\tilde{a}} \\ \frac{D\tilde{F}(\tilde{b}).u}{|D\tilde{F}(\tilde{b}).u|} & \text{when } u \in S_{\tilde{b}}. \end{cases}$$

The universal cover of $\bar{A}_{\tilde{a},\tilde{b}}$ is $\mathbb{R} \times [0, 1]$. We suppose that \hat{f} is the lift of f fixing the preimages of ∞ by the covering projection $\hat{\pi}_{\tilde{a},\tilde{b}}$. For any $u \in \bar{A}_{\tilde{a},\tilde{b}}$, we have that $p_1(\hat{f}(\hat{u})) - p_1(\hat{u})$ is uniformly bounded because $\bar{A}_{\tilde{a},\tilde{b}}$ is compact, where \hat{u} is any lift of u . There exists N , depending on I , such that for every $\hat{z} \in \hat{A}_{\tilde{a},\tilde{b}}$, one has $|p_1(\hat{F}_{\tilde{a},\tilde{b}}(\hat{z})) - p_1(\hat{z})| \leq N$. Moreover, for every $n \geq 1$, we have

$$(1.5) \quad \left| \frac{p_1 \circ \hat{F}_{\tilde{a},\tilde{b}}^n(\hat{z}) - p_1(\hat{z})}{n} \right| \leq \frac{1}{n} \sum_{i=0}^{n-1} |p_1 \circ \hat{F}_{\tilde{a},\tilde{b}}^{i+1}(\hat{z}) - p_1 \circ \hat{F}_{\tilde{a},\tilde{b}}^i(\hat{z})| \leq N.$$

If $\tilde{z} \in \text{Rec}^+(\tilde{F}_{\tilde{a},\tilde{b}})$ and $\rho_{A_{\tilde{a},\tilde{b}}, \hat{F}_{\tilde{a},\tilde{b}}}(\tilde{z})$ exists, by the definition of rotation number (see Section 1.3.3), we deduce that $|\rho_{A_{\tilde{a},\tilde{b}}, \hat{F}_{\tilde{a},\tilde{b}}}(\tilde{z})| \leq N$. We have completed the proof. \square

Observe that the proof of Lemma 1.8 gives us an information about how rotate not only the positively recurrent points of $\tilde{F}_{\tilde{a},\tilde{b}}$ but in fact every point in $A_{\tilde{a},\tilde{b}}$, we will use this fact in Section 5.

By Lemma 1.7 and Lemma 1.8, we have the following proposition immediately.

Proposition 1.9. *The WB-property is satisfied if $F \in \text{Diff}(M)$.*

Obviously, I satisfies the B-property if $\sharp \text{Fix}_{\text{Cont}, I}(F) < +\infty$. In Example 7.1 of Appendix, we construct an isotopy $I = (F_t)_{0 \leq t \leq 1}$ such that $F = F_1$ is a diffeomorphism of M but does not satisfy the B-property. In that example, we show that F is not a C^1 -diffeomorphism of M . If F is a C^1 -diffeomorphism of M , we have the following result

Proposition 1.10. *The B-property is satisfied if $F \in \text{Diff}^1(M)$.*

Before proving Proposition 1.10, we need the following lemma ([BFLM13, Lemma 5.6]).

Lemma 1.11. *Let h be a C^1 -diffeomorphism of \mathbf{S}^2 and $a \in \text{Fix}(h)$. For all point $z \in \mathbf{S}^2$ different from a and its antipodal point, denote γ_z the unique great circle that passes through them and a , and denote γ_z^- (resp. γ_z^+) the small (resp. large) arc of γ_z joining a and z . Then there exists a neighborhood W of a on \mathbf{S}^2 such that for all $z \in \text{Fix}(h) \cap W$, we have $h(\gamma_z^-) \cap \gamma_z^+ = \{z, a\}$.*

Proof of Proposition 1.10. We only need to consider the case where

$$\sharp \text{Fix}_{\text{Cont}, I}(F) = +\infty.$$

To get a proof by contradiction, according to Definition 1.6, we suppose that there exist a sequence of pairs $\{(\tilde{a}_n, \tilde{b}_n)\}_{n \geq 1} \subset (\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta}$ such that $\lim_{n \rightarrow +\infty} i(\tilde{F}; \tilde{a}_n, \tilde{b}_n) = +\infty$

(the case where $\lim_{n \rightarrow +\infty} i(\tilde{F}; \tilde{a}_n, \tilde{b}_n) = -\infty$ is similar). By the property P2, we can suppose that the sequence $\{\tilde{a}_n\}_{n \geq 1}$ is bounded by replacing \tilde{a}_n and \tilde{b}_n with $\alpha_n(\tilde{a}_n)$ and $\alpha_n(\tilde{b}_n)$ where $\alpha_n \in G$ if necessary. The property P4 implies that the sequence $\{\tilde{b}_n\}_{n \geq 1}$ is also bounded. Therefore, by continuity, we can suppose that $\lim_{n \rightarrow +\infty} \tilde{a}_n = \tilde{a}$ and $\lim_{n \rightarrow +\infty} \tilde{b}_n = \tilde{b}$ where $\tilde{a} \in \text{Fix}(\tilde{F})$ and $\tilde{b} \in \text{Fix}(\tilde{F})$ by extracting subsequences if necessary. According to the property P1, we deduce that $\tilde{a} = \tilde{b}$. Moreover, as F is a diffeomorphism, so I satisfies the WB-property at \tilde{a} . That is, there is a number $N_{\tilde{a}} \geq 0$ such that $|i(\tilde{F}; \tilde{a}, \tilde{z})| \leq N_{\tilde{a}}$ for all $\tilde{z} \in \text{Fix}(\tilde{F}) \setminus \{\tilde{a}\}$. Hence, we can suppose that $\tilde{a}_n \neq \tilde{a}$ and $\tilde{b}_n \neq \tilde{a}$ for all n by taking n large enough.

For every $n \geq 1$, let \tilde{I}_n be an isotopy that fixes \tilde{a} and \tilde{a}_n (see Corollary 1.3). Then there exists k_n such that

$$(1.6) \quad i_{\tilde{I}_n}(\tilde{z}, \tilde{z}') = i(\tilde{F}; \tilde{z}, \tilde{z}') + k_n$$

for every two distinct fixed points \tilde{z} and \tilde{z}' of \tilde{F} (see Section 1.4.2). Observing that $i_{\tilde{I}_n}(\tilde{a}, \tilde{a}_n) = 0$ for every n , Equation 1.6 implies that $|k_n| \leq N_{\tilde{a}}$ and $\lim_{n \rightarrow +\infty} i_{\tilde{I}_n}(\tilde{a}_n, \tilde{b}_n) = +\infty$. Moreover, we have $i_{\tilde{I}_n}(\tilde{a}, \tilde{b}_n) = i(\tilde{F}; \tilde{a}, \tilde{b}_n) + k_n$, hence $|i_{\tilde{I}_n}(\tilde{a}, \tilde{b}_n)| \leq 2N_{\tilde{a}}$.

Consider the annulus $A_{\tilde{a}, \tilde{a}_n} = \mathbf{S} \setminus \{\tilde{a}, \tilde{a}_n\}$ and $\tilde{F}_{\tilde{a}, \tilde{a}_n}$. By the proof of Lemma 1.5, we know that

$$\rho_{A_{\tilde{a}, \tilde{a}_n}, \tilde{F}_{\tilde{a}, \tilde{a}_n}}(\tilde{b}_n) = i_{\tilde{I}_n}(\tilde{a}, \tilde{b}_n) - i_{\tilde{I}_n}(\tilde{a}_n, \tilde{b}_n).$$

Therefore,

$$(1.7) \quad \lim_{n \rightarrow +\infty} \rho_{A_{\tilde{a}, \tilde{a}_n}, \tilde{F}_{\tilde{a}, \tilde{a}_n}}(\tilde{b}_n) = -\infty.$$

Fix $q \geq 1$. We apply Lemma 1.11 to $\tilde{F}_{\tilde{a}, \tilde{a}_n}$. When n is large enough, there are two arcs $\tilde{\gamma}^-$ and $\tilde{\gamma}^+$ in $A_{\tilde{a}, \tilde{a}_n}$ joining \tilde{a} and \tilde{a}_n that are disjoint and $\tilde{F}_{\tilde{a}, \tilde{a}_n}^q(\tilde{\gamma}^-) \cap \tilde{\gamma}^+ = \emptyset$. Recall that $\hat{\pi}_{\tilde{a}, \tilde{a}_n} : \hat{A}_{\tilde{a}, \tilde{a}_n} \rightarrow A_{\tilde{a}, \tilde{a}_n}$ is the universal cover of $A_{\tilde{a}, \tilde{a}_n}$, $\hat{F}_{\tilde{a}, \tilde{a}_n}$ is the lift of $\tilde{F}_{\tilde{a}, \tilde{a}_n}$ that fixes the preimages of ∞ by $\hat{\pi}_{\tilde{a}, \tilde{a}_n}$ and $T_{\tilde{a}, \tilde{a}_n}$ is the generator of $H_1(A_{\tilde{a}, \tilde{a}_n}, \mathbb{R})$ defined by the oriented boundary of small disk centered at \tilde{a} . Choose a connected component $\hat{\gamma}^-$ of $\hat{\pi}_{\tilde{a}, \tilde{a}_n}^{-1}(\tilde{\gamma}^-)$ and endow $\hat{\gamma}^-$ with an orientation from the lower end to the upper end. The arc $\hat{F}_{\tilde{a}, \tilde{a}_n}^q(\hat{\gamma}^-)$ does not meet any connected component of $\hat{\pi}_{\tilde{a}, \tilde{a}_n}^{-1}(\tilde{\gamma}^+)$ and thus meets at most a translated $T_{\tilde{a}, \tilde{a}_n}^k(\hat{\gamma}^-)$. As $\hat{F}_{\tilde{a}, \tilde{a}_n}$ has a fixed point (the lift $\hat{\infty}$ of ∞), the arc $\hat{F}_{\tilde{a}, \tilde{a}_n}^q(\hat{\gamma}^-)$ can not be on the right of $T_{\tilde{a}, \tilde{a}_n}(\hat{\gamma}^-)$ (otherwise, $\hat{F}_{\tilde{a}, \tilde{a}_n}^q$ has no fixed point). Therefore, it is on the left of the arc $T_{\tilde{a}, \tilde{a}_n}^2(\hat{\gamma}^-)$. For the same reason, it is on the right of the arc $T_{\tilde{a}, \tilde{a}_n}^{-2}(\hat{\gamma}^-)$. As $\hat{F}_{\tilde{a}, \tilde{a}_n}$ and $T_{\tilde{a}, \tilde{a}_n}$ commute, it implies that the arc $\hat{F}_{\tilde{a}, \tilde{a}_n}^q(T(\hat{\gamma}^-))$ is on the left of $T_{\tilde{a}, \tilde{a}_n}^3(\hat{\gamma}^-)$ and on the right of $T_{\tilde{a}, \tilde{a}_n}^{-1}(\hat{\gamma}^-)$. Consider a point $\tilde{z} \in \text{Rec}^+(\tilde{F}) \setminus \{\tilde{a}, \tilde{a}_n\}$ such that the rotation number $\rho_{A_{\tilde{a}, \tilde{a}_n}, \tilde{F}_{\tilde{a}, \tilde{a}_n}}(\tilde{z})$ is well defined. There exists a unique lift \hat{z} of z that is in the region between $\hat{\gamma}^-$ and $T_{\tilde{a}, \tilde{a}_n}(\hat{\gamma}^-)$. By induction, we deduce that the point $\hat{F}_{\tilde{a}, \tilde{a}_n}^{qm}(\hat{z})$ is in the region between $T_{\tilde{a}, \tilde{a}_n}^{-2m}(\hat{\gamma}^-)$ and $T_{\tilde{a}, \tilde{a}_n}^{3m}(\hat{\gamma}^-)$ for all $m \geq 1$. By the definition of the rotation number (see Section 1.3.3), we have $|\rho_{A_{\tilde{a}, \tilde{a}_n}, \tilde{F}_{\tilde{a}, \tilde{a}_n}}(\tilde{z})| \leq 3/q$. As q can be choose arbitrarily large, we have

$$(1.8) \quad \lim_{n \rightarrow +\infty} \rho_{A_{\tilde{a}, \tilde{a}_n}, \tilde{F}_{\tilde{a}, \tilde{a}_n}}(\tilde{z}) = 0.$$

In particular, we have

$$\lim_{n \rightarrow +\infty} \rho_{A_{\tilde{a}_n}, \tilde{F}_{\tilde{a}_n}}(\tilde{b}_n) = 0,$$

which conflicts with the limit 1.7. We have completed the proof. \square

2. DISK CHAINS AND FRANKS' LEMMA

In this section, we will recall some classical results of plane or annulus homeomorphism, and extend some results of Franks so that we can use them in Section 5.

2.1. Disk Chain. Let M be a surface and h be a homeomorphism of M . A *disk chain* C of h is a family $\{D_i\}_{1 \leq i \leq n}$ of embedded open disks of M such that there are positive integers $\{m_i\}_{1 \leq i < n}$ satisfying

- (1) if $i \neq j$, then either $D_i = D_j$ or $D_i \cap D_j = \emptyset$;
- (2) for $1 \leq i < n$, $h^{m_i}(D_i) \cap D_{i+1} \neq \emptyset$.

We write $C = \{D_i\}_{1 \leq i \leq n}$ or $C = (\{D_i\}_{1 \leq i \leq n}, \{m_i\}_{1 \leq i \leq n})$ in a more detailed way. We define the *length* of the chain C to be the integer $l(C) = \sum_{i=1}^{n-1} m_i$. If $D_1 = D_n$ we say that $\{D_i\}_{1 \leq i \leq n}$ is a *periodic disk chain*.

A *free disk* of h is a disk in M which does not meet its image by h . A *free disk chain* of h is a disk chain $C = \{D_i\}_{1 \leq i \leq n}$ such that every D_i is a free disk of h .

2.2. Franks' Lemma.

Proposition 2.1 (Franks' Lemma [Fra88]). *Let $H : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an orientation preserving homeomorphism. If H possesses a periodic free disk chain, then H has at least one fixed point.*

Recall that $\mathbb{A} = \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ is the open annulus and $T : (x, y) \mapsto (x+1, y)$ is the generator of the covering transformation group. Let $h \in \text{Homeo}_*(\mathbb{A})$ and H be a lift of h to \mathbb{R}^2 . We say that $\tilde{D} \subset \mathbb{R}^2$ is a *positively returning disk* if all the following conditions hold:

- $T^k(\tilde{D}) \cap \tilde{D} = \emptyset$ for all $k \in \mathbb{Z} \setminus \{0\}$;
- $H(\tilde{D}) \cap \tilde{D} = \emptyset$;
- there exist $n > 0$ and $k > 0$ such that $H^n(\tilde{D}) \cap T^k(\tilde{D}) \neq \emptyset$.

A *negatively returning disk* is defined similarly but with $k < 0$.

If there exists an open disk that is both positively and negatively returning, then it is easy to construct a periodic free disk chain of H . Hence, by Franks' Lemma, we have the following result:

Corollary 2.2 ([Fra88]). *If H has an open disk $\tilde{D} \subset \mathbb{R}^2$ which is both positively and negatively returning, then there is a fixed point of H .*

Suppose that $D \subset \mathbb{A}$ is a free disk of h . We define the following set:

$$\text{Rot}_D(H) = \text{Conv}\{p/q \mid p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \setminus \{0\}, H^q(\tilde{D}) \cap T^p(\tilde{D}) \neq \emptyset\}$$

where $\text{Conv}(A)$ represents the convex hull of a set A and \tilde{D} is an arbitrary connected component of $\pi^{-1}(D)$. Observe that $\text{Rot}_D(H)$ does not depend on the choice of \tilde{D} . By Corollary 2.2, it holds:

Corollary 2.3. *For every $k \in \text{Rot}_D(H) \cap \mathbb{Z}$, there exists a point \tilde{z}_0 such that $H(\tilde{z}_0) = T^k(\tilde{z}_0)$.*

Proof. Choose any connected component \tilde{D} of $\pi^{-1}(D)$. We first suppose that there is an integer k such that $H^q(\tilde{D}) \cap T^{kq}(\tilde{D}) \neq \emptyset$. Note that this case covers the case where k is a boundary point of $\text{Rot}_D(H)$. Denote by H' the lift $H' = T^{-k} \circ H$ of h . We have $H'^q(\tilde{D}) \cap \tilde{D} \neq \emptyset$ and $H'(\tilde{D}) \cap \tilde{D} = \emptyset$ since D is free. According to Proposition 2.1, H' has a fixed point \tilde{z}_0 , that is, $H(\tilde{z}_0) = T^k(\tilde{z}_0)$.

We now suppose that there are two rational numbers p_i/q_i ($i = 1, 2$) and an integer k such that

- $p_1/q_1 < k < p_2/q_2$;
- $H^{q_1}(\tilde{D}) \cap T^{p_1}(\tilde{D}) \neq \emptyset$;
- $H^{q_2}(\tilde{D}) \cap T^{p_2}(\tilde{D}) \neq \emptyset$.

Considering the lift $H' = T^{-k} \circ H$, we have

$$H'^{q_1}(\tilde{D}) \cap T^{p_1 - q_1 k}(\tilde{D}) \neq \emptyset$$

and

$$H'^{q_2}(\tilde{D}) \cap T^{p_2 - q_2 k}(\tilde{D}) \neq \emptyset.$$

Therefore, \tilde{D} is a both positively and negatively returning disk of H' . By Corollary 2.2, H' has a fixed point. We have completed the proof. \square

Let $C = (\{D_i\}_{1 \leq i \leq n}, \{m_i\}_{1 \leq i < n})$ be a periodic disk chain of h in \mathbb{A} . A lift of C for H in \mathbb{R}^2 is a disk chain $\tilde{C} = (\{\tilde{D}_i\}_{1 \leq i \leq n}, \{m_i\}_{1 \leq i < n})$ in \mathbb{R}^2 such that $\pi(\tilde{D}_i) = D_i$ for every i .

We define the *width* of the lift \tilde{C} of C to be the integer $w(H; \tilde{C}) = k$ such that $\tilde{D}_n = T^k(\tilde{D}_1)$. For every $p \in \mathbb{Z}$, the disk chain $T^p(\tilde{C}) = (\{T^p(\tilde{D}_i)\}_{1 \leq i \leq n}, \{m_i\}_{1 \leq i < n})$ is also a lift of C for H since H commutes with T . The disk chain

$$T^p \cdot \tilde{C} = \{\tilde{D}_1, T^{pm_1}(\tilde{D}_2), T^{p(m_1+m_2)}(\tilde{D}_3), \dots, T^{pl(C)}(\tilde{D}_n)\}$$

is a lift of C for $T^p \circ H$. Therefore, the width of \tilde{C} satisfies

$$w(H; \tilde{C}) = w(H; T^p(\tilde{C}))$$

and

$$w(T^p \circ H; T^p \cdot \tilde{C}) = p l(C) + w(H; \tilde{C})$$

for every $p \in \mathbb{Z}$.

Using Corollary 2.2 and Corollary 2.3, we have the following lemma.

Lemma 2.4. *Let $h \in \text{Homeo}_*(\mathbb{A})$ and H be a lift of h to \mathbb{R}^2 . Suppose that $\text{Rot}_{\text{Fix}(h)}(H) \subset [-N, N]$ for some $N \in \mathbb{N}$, and that there is a disk D in \mathbb{A} satisfying $H(\tilde{D}) \cap T^k(\tilde{D}) \neq \emptyset$ if and only if $k = 0$, where \tilde{D} is any connected component of $\pi^{-1}(D)$, and that a periodic disk chain $C = (\{D_i\}_{1 \leq i \leq n}, \{m_i\}_{1 \leq i < n})$ of h such that*

- (1) $D_1 = D$;
- (2) if $D_i \neq D$ then D_i is a free disk of h .

Then, we have

- $|w(H; \tilde{C})| < (N + 1)l(C)$ for all lift \tilde{C} of C ;
- $\text{Rot}_{D_i}(H) \subset] - (N + 1), N + 1[$ if $D_i \neq D$.

Proof. Obviously, $C' = (\{D, D\}, \{1\})$ is a periodic disk chain of h .

Fix a connected component \tilde{D} of $\pi^{-1}(D)$ and a lift $\tilde{C} = \{\tilde{D}_i\}_{1 \leq i \leq n}$ of C for H that satisfies $\tilde{D}_1 = \tilde{D}$. Define \mathcal{D} as the family of all connected components of $\pi^{-1}(D_i)$, $1 \leq i \leq n$.

Suppose first that $w(H; \tilde{C}) \geq 0$, consider the lift $H' = H \circ T^{-(N+1)}$, we have the following facts

- $\text{Fix}(H') = \emptyset$;
- $H'(\tilde{D}) \cap \tilde{D} = \emptyset$;
- there is a free disk chain \tilde{C}' in \mathcal{D} of length 1 from \tilde{D} to $T^{-(N+1)}(\tilde{D})$ for H' (indeed, this disk chain is a lift of C' for H');
- there is a free disk chain \tilde{C} in \mathcal{D} of length $l(C)$ from \tilde{D} to $T^{-(N+1)l(C)+w(H; \tilde{C})}(\tilde{D})$ for H' (indeed, this disk chain is a lift of C for H').

The first item follows from $\text{Rot}_{\text{Fix}(h)}(H) \subset [-N, N]$. The second and third items hold by the hypothesis of D . The last one follows from the hypothesis (1) and the property of $w(H; \tilde{C})$.

If $-(N+1)l(C) + w(H; \tilde{C}) = 0$, then \tilde{C} is a periodic free disk chain for H' . By Proposition 2.1, H' has a fixed point, which conflicts with the first item. If $r = -(N+1)l(C) + w(H; \tilde{C}) > 0$, then the disk chain

$$\tilde{C} \cup T^r(\tilde{C}) \cup \dots \cup T^{Nr}(\tilde{C}) \cup T^{(N+1)r}(\tilde{C}') \cup \dots \cup T^{N+1}(\tilde{C}')$$

is a periodic free disk chain for H' . By Proposition 2.1 again, H' has a fixed point, which still conflicts with the first item. Hence $w(H; \tilde{C}) < (N+1)l(C)$.

In the case where $w(H; \tilde{C}) < 0$, replacing $H' = H \circ T^{-(N+1)}$ by $H' = H \circ T^{N+1}$, similarly to the case $w(H; \tilde{C}) \geq 0$, we get $w(H; \tilde{C}) > -(N+1)l(C)$. The first conclusion is proven.

Fix a disk $D_i \neq D$ and $p/q \in \text{Rot}_{D_i}(H)$. For every $s \geq 1$, consider the following periodic disk chain of h

$$C_s = \{D_1, \dots, \underbrace{D_i, \dots, D_i}_{s+1}, \dots, D_n\}$$

with

$$\{m_1, \dots, m_{i-1}, \underbrace{q, \dots, q}_s, m_i, \dots, m_{n-1}\}$$

and its lift for H

$$\tilde{C}_s = \{\tilde{D}_1, \dots, \tilde{D}_i, T^p(\tilde{D}_i), \dots, T^{sp}(\tilde{D}_i), T^{sp}(\tilde{D}_{i+1}), \dots, T^{sp}(\tilde{D}_n)\}.$$

Then we have $l(C_s) = l(C) + sq$ and $w(H; \tilde{C}_s) = w(H; \tilde{C}) + sp$. By the first conclusion, we get $|w(H; \tilde{C}_s)| < (N+1)l(C_s)$. Letting s tend to $+\infty$, we get $|p/q| \leq N+1$. Moreover, if $p/q = N+1$ (resp. $p/q = -(N+1)$), according to Corollary 2.3, then there exists a fixed point of h with rotation number $N+1$ (resp. $-(N+1)$) for H , which conflicts with the hypothesis $\text{Rot}_{\text{Fix}(h)}(H) \subset [-N, N]$. Therefore $|p/q| < N+1$. We have completed the proof. \square

The following Theorem is due to Franks [Fra88] when \mathbb{A} is a closed annulus and h has no wandering point, and it was improved by Le Calvez [Lec05] to the case where \mathbb{A} is an open annulus and h satisfies the intersection property (see also [Wang14, Proposition 12]):

Theorem 2.5. *Let $h \in \text{Homeo}_*(\mathbb{A})$ and H be a lift of h to \mathbb{R}^2 . Suppose that there exist two positively recurrent points of rotation numbers ν^- and ν^+ (eventually equal to $\pm\infty$)*

with $\nu^- < \nu^+$, and suppose that h satisfies the following intersection property: any simple closed curve of \mathbb{A} which is not null-homotopic meets its image by h . Then for any rational number $p/q \in]\nu^-, \nu^+[$ written in an irreducible way, there exists a periodic point of period q whose rotation number is p/q .

3. SYMPLECTIC ACTION

The action is a classical object in symplectic geometry. We will first recall it in this Section. Then, we will explain how to generalize the action to a simple case where the time-one map F of I is a diffeomorphism, the set $\text{Fix}_{\text{Cont}, I}(F)$ of contractible fixed points is finite and unlinked (we will define what it means), and $\rho_{M, I}(\mu) = 0$ where $\mu \in \mathcal{M}(F)$. At the end of the section, our main theorem will be stated.

3.1. The classical action function. Let us recall what is the action function. In this section, we suppose that (M, ω) is a symplectic manifold (not necessarily closed).

3.1.1. Symplectic and Hamiltonian. A diffeomorphism $F : M \rightarrow M$ is called *symplectic* if it preserves the form ω . Symplectic diffeomorphisms form a group denoted by $\text{Symp}(M, \omega)$. Let $\text{Symp}_*(M, \omega)$ denote the path-connected component of the Id_M in $\text{Symp}(M, \omega)$.

Consider a smooth isotopy $I = (F_t)_{t \in [0, 1]}$ in $\text{Symp}_*(M, \omega)$ with $F_0 = \text{Id}_M$ and $F_1 = F$. Let ξ_t be the corresponding time-dependent vector field on M :

$$\frac{d}{dt}F_t(x) = \xi_t(F_t(x)) \quad \text{for all } x \in M, \quad t \in [0, 1].$$

Since the Lie derivative $L_{\xi_t}\omega$ vanishes, we get that the 1-forms $\lambda_t = -i_{\xi_t}\omega$ are closed. Write $[\lambda_t]$ for the cohomology class of λ_t . The quantity

$$\text{Flux}(I) = \int_0^1 [\lambda_t] dt \in H^1(M, \mathbb{R}),$$

is called the *flux* of the isotopy I . It is well known that $\text{Flux}(I)$ does not change under a homotopy of the path I with fixed end points (see [MS95, Chapter 10]).

An isotopy I is called *Hamiltonian* if the 1-forms λ_t are exact for all t . In this case there exists a smooth function $H : [0, 1] \times M \rightarrow \mathbb{R}$ so that $\lambda_t = dH_t$, where $H_t(x)$ stands for $H(t, x)$. The function H is called the Hamiltonian function generating the flow I . Note that H_t is defined uniquely up to an additive time-dependent constant.

A symplectic diffeomorphism $F : M \rightarrow M$ is called *Hamiltonian* if there exists a Hamiltonian isotopy $I = (F_t)_{t \in [0, 1]}$ with $F_0 = \text{Id}_M$ and $F_1 = F$. Hamiltonian diffeomorphisms form a group denoted by $\text{Ham}(M, \omega)$. The following theorem characterizes the relation between flux and Hamiltonian diffeomorphisms (see [MS95, Theorem 10.12]).

Theorem 3.1. *Let $F \in \text{Symp}_*(M, \omega)$. Then F is Hamiltonian if and only if there exists an isotopy $I = (F_t)_{t \in [0, 1]}$ in $\text{Symp}_*(M, \omega)$ such that $F_0 = \text{Id}_M$, $F_1 = F$ and $\text{Flux}(I) = 0$. In that case, I is isotopic with fixed endpoints to a Hamiltonian isotopy.*

Suppose that (M, ω) is a closed symplectic surface and $I = (F_t)_{t \in [0, 1]}$ is a smooth isotopy in $\text{Symp}_*(M, \omega)$. Let denote by μ the measure induced by ω . We have the following relation between the $\text{Flux}(I)$ and $\rho_{M, I}(\mu)$ (see [FH03, Proposition 2.11]): for any smooth loop σ on M , we have

$$\langle \text{Flux}(I), [\sigma]_M \rangle = \rho_{M, I}(\mu) \wedge [\sigma]_M.$$

Hence, I is Hamiltonian if and only if $\rho_{M, I}(\mu) = 0$.

3.1.2. *Action function and action difference.* In this section, we suppose that (M, ω) is a symplectic manifold with $\pi_2(M) = 0$ (for example, a closed oriented surface with genus $g \geq 1$).

Let $I = (F_t)_{t \in [0,1]}$ be a Hamiltonian isotopy on M with $F_0 = \text{Id}_M$ and $F_1 = F$. Suppose that the function H is the Hamiltonian function generating the flow I .

Let x be a contractible fixed point of F . Take any immersed disk $D_x \subset M$ with $\partial D_x = I(x)$, and define the *action function* (or *action* for short)

$$\mathcal{A}_H(x) = \int_{D_x} \omega - \int_0^1 H_t(F_t(x)) dt.$$

The definition is well defined, that is $\mathcal{A}_H(x)$ does not depend on the choice of D_x . It is sufficient to prove the integral $\int_{D_x} \omega$ does not depend on the choice of D_x . Indeed, let D'_x be another choice, the 2-chain $\Pi = D_x - D'_x$ represents an immersed 2-sphere in M , and hence $\int_{\Pi} \omega = 0$ since $\pi_2(M) = 0$. Hence the claim follows.

Given two contractible fixed points x and y of F , take a path $\gamma : [0,1] \rightarrow M$ with $\gamma(0) = x$ and $\gamma(1) = y$. Choose two immersed disks D_x and D_y so that $\partial D_x = I(x)$ and $\partial D_y = I(y)$. Let us define $\Delta : [0,1] \times [0,1] \rightarrow M$ by $\Delta(t, s) = F_t(\gamma(s))$ where we assume that the boundary of the square $[0,1] \times [0,1]$ is oriented counter-clockwise and observe that $\partial\Delta = -\gamma + F\gamma - I(y) + I(x)$. So $F\gamma - \gamma = \partial\Delta + \partial D_y - \partial D_x$ is a 1-cycle and is the boundary of Σ where Σ is a 2-chain.

Define the *action difference* for x and y :

$$(3.1) \quad \delta(F; x, y) = \int_{\Sigma} \omega.$$

Since $\pi_2(M) = 0$, it does not depend on the choice of Σ , and hence not on D_x and D_y . Let us prove that it does not depend on the choice of γ .

Denote by ξ_t the vector field of the flow F_t . Then

$$\begin{aligned} \Delta^* \omega &= \omega \left(\xi_t(F_t \gamma(s)), \frac{\partial}{\partial s} F_t \gamma(s) \right) dt \wedge ds \\ &= -dH_t \left(\frac{\partial}{\partial s} F_t \gamma(s) \right) dt \wedge ds. \end{aligned}$$

Hence,

$$\begin{aligned} \int_{\Delta} \omega &= \int_{[0,1]^2} \Delta^* \omega = - \int_0^1 dt \int_0^1 dH_t \left(\frac{\partial}{\partial s} F_t \gamma(s) \right) ds \\ &= \int_0^1 H_t(F_t(x)) dt - \int_0^1 H_t(F_t(y)) dt. \end{aligned}$$

Finally, we have

$$(3.2) \quad \delta(F; x, y) = \int_{\Sigma} \omega = \int_{\Delta} \omega + \int_{D_y} \omega - \int_{D_x} \omega = \mathcal{A}_H(y) - \mathcal{A}_H(x).$$

Equation 3.2 shows that the action difference does not depend on the choice of γ , we have completed our claim. Moreover, we also give a relation between the action difference and the action function.

3.1.3. The action function and action difference on the universal covering space. When $I = (F_t)_{t \in [0,1]} \subset \text{Symp}_*(M, \omega) \setminus \text{Ham}(M, \omega)$, the action function (see Definition 3.1.2) is not meaningful. However, observing that the universal cover \widetilde{M} of M is simply connected, the lifted identity isotopy $\widetilde{I} = (\widetilde{F}_t)_{t \in [0,1]} \subset \text{Symp}_*(\widetilde{M}, \widetilde{\omega})$ of I to \widetilde{M} where $\widetilde{\omega}$ is the lift of the symplectic structure ω to \widetilde{M} is automatically Hamiltonian since $H^1(\widetilde{M}, \mathbb{R}) = 0$ (see Theorem 3.1). Let \widetilde{H} be the Hamiltonian function generating the flow \widetilde{I} . As before, we can define the action function $\mathcal{A}_{\widetilde{H}}(\widetilde{x})$ for any fixed point \widetilde{x} of $\widetilde{F} = \widetilde{F}_1$ and the action difference $\delta(\widetilde{F}; \widetilde{x}, \widetilde{y})$ for any two distinct fixed points \widetilde{x} and \widetilde{y} of \widetilde{F} , and we have the relation $\delta(\widetilde{F}; \widetilde{x}, \widetilde{y}) = \mathcal{A}_{\widetilde{H}}(\widetilde{y}) - \mathcal{A}_{\widetilde{H}}(\widetilde{x})$.

Let us see what happens in the particular case where I is Hamiltonian. Suppose that H is the Hamiltonian function generating the flow I and \widetilde{H} is its lift to \widetilde{M} . For any contractible fixed point x of F and its lift \widetilde{x} , we have $\mathcal{A}_{\widetilde{H}}(\widetilde{x}) = \mathcal{A}_H(x)$ (see [Pol02, Theorem 2.1.C] and [FH03, Remark 2.7]). Hence, for any two distinct contractible fixed points x and y of F , and their lifts \widetilde{x} and \widetilde{y} , we have

$$(3.3) \quad \delta(\widetilde{F}; \widetilde{x}, \widetilde{y}) = \mathcal{A}_{\widetilde{H}}(\widetilde{y}) - \mathcal{A}_{\widetilde{H}}(\widetilde{x}) = \mathcal{A}_H(y) - \mathcal{A}_H(x).$$

3.2. A generalization of the action function in a simple case. The action difference of two contractible fixed points x, y of F equals to the algebraic area of any path γ connecting x and y along the isotopy I , that is, the area of the path γ along I swept out. By this observation, we would like to generalize such an object to the case where ω is replaced by a finite Borel measure μ and the Hamiltonian isotopy by an identity isotopy I with $\rho_{M,I}(\mu) = 0$.

There is a case where this can be done easily (see [Lec05]). Suppose that $I = (F_t)_{t \in [0,1]}$ is an identity isotopy of M , the time-one map F of I is a diffeomorphism, the set $\text{Fix}_{\text{Cont},I}(F)$ of contractible fixed points is finite and unlinked, that means that there exists an isotopy $I' = (F'_t)_{t \in [0,1]}$ homotopic to I that fixes every point of $\text{Fix}_{\text{Cont},I}(F)$, and the measure $\mu \in \mathcal{M}(F)$ satisfies $\rho_{M,I}(\mu) = 0$.

Let $N = M \setminus \text{Fix}_{\text{Cont},I}(F)$, by the method of blowing-up, we can naturally get a compactification \overline{N} of N if we replace each point $x \in \text{Fix}_{\text{Cont},I}(F)$ by S_x , the tangent unit circle at x . The diffeomorphism $F|_N$ can be extended to a homeomorphism \overline{F} on \overline{N} which is isotopic to identity and induces the natural action by the linear map $DF(x)$ on S_x . As μ does not charge any point of $\text{Fix}_{\text{Cont},I}(F)$, we can define a measure on \overline{N} which is invariant by \overline{F} , denoted also μ . Therefore, we can define the rotation vector in $H_1(\overline{N}, \mathbb{R})$. The inclusion $\iota : N \hookrightarrow \overline{N}$ induces an isomorphism $\iota_* : H_1(N, \mathbb{R}) \rightarrow H_1(\overline{N}, \mathbb{R})$. We denote by $\rho_{N,I}(\mu) \in H_1(N, \mathbb{R})$ the rotation vector transported by this isomorphism. Let γ be a simple path in N joining $a \in \text{Fix}_{\text{Cont},I}(F)$ and $b \in \text{Fix}_{\text{Cont},I}(F)$. We can define the algebraic intersection number $\gamma \wedge \rho_{N,I}(\mu)$. Remark here that $\gamma \wedge \rho_{N,I}(\mu)$ is independent on the chosen γ because the rotation vector $\rho_{M,I}(\mu) \in H_1(M, \mathbb{R})$ is zero. Moreover, we can write

$$\gamma \wedge \rho_{N,I}(\mu) = L(b) - L(a),$$

where $L : \text{Fix}_{\text{Cont},I}(F) \rightarrow \mathbb{R}$ is a function, defined up to an additive constant. We call that L is the *action function*. In the proof of Theorem 0.1, you will find the reason why we call it as action function.

3.3. Our main theorem. It is natural to ask if we can generalize the action to a more general case. Let us first analyze what has been done above. The key points of his

generalization are that F is a diffeomorphism of M and that there is another identity isotopy I' homotopic to I that fixes all contractible fixed points of F . The differentiability hypothesis prevents the dynamics to be too wild in a neighborhood of a contractible fixed point so that it provides some boundedness condition, which means one can compactify the sub-manifold $N = M \setminus \text{Fix}_{\text{Cont}, I}(F)$ by blowing-up. It seems to us that keeping the boundedness condition is necessary and that is why we define the boundedness properties in Section 1.5. However, in general case, there may not exist such an isotopy I' that fixes all contractible fixed points of F . How to deal with this obstacle? The section 3.1.3 reminds us that it will be a good idea if we consider the universal covering space \widetilde{M} . A key point is that we can always find an isotopy \widetilde{I}' from $\text{Id}_{\widetilde{M}}$ to \widetilde{F} that fixes any two fixed points of \widetilde{F} , where \widetilde{F} is the time-one map of the lifted identity isotopy \widetilde{I} of I to \widetilde{M} (see Corollary 1.3). It makes us able to define the action difference for every two fixed points of \widetilde{F} and generalize the classical action.

Theorem 0.1 *Let M be a closed oriented surface with genus $g \geq 1$ and F be the time-one map of an identity isotopy I on M . Suppose that $\mu \in \mathcal{M}(F)$ and $\rho_{M, I}(\mu) = 0$. In each of the following cases*

- $F \in \text{Diff}(M)$;
- I satisfies the WB-property and the measure μ has full support;
- I satisfies the WB-property and the measure μ is ergodic,

an action function can be defined which generalizes the classical case.

4. EXTENSION OF THE LINKING NUMBER

In this section, we will first extend the notion of linking number defined in Section 1.4.4, then state some elementary properties about it.

4.1. Extension of the linking number for a positively recurrent point.

Recall that F is the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$ on a closed oriented surface M of genus $g \geq 1$ and \widetilde{F} is the time-one map of the lifted identity isotopy $\widetilde{I} = (\widetilde{F}_t)_{t \in [0,1]}$ on the universal cover \widetilde{M} of M . For every distinct fixed points \widetilde{a} and \widetilde{b} of \widetilde{F} , by Lemma 1.2, we can choose an isotopy \widetilde{I}_1 from $\text{Id}_{\widetilde{M}}$ to \widetilde{F} that fixes \widetilde{a} and \widetilde{b} .

Let us fix $z \in \text{Rec}^+(F) \setminus \pi(\{\widetilde{a}, \widetilde{b}\})$ and consider an open disk $U \subset M \setminus \pi(\{\widetilde{a}, \widetilde{b}\})$ containing z . For every pair $(z', z'') \in U^2$, choose an oriented simple path $\gamma_{z', z''}$ in U from z' to z'' . Denote by $\widetilde{\Phi}$ the lift of the first return map Φ :

$$\begin{aligned} \widetilde{\Phi} : \pi^{-1}(\text{Rec}^+(F)) \cap \pi^{-1}(U) &\rightarrow \pi^{-1}(\text{Rec}^+(F)) \cap \pi^{-1}(U) \\ \widetilde{z} &\mapsto \widetilde{F}^{\tau(z)}(\widetilde{z}), \end{aligned}$$

where $z = \pi(\widetilde{z})$ and $\tau(z)$ is the first return time in U .

For any $\widetilde{z} \in \pi^{-1}(U)$, write $U_{\widetilde{z}}$ the connected component of $\pi^{-1}(U)$ that contains \widetilde{z} . For every $j \geq 1$, recall that $\tau_j(z) = \sum_{i=0}^{j-1} \tau(\Phi^i(z))$. For every $n \geq 1$, consider the following curves in \widetilde{M} :

$$\widetilde{\Gamma}_{\widetilde{I}_1, \widetilde{z}}^n = \widetilde{I}_1^{\tau_n(z)}(\widetilde{z}) \widetilde{\gamma}_{\widetilde{\Phi}^n(\widetilde{z}), \widetilde{z}_n},$$

where $\tilde{z}_n \in \pi^{-1}(\{z\}) \cap \tilde{U}_{\tilde{\Phi}^n(\tilde{z})}$, and $\tilde{\gamma}_{\tilde{\Phi}^n(\tilde{z}), \tilde{z}_n}$ is the lift of $\gamma_{\Phi^n(z), z}$ that is contained in $\tilde{U}_{\tilde{\Phi}^n(\tilde{z})}$. We can define the following infinite product (see Section 1.1):

$$\tilde{\Gamma}_{\tilde{I}_1, z}^n = \prod_{\pi(\tilde{z})=z} \tilde{\Gamma}_{\tilde{I}_1, \tilde{z}}^n.$$

In particular, when $z \in \text{Fix}(F)$, $\tilde{\Gamma}_{\tilde{I}_1, z}^1 = \prod_{\pi(\tilde{z})=z} \tilde{I}_1(\tilde{z})$.

When $\tilde{U}_{\tilde{\Phi}^n(\tilde{z})} = \tilde{U}_{\tilde{z}}$, the curve $\tilde{\Gamma}_{\tilde{I}_1, \tilde{z}}^n$ is a loop and hence $\tilde{\Gamma}_{\tilde{I}_1, z}^n$ is an infinite family of loops, that will be called a *multi-loop*. When $\tilde{U}_{\tilde{\Phi}^n(\tilde{z})} \neq \tilde{U}_{\tilde{z}}$, the curve $\tilde{\Gamma}_{\tilde{I}_1, \tilde{z}}^n$ is a compact path and hence $\tilde{\Gamma}_{\tilde{I}_1, z}^n$ is an infinite family of paths (it can be seen as a family of proper paths, that means all of two ends of these paths going to ∞), that will be called a *multi-path*.

In the both cases, for every neighborhood \tilde{V} of ∞ , there are finitely many loops or paths $\tilde{\Gamma}_{\tilde{I}_1, \tilde{z}}^n$ that are not included in \tilde{V} . By adding the point ∞ at infinity, we get a multi-loop on the sphere $\mathbf{S} = \tilde{M} \sqcup \{\infty\}$.

In fact, $\tilde{\Gamma}_{\tilde{I}_1, z}^n$ can be seen as a multi-loop in the annulus $A_{\tilde{a}, \tilde{b}}$ with a finite homology. As a consequence, if $\tilde{\gamma}$ is a path from \tilde{a} to \tilde{b} , the intersection number $\tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}_1, z}^n$ is well defined and does not depend on $\tilde{\gamma}$. By Remark 1.4 and the properties of intersection number, the intersection number is also independent of the choice of the identity isotopy \tilde{I}_1 but depends on U . Moreover, observe that the path $(\prod_{i=0}^{n-1} \gamma_{\Phi^{n-i}(z), \Phi^{n-i-1}(z)}) (\gamma_{\Phi^n(z), z})^{-1}$ is a loop in U , we have

$$(4.1) \quad \tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}_1, z}^n = \tilde{\gamma} \wedge \prod_{j=0}^{n-1} \tilde{\Gamma}_{\tilde{I}_1, \Phi^j(z)}^1 = \sum_{j=0}^{n-1} \tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}_1, \Phi^j(z)}^1.$$

For $n \geq 1$, we can define the function

$$L_n : ((\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta}) \times (\text{Rec}^+(F) \cap U) \rightarrow \mathbb{Z},$$

$$L_n(\tilde{F}; \tilde{a}, \tilde{b}, z) = \tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}_1, z}^n = \sum_{j=0}^{n-1} L_1(\tilde{F}; \tilde{a}, \tilde{b}, \Phi^j(z))$$

where $U \subset M \setminus \pi(\{\tilde{a}, \tilde{b}\})$. The last equation follows from Equation 4.1. The function L_n depends on U but not on the choice of $\gamma_{\Phi^n(z), z}$.

Definition 4.1. Fix $z \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$. Let us say that the linking number $i(\tilde{F}; \tilde{a}, \tilde{b}, z) \in \mathbb{R}$ is defined, if

$$\lim_{k \rightarrow +\infty} \frac{L_{n_k}(\tilde{F}; \tilde{a}, \tilde{b}, z)}{\tau_{n_k}(z)} = i(\tilde{F}; \tilde{a}, \tilde{b}, z)$$

for any subsequence $\{\Phi^{n_k}(z)\}_{k \geq 1}$ of $\{\Phi^n(z)\}_{n \geq 1}$ which converges to z .

Note here that the linking number $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ does not depend on U since if U and U' are open disks containing z , there exists a disk containing z that is contained in $U \cap U'$. In particular, when $z \in \text{Fix}(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$, the linking number $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ always exists and is equal to $L_1(\tilde{F}; \tilde{a}, \tilde{b}, z)$.

Remark 4.2. When $z \in \text{Rec}^+(F) \setminus \text{Fix}_{\text{Cont}, I}(F)$, the linking number $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ depends on the choice of \tilde{a} and \tilde{b} if it exists. Indeed, consider the following smooth identity isotopy on \mathbb{R}^2 : $\tilde{I} = (\tilde{F}_t)_{t \in [0,1]} : (x, y) \mapsto (x, y + t \sin(2\pi x))$. It induces an identity smooth isotopy $I = (F_t)_{t \in [0,1]}$ on $\mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. Obviously $\text{Fix}(\tilde{F}) = \{(x, y) \mid x = k, x = k + 1/2, k \in \mathbb{Z}\}$ and $z = (1/4, 0) \in \mathbb{T}^2$ is a fixed point of F but not a contractible fixed point of F . Let $\tilde{a}_k = (k, 1/2) \in \mathbb{R}^2$ where $k \in \mathbb{Z}$. It is easy to see that $i(\tilde{F}; \tilde{a}_0, \tilde{a}_k, z) = k$ and $\pi(\tilde{a}_k) = \pi(\tilde{a}_{k'})$ where $k, k' \in \mathbb{Z}$.

4.2. Some elementary properties of the linking number.

For any $q \geq 1$, F^q is the time-one map of the identity isotopy I^q on M (see Formula 1.1). We know that a positively recurrent point of F is also a positively recurrent point of F^q , so we can define the linking number $i(\tilde{F}^q; \tilde{a}, \tilde{b}, z)$.

Proposition 4.3. *If $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ exists, then $i(\tilde{F}^q; \tilde{a}, \tilde{b}, z)$ exists for every $q \geq 1$ and $i(\tilde{F}^q; \tilde{a}, \tilde{b}, z) = qi(\tilde{F}; \tilde{a}, \tilde{b}, z)$.*

Proof. Let $\tilde{\gamma}$ be any simple path from \tilde{a} to \tilde{b} and \tilde{I}_1 be an isotopy that fixes \tilde{a} and \tilde{b} . We suppose that $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ exists. Let U be an open disk containing z . For every $q \geq 1$, write respectively $\tau'(z)$ and $\Phi'(z)$ for the first return time and the first return map of F^q in this proof. Recall that

$$\tau'_n(z) = \sum_{i=0}^{n-1} \tau'(\Phi'^i(z))$$

and

$$\tilde{\Gamma}_{\tilde{I}_1^q, \tilde{z}}^n = \tilde{I}_1^{q\tau'_n(z)}(\tilde{z}) \tilde{\gamma}_{\tilde{\Phi}^m(\tilde{z}), \tilde{z}_n}, \quad \tilde{\Gamma}_{\tilde{I}_1^q, z}^n = \prod_{\pi(\tilde{z})=z} \tilde{\Gamma}_{\tilde{I}_1^q, \tilde{z}}^n$$

where $\tilde{\Phi}'$ is the lift of Φ' to $\pi^{-1}(U)$, $\tilde{z}_n \in \pi^{-1}(\{z\}) \cap \tilde{U}_{\tilde{\Phi}^m(\tilde{z})}$ and $\tilde{\gamma}_{\tilde{\Phi}^m(\tilde{z}), \tilde{z}_n}$ is the lift of $\gamma_{\Phi^m(z), z}$ that is in $\tilde{U}_{\tilde{\Phi}^m(\tilde{z})}$.

We suppose that the subsequence $\{\Phi'^{n_k}(z)\}_{k \geq 1}$ converges to z . For every k , there is $n'_k \in \mathbb{N}$ such that $\tau'_{n'_k}(z) = q\tau'_{n_k}(z)$. By Definition 4.1, for any subsequence $\{\Phi'^{n_k}(z)\}_{k \geq 1}$ which converges to z , we have

$$\begin{aligned} & \lim_{k \rightarrow +\infty} \frac{L_{n_k}(\tilde{F}^q; \tilde{a}, \tilde{b}, z)}{\tau'_{n_k}(z)} \\ &= \lim_{k \rightarrow +\infty} \frac{\tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}_1^q, z}^{n_k}}{\tau'_{n_k}(z)} \\ &= q \cdot \lim_{k \rightarrow +\infty} \frac{\tilde{\gamma} \wedge \prod_{\pi(\tilde{z})=z} \tilde{I}_1^{q\tau'_{n_k}(z)}(\tilde{z}) \tilde{\gamma}_{\tilde{\Phi}^m(\tilde{z}), \tilde{z}_{n_k}}}{q\tau'_{n_k}(z)} \\ &= q \cdot \lim_{k \rightarrow +\infty} \frac{L_{n'_k}(\tilde{F}; \tilde{a}, \tilde{b}, z)}{\tau'_{n'_k}(z)} \\ &= qi(\tilde{F}; \tilde{a}, \tilde{b}, z). \end{aligned}$$

□

Let H be an orientation preserving homeomorphism of M and \tilde{H} be a lift of H to \tilde{M} . Consider the time-one map $H \circ F \circ H^{-1}$ of the isotopy $I' = H \circ I \circ H^{-1}$ and write the time-one map of the identity isotopy \tilde{I}' as $\tilde{H} \circ \tilde{F} \circ \tilde{H}^{-1}$, where \tilde{I}' is the lift of I' to \tilde{M} .

Proposition 4.4. *For every distinct fixed points \tilde{a}, \tilde{b} of \tilde{F} and every $z \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$, we have $L_n(\tilde{H} \circ \tilde{F} \circ \tilde{H}^{-1}; \tilde{H}(\tilde{a}), \tilde{H}(\tilde{b}), H(z)) = L_n(\tilde{F}; \tilde{a}, \tilde{b}, z)$ for every n . If $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ exists, then $i(\tilde{H} \circ \tilde{F} \circ \tilde{H}^{-1}; \tilde{H}(\tilde{a}), \tilde{H}(\tilde{b}), H(z))$ also exists and*

$$i(\tilde{H} \circ \tilde{F} \circ \tilde{H}^{-1}; \tilde{H}(\tilde{a}), \tilde{H}(\tilde{b}), H(z)) = i(\tilde{F}; \tilde{a}, \tilde{b}, z).$$

In particular, $i(\tilde{F}; \alpha(\tilde{a}), \alpha(\tilde{b}), z) = i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ for all $\alpha \in G$.

Proof. Let $\tilde{\gamma}$ be any simple path from \tilde{a} to \tilde{b} . Observe that the isotopy $\tilde{I}'_1 = \tilde{H} \circ \tilde{I}_1 \circ \tilde{H}^{-1}$ fixes $\tilde{H}(\tilde{a})$ and $\tilde{H}(\tilde{b})$, $\tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}_1, \tilde{z}}^n = \tilde{H}(\tilde{\gamma}) \wedge \tilde{\Gamma}_{\tilde{I}'_1, \tilde{H}(\tilde{z})}^n$ for every n . Hence

$$L_n(\tilde{H} \circ \tilde{F} \circ \tilde{H}^{-1}; \tilde{H}(\tilde{a}), \tilde{H}(\tilde{b}), H(z)) = L_n(\tilde{F}; \tilde{a}, \tilde{b}, z).$$

The proposition follows from Definition 4.1. \square

Proposition 4.5. *For every distinct fixed points \tilde{a}, \tilde{b} and \tilde{c} of \tilde{F} , and every $z \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}, \tilde{b}, \tilde{c}\})$, we have $L_n(\tilde{F}; \tilde{a}, \tilde{b}, z) + L_n(\tilde{F}; \tilde{b}, \tilde{c}, z) + L_n(\tilde{F}; \tilde{c}, \tilde{a}, z) = 0$ for all n . Moreover, if two among the three linking numbers $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$, $i(\tilde{F}; \tilde{b}, \tilde{c}, z)$ and $i(\tilde{F}; \tilde{c}, \tilde{a}, z)$ exist, then the last one also exists and we have*

$$i(\tilde{F}; \tilde{a}, \tilde{b}, z) + i(\tilde{F}; \tilde{b}, \tilde{c}, z) + i(\tilde{F}; \tilde{c}, \tilde{a}, z) = 0.$$

Before proving Proposition 4.5, we introduce some notations and recall some results of annulus homeomorphism.

If $\{\gamma_i\}_{1 \leq i \leq k}$ and $\{\gamma'_j\}_{1 \leq j \leq k'}$ are two finite families of loops or compact paths in $\mathbf{S} = \tilde{M} \cup \{\infty\}$ such that $\prod_{i=1}^k \gamma_i$ and $\prod_{j=1}^{k'} \gamma'_j$ are well defined (in the concatenation sense, see Section 1.1) and the algebraic intersection number $\left(\prod_{i=1}^k \gamma_i\right) \wedge \left(\prod_{j=1}^{k'} \gamma'_j\right)$ is well defined (see Section 1.2), then we formally write

$$\left(\prod_{i=1}^k \gamma_i\right) \wedge \left(\prod_{j=1}^{k'} \gamma'_j\right) = \sum_{i,j} \gamma_i \wedge \gamma'_j.$$

Recall that $\mathbb{A} = \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ is the open annulus and $T : (x, y) \mapsto (x+1, y)$ is the generator of the covering transformation group. If $I = (h_t)_{t \in [0,1]}$ with $h_0 = h_1 = \text{Id}_{\mathbb{A}}$ is a loop in $\text{Homeo}_*(\mathbb{A})$, write $[I]_1 \in \pi_1(\text{Homeo}_*(\mathbb{A}))$ for the homotopy class of I . Recall that $\pi_1(\text{Homeo}_*(\mathbb{A})) \simeq \mathbb{Z}$. Therefore, we may write $\pi_1(\text{Homeo}_*(\mathbb{A})) = \bigcup_{k \in \mathbb{Z}} \mathcal{C}_k$ where \mathcal{C}_k is the class which satisfies that for every $[I]_1 \in \mathcal{C}_k$, any lift \tilde{I} of I to the universal covering space $\tilde{\mathbb{A}}$ satisfies $\tilde{h}_1(\tilde{z}) - \tilde{h}_0(\tilde{z}) = T^k(\tilde{z})$ for every $\tilde{z} \in \tilde{\mathbb{A}}$.

Proof of Proposition 4.5. Suppose that $\tilde{\gamma}_1, \tilde{\gamma}_2$ and $\tilde{\gamma}_3$ are oriented simple paths from \tilde{a} to \tilde{b} , \tilde{b} to \tilde{c} and \tilde{c} to \tilde{a} , respectively. We choose isotopies \tilde{I}_j ($j = 0, 1, 2, 3$) such that \tilde{I}_1 fixes \tilde{a}, \tilde{b} and ∞ , \tilde{I}_2 fixes \tilde{b}, \tilde{c} and ∞ , \tilde{I}_3 fixes \tilde{c}, \tilde{a} and ∞ , and \tilde{I}_0 fixes \tilde{a}, \tilde{b} and \tilde{c} .

For every $z \in M \setminus \pi(\{\tilde{a}, \tilde{b}, \tilde{c}\})$, every lift \tilde{z} of z , every $j \in \{1, 2, 3\}$ and every $n \geq 1$, the path $\tilde{I}_0^n(\tilde{z})(\tilde{I}_j^n(\tilde{z}))^{-1}$ is a loop where $(\tilde{I}_j^n(\tilde{z}))^{-1}$ is the inverse of the path $\tilde{I}_j^n(\tilde{z})$. We claim that

$$(4.2) \quad \tilde{\gamma}_j \wedge \left(\tilde{I}_0^n(\tilde{z})(\tilde{I}_j^n(\tilde{z}))^{-1}\right) = \tilde{\gamma}_j \wedge \tilde{I}_0^n(\tilde{z}) - \tilde{\gamma}_j \wedge \tilde{I}_j^n(\tilde{z}) = n \cdot (\tilde{\gamma}_j \wedge \tilde{I}_0(\infty)).$$

Indeed, let \mathbb{A}_j ($j = 1, 2, 3$) be respectively $\mathbf{S} \setminus \{\tilde{a}, \tilde{b}\}$, $\mathbf{S} \setminus \{\tilde{b}, \tilde{c}\}$ and $\mathbf{S} \setminus \{\tilde{c}, \tilde{a}\}$. For every $n \in \mathbb{N}$, considering the loops $\tilde{I}_j^{-n} \tilde{I}_0^n \subset \text{Homeo}_*(\mathbb{A}_j)$ (see Formula 1.1) where \tilde{I}_j^{-1} is the inverse of \tilde{I}_j , we have $[\tilde{I}_j^{-n} \tilde{I}_0^n]_1 \in \mathcal{C}_{n \cdot (\tilde{\gamma}_j \wedge \tilde{I}_0(\infty))}^j$ ($j = 1, 2, 3$) where \mathcal{C}_k^j is a class in $\pi_1(\text{Homeo}_*(\mathbb{A}_j))$. Observing that $(\tilde{I}_j^{-n} \tilde{I}_0^n)(\tilde{z}) = \tilde{I}_0^n(\tilde{z})(\tilde{I}_j^n(\tilde{z}))^{-1}$, the claim (4.2) follows.

In the case where $z \in \text{Fix}(F) \setminus \pi(\{\tilde{a}, \tilde{b}, \tilde{c}\})$, for every lift \tilde{z} of z , we have

$$\tilde{\gamma}_j \wedge \tilde{I}_0(\tilde{z}) - \tilde{\gamma}_j \wedge \tilde{I}_j(\tilde{z}) = \tilde{\gamma}_j \wedge \tilde{I}_0(\infty) \quad (j = 1, 2, 3).$$

Write C_z for the set of points $\tilde{z} \in \pi^{-1}(\{z\})$ such that $\tilde{I}_j(\tilde{z}) \cap \bigcup_{j'=1}^3 \tilde{\gamma}_{j'} \neq \emptyset$ for every j .

As all \tilde{I}_j fix ∞ , we know that C_z is finite.

Recall that

$$i(\tilde{F}; \tilde{a}, \tilde{b}, z) = \tilde{\gamma}_1 \wedge \tilde{\Gamma}_{\tilde{I}_1, z}^1, \quad i(\tilde{F}; \tilde{b}, \tilde{c}, z) = \tilde{\gamma}_2 \wedge \tilde{\Gamma}_{\tilde{I}_2, z}^1 \quad \text{and} \quad i(\tilde{F}; \tilde{c}, \tilde{a}, z) = \tilde{\gamma}_3 \wedge \tilde{\Gamma}_{\tilde{I}_3, z}^1$$

where

$$\tilde{\Gamma}_{\tilde{I}_j, z}^1 = \prod_{\pi(\tilde{z})=z} \tilde{I}_j(\tilde{z}) \quad (j = 1, 2, 3).$$

Observe that

$$\sum_{j=1}^3 \sum_{\tilde{z} \in C_z} \tilde{\gamma}_j \wedge \tilde{I}_0(\tilde{z}) = \sum_{\tilde{z} \in C_z} \sum_{j=1}^3 \tilde{\gamma}_j \wedge \tilde{I}_0(\tilde{z}) = 0, \quad \sum_{j=1}^3 \tilde{\gamma}_j \wedge \tilde{I}_0(\infty) = 0$$

and

$$\tilde{\gamma}_j \wedge \tilde{\Gamma}_{\tilde{I}_j, z}^1 = \tilde{\gamma}_j \wedge \prod_{\pi(\tilde{z})=z} \tilde{I}_j(\tilde{z}) = \sum_{\tilde{z} \in C_z} \tilde{\gamma}_j \wedge \tilde{I}_j(\tilde{z}) \quad (j = 1, 2, 3).$$

We get

$$\begin{aligned} i(\tilde{F}; \tilde{a}, \tilde{b}, z) + i(\tilde{F}; \tilde{b}, \tilde{c}, z) + i(\tilde{F}; \tilde{c}, \tilde{a}, z) &= \sum_{j=1}^3 \left(\tilde{\gamma}_j \wedge \tilde{\Gamma}_{\tilde{I}_j, z}^1 \right) \\ &= - \sum_{j=1}^3 \sum_{\tilde{z} \in C_z} \left(\tilde{\gamma}_j \wedge \tilde{I}_0(\tilde{z}) - \tilde{\gamma}_j \wedge \tilde{I}_j(\tilde{z}) \right) \\ &= - \sum_{\tilde{z} \in C_z} \sum_{j=1}^3 \tilde{\gamma}_j \wedge \tilde{I}_0(\infty) \\ &= 0. \end{aligned}$$

Hence we have proved the proposition in this case.

In the case where $z \in \text{Rec}^+(F) \setminus \text{Fix}(F)$, recall that

$$\tilde{\Gamma}_{\tilde{I}_j, \tilde{z}}^n = \tilde{I}_j^{\tau_n(z)}(\tilde{z}) \tilde{\gamma}_{\tilde{\Phi}^n(\tilde{z}), \tilde{z}_n} \quad (0 \leq j \leq 3),$$

where $\tilde{z}_n \in \pi^{-1}(\{z\}) \cap \tilde{U}_{\tilde{\Phi}^n(\tilde{z})}$ and $\tilde{\gamma}_{\tilde{\Phi}^n(\tilde{z}), \tilde{z}_n}$ is the lift of $\gamma_{\tilde{\Phi}^n(z), z}$ in $\tilde{U}_{\tilde{\Phi}^n(\tilde{z})}$. For every $1 \leq j \leq 3$, we have $\tilde{\Gamma}_{\tilde{I}_0, \tilde{z}}^n (\tilde{\Gamma}_{\tilde{I}_j, \tilde{z}}^n)^{-1}$ is a loop where $(\tilde{\Gamma}_{\tilde{I}_j, \tilde{z}}^n)^{-1}$ is the inverse of the path $\tilde{\Gamma}_{\tilde{I}_j, \tilde{z}}^n$. Therefore, for every lift \tilde{z} of z and $n \geq 1$, we have

$$\tilde{\gamma}_j \wedge \left(\tilde{\Gamma}_{\tilde{I}_0, \tilde{z}}^n (\tilde{\Gamma}_{\tilde{I}_j, \tilde{z}}^n)^{-1} \right) = \tilde{\gamma}_j \wedge \tilde{\Gamma}_{\tilde{I}_0, \tilde{z}}^n - \tilde{\gamma}_j \wedge \tilde{\Gamma}_{\tilde{I}_j, \tilde{z}}^n = \tau_n(z) \cdot (\tilde{\gamma}_j \wedge \tilde{I}_0(\infty)) \quad (j = 1, 2, 3).$$

For every n , write C_z^n for the set of points $z \in \pi^{-1}(\{z\})$ such that $\tilde{\Gamma}_{\tilde{I}_j, \tilde{z}}^n \cap \bigcup_{j=1}^3 \tilde{\gamma}_j \neq \emptyset$.

Here again, we know that C_z^n is finite.

Recall that

$$L_n(\tilde{F}; \tilde{a}, \tilde{b}, z) = \tilde{\gamma}_1 \wedge \tilde{\Gamma}_{\tilde{I}_1, z}^n, \quad L_n(\tilde{F}; \tilde{b}, \tilde{c}, z) = \tilde{\gamma}_2 \wedge \tilde{\Gamma}_{\tilde{I}_1, z}^n \quad \text{and} \quad L_n(\tilde{F}; \tilde{c}, \tilde{a}, z) = \tilde{\gamma}_3 \wedge \tilde{\Gamma}_{\tilde{I}_1, z}^n$$

where

$$\tilde{\Gamma}_{\tilde{I}_1, z}^n = \prod_{\pi(\tilde{z})=z} \tilde{\Gamma}_{\tilde{I}_1, \tilde{z}}^n.$$

Similarly to the fixed point case, we have $L_n(\tilde{F}; \tilde{a}, \tilde{b}, z) + L_n(\tilde{F}; \tilde{b}, \tilde{c}, z) + L_n(\tilde{F}; \tilde{c}, \tilde{a}, z) = 0$. Hence for any subsequence $\{\Phi^{n_k}(z)\}_{k \geq 1}$ which converges to z , we get

$$\begin{aligned} (4.3) \quad & \frac{L_{n_k}(\tilde{F}; \tilde{a}, \tilde{b}, z) + L_{n_k}(\tilde{F}; \tilde{b}, \tilde{c}, z) + L_{n_k}(\tilde{F}; \tilde{c}, \tilde{a}, z)}{\tau_{n_k}(z)} \\ &= \frac{1}{\tau_{n_k}(z)} \sum_{j=1}^3 \left(\tilde{\gamma}_j \wedge \tilde{\Gamma}_{\tilde{I}_j, z}^{n_k} \right) \\ &= -\frac{1}{\tau_{n_k}(z)} \sum_{j=1}^3 \sum_{\tilde{z} \in C_z^{n_k}} \left(\tilde{\gamma}_j \wedge \tilde{\Gamma}_{\tilde{I}_0, \tilde{z}}^{n_k} - \tilde{\gamma}_j \wedge \tilde{\Gamma}_{\tilde{I}_j, \tilde{z}}^{n_k} \right) \\ &= -\sum_{\tilde{z} \in C_z^{n_k}} \sum_{j=1}^3 \tilde{\gamma}_j \wedge \tilde{I}_0(\infty) \\ &= 0. \end{aligned}$$

Letting $k \rightarrow +\infty$ in Equation 4.3, we have completed the proposition. \square

5. BOUNDEDNESS AND EXISTENCE OF THE LINKING NUMBER

This Section is divided into two parts. In the first part, we study the boundedness of the linking number when it exists. In the second part, we study the existence and boundedness of the linking number if the map F preserves a Borel finite measure on M .

5.1. Boundedness.

In this section, let \tilde{a} and \tilde{b} be two distinct fixed points of \tilde{F} . We suppose that I satisfies WB-property at \tilde{a} and \tilde{b} . By Lemma 1.7, there is a positive number $N_{\tilde{a}, \tilde{b}}$ such that $\text{Rot}_{\text{Fix}(\tilde{F}_{\tilde{a}, \tilde{b}})}(\tilde{F}_{\tilde{a}, \tilde{b}}) \subset [-N_{\tilde{a}, \tilde{b}}, N_{\tilde{a}, \tilde{b}}]$.

Fix an isotopy \tilde{I}_1 from $\text{Id}_{\tilde{M}}$ to \tilde{F} which fixes \tilde{a} and \tilde{b} . Let $\tilde{\gamma}$ be any oriented path in \tilde{M} from \tilde{a} to \tilde{b} . Fix an open disk \tilde{W} that contains ∞ and is disjoint from $\tilde{\gamma}$. We choose an open disk $\tilde{V} \subset \tilde{W}$ that contains ∞ such that for every $\tilde{z} \in \tilde{V}$, we have $\tilde{I}_1(\tilde{z}) \subset \tilde{W}$. Observe that if $\widehat{\infty}$ is a given lift of ∞ in $\widehat{A}_{\tilde{a}, \tilde{b}}$, if \widehat{W} (resp. \widehat{V}) is the connected component of $\pi^{-1}(\tilde{W})$ (resp. $\pi^{-1}(\tilde{V})$) that contains $\widehat{\infty}$, then we have $\widehat{F}_{\tilde{a}, \tilde{b}}(\widehat{V}) \subset \widehat{W}$, which implies that \widehat{V} is free for every other lift $\widehat{F}_{\tilde{a}, \tilde{b}} \circ T_{\tilde{a}, \tilde{b}}^k$, where $k \in \mathbb{Z} \setminus \{0\}$. Let A^c denote the complement of a set A . For every $z \in M \setminus \pi(\{\tilde{a}, \tilde{b}\})$, write $X_z = \pi^{-1}(\{z\}) \cap (\tilde{V} \cap \widehat{F}_{\tilde{a}, \tilde{b}}^{-1}(\widehat{V}))^c$. Observe that there exists $K_{\tilde{a}, \tilde{b}} \in \mathbb{N}$ such that $\#X_z \leq K_{\tilde{a}, \tilde{b}}$ for every $z \in M \setminus \pi(\{\tilde{a}, \tilde{b}\})$.

In the case where $z \in \text{Rec}^+(F) \setminus \text{Fix}(F)$, we choose an open disk U that contains z and is free for F . As the value $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ depends neither on $\tilde{\gamma}$ nor on U , we can always suppose that $\tilde{\gamma} \cap \pi^{-1}(U) = \emptyset$ by perturbing $\tilde{\gamma}$ a little and shrinking U if necessary. For every $n \geq 1$, write

$$X_z^n = \pi^{-1}(\{z, F(z), \dots, F^{\tau_n(z)-1}(z)\}) \cap (\tilde{V} \cap \tilde{F}_{\tilde{a}, \tilde{b}}^{-1}(\tilde{V}))^c.$$

Observe that $\#X_z^n \leq \tau_n(z)K_{\tilde{a}, \tilde{b}}$.

The following result is the main proposition of this section.

Proposition 5.1. *The following two statements hold:*

- If $z \in \text{Fix}(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$, we have $|i(\tilde{F}; \tilde{a}, \tilde{b}, z)| < K_{\tilde{a}, \tilde{b}}(N_{\tilde{a}, \tilde{b}} + 1)$.
- If $z \in \text{Rec}^+(F) \setminus \text{Fix}(F)$ and $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ is defined, then $|i(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq K_{\tilde{a}, \tilde{b}}K_U$, where $K_U \in \mathbb{N}$ depends only on U .

In order to prove Proposition 5.1, we consider two cases: the fixed point case and the non-fixed point case. The first case is more easy to deal with and the second case is a little more complicated, but the ideas are similar.

The fixed point case.

When $z \in \text{Fix}(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$, then $\tau(z) = 1$ and $i(\tilde{F}; \tilde{a}, \tilde{b}, z) = L_1(\tilde{F}; \tilde{a}, \tilde{b}, z)$, we have the following results.

Lemma 5.2. *If $z \in \text{Fix}_{\text{Cont}, I}(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$, then $|i(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq K_{\tilde{a}, \tilde{b}}N_{\tilde{a}, \tilde{b}}$.*

Proof. By Definition 4.1 and Lemma 1.5, we have

$$i(\tilde{F}; \tilde{a}, \tilde{b}, z) = \sum_{\pi(\tilde{z})=z} \rho_{A_{\tilde{a}, \tilde{b}}, \tilde{F}_{\tilde{a}, \tilde{b}}}(\tilde{z}) = \sum_{\tilde{z} \in X_z} \rho_{A_{\tilde{a}, \tilde{b}}, \tilde{F}_{\tilde{a}, \tilde{b}}}(\tilde{z}).$$

The lemma follows from the fact that $\#X_z \leq K_{\tilde{a}, \tilde{b}}$ and that $\text{Rot}_{\text{Fix}(\tilde{F}_{\tilde{a}, \tilde{b}})}(\tilde{F}_{\tilde{a}, \tilde{b}}) \subset [-N_{\tilde{a}, \tilde{b}}, N_{\tilde{a}, \tilde{b}}]$. \square

Lemma 5.3. *If $z \in \text{Fix}(F) \setminus \text{Fix}_{\text{Cont}, I}(F)$, then $|i(\tilde{F}; \tilde{a}, \tilde{b}, z)| < K_{\tilde{a}, \tilde{b}}(N_{\tilde{a}, \tilde{b}} + 1)$.*

Proof. We have

$$i(\tilde{F}; \tilde{a}, \tilde{b}, z) = \tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}_1, z}^1 = \sum_{\tilde{z} \in X_z} \tilde{\gamma} \wedge \tilde{I}_1(\tilde{z}).$$

Observe that if $\tilde{z} \in X_z$, then the trajectory of $\tilde{I}_1(\tilde{z})$ is not included in \tilde{V} . Therefore we can write the multi-path $\prod_{\tilde{z} \in X_z} \tilde{I}_1(\tilde{z})$ as finitely many sub-paths:

$$\prod_{\tilde{z} \in X_z} \tilde{I}_1(\tilde{z}) = \prod_{1 \leq i \leq P(z)} \tilde{\Gamma}_i(z),$$

where

$$\tilde{\Gamma}_i(z) = \prod_{0 \leq j < m^i(z)} \tilde{I}_1(\tilde{F}_{\tilde{a}, \tilde{b}}^j(\tilde{z}_i))$$

is a path with $\tilde{z}_i \in X_z \cap \tilde{V}$, $\tilde{F}_{\tilde{a}, \tilde{b}}^j(\tilde{z}_i) \in X_z \cap \tilde{V}^c$ for $1 \leq j < m^i$ and $\tilde{F}_{\tilde{a}, \tilde{b}}^{m^i}(\tilde{z}_i) \in \tilde{V}$. For every i , we get a periodic disk chain $C_i = (\{\tilde{V}, \tilde{V}\}, \{m^i\})$ whose length $l(C_i)$ is equal to m^i (see Section 2).

Obviously, $\sum_i m^i \leq K_{\tilde{a}, \tilde{b}}$. Let $k^i(z) = \tilde{\gamma} \wedge \tilde{\Gamma}_i$. We have $i(\tilde{F}; \tilde{a}, \tilde{b}, z) = \tilde{\gamma} \wedge \tilde{\Gamma}_{1,z}^1 = \sum_i k^i$. Therefore, to get the lemma, it is sufficient to prove that $|k^i| < m^i(N_{\tilde{a}, \tilde{b}} + 1)$.

For every i , the path $\tilde{\Gamma}_i$ is lifted to a path from a point $\tilde{z}_i \in \tilde{V}$ to $\tilde{F}_{\tilde{a}, \tilde{b}}^{m^i}(\tilde{z}_i) \in T_{\tilde{a}, \tilde{b}}^{k^i}(\tilde{V})$ and hence we get a lift $\tilde{C}_i = (\{\tilde{V}, T_{\tilde{a}, \tilde{b}}^{k^i}(\tilde{V})\}, \{m^i\})$ of C_i for $\tilde{F}_{\tilde{a}, \tilde{b}}$ with width $w(\tilde{F}_{\tilde{a}, \tilde{b}}; \tilde{C}_i) = k^i$. By the construction of \tilde{V} , replacing \mathbb{A} by $\tilde{A}_{\tilde{a}, \tilde{b}}$, h by $\tilde{F}_{\tilde{a}, \tilde{b}}$, H by $\tilde{F}_{\tilde{a}, \tilde{b}}$, D by \tilde{V} and C by C_i in Lemma 2.4, we get $|k^i| < m^i(N_{\tilde{a}, \tilde{b}} + 1)$. We have completed the proof. \square

The non-fixed point case.

Let $z \in \text{Rec}^+(F) \setminus \text{Fix}(F)$ and U be an open free disk for F that contains z . Recall that, for every lift \tilde{z} of z and every $n \geq 0$, there is a unique connected component $\tilde{U}_{\tilde{\Phi}^n(\tilde{z})}$ of $\pi^{-1}(U)$ such that $\tilde{\Phi}^n(\tilde{z}) \in \tilde{U}_{\tilde{\Phi}^n(\tilde{z})}$ and a unique $\alpha_{z,n} \in G$ such that $\tilde{U}_{\tilde{\Phi}^n(\tilde{z})} = \alpha_{z,n}(\tilde{U}_{\tilde{z}})$. For convenience, we define

$$\tilde{F}_{\tilde{a}, \tilde{b}}^*(\tilde{z}') = \begin{cases} \tilde{F}_{\tilde{a}, \tilde{b}}(\tilde{z}') & \text{if } \pi(\tilde{z}') \in \{z, \dots, F^{\tau_n(z)-2}(z)\}; \\ \alpha_{z,n}(\tilde{z}) & \text{if } \pi(\tilde{z}') = F^{\tau_n(z)-1}(z) \text{ and } \tilde{F}_{\tilde{a}, \tilde{b}}(\tilde{z}') \in \tilde{U}_{\alpha_{z,n}(\tilde{z})}. \end{cases}$$

and

$$\tilde{I}_1^*(\tilde{z}') = \begin{cases} \tilde{I}_1(\tilde{z}') & \text{if } \pi(\tilde{z}') \in \{z, \dots, F^{\tau_n(z)-2}(z)\}; \\ \tilde{I}_1(\tilde{z}') \tilde{\gamma}_{\tilde{F}_{\tilde{a}, \tilde{b}}(\tilde{z}'), \alpha_{z,n}(\tilde{z})} & \text{if } \pi(\tilde{z}') = F^{\tau_n(z)-1}(z) \text{ and } \tilde{F}_{\tilde{a}, \tilde{b}}(\tilde{z}') \in \tilde{U}_{\alpha_{z,n}(\tilde{z})}, \end{cases}$$

where $\tilde{\gamma}_{\tilde{F}_{\tilde{a}, \tilde{b}}(\tilde{z}'), \alpha_{z,n}(\tilde{z})}$ is the lift of $\gamma_{\tilde{\Phi}^n(z), z}$ that is in $\tilde{U}_{\alpha_{z,n}(\tilde{z})}$.

We have to consider two cases: $\alpha_{z,n} = e$ and $\alpha_{z,n} \neq e$. First, we consider the case where $\alpha_{z,n} \neq e$. We have the following lemma.

Lemma 5.4. *If $\alpha_{z,n} \neq e$, then $|L_n(\tilde{F}; \tilde{a}, \tilde{b}, z)| < \tau_n(z)K_{\tilde{a}, \tilde{b}}(N_{\tilde{a}, \tilde{b}} + 1)$.*

Proof. In this case, the curve $\tilde{\Gamma}_{1,z}^n$ is a multi-path in \tilde{M} . By the definition of $L_n(\tilde{F}; \tilde{a}, \tilde{b}, z)$, we have

$$L_n(\tilde{F}; \tilde{a}, \tilde{b}, z) = \tilde{\gamma} \wedge \tilde{\Gamma}_{1,z}^n = \sum_{\tilde{z}' \in X_z^n} \tilde{\gamma} \wedge \tilde{I}_1^*(\tilde{z}').$$

We can write the multi-path

$$(5.1) \quad \prod_{\tilde{z}' \in X_z^n} \tilde{I}_1^*(\tilde{z}') = \prod_{1 \leq i \leq P_n(z)} \tilde{\Gamma}_i^n(z),$$

where

$$(5.2) \quad \tilde{\Gamma}_i^n(z) = \prod_{0 \leq j < m_n^i(z)} \tilde{I}_1^*(\tilde{F}_{\tilde{a}, \tilde{b}}^{*j}(\tilde{z}_i))$$

is a path with $\tilde{z}_i \in X_z^n \cap \tilde{V}$, $\tilde{F}_{\tilde{a}, \tilde{b}}^{*j}(\tilde{z}_i) \in X_z^n \cap \tilde{V}^c$ for $1 \leq j < m_n^i$ and $\tilde{F}_{\tilde{a}, \tilde{b}}^{*m_n^i}(\tilde{z}_i) \in \tilde{V}$. Hence, for every i , we get a periodic disk chain C_i that satisfies the hypothesis of Lemma 2.4 with length m_n^i . When we lift the path $\tilde{\Gamma}_i^n$, we can get a lift of C_i for $\tilde{F}_{\tilde{a}, \tilde{b}}$ with width k_n^i .

Obviously, we have $\sum_i m_n^i < \tau_n K_{\tilde{a}, \tilde{b}}$. Let $k_n^i(z) = \tilde{\gamma} \wedge \tilde{\Gamma}_i^n$. Hence $L_n(\tilde{F}; \tilde{a}, \tilde{b}, z) = \sum_i k_n^i$. It is sufficient to prove that $|k_n^i| < m_n^i(N_{\tilde{a}, \tilde{b}} + 1)$.

Similarly to the proof of Lemma 5.3, replacing \mathbb{A} by $\tilde{A}_{\tilde{a},\tilde{b}}$, h by $\tilde{F}_{\tilde{a},\tilde{b}}$, H by $\hat{F}_{\tilde{a},\tilde{b}}$, D by \tilde{V} and C by C_i in Lemma 2.4, we get $|k_n^i| < m_n^i(N_{\tilde{a},\tilde{b}} + 1)$. This proves the first case. \square

As a consequence, we have the following proposition.

Proposition 5.5. *We suppose that $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ and $\rho_{M,I}(z)$ exist, then*

$$|i(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq K_{\tilde{a},\tilde{b}}(N_{\tilde{a},\tilde{b}} + 1) \quad \text{if } \rho_{M,I}(z) \neq 0.$$

Proof. If $z \in \text{Fix}(F)$ and $\rho_{M,I}(z) \neq 0$, then z is not a contractible fixed point and the conclusion follows from Lemma 5.3. Suppose now that $z \in \text{Rec}^+(F) \setminus \text{Fix}(F)$ and $U \subset M \setminus \text{Fix}(F)$ is a free open disk containing z . If $\rho_{M,I}(z) \neq 0$, then there exists a positive number N such that $\alpha_{z,n} \neq e$ when $n \geq N$ (refer to Section 1.3.2). In that case, the conclusion follows from Lemma 5.4. \square

Let us study the case where $\alpha_{z,n} = e$.

Lemma 5.6. *There exists a positive integer K_U which depends on U such that*

$$|L_n(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq \tau_n(z) K_{\tilde{a},\tilde{b}} K_U \quad \text{if } \alpha_{z,n} = e.$$

Before proving Lemma 5.6, we require the following lemma.

Lemma 5.7. *Let \tilde{U} be any connected component of $\pi^{-1}(U)$ in \tilde{V}^c . If*

$$\text{Rot}_{\tilde{U}}(\hat{F}_{\tilde{a},\tilde{b}}) \not\subset] - (N_{\tilde{a},\tilde{b}} + 1), N_{\tilde{a},\tilde{b}} + 1[,$$

then we have

- (1) $\alpha_{z',n} = e$ for all $z' \in \text{Rec}^+(F) \cap U$ and all $n \geq 1$;
- (2) $\bigcup_{k \geq 1} \tilde{F}^k(\pi^{-1}(\text{Rec}^+(F)) \cap \tilde{U}) \subset \tilde{V}^c$;
- (3) $\text{Rot}_{\tilde{U}}(\hat{F}_{\tilde{a},\tilde{b}}) \subset]l, l+1[$ for some integer l with $l \geq N_{\tilde{a},\tilde{b}} + 1$ or $l \leq -(N_{\tilde{a},\tilde{b}} + 2)$ where l depends on \tilde{U} .

Let us prove now Lemma 5.6 supposing Lemma 5.7 whose proof will be given later.

Proof of Lemma 5.6. As $\alpha_{z,n} = e$, the curve $\tilde{\Gamma}_{I_1,z}^n$ is a multi-loop in \tilde{M} . Let $p_n(\tilde{z}) = \tilde{\gamma} \wedge \tilde{\Gamma}_{I_1,\tilde{z}}^n$ where $\tilde{z} \in \pi^{-1}(z)$. Obviously, $p_n(\tilde{z})/\tau_n(z) \in \text{Rot}_{\tilde{U}_{\tilde{z}}}(\hat{F}_{\tilde{a},\tilde{b}})$.

Let us first analyze the possible cases that need to be considered in the proof. The set X_z^n maybe contain a “whole orbit” of some lift \tilde{z} of z , that means $\tilde{F}^j(\tilde{z}) \in X_z^n$ for all $0 \leq j < \tau_n(z)$, or a “partial orbit” of \tilde{z} . In the case where a “partial orbit” of \tilde{z} is contained in X_z^n , similarly to the proof of Lemma 5.3, we can get a periodic disk chain of $\tilde{F}_{\tilde{a},\tilde{b}}$ that satisfies the hypothesis of Lemma 2.4 and hence we can estimate the intersection number of $\tilde{\gamma}$ and the path on which the “partial orbit” of \tilde{z} lies. In the case where the “whole orbit” of \tilde{z} is contained in X_z^n , we can use Lemma 5.7 to get either $|p_n(\tilde{z})/\tau_n(z)| < N_{\tilde{a},\tilde{b}} + 1$, or $l < p_n(\tilde{z})/\tau_n(z) < l + 1$ where $l \in \mathbb{Z}$ depends on \tilde{U} and satisfies $l \geq N_{\tilde{a},\tilde{b}} + 1$ or $l \leq -(N_{\tilde{a},\tilde{b}} + 2)$. Finally, we only need to sum the intersection numbers of all the cases above.

Let us begin the rigorous proof. Write

$$S_z^n = \{\tilde{z} \in \pi^{-1}(z) \mid \tilde{F}^j(\tilde{z}) \in \tilde{V}^c \text{ for all } 0 \leq j < \tau_n(z)\}$$

and

$$Y_z^n = \{\tilde{F}^j(\tilde{z}) \mid \tilde{z} \in S_z^n, 0 \leq j < \tau_n(z)\}.$$

As before, we write

$$L_n(\tilde{F}; \tilde{a}, \tilde{b}, z) = \tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}_1, z}^n = \sum_{\tilde{z}' \in X_z^n} \tilde{\gamma} \wedge \tilde{I}_1^*(\tilde{z}').$$

We can write the multi-path as follows

$$(5.3) \quad \prod_{\tilde{z}' \in X_z^n} \tilde{I}_1^*(\tilde{z}') = \prod_{\tilde{z}' \in Y_z^n} \tilde{I}_1^*(\tilde{z}') \cdot \prod_{\tilde{z}' \in X_z^n \setminus Y_z^n} \tilde{I}_1^*(\tilde{z}') = \prod_{1 \leq i \leq P'_n(z)} \tilde{\Gamma}_i^n(z) \cdot \prod_{P'_n(z) < i \leq P_n(z)} \tilde{\Gamma}_i^n(z),$$

where

$$(5.4) \quad \tilde{\Gamma}_i^n(z) = \tilde{\Gamma}_{\tilde{I}_1, \tilde{z}_i}^n = \prod_{0 \leq j < m_n^i(z)} \tilde{I}_1^*(\tilde{F}_{\tilde{a}, \tilde{b}}^{*j}(\tilde{z}_i))$$

for $1 \leq i \leq P'_n$ with $\tilde{z}_i \in S_z^n$ and $m_n^i = \tau_n$; and

$$(5.5) \quad \tilde{\Gamma}_i^n(z) = \prod_{0 \leq j < m_n^i(z)} \tilde{I}_1^*(\tilde{F}_{\tilde{a}, \tilde{b}}^{*j}(\tilde{z}_i))$$

for $P'_n < i \leq P_n$ with $\tilde{z}_i \in X_z^n \cap \tilde{V}$, $\tilde{F}_{\tilde{a}, \tilde{b}}^{*j}(\tilde{z}_i) \in X_z^n \cap \tilde{V}^c$ for $1 \leq j < m_n^i$ and $\tilde{F}_{\tilde{a}, \tilde{b}}^{*m_n^i}(\tilde{z}_i) \in \tilde{V}$.

Obviously, $\sum_i m_n^i \leq \tau_n(z) K_{\tilde{a}, \tilde{b}}$. Let $k_n^i(z) = \tilde{\gamma} \wedge \tilde{\Gamma}_i^n$. Hence $L_n(\tilde{F}; \tilde{a}, \tilde{b}, z) = \sum_i k_n^i$. To prove Lemma 5.6, it is sufficient to prove that there exists a positive integer K_U which depends only on U such that $|k_n^i| \leq m_n^i K_U$.

When $1 \leq i \leq P'_n$, by Lemma 5.7 and the fact that $P'_n \leq K_{\tilde{a}, \tilde{b}}$, there exists a positive integer r that depends on U such that $\text{Rot}_{\tilde{U}_{\tilde{z}_i}}(\hat{F}_{\tilde{a}, \tilde{b}}) \subset [-r, r]$. Observing that $k_n^i = p_n(\tilde{z}_i) = \tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}_1, \tilde{z}_i}^n$, $m_n^i = \tau_n$, and $k_n^i/m_n^i = p_n(\tilde{z}_i)/\tau_n(z) \in \text{Rot}_{\tilde{U}_{\tilde{z}_i}}(\hat{F}_{\tilde{a}, \tilde{b}})$, we have $|k_n^i| \leq m_n^i r$.

When $P'_n < i \leq P_n$, similarly to the proof of Lemma 5.3, we can get $|k_n^i| < m_n^i (N_{\tilde{a}, \tilde{b}} + 1)$.

Let $K_U = \max\{N_{\tilde{a}, \tilde{b}} + 1, r\}$. We have $|k_n^i| \leq m_n^i K_U$ for every $1 \leq i \leq P_n$ and hence

$$|L_n(\tilde{F}; \tilde{a}, \tilde{b}, z)| = \left| \sum_i k_n^i \right| \leq \tau_n(z) K_{\tilde{a}, \tilde{b}} K_U.$$

□

Proof of Lemma 5.7. (1). Suppose that there is a point $z' \in \text{Rec}^+(F) \cap U$ and some $n_0 \geq 1$ such that $\alpha_{z', n_0} \neq e$. Let \tilde{z}' be the lift of z' that is in \tilde{U} . Similarly to the proof of Lemma 5.4, we can find a path

$$\tilde{\Gamma}_i^{n_0}(z') = \prod_{0 \leq j < m_{n_0}^i(z')} \tilde{I}_1^*(\tilde{F}_{\tilde{a}, \tilde{b}}^{*j}(\tilde{z}_i))$$

which satisfies $\tilde{z}_i \in X_{z'}^{n_0} \cap \tilde{V}$, $\tilde{F}_{\tilde{a}, \tilde{b}}^{*j}(\tilde{z}_i) \in X_{z'}^{n_0} \cap \tilde{V}^c$ for all $1 \leq j < m_{n_0}^i$, $\tilde{z}' = \tilde{F}_{\tilde{a}, \tilde{b}}^{*j_0}(\tilde{z}_i)$ for some $1 \leq j_0 < m_{n_0}^i$, and $\tilde{F}_{\tilde{a}, \tilde{b}}^{*m_{n_0}^i}(\tilde{z}_i) \in \tilde{V}$. Hence, we get a periodic disk chain C' that contains \tilde{U} as an element and satisfies the hypothesis of Lemma 2.4. Replacing \mathbb{A} by $\tilde{A}_{\tilde{a}, \tilde{b}}$, h by $\tilde{F}_{\tilde{a}, \tilde{b}}$, H by $\hat{F}_{\tilde{a}, \tilde{b}}$, D by \tilde{V} and C by C' in Lemma 2.4 (the second conclusion), we get $\text{Rot}_{\tilde{U}}(\hat{F}_{\tilde{a}, \tilde{b}}) \subset] - (N_{\tilde{a}, \tilde{b}} + 1), N_{\tilde{a}, \tilde{b}} + 1[$. We have a contradiction.

(2). Suppose that there is a point $\tilde{z}' \in \pi^{-1}(z') \cap \tilde{U}$ where $z' \in \text{Rec}^+(F)$ and an integer $n_0 \geq 1$ such that $\tilde{F}^{n_0}(\tilde{z}') \in \tilde{V}$. By (1), it is sufficient to consider the case where $\alpha_{z',n} = e$ for all $n \geq 1$, that means, $\tilde{F}^{\tau_n(z')}(\tilde{z}') \in \tilde{U}$ for all $n \geq 1$. We choose a positive integer n_1 large enough such that $\tau_{n_1}(z') > n_0$. We have $\tilde{F}^{\tau_{n_1}(z')-n_0}(\tilde{F}^{n_0}(\tilde{z}')) \in \tilde{U}$. Then we get $\tilde{F}^{\tau_{n_1}(z')-n_0}(\tilde{V}) \cap \tilde{U} \neq \emptyset$ and $\tilde{F}^{n_0}(\tilde{U}) \cap \tilde{V} \neq \emptyset$. Therefore, the disk chain $(\{\tilde{V}, \tilde{U}, \tilde{V}\}, \{\tau_{n_1}(z') - n_0, n_0\})$ is a periodic disk chain that satisfies the hypothesis of Lemma 2.4. Applying Lemma 2.4 again, we get $\text{Rot}_{\tilde{U}}(\hat{F}_{\tilde{a},\tilde{b}}) \subset] - (N_{\tilde{a},\tilde{b}} + 1), N_{\tilde{a},\tilde{b}} + 1[$. It is still a contradiction.

(3). This follows from Corollary 2.3 and the hypothesis $\text{Rot}_{\text{Fix}(\tilde{F}_{\tilde{a},\tilde{b}})}(\hat{F}_{\tilde{a},\tilde{b}}) \subset [-N_{\tilde{a},\tilde{b}}, N_{\tilde{a},\tilde{b}}]$ immediately. \square

Proof of Proposition 5.1. This follows from Lemma 5.2, 5.3, 5.4 and 5.6. \square

At the end of this section, we study the boundedness in the case where the time-one map F of I satisfies the differential conditions.

Proposition 5.8. *For any two distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} , if F and F^{-1} are differentiable at $\pi(\tilde{a})$ and $\pi(\tilde{b})$, then there exists $N_{\tilde{a},\tilde{b}} \in \mathbb{R}$ such that $|i(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq N_{\tilde{a},\tilde{b}}$ if $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ exists.*

Proof. We make a proof by contradiction. If it is not true, without loss of generality, we suppose that there is a sequence $\{z_k\}_{k \geq 1} \subset \text{Rec}^+(F)$ such that $\lim_{k \rightarrow +\infty} i(\tilde{F}; \tilde{a}, \tilde{b}, z_k) = +\infty$. By the proof of Lemma 5.6 and the conclusion (1) of Lemma 5.7, we have $\alpha_{z_k,n} = e$ for every $n \geq 1$ when k is large enough. Hence $\tilde{z}_k \in \text{Rec}^+(\tilde{F}) \setminus \text{Fix}(\tilde{F})$ when k is large enough where $\tilde{z}_k \in \pi^{-1}(z_k)$. By the proof of Lemma 5.6 and the conclusion (2) of Lemma 5.7, we only need consider the lifts \tilde{z}_k of z_k whose whole orbit is in \tilde{V}^c when k is large enough. However, such lifts are finite (at most $K_{\tilde{a},\tilde{b}}$). This implies that there exists a sequence $\{\tilde{z}_k\}_{k \geq 1}$ with $\tilde{z}_k \in \pi^{-1}(z_k)$ such that $\lim_{k \rightarrow +\infty} \rho_{A_{\tilde{a},\tilde{b}}, \hat{F}_{\tilde{a},\tilde{b}}}(\tilde{z}_k) = +\infty$, which conflicts with Lemma 1.8. \square

In Example 7.2 of Appendix, we will construct an identity isotopy I of a closed surface such that I satisfies the B-property but its time-one map is not a diffeomorphism and there are two different fixed points \tilde{z}_0 and \tilde{z}_1 of \tilde{F} such that the linking number $i(\tilde{F}; \tilde{z}_0, \tilde{z}_1, z)$ is not uniformly bounded for $z \in \text{Rec}^+(F) \setminus \pi(\{\tilde{z}_0, \tilde{z}_1\})$.

5.2. Existence and Boundedness in the conservative case.

Proposition 5.9. *Suppose that I satisfies the WB-property at \tilde{a} and \tilde{b} . If $\mu \in \mathcal{M}(F)$, then μ -almost every point $z \in \text{Rec}^+(F)$ has a rotation vector $\rho_{M,I}(z) \in H_1(M, \mathbb{R})$ and has a linking number $i(\tilde{F}; \tilde{a}, \tilde{b}, z) \in \mathbb{R}$. Moreover, for all $z \in \text{Rec}^+(F)$ satisfying that $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$, $\rho_{M,I}(z)$ exist and $\rho_{M,I}(z) \neq 0$, there exists $C > 0$ such that $|i(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq C$.*

Proof. According to Poincaré Recurrence Theorem, we have $\mu(\text{Rec}^+(F)) = \mu(M)$.

When $z \in \text{Fix}(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$, by Section 1.3.2 and Section 5.1, $\rho_{M,I}(z)$ and $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ exist and are bounded. Thus we only need to consider the non-fixed point case.

Fix a free open disk $U \subset M \setminus \pi(\{\tilde{a}, \tilde{b}\})$ with $\mu(U) > 0$. For any $z \in \text{Rec}^+(F) \cap U$, by Lemma 5.4 and Lemma 5.6, we have $|L_1(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq \tau(z) K_{\tilde{a},\tilde{b}} (N_{\tilde{a},\tilde{b}} + 1)$ if $\alpha_{z,1} \neq e$ and

$|L_1(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq \tau(z)K_{\tilde{a}, \tilde{b}}K_U$ if $\alpha_{z,1} = e$. This implies that $L_1(\tilde{F}; \tilde{a}, \tilde{b}, z) \in L^1(U, \mathbb{R}, \mu)$. By Birkhoff Ergodic Theorem, we deduce that the sequence $\{L_n(\tilde{F}; \tilde{a}, \tilde{b}, z)/n\}_{n=1}^{+\infty}$ converges to a real number $L^*(\tilde{F}; \tilde{a}, \tilde{b}, z)$ for μ -almost every point on $\text{Rec}^+(F) \cap U$. Recall that, for μ -almost every point on $\text{Rec}^+(F) \cap U$, the sequence $\{\tau_n(z)/n\}_{n=1}^{+\infty}$ converges to a real number $\tau^*(z)$ (see Section 1.3.2).

We can define the linking number on U as follows (modulo sets of measure zero):

$$(5.6) \quad i(\tilde{F}; \tilde{a}, \tilde{b}, z) = \lim_{n \rightarrow +\infty} \frac{L_n(\tilde{F}; \tilde{a}, \tilde{b}, z)}{\tau_n(z)} = \frac{L^*(\tilde{F}; \tilde{a}, \tilde{b}, z)}{\tau^*(z)}.$$

By Proposition 5.1, the linking number $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ has a bound K_U for μ -almost every point $z \in \text{Rec}^+(F) \cap U$. As U is arbitrarily chosen, this implies that we can define the function $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ for μ -almost every point $z \in M \setminus \pi(\{\tilde{a}, \tilde{b}\})$.

Finally, by Proposition 5.5, we can uniformly bound $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ if $\rho_{M,I}(z) \neq 0$. \square

Remark here that, under the hypothesis of Proposition 5.9, $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ is bounded on U , but does not necessarily possess a uniform bound on $M \setminus \pi(\{\tilde{a}, \tilde{b}\})$ (see Example 7.2). However, when F is a diffeomorphism of M (see Proposition 5.8), we can get a uniform bound. Moreover, we can get a uniform bound in the case where the support of the measure is the whole space, as stated in the following proposition.

Proposition 5.10. *With the same hypotheses as Proposition 5.9 and if furthermore $\mu \in \mathcal{M}(F)$ has full support, we have $|i(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq K_{\tilde{a}, \tilde{b}}(N_{\tilde{a}, \tilde{b}} + 1)$ if it exists.*

Proof. The measure μ may naturally be lifted to a (non finite) measure $\tilde{\mu}$ on \tilde{M} . Since μ does not charge $\pi(\tilde{a})$ and $\pi(\tilde{b})$, $\tilde{\mu}$ can be seen as a measure on $A_{\tilde{a}, \tilde{b}}$ invariant by $\tilde{F}_{\tilde{a}, \tilde{b}}$ satisfying $\tilde{\mu}(A_{\tilde{a}, \tilde{b}}) = +\infty$. As the support of $\tilde{\mu}$ is \tilde{M} and $\tilde{F}_{\tilde{a}, \tilde{b}}$ preserves the measure $\tilde{\mu}$, the homeomorphism $\tilde{F}_{\tilde{a}, \tilde{b}}$ satisfies the intersection property, that is, any simple closed curve of $A_{\tilde{a}, \tilde{b}}$ which is not null-homotopic meets its image by $\tilde{F}_{\tilde{a}, \tilde{b}}$. Indeed, any closed curve which goes through ∞ will meet its image by $\tilde{F}_{\tilde{a}, \tilde{b}}$ since $\tilde{F}_{\tilde{a}, \tilde{b}}$ fixes the point ∞ . If the closed curve does not pass through ∞ , we may go back to \tilde{M} and consider a component enclosed by the closed curve which contains \tilde{a} or \tilde{b} and which has finite measure, then it will meet its image since \tilde{F} preserves the measure $\tilde{\mu}$.

In the case where $z \in \text{Fix}(F)$, it is obvious that $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ is uniformly bounded.

Choose any free open disk $U \subset M \setminus \text{Fix}(F)$, according to Lemma 5.4, we only need to consider the points $z \in \text{Rec}^+(F) \cap U$ such that $\alpha_{z,n} = e$ for n large enough. We suppose that z is a such point and $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ exists. We go to the annulus $A_{\tilde{a}, \tilde{b}}$, for any lift \tilde{z} of z , then we have $\rho_{A_{\tilde{a}, \tilde{b}}, \tilde{F}_{\tilde{a}, \tilde{b}}}(\tilde{z}) = \lim_{n \rightarrow +\infty} \frac{\tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{a}, \tilde{b}}^n}{\tau_n(z)}$.

We claim that, for any $\epsilon > 0$, $|i(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq (N_{\tilde{a}, \tilde{b}} + 1 + \epsilon)K_{\tilde{a}, \tilde{b}}$. Otherwise, without loss of generality, we can suppose that $i(\tilde{F}; \tilde{a}, \tilde{b}, z) > (N_{\tilde{a}, \tilde{b}} + 1 + \epsilon)K_{\tilde{a}, \tilde{b}}$. Then there exists a number N large enough such that for every $n \geq N$, there is a lift \tilde{z}_n of z in \tilde{V}^c satisfying $\frac{\tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{a}, \tilde{b}}^n}{\tau_n(z)} > N_{\tilde{a}, \tilde{b}} + 1 + \epsilon$. This implies that there exists a lift \tilde{z} of z in \tilde{V}^c such that $\rho_{A_{\tilde{a}, \tilde{b}}, \tilde{F}_{\tilde{a}, \tilde{b}}}(\tilde{z}) \geq N_{\tilde{a}, \tilde{b}} + 1 + \epsilon > N_{\tilde{a}, \tilde{b}} + 1$. By the fact $\rho_{A_{\tilde{a}, \tilde{b}}, \tilde{F}_{\tilde{a}, \tilde{b}}}(\infty) = 0$ and according

to Theorem 2.5, $\tilde{F}_{\tilde{a}, \tilde{b}}$ has a fixed point whose rotation number is $N_{\tilde{a}, \tilde{b}} + 1$, which is a contradiction. This proves the claim.

As ϵ is arbitrarily chosen, we get $|i(\tilde{F}; \tilde{a}, \tilde{b}, z)| \leq K_{\tilde{a}, \tilde{b}}(N_{\tilde{a}, \tilde{b}} + 1)$. \square

The function $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ is not necessarily μ -integrable (see Example 7.2). But in some cases, as we have stated above, where the time-one map F is a diffeomorphism of M , or I satisfies the WB-property at \tilde{a} and \tilde{b} , and μ is ergodic (because it is constant μ -a.e.) or the support of μ is the whole space, the function $i_\mu(\tilde{F}; \tilde{a}, \tilde{b}, z)$ is μ -integrable.

Suppose now the function $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ is μ -integrable. We can define the *action difference* of \tilde{a} and \tilde{b} as follows

$$(5.7) \quad i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) = \int_{M \setminus \pi(\{\tilde{a}, \tilde{b}\})} i(\tilde{F}; \tilde{a}, \tilde{b}, z) d\mu.$$

From Proposition 4.3 and Proposition 4.4, we get the following corollaries immediately:

Corollary 5.11. *We have $i_\mu(\tilde{F}^q; \tilde{a}, \tilde{b}) = q i_\mu(\tilde{F}; \tilde{a}, \tilde{b})$ for all $q \geq 1$.*

Corollary 5.12. *Let H be an orientation preserving homeomorphism of M and \tilde{H} be a lift of H to \tilde{M} . We have $i_{H_*(\mu)}(\tilde{H} \circ \tilde{F} \circ \tilde{H}^{-1}; \tilde{H}(\tilde{a}), \tilde{H}(\tilde{b})) = i_\mu(\tilde{F}; \tilde{a}, \tilde{b})$. In particular, $i_\mu(\tilde{F}; \alpha(\tilde{a}), \alpha(\tilde{b})) = i_\mu(\tilde{F}; \tilde{a}, \tilde{b})$ for all $\alpha \in G$.*

At the end of this section, we will give the integral (5.7) a geometric description when F and F^{-1} are differentiable at $\pi(\tilde{a})$ and $\pi(\tilde{b})$. Before that, let us introduce a definition.

Let $\mathbb{A} = \mathbf{T}^1 \times [0, 1]$ be a closed annulus and let T be the generator of the covering transformation group $\pi : \tilde{\mathbb{A}} \rightarrow \mathbb{A}$ where $\tilde{\mathbb{A}} = \mathbb{R} \times [0, 1]$. Suppose that $J = (h_t)_{t \in [0, 1]}$ is an isotopy of \mathbb{A} from $\text{Id}_{\mathbb{A}}$ to h , ν is a Borel measure (ν is admitted to be an infinite measure here) invariant by h on \mathbb{A} . Let $\gamma : [0, 1] \rightarrow \mathbb{A}$ be a simple oriented path which satisfies $\gamma(0) \in \mathbf{T}^1 \times \{0\}$, $\gamma(1) \in \mathbf{T}^1 \times \{1\}$ and $\text{Int}(\gamma) \subset \text{Int}(\mathbb{A})$. Denote by $\Sigma' : [0, 1] \times [0, 1] \rightarrow \mathbb{A}$ the 2-chain $\Sigma'(s, t) = h_s^{-1}(\gamma(t))$ and by $|\Sigma'| = \{z \in \mathbb{A} \mid z = h_s^{-1}(\gamma(t)), (s, t) \in [0, 1] \times [0, 1]\}$ the support of Σ' . When $\nu(\gamma) = 0$, the intersection number $\gamma \wedge J(z)$ is well defined for ν -almost every z on \mathbb{A} . Define the *algebraic area of the 2-chain Σ' in \mathbb{A}* , that is, the algebraic area (for ν) “swept out” by $\bigcup_{s \in [0, 1]} h_s^{-1}(\gamma)$, as follows

$$\int_{\Sigma'} d\nu = \int_{\mathbb{A}} \gamma \wedge J(z) d\nu.$$

When $\nu(|\Sigma'|) < +\infty$, the integral is well defined. Indeed, there exist a number $N \geq 0$ such that $|\gamma \wedge J(z)| \leq N$ since \mathbb{A} is compact. Obviously, $\gamma \cap J(z) = \emptyset$ if $z \notin \bigcup_{s \in [0, 1]} h_s^{-1}(\gamma(t))$. Therefore,

$$\left| \int_{\Sigma'} d\nu \right| \leq \int_{\mathbb{A}} |\gamma \wedge J(z)| d\nu \leq \nu(|\Sigma'|)N < +\infty.$$

Let H be the lift of h that is the time-one map of the lifted identity isotopy \tilde{J} of J , $\tilde{\gamma}$ be a connected component of γ in $\tilde{\mathbb{A}}$ and $\tilde{\nu}$ be the lift of ν to $\tilde{\mathbb{A}}$. Let \tilde{D}' be the closed region between $H^{-1}(\tilde{\gamma})$ and $T(H^{-1}(\tilde{\gamma}))$ which is a fundamental domain of T . We have

$$(5.8) \quad \int_{\Sigma'} d\nu = \int_{\mathbb{A}} \gamma \wedge J(z) d\nu = \int_{\tilde{D}'} \tilde{\gamma} \wedge \tilde{J}(\tilde{z}) d\tilde{\nu},$$

which does not depend on the choice of $\tilde{\gamma}$.

Denote by $\Sigma = h * \Sigma' : [0, 1] \times [0, 1] \rightarrow \mathbb{A}$ the 2-chain $\Sigma(s, t) = h_s^{-1}(h(\gamma(t)))$ and suppose that $\nu(|\Sigma|) < +\infty$. Let $\tilde{D} = H(\tilde{D}')$ be the closed region between $\tilde{\gamma}$ and $T(\tilde{\gamma})$ which is also a fundamental domain of T . By Equation 5.8, we have

$$(5.9) \quad \int_{\Sigma} d\nu = \int_{\mathbb{A}} h(\gamma) \wedge J(z) d\nu = \int_{\tilde{D}} H(\tilde{\gamma}) \wedge \tilde{J}(\tilde{z}) d\tilde{\nu}.$$

Equation 5.9 tell us that the value $\int_{\Sigma} d\nu$ is equal to the algebraic area (for $\tilde{\nu}$) of the region of $\tilde{\mathbb{A}}$ situated between $\tilde{\gamma}$ and its image $H(\tilde{\gamma})$. Furthermore, if we suppose that J fixes a point ∞ in \mathbb{A} , we have

$$(5.10) \quad \begin{aligned} \int_{\Sigma} d\nu &= \int_{\mathbb{A}} h(\gamma) \wedge J(z) d\nu \\ &= \int_{\mathbb{A}} \gamma \wedge (h^{-1} \circ J)(z) d\nu \\ &= \int_{\mathbb{A}} \gamma \wedge (h^{-1} \circ J \circ h)(z) d\nu \\ &= \int_{\mathbb{A}} \gamma \wedge J(z) d\nu. \end{aligned}$$

Indeed, write the isotopy $J' = h^{-1} \circ J \circ h = (h^{-1} \circ h_t \circ h)_{0 \leq t \leq 1}$. The third equation holds because h is a homeomorphism of \mathbb{A} and preserves the measure ν . Observing that the isotopy $J^{-1}J'$ is a loop (whose base point is $\text{Id}_{\mathbb{A}}$) in $\text{Homeo}_*(\mathbb{A})$ and fixes the point ∞ , recall that $\pi_1(\text{Homeo}_*(\mathbb{A})) = \bigcup_{k \in \mathbb{Z}} \mathcal{C}_k$ (see the proof of Proposition 4.5), we get $[J^{-1}J']_1 \in \mathcal{C}_0$. Hence, we get the last equation. It is easy to prove that, by induction and Equation 5.10, $\int_{\Sigma} d\nu$ is equal to $\int_{h^{k*}\Sigma'} d\nu$ for every $k \in \mathbb{Z}$.

Remark that we can also define the algebraic area of the 2-chain Σ when γ is not simple if we consider the oriented domain enclosed by $\tilde{\gamma}$, $H(\tilde{\gamma})$ and $\partial\tilde{\mathbb{A}}$ in $\tilde{\mathbb{A}}$. However, to prove Theorem 0.1 in the next section, it is enough to merely consider the case of a simple oriented path.

Suppose now the measure ν is defined by a symplectic form ω , that is, $\nu(A) = \int_A \omega$ for all measurable sets $A \subset \mathbb{A}$. Observe that $\tilde{\omega}$ is exact in $\tilde{\mathbb{A}}$ where $\tilde{\omega}$ is the lift of ω to $\tilde{\mathbb{A}}$. Equation 5.9 and Stokes' theorem imply that $\int_{\Sigma} \omega$ (defined by the integral of differential 2-form on 2-chain) is nothing else but the algebraic area of the 2-chain Σ in \mathbb{A} , $\int_{\Sigma} d\nu$ (defined by Equation 5.9).

We now suppose that the time-one map F of I and its inverse F^{-1} are differentiable at $\pi(\tilde{a})$ and $\pi(\tilde{b})$. Let $\tilde{I}_1 = (\tilde{F}'_t)_{t \in [0, 1]}$ be an isotopy from $\text{Id}_{\tilde{M}}$ to \tilde{F} that fixes \tilde{a} and \tilde{b} , and $\tilde{\mu}$ be the lift of μ to \tilde{M} . Let $\tilde{\gamma} : [0, 1] \rightarrow \tilde{M}$ be a simple oriented path from \tilde{a} to \tilde{b} with $\tilde{\gamma}(0) = \tilde{a}$ and $\tilde{\gamma}(1) = \tilde{b}$. Consider the annulus $A_{\tilde{a}, \tilde{b}}$ and the annulus map $\tilde{F}_{\tilde{a}, \tilde{b}}$. Recall that, in the proof of Lemma 1.8, $\bar{A}_{\tilde{a}, \tilde{b}} = S_{\tilde{a}} \sqcup A_{\tilde{a}, \tilde{b}} \sqcup S_{\tilde{b}}$ is the natural compactification of $A_{\tilde{a}, \tilde{b}}$ where $S_{\tilde{a}}$ and $S_{\tilde{b}}$ are the tangent unit circles at \tilde{a} and \tilde{b} . We can identify $\tilde{\gamma}$ as an oriented path in $\bar{A}_{\tilde{a}, \tilde{b}}$ and \tilde{I}_1 as an identity isotopy of $\bar{A}_{\tilde{a}, \tilde{b}}$. As the measure $\tilde{\mu}$ is invariant by \tilde{F} and $\tilde{\mu}(\tilde{a}) = \tilde{\mu}(\tilde{b}) = 0$, it naturally induces a measure on $\bar{A}_{\tilde{a}, \tilde{b}}$, denoted still by $\tilde{\mu}$.

Suppose that $\tilde{\Sigma}$ is the 2-chain $\tilde{\Sigma} : [0, 1] \times [0, 1] \rightarrow \tilde{M}$ defined by $\tilde{\Sigma}(s, t) = \tilde{F}'_s^{-1}(\tilde{F}(\tilde{\gamma}(t)))$ whose boundary is $\tilde{F}(\tilde{\gamma})\tilde{\gamma}^{-1}$ with the boundary of the square $[0, 1] \times [0, 1]$ oriented counter-clockwise. As \tilde{I}_1 fixes ∞ , the intersection number $\tilde{\gamma} \wedge \tilde{I}_1(\tilde{z})$ is zero when \tilde{z} belongs to a

neighborhood of ∞ . Therefore, if $\tilde{\mu}(\tilde{\gamma}) = 0$, we can define *the algebraic area of the 2-chain $\tilde{\Sigma}$ in $\tilde{M} \setminus \{\tilde{a}, \tilde{b}\}$* as follows

$$\int_{\tilde{\Sigma}} d\tilde{\mu} = \int_{\tilde{M} \setminus \{\tilde{a}, \tilde{b}\}} \tilde{\gamma} \wedge \tilde{I}_1(\tilde{z}) d\tilde{\mu} = \int_{\tilde{A}_{\tilde{a}, \tilde{b}}} \tilde{\gamma} \wedge \tilde{I}_1(\tilde{z}) d\tilde{\mu}.$$

Remark here that if the measure μ is defined by a symplectic form ω , then $\int_{\tilde{\Sigma}} \tilde{\omega}$ (see Equation 3.1 and Equation 3.3) is nothing else but $\int_{\tilde{\Sigma}} d\tilde{\mu}$ where $\tilde{\omega}$ is the lift of ω to \tilde{M} . Moreover, we have the following result which is a key step to prove our generalized action function defined in the next section.

Lemma 5.13. *If $\tilde{\mu}(\tilde{\gamma}) = 0$, then we have*

$$i_{\mu}(\tilde{F}, \tilde{a}, \tilde{b}) = \int_{\tilde{\Sigma}} d\tilde{\mu}.$$

Proof. From Proposition 5.8, we know that $i_{\mu}(\tilde{F}, \tilde{a}, \tilde{b})$ is well defined. Let

$$Z = \bigcup_{k=0}^{+\infty} (F^{-k}(\pi(\tilde{\gamma}))).$$

Observe that $\mu(\text{Rec}^+(F) \setminus Z) = \mu(M)$. For every $z \in \text{Rec}^+(F) \setminus Z$ and every $n \geq 1$, consider the following infinite family of paths in \tilde{M} :

$$\tilde{\Gamma}'_{I_1, z}{}^n = \prod_{\pi(\tilde{z})=z} \tilde{I}_1^n(\tilde{z}).$$

Define the function

$$G_n(\tilde{F}; \tilde{a}, \tilde{b}, z) = \tilde{\gamma} \wedge \tilde{\Gamma}'_{I_1, z}{}^n.$$

Let us verify that this is well defined. Consider the annulus $A_{\tilde{a}, \tilde{b}}$ and the annulus map $\tilde{F}_{\tilde{a}, \tilde{b}}$. For any $z \in \text{Rec}^+(F) \setminus Z$, let \tilde{z} be any lift of z to \tilde{M} (we also write \tilde{z} in $A_{\tilde{a}, \tilde{b}}$), and \hat{z} be any lift of \tilde{z} to $\hat{A}_{\tilde{a}, \tilde{b}}$. In the proof of Lemma 1.8, we have proved that $|p_1(\tilde{F}_{\tilde{a}, \tilde{b}}(\hat{z})) - p_1(\hat{z})|$ is uniformly bounded for any $\hat{z} \in \hat{A}_{\tilde{a}, \tilde{b}}$, say N as a bound, and depends on the isotopy I but not on the choice of \hat{z} . Fix an open disk \tilde{W} containing ∞ and disjoint from $\tilde{\gamma}$. As $\tilde{I}_1(\infty) = \infty$, for every $n \geq 1$, we can choose an open disk $\tilde{V}_n \subset \tilde{W}$ containing ∞ such that for every $\tilde{z} \in \tilde{V}_n$, we have $\tilde{I}_1^n(\tilde{z}) \in \tilde{W}$. Write $X_z'^n = \pi^{-1}(\{z\}) \cap \tilde{V}_n^c$. We deduce that there is a positive integer K'_n such that $\sharp X_z'^n \leq K'_n$ and

$$\left| G_n(\tilde{F}; \tilde{a}, \tilde{b}, z) \right| = \left| \tilde{\gamma} \wedge \tilde{\Gamma}'_{I_1, z}{}^n \right| = \left| \sum_{\tilde{z} \in X_z'^n} \tilde{\gamma} \wedge \tilde{I}_1^n(\tilde{z}) \right| \leq K'_n N.$$

Hence we complete the claim. As a consequence, $G_1(\tilde{F}; \tilde{a}, \tilde{b}, z) \in L^1(M \setminus \pi(\{\tilde{a}, \tilde{b}\}), \mathbb{R}, \mu)$.

Moreover, we can write $G_n(\tilde{F}; \tilde{a}, \tilde{b}, z)$ as a Birkhoff sum:

$$G_n(\tilde{F}; \tilde{a}, \tilde{b}, z) = \tilde{\gamma} \wedge \tilde{\Gamma}'_{I_1, z}{}^n = \tilde{\gamma} \wedge \prod_{i=0}^{n-1} \tilde{\Gamma}'_{I_1, F^i(z)} = \sum_{j=0}^{n-1} G_1(\tilde{F}; \tilde{a}, \tilde{b}, F^j(z)).$$

According to Birkhoff Ergodic theorem, the limit

$$\lim_{n \rightarrow +\infty} \frac{G_n(\tilde{F}; \tilde{a}, \tilde{b}, z)}{n} = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} G_1(\tilde{F}; \tilde{a}, \tilde{b}, F^j(z))$$

exists for μ -almost everywhere on $M \setminus \pi(\{\tilde{a}, \tilde{b}\})$. We know that

$$i(\tilde{F}; \tilde{a}, \tilde{b}, z) = \lim_{n \rightarrow +\infty} \frac{L_n(\tilde{F}; \tilde{a}, \tilde{b}, z)}{\tau_n(z)} = \frac{L^*(\tilde{F}; \tilde{a}, \tilde{b}, z)}{\tau^*(z)}$$

for μ -almost every point $z \in M \setminus \pi(\{\tilde{a}, \tilde{b}\})$ exists (see Proposition 5.9). As $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ does not depend on the choice of U (see Definition 4.1), when $z \notin \pi(\tilde{\gamma})$, we can suppose that the disk U is small enough such that $U \cap \pi(\tilde{\gamma}) = \emptyset$. Therefore, $\{L_n(\tilde{F}; \tilde{a}, \tilde{b}, z)/\tau_n(z)\}_{n \geq 1}$ is a subsequence of $\{G_n(\tilde{F}; \tilde{a}, \tilde{b}, z)/n\}_{n \geq 1}$. We get

$$i(\tilde{F}; \tilde{a}, \tilde{b}, z) = \lim_{n \rightarrow +\infty} \frac{G_n(\tilde{F}; \tilde{a}, \tilde{b}, z)}{n}$$

for μ -almost everywhere on $M \setminus \pi(\{\tilde{a}, \tilde{b}\})$.

By Birkhoff Ergodic theorem, we have

$$\begin{aligned} i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) &= \int_{M \setminus \pi(\{\tilde{a}, \tilde{b}\})} i(\tilde{F}; \tilde{a}, \tilde{b}, z) \, d\mu \\ &= \int_{M \setminus \pi(\{\tilde{a}, \tilde{b}\})} G_1(\tilde{F}; \tilde{a}, \tilde{b}, z) \, d\mu \\ &= \int_{\tilde{M} \setminus \pi^{-1}(\pi(\{\tilde{a}, \tilde{b}\}))} \tilde{\gamma} \wedge \tilde{I}_1(\tilde{z}) \, d\tilde{\mu} \\ &= \int_{\tilde{\Sigma}} d\tilde{\mu}, \end{aligned}$$

□

6. ACTION FUNCTION

This section will be divided into three parts. In the first part, we will define a new action function and prove Theorem 0.1. In the second part, we will define the action spectrum which is invariant under conjugation by an orientation and measure preserving homeomorphism.

6.1. Definition of the action function. In this section, we suppose that the action difference $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ is well defined for every two distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} .

We define the *action difference* as follows:

$$\begin{aligned} i_\mu : (\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta} &\rightarrow \mathbb{R} \\ (\tilde{a}, \tilde{b}) &\mapsto i_\mu(\tilde{F}; \tilde{a}, \tilde{b}). \end{aligned}$$

From Proposition 4.5, we have the following corollary immediately:

Corollary 6.1. *For any distinct fixed points \tilde{a} , \tilde{b} and \tilde{c} of \tilde{F} , we have*

$$i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) + i_\mu(\tilde{F}; \tilde{b}, \tilde{c}) + i_\mu(\tilde{F}; \tilde{c}, \tilde{a}) = 0.$$

That is, i_μ is a coboundary on $\text{Fix}(\tilde{F})$. So there is a function $l_\mu : \text{Fix}(\tilde{F}) \rightarrow \mathbb{R}$, defined up to an additive constant, such that

$$i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) = l_\mu(\tilde{F}; \tilde{b}) - l_\mu(\tilde{F}; \tilde{a}).$$

We call the function l_μ the *action function* (or *action* for short) on $\text{Fix}(\tilde{F})$ defined by the measure μ .

By Section 1.4.3, the properties of $i_\mu(\tilde{F}; \tilde{a}, \tilde{b}, z)$ (see Section 4.1) and Corollary 6.1, we have the following proposition:

Proposition 6.2. *The action difference i_μ (hence the action l_μ) only depends on the homotopic class with fixed endpoints of I . Moreover, i_μ only depends on the time-one map F when $g > 1$ and i_μ depends on the homotopic class of I when $g = 1$. The same property holds for I_μ (hence L_μ) which defines in the case where $\rho_{M,I}(\mu) = 0$ (see Formula 6.1 and Proposition 6.3 below).*

Proposition 6.3. *If $\rho_{M,I}(\mu) = 0$, then $i_\mu(\tilde{F}; \tilde{a}, \alpha(\tilde{a})) = 0$ for every $\tilde{a} \in \text{Fix}(\tilde{F})$ and every $\alpha \in G^*$. As a consequence, there exists a function L_μ defined on $\text{Fix}_{\text{Cont},I}(F)$ such that for every two distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} , we have*

$$i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) = L_\mu(\tilde{F}; \pi(\tilde{b})) - L_\mu(\tilde{F}; \pi(\tilde{a})).$$

Proof. There exists an isotopy I' homotopic to I that fixes $\pi(\tilde{a})$. It is lifted to an isotopy \tilde{I}' that fixes \tilde{a} and $\alpha(\tilde{a})$. Observe that if $\tilde{\gamma}$ is an oriented path from \tilde{a} to $\alpha(\tilde{a})$, then the intersection number $\tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}',z}^n$ (see Section 4.1) is equal to the intersection between the loop $\pi(\tilde{\gamma})$ and the loop $I'^{\tau_n(z)}(z)\gamma_{\Phi^n(z),z}$ (see Section 1.3.2). As $\rho_{M,I}(\mu) = \rho_{M,I'}(\mu) = 0$ and $\pi(\tilde{a}) \in \text{Fix}_{\text{Cont},I}(F)$ (or $\mu(\pi(\tilde{a})) = 0$), we have

$$\begin{aligned} i_\mu(\tilde{F}; \tilde{a}, \alpha(\tilde{a})) &= \int_{M \setminus \{\pi(\tilde{a})\}} i(\tilde{F}; \tilde{a}, \alpha(\tilde{a}), z) \, d\mu \\ &= \int_{M \setminus \{\pi(\tilde{a})\}} \lim_{n \rightarrow +\infty} \frac{L_n(\tilde{F}; \tilde{a}, \alpha(\tilde{a}), z)}{\tau_n(z)} \, d\mu \\ &= \int_{M \setminus \{\pi(\tilde{a})\}} \lim_{n \rightarrow +\infty} \frac{\tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}',z}^n}{\tau_n(z)} \, d\mu \\ &= \pi(\tilde{\gamma}) \wedge \rho_{M,I'}(\mu) \\ &= 0. \end{aligned}$$

The second conclusion follows from Corollary 6.1. \square

We call the function L_μ the *action* on $\text{Fix}_{\text{Cont},I}(F)$ defined by the measure μ . We note that the results above hold for the set of all such pairs $(\tilde{a}, \tilde{b}) \in (\text{Fix}(\tilde{F}) \times \tilde{F}) \setminus \tilde{\Delta}$ which the action difference can be defined on.

As a consequence, if F is a diffeomorphism of M (by Proposition 5.8 and 5.9), or the isotopy I satisfies the WB-property and $\text{Supp}(\mu) = M$ (by Proposition 5.9 and 5.10) or μ is ergodic (by Proposition 5.9 and Birkhoff Ergodic theorem), then the action function is well defined on $\text{Fix}(\tilde{F})$, but the action can be unbounded (See Example 7.1 of Appendix).

Proof of Theorem 0.1. From Corollary 6.1 and Proposition 6.3, we define the *action difference* $I_\mu : (\text{Fix}_{\text{Cont},I}(F) \times \text{Fix}_{\text{Cont},I}(F)) \setminus \Delta \rightarrow \mathbb{R}$ and the action $L_\mu : \text{Fix}_{\text{Cont},I}(F) \rightarrow \mathbb{R}$ as follows

$$(6.1) \quad I_\mu(\tilde{F}; a, b) = i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) = L_\mu(\tilde{F}; b) - L_\mu(\tilde{F}; a),$$

where \tilde{a} and \tilde{b} are any lifts of a and b . We only need to prove that the function L_μ defined in this section is a generalization of the action difference in Section 3.1.2.

Observe that, in the classical case, $I = (F_t)_{t \in [0,1]} \subset \text{Diff}_*(M)$ where $\text{Diff}_*(M)$ is the set of diffeomorphisms that are isotopic to the identity. The measure μ is defined by a symplectic form ω . Therefore, μ is non-atomic. Comparing Equation 3.3 with Equation 6.1, it sufficient to prove that $I_\mu(\tilde{F}; a, b) = i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) = \delta(\tilde{F}, \tilde{a}, \tilde{b})$.

Let $\tilde{\gamma}$ be any oriented path from \tilde{a} to \tilde{b} . By Lemma 5.13, we have

$$i_\mu(\tilde{F}, \tilde{a}, \tilde{b}) = \int_{\tilde{\Sigma}} d\tilde{\mu}$$

where $\tilde{\Sigma}$ is the 2-chain whose boundary is $\tilde{F}(\tilde{\gamma}) - \tilde{\gamma}$ (i.e., identify $\tilde{F}(\tilde{\gamma})\tilde{\gamma}^{-1}$ as a 1-chain) as defined in Lemma 5.13. As $\delta(\tilde{F}, \tilde{a}, \tilde{b})$ does not depend on the choices of $\tilde{\gamma}$ and $\tilde{\Sigma}$ (see Section 3.1.2), we have $i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) = \delta(\tilde{F}, \tilde{a}, \tilde{b})$. \square

From Theorem 0.1 and Corollary 5.11, we get the following iteration formula of the action function with regard to \tilde{F} immediately:

Proposition 0.2 *Under the same hypotheses as Theorem 0.1, for every two distinct contractible fixed points a and b of F , we have $I_\mu(\tilde{F}^q; a, b) = qI_\mu(\tilde{F}; a, b)$ for all $q \geq 1$.*

6.2. Action spectrum. We suppose that the action l_μ is well defined. Write \tilde{F} as the lift of F obtained by lifting I to an isotopy \tilde{I} to \tilde{M} starting $\text{Id}_{\tilde{M}}$.

Define the *action spectrum of I* as follows (up to an additive constant):

$$\sigma(\tilde{F}) = \{l_\mu(\tilde{F}; \tilde{z}) \mid z \in \text{Fix}(\tilde{F})\} \subset \mathbb{R}.$$

Moreover, if $\rho_{M,I}(\mu) = 0$, we can write the action spectrum of I as (up to an additive constant):

$$\sigma(\tilde{F}) = \{L_\mu(\tilde{F}; z) \mid z \in \text{Fix}_{\text{Cont},I}(F)\} \subset \mathbb{R}.$$

Recall that $\text{Homeo}^+(M, \mu)$ is the subgroup of $\text{Homeo}(M)$ whose elements preserve the measure μ and the orientation. By Corollary 5.12, we have the following conjugation invariance property:

Proposition 0.3 *The action spectrum is invariant by conjugation in $\text{Homeo}^+(M, \mu)$.*

7. APPENDIX

We fix a closed surface M of genus $g \geq 1$ and a topological closed disk D on M all examples will coincide with the identity outside of D including isotopies. Up to a diffeomorphism, we may suppose that D is the closed unit Euclidean disk. We will construct an identity isotopy $I = (F_t)_{t \in [0,1]}$, we will write $F = F_1$ and $\tilde{F} = \tilde{F}_1$ the time-one map of $\tilde{I} = (\tilde{F}_t)_{t \in [0,1]}$ that is the lifted identity isotopy of I on the universal covering space $\pi: \tilde{M} \rightarrow M$.

Example 7.1. We construct an identity isotopy I of M and a measure $\mu \in \mathcal{M}(F)$ such that

- $\rho_{M,I}(\mu) = 0$;
- $F \in \text{Diff}(M)$ (and hence I satisfies the WB-property);
- I does not satisfy the B-property (and hence $F \notin \text{Diff}^1(M)$);
- there is a compact set $\tilde{P} \subset \tilde{M}$ and $\{(\tilde{z}_k, \tilde{z}'_k)\}_{k \geq 1} \subset (\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta}$ in $\tilde{P} \times \tilde{P}$, such that the linking numbers $i(\tilde{F}; \tilde{z}_k, \tilde{z}'_k, z)$ are not uniformly bounded;
- the action L_μ (see Section 6.1) is not bounded.

Use the Cartesian (x, y) -coordinate system in D and suppose $z_0 = (0, 0)$. On the x -axis, we suppose that B_k ($k \geq 1$) is a ball whose center is on $z_k = 1/(k+1) + 1/(2k(k+1))$ and whose radius is $r_k = 1/2(k+1)^2$.

Consider a family of smooth functions $\alpha_k : [0, r_k] \rightarrow \mathbb{R}$ such that $\alpha_k = 0$ on neighborhoods of 0 and r_k , $\alpha_k(r_k/2) = 2(-1)^k(k+1)^5$ and

$$2\pi \int_0^{r_k} \alpha_k(r) r \, dr = (-1)^k k.$$

Consider the following diffeomorphism F of D which is defined by the formula:

$$(7.1) \quad F(z_k + re^{2i\pi\theta}) = \begin{cases} z_k + re^{2i\pi(\theta + \alpha_k(r))} & \text{on } B_k; \\ \text{Id} & \text{on } D \setminus \bigcup_{k \geq 1} B_k. \end{cases}$$

We construct an isotopy $I = (F_t)_{t \in [0,1]}$ on D by replacing $\alpha_k(r)$ with $t\alpha_k(r)$ in Formula 7.1.

Obviously, z_k and $z'_k = z_k + r_k/2$ are fixed points of F and we have

$$i(\tilde{F}; \tilde{z}_k, \tilde{z}'_k) = 2(-1)^k(k+1)^5$$

and

$$i(\tilde{F}; \tilde{z}_0, \tilde{z}_k, z'_k) = \rho_{A_{\tilde{z}_0, \tilde{z}_k}, \tilde{F}_{\tilde{z}_0, \tilde{z}_k}}(\tilde{z}'_k) = 2(-1)^{k+1}(k+1)^5$$

where \tilde{z}_0, \tilde{z}_k and \tilde{z}'_k are contained in a connected component \tilde{D} of $\pi^{-1}(D)$. Hence I does not satisfy the B-property and there is a compact set $\text{Cl}(\tilde{D})$ and $\{\tilde{z}_k\}_{k \geq 1} \subset \text{Fix}(F) \setminus \{\tilde{z}_0\}$ in $\text{Cl}(\tilde{D})$, such that the linking numbers $i(\tilde{F}; \tilde{z}_0, \tilde{z}_k, z)$ are not uniformly bounded.

It is easy to prove that F is a diffeomorphism of M but it is not a C^1 -diffeomorphism of M : its differential DF is not continuous at z_0 .

Consider a finite measure μ on M satisfying that

- μ has full support;
- μ is non-atomic;
- μ restricted on B_k is the Lebesgue measure with $\mu(B_k) = \pi r_k^2$ for every $k \geq 1$.

Obviously, $\mu \in \mathcal{M}(F)$ and $\rho_{M,I}(\mu) = 0$. Furthermore, we have

$$I_\mu(\tilde{F}; z_{k+1}, z_k) = i_\mu(\tilde{F}; \tilde{z}_{k+1}, \tilde{z}_k) = (-1)^{k+1}(2k+1)$$

and

$$I_\mu(\tilde{F}; z_0, z_k) = i_\mu(\tilde{F}; \tilde{z}_0, \tilde{z}_k) = (-1)^{k+1}k.$$

Therefore, the action L_μ is not bounded.

Example 7.2. We construct an isotopy I of M and a measure $\mu \in \mathcal{M}(F)$ such that

- $F \notin \text{Diff}(M)$;
- I satisfies the B-property;
- there are two different fixed points \tilde{z}_0 and \tilde{z}_1 of \tilde{F} such that the linking number $i(\tilde{F}; \tilde{z}_0, \tilde{z}_1, z)$ is not bounded;
- there are two different fixed points \tilde{z}_0 and \tilde{z}_1 of \tilde{F} such that the linking number $i(\tilde{F}; \tilde{z}_0, \tilde{z}_1, z)$ is not μ -integrable.

Now we consider the polar coordinate for D with the center $z_0 = (0, 0)$ and suppose $z_1 = (4/5, 0)$. Let $D_{p/q} = \{(r, \theta) \mid r \in]0, p/q[\}$ where $p/q \in]0, 1[\cap \mathbb{Q}$. Consider a smooth decreasing function $\alpha : [0, 3/4] \rightarrow \mathbb{R}$ such that $\alpha|_{[0, 1/2]} \equiv 1$ and $\alpha = 0$ on neighborhood of $3/4$. Take a C^∞ -diffeomorphism $\rho(r)$ of $]0, 3/4[$ as follows

- $\rho(r)$ fixes the point $1/k$ for every $k > 1$ and $\rho(r) = r$ when $r \in [1/2, 3/4[$;
- $\rho^n(r) \rightarrow 1/(k+1)$ when $n \rightarrow -\infty$ for every $k > 1$ and $r \in]1/(k+1), 1/k[$;
- $\rho^n(r) \rightarrow 1/k$ when $n \rightarrow +\infty$ for every $k > 1$ and $r \in]1/(k+1), 1/k[$.

Consider the following homeomorphism F of D defined on D by the formula:

$$(7.2) \quad F(re^{2i\pi\theta}) = \begin{cases} \rho(r)e^{2i\pi(\theta + \alpha(r)(2^{\frac{1}{r}} + \frac{1}{2}))} & \text{on } D_{3/4}; \\ \text{Id} & \text{on } D \setminus D_{3/4}. \end{cases}$$

We construct an isotopy $I = (F_t)_{t \in [0, 1]}$ on D by replacing $\alpha(r)(2^{\frac{1}{r}} + \frac{1}{2})$ with $t\alpha(r)(2^{\frac{1}{r}} + \frac{1}{2})$ and $\rho(r)$ with $(1-t)r + t\rho(r)$ in Formula 7.2. It is easy to see that F is not differentiable at z_0 .

Let $B_k = \{(r, \theta) \mid r \in]1/(k+1), 1/k[\}$ and $C_k = \{z \in D \mid |z| = 1/k\}$ ($k \geq 2$). Consider a finite measure μ on M that is invariant by F as follows

$$\mu = \sum_{k \geq 2} 2^{-(k-1)} \mu_k$$

where μ_k is the Lebesgue probability measure on C_k .

Fix one point $z_k \in C_k$ for every $k \geq 2$. Let \tilde{z}_k ($k \geq 0$) be any lift of z_k contained in a connected component of $\pi^{-1}(D)$. For any point $z \in B_k$, the ω -limit set of z is included in C_k and the α -limit set of z is included in C_{k+1} . When $z \in C_k$, the angle of the trajectory of $I(z)$ rotating around z_0 is $(2^{k+1} + 1)\pi$. Hence F has no contractible fixed points on $D_{1/2}$. When $z \in D_{3/4} \setminus D_{1/2}$, the angle of the trajectory of $I(z)$ rotating around z_0 is uniformly bounded. Therefore, I satisfies the B-property. However, $i(\tilde{F}; \tilde{z}_0, \tilde{z}_1, z_k) = 2^k + 1/2$ and $i(\tilde{F}; \tilde{z}_0, \tilde{z}_1, z)$ is not μ -integrable. Remark that the support of μ is not the whole space.

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