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Some properties of Hamiltonian homeomorphisms on aspherical closed surfaces

by

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SOME PROPERTIES OF HAMILTONIAN HOMEOMORPHISMS ON ASPHERICAL CLOSED SURFACES

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ABSTRACT. In this article, we generalize Schwarz's theorem to C^0-case on aspherical closed surfaces and prove that a nontrivial Hamiltonian homeomorphism has no connected contractible fixed points set, thus no connected fixed points set.

1. INTRODUCTION

1.1. **Background.** The famous Gromov-Eliashberg Theorem, that the group of symplectic diffeomorphisms is C^0-closed in the full group of diffeomorphisms, makes us interested in defining a symplectic homeomorphism as a homeomorphism which is a C^0-limit of symplectic diffeomorphisms. This becomes a central theme of what is now called "C^0-symplectic topology". There is a family of problems in symplectic topology that are interesting to be extended to the continuous analogs of classical smooth objects of the symplectic world (see, e.g., [Lec05, Lec06, Vit06, OM07, Hum11, BS13, Se13, HLecS15, HLeRS15, BHS16]). In the theme of C^0-symplectic topology, there are many questions still open, e.g., the C^0-flux conjecture (see [LMP98, On06, B14]), the simplicity of the group of Hamiltonian homeomorphisms of surfaces (see [Fat80, OM07]).

Suppose that (M,ω) is a symplectic manifold. Let $I=(F_t)_{t\in\mathbb{R}}$ be a Hamiltonian flow on M with $F_0=\mathrm{Id}_M$ and $F_1=F$. When M is compact, among the properties of F, one may notice the fact that it preserves the volume form $\omega^n=\omega\wedge\cdots\wedge\omega$ and that the "rotation vector" $\rho_{M,I}(\mu)\in H_1(M,\mathbb{R})$ (see Section 2.3) of the finite measure μ induced by ω^n vanishes. Let M be a closed oriented surface with genus $g\geq 1$. In this case, M is an aspherical closed surface with the property $\pi_2(M)=0$. Let $I=(F_t)_{t\in[0,1]}$ be an identity isotopy on M, that is, a continuous path in $\mathrm{Homeo}(M)$ with $F_0=\mathrm{Id}_M$. We suppose that its time-one map F preserves the measure μ induced by ω. It is well known that the condition $\rho_{M,I}(\mu)=0$ is equivalent to saying that the homeomorphism F is in the C^0-closure of $\mathrm{Ham}(M,\omega)$. In this sense, we call such I a *Hamiltonian isotopy* and such F a *Hamiltonian homeomorphism.* In this article, we carry out some foundational studies of Hamiltonian homeomorphisms (and a more general notion) on aspherical closed surfaces.

Let (M,ω) be a symplectic manifold with $\pi_2(M)=0$. Suppose that $H:\mathbb{R}\times M\to\mathbb{R}$, one-periodic in time, is the Hamiltonian function generating the flow I. Denote by $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ the set of contractible fixed points of F, that is, $x\in\mathrm{Fix}_{\mathrm{Cont},I}(F)$ if and only if x is a fixed point of F and the oriented loop $I(x):t\mapsto F_t(x)$ defined on $[0,1]$ is contractible on M. The classical action function is defined, up to an additive constant, on

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$\mathrm{Fix}_{\mathrm{Cont},I}(F)$ as follows

$$\mathcal{A}_H(x) = \int_{D_x} \omega - \int_0^1 H(t, F_t(x))\,\mathrm{d}t,$$

where $x \in \mathrm{Fix}_{\mathrm{Cont},I}(F)$ and $D_x \subset M$ is any 2-simplex with $\partial D_x = I(x)$. The following deep result was proved [Sz00] by using Floer homology with the real filtration induced by the action function.

Theorem 1.1 (Schwarz). *Let (M, ω) be a closed symplectic manifold with $\pi_2(M) = 0$. Let $I = (F_t)_{t\in\mathbb{R}}$ be a Hamiltonian flow on M with $F_0 = \mathrm{Id}_M$ and $F_1 = F$ generated by a Hamiltonian function H. Assume that $F \neq \mathrm{Id}_M$. Then there are $x, y \in \mathrm{Fix}_{\mathrm{Cont},I}(F)$ such that $\mathcal{A}_H(x) \neq \mathcal{A}_H(y)$.*

Let M be a closed oriented surface with genus $g \geq 1$ and F be a time-one map of an identity isotopy I on M. Denote by $\mathcal{M}(F)$ the set of Borel finite measures on M that are invariant by F and have no atoms on $\mathrm{Fix}_{\mathrm{Cont},I}(F)$. Through the WB-property [Wang11a] (see Section 2.4 for details), the classical action of Hamiltonian diffeomorphism has been generalized to Hamiltonian homeomorphism (and more general notions) [Wang11a, page 86] (or see [Wang11b]):

Theorem 1.2. *Let $F \in \mathrm{Homeo}(M)$ be the time-one map of an identity isotopy I on M. Suppose that $\mu \in \mathcal{M}(F)$ and $\rho_{M,I}(\mu) = 0$. In each of the following cases:*

- *$F \in \mathrm{Diff}(M)$ (not necessarily C^1);*
- *I satisfies the WB-property and the measure μ has full support;*
- *I satisfies the WB-property and the measure μ is ergodic,*

an action function can be defined, which generalizes the classical case.

In the classical case, one can prove that the action function is a constant on a connected set of contractible fixed points by applying the smoothness of the action function and the nowhere dense property of the action spectrum. In all the generalized cases given in Theorem 1.2, we show in this article that this property still holds.

One may ask whether Schwarz's theorem is still true in the three cases above. We show in this article that it is true in the second case but no longer when the measure μ has no full support.

As an application, we show that a nontrivial Hamiltonian homeomorphism has no connected contractible fixed points set, thus no connected fixed points set. We remark that this property is a 2-dimension phenomenon. Recently, Buhovsky, Humilière and Seyfaddini [BHS16] have constructed a Hamiltonian homeomorphism with only one fixed point on any closed symplectic manifold of dimension at least 4.

1.2. **Statement of results.** We say that a homeomorphism F is *μ-symplectic* if $\mu \in \mathcal{M}(F)$ and it has full support. An identity isotopy I is *μ-Hamiltonian* if the time-one map F is μ-symplectic and moreover $\rho_{M,I}(\mu) = 0$.

The main results of this article are summarized as follows.

Proposition 1.3. *Under the hypotheses of Theorem 1.2, the action function defined in Theorem 1.2 is a constant on each connected component of* $\mathrm{Fix}_{\mathrm{Cont},I}(F)$.

Theorem 1.4. *Let F be the time-one map of a μ-Hamiltonian isotopy I. If I satisfies the WB-property and $F \neq \mathrm{Id}_M$, the action function defined in Theorem 1.2 is not constant.*

Theorem 1.4 is a generalization of Schwarz's theorem on closed oriented surfaces. The main tools we use in its proof are the theory of transverse foliations for dynamical systems of surfaces inspired by Le Calvez [Lec05, Lec06] and its recent progress [Jau14].

Recall the classical version of Arnold conjecture for surface homeomorphisms due to Matsumoto [Mat00] (see also [Lec05]): any Hamiltonian homeomorphism has at least three contractible fixed points (see Theorem 5.1 below).

As a consequence of Proposition 1.3 and Theorem 1.4, we have the following theorem:

Theorem 1.5. *Let F be the time-one map of a μ-Hamiltonian isotopy I. If the set* $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ *is connected, the time-one map F must be* Id_M*. In particular, if* $\mathrm{Fix}(F)$ *is connected, F must be* Id_M*.*

Proof. By Theorem 5.1, $\mathrm{Fix}_{\mathrm{Cont},I}(F) \neq \emptyset$. Moreover, if the set $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ is connected, the isotopy must satisfy the WB-property (see Lemma 2.8). Therefore, the action function is well defined by Theorem 1.2. The conclusion follows by Proposition 1.3 and Theorem 1.4. Note that the connectedness of $\mathrm{Fix}(F)$ implies that $\mathrm{Fix}_{\mathrm{Cont},I}(F) = \mathrm{Fix}(F)$ because $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ is an open and closed subset of $\mathrm{Fix}(F)$. □

If $F \neq \mathrm{Id}_M$, Theorem 1.5 implies that the number of connected components of $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ is at least 2, which is optimal by the following example.

Example 1.6. Let μ be the measure induced by the area form ω and D be a topological closed disk on M. Up to a diffeomorphism, we may suppose that D is the closed unit Euclidean disk. Let us consider the polar coordinate for D with the center $z_0 = (0,0)$. We suppose that the area form $\omega|_D = rdr \wedge d\theta$. Consider the following isotopy $(F_t)_{t\in[0,1]}$ on M which defined on D by the formula

$$\begin{array}{rcl} F_t : D & \to & D \\ (r,\theta) & \mapsto & (r, \theta + 2\pi r t), \end{array}$$

and $F_t|_{M\setminus D} = \mathrm{Id}_{M\setminus D}$ for all $t \in [0,1]$. Obviously, $\rho_{M,I}(\mu) = 0$ and $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ has exactly two connected components: $\{z_0\}$ and $M \setminus \mathrm{Int}(D)$, where $\mathrm{Int}(D)$ is the interior of D.

By Theorem 1.5 and Theorem 5.1, if $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ has exactly two connected components, its cardinality must be infinite.

Remark that neither Theorem 1.4 nor 1.5 is valid when the measure does not have full support. We refer to Example 6.3 and Example 6.4 for counterexamples of Theorem 1.4. Example 6.3 is also a counterexample of Theorem 1.5 on a torus. When the genus of M is more than 2, one can choose an identity isotopy on M with exactly one contractible fixed point z (such isotopy exists by Lefschetz-Nielsen's formula) and the Dirac measure δ_z.

The article is organized as follows. In Section 2, we first introduce some notations, and recall the linking number on contractible fixed points and the boundedness properties. In Section 3, we explain the approach to defining the generalized action function and study the continuity of this action function. Our main results Proposition 1.3 and Theorem 1.4 will be proved in Section 4 and Section 5, respectively. In Appendix, we provide the proofs of the lemmas which are not given in the main sections and construct Example 6.3 and Example 6.4.

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2. Notations

We denote by $|\cdot|$ the usual Euclidean metric on $\mathbb{R}^k$ or $\mathbb{C}^k$ and by $\mathbf{S}^{k-1}=\{x\in\mathbb{R}^k \mid |x|=1\}$ the unit sphere.

If A is a set, we write $\sharp A$ for the cardinality of A. If (S,σ,μ) is a measure space and V is any finite dimensional linear space, denote by $L^1(S,V,\mu)$ the set of μ-integrable functions from S to V. If X is a topological space and A is a subset of X, denote by $\mathrm{Int}_X(A)$ and $\mathrm{Cl}_X(A)$ respectively the interior and the closure of A. We will omit the subscript X if there is no any confusion. If M is a manifold and N is a submanifold of M, we denote by ∂N the boundary of N on M. If M is a smooth manifold (with boundary or not), we denote by $\mathrm{Homeo}(M)$ (resp. $\mathrm{Diff}(M)$, $\mathrm{Diff}^1(M)$) the set of all homeomorphisms (resp. diffeomorphisms, C^1-diffeomorphisms) of M.

2.1. **Identity isotopies.** An *identity isotopy* $I=(F_t)_{t\in[0,1]}$ on M is a continuous path

$$\begin{array}{ccc}[0,1] & \to & \mathrm{Homeo}(M)\\ t & \mapsto & F_t\end{array}$$

such that $F_0=\mathrm{Id}_M$, the last set being endowed with the compact-open topology. We naturally extend this map to $\mathbb{R}$ by writing $F_{t+1}=F_t\circ F_1$. We can also define the inverse isotopy of I as $I^{-1}=(F_{-t})_{t\in[0,1]}=(F_{1-t}\circ F_1^{-1})_{t\in[0,1]}$. We denote by $\mathrm{Homeo}_*(M)$ the set of all homeomorphisms of M that are isotopic to the identity.

A *path* on a manifold M is a continuous map $\gamma: J\to M$ defined on a nontrivial interval J (up to an increasing reparametrization). We can talk of a proper path (i.e. $\gamma^{-1}(K)$ is compact for any compact set K) or a compact path (i.e. J is compact). When γ is a compact path, $\gamma(\inf J)$ and $\gamma(\sup J)$ are the *ends* of γ. We say that a compact path γ is a *loop* if the two ends of γ coincide. The inverse of the path γ is defined by $\gamma^{-1}: t\mapsto\gamma(-t)$, $t\in -J$. If $\gamma_1: J_1\to M$ and $\gamma_2: J_2\to M$ are two paths such that

$$b_1=\sup J_1\in J_1,\quad a_2=\inf J_2\in J_2\quad\text{and}\quad\gamma_1(b_1)=\gamma_2(a_2),$$

then the *concatenation* γ_1 *and* γ_2 is defined on $J=J_1\cup(J_2+(b_1-a_2))$ in the classical way, where $(J_2+(b_1-a_2))$ represents the translation of J_2 by (b_1-a_2):

$$\gamma_1\gamma_2(t)=\begin{cases}\gamma_1(t) & \text{if}\quad t\in J_1;\\ \gamma_2(t+a_2-b_1) & \text{if}\quad t\in J_2+(b_1-a_2).\end{cases}$$

Let $\mathcal{I}$ be an interval (maybe infinite) of $\mathbb{Z}$. If $\{\gamma_i: J_i\to M\}_{i\in\mathcal{I}}$ is a family of compact paths satisfying that $\gamma_i(\sup(J_i))=\gamma_{i+1}(\inf(J_{i+1}))$ for every $i\in\mathcal{I}$, then we can define their concatenation $\prod_{i\in\mathcal{I}}\gamma_i$.

If $\{\gamma_i\}_{i\in\mathcal{I}}$ is a family of compact paths where $\mathcal{I}=\bigsqcup_{j\in\mathcal{J}}\mathcal{I}_j$ and $\mathcal{I}_j$ is an interval of $\mathbb{Z}$ such that $\prod_{i\in\mathcal{I}_j}\gamma_i$ is well defined (in the concatenation sense) for every $j\in\mathcal{J}$, we define their *product* by abusing notations:

$$\prod_{i\in\mathcal{I}}\gamma_i=\prod_{j\in\mathcal{J}}\prod_{i\in\mathcal{I}_j}\gamma_i.$$

The *trajectory* of a point z for the isotopy $I=(F_t)_{t\in[0,1]}$ is the oriented path $I(z): t\mapsto F_t(z)$ defined on $[0,1]$. Suppose that $\{I_k\}_{1\le k\le k_0}$ is a family of identity isotopies on M. Write $I_k=(F_{k,t})_{t\in[0,1]}$. We can define a new identity isotopy $I_{k_0}\cdots I_2 I_1=(F_t)_{t\in[0,1]}$ by concatenation as follows

$$F_t(z)=F_{k,\,k_0t-(k-1)}(F_{k-1,1}\circ F_{k-2,1}\circ\cdots\circ F_{1,1}(z))\quad\text{if}\quad\frac{k-1}{k_0}\le t\le\frac{k}{k_0}. \tag{2.1}$$

In particular, $I^{k_0}(z)=\prod_{k=0}^{k_0-1} I(F^k(z))$ when $I_k = I$ for all $1\leq k\leq k_0$.

We write $\mathrm{Fix}(F)$ for the set of fixed points of F. A fixed point z of $F=F_1$ is *contractible* if $I(z)$ is homotopic to zero. We write $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ for the set of contractible fixed points of F, which obviously depends on I.

2.2. **The algebraic intersection number.** The choice of an orientation on M permits us to define the algebraic intersection number $\Gamma\wedge\Gamma'$ between two loops. We keep the same notation $\Gamma\wedge\gamma$ for the algebraic intersection number between a loop and a path γ when it is defined, for example, when γ is proper or when γ is compact path whose extremities are not in Γ. Similarly, we write $\gamma\wedge\gamma'$ for the algebraic intersection number of two path γ and γ' when it is defined, for example, when γ and γ' are compact paths and the ends of γ (resp. γ') are not on γ' (resp. γ). If Γ is a loop on a smooth manifold M, write $[\Gamma]\in H_1(M,\mathbb{Z})$ for the homology class of Γ. It is clear that the value $\Gamma\wedge\gamma$ does not depends on the choice of the path γ that fixes its endpoints when $[\Gamma]=0$.

2.3. **Rotation vector.** Let us introduce the classical notion of rotation vector which was defined originally in [St57]. Suppose that F is the time-one map of an identity isotopy $I=(F_t)_{t\in[0,1]}$. Let $\mathrm{Rec}^+(F)$ be the set of positively recurrent points of F. If $z\in\mathrm{Rec}^+(F)$, fix an open disk $U\subset M$ containing z, and write $\{F^{n_k}(z)\}_{k\geq 1}$ for the subsequence of the positive orbit of z obtained by keeping the points that are in U. For any $k\geq 0$, choose a simple path $\gamma_{F^{n_k}(z),z}$ in U joining $F^{n_k}(z)$ to z. The homology class $[\Gamma_k]\in H_1(M,\mathbb{Z})$ of the loop $\Gamma_k=I^{n_k}(z)\gamma_{F^{n_k}(z),z}$ does not depend on the choice of $\gamma_{F^{n_k}(z),z}$. Say that z has a *rotation vector* $\rho_{M,I}(z)\in H_1(M,\mathbb{R})$ if

$$\lim_{l\to+\infty}\frac{1}{n_{k_l}}[\Gamma_{k_l}]=\rho_{M,I}(z)$$

for any subsequence $\{F^{n_{k_l}}(z)\}_{l\geq 1}$ which converges to z. Neither the existence nor the value of the rotation vector depends on the choice of U.

Suppose that M is compact and that F is the time-one map of an identity isotopy $I=(F_t)_{t\in[0,1]}$ on M. Recall that $\mathcal{M}(F)$ is the set of Borel finite measures on M whose elements are invariant by F. If $\mu\in\mathcal{M}(F)$, we can define the rotation vector $\rho_{M,I}(z)$ for μ-almost every positively recurrent point. Moreover, we can prove that the rotation vector is uniformly bounded if it exists (See [Wang11a, page 52]). Therefore, we define the *rotation vector of the measure*

$$\rho_{M,I}(\mu)=\int_M \rho_{M,I}\,\mathrm{d}\mu\in H_1(M,\mathbb{R}).$$

2.4. **The weak boundedness property and the boundedness property.**

2.4.1. We begin by recalling some results about identity isotopies, which will be often used in the article.

Remark 2.1. Suppose that M is an oriented compact surface and that F is the time-one map of an identity isotopy $I=(F_t)_{t\in[0,1]}$ on M. When $z\in\mathrm{Fix}_{\mathrm{Cont},I}(F)$, there is another identity isotopy $I'=(F'_t)_{t\in[0,1]}$ homotopic to I with fixed endpoints such that I' fixes z (see, e.g., [Jau14, Proposition 2.15]), that is, there is a continuous map $H:[0,1]\times[0,1]\times M\to M$ such that

- $H(0,t,z)=F_t(z)$ and $H(1,t,z)=F'_t(z)$ for all $t\in[0,1]$;
- $H(s,0,z)=\mathrm{Id}_M(z)$ and $H(s,1,z)=F(z)$ for all $s\in[0,1]$;
- $F'_t(z)=z$ for all $t\in[0,1]$.

Lemma 2.2 ([Wang11a], page 54). *Let $\mathbf{S}^2$ be the 2-sphere and $I = (F_t)_{t\in[0,1]}$ be an identity isotopy on $\mathbf{S}^2$. For every three different fixed points z_i ($i = 1, 2, 3$) of F_1, there exists another identity isotopy $I' = (F'_t)_{t\in[0,1]}$ from $\mathrm{Id}_{\mathbf{S}^2}$ to F_1 such that I' fixes z_i ($i = 1, 2, 3$).*

As a consequence, we have the following corollary.

Corollary 2.3. *Let $I = (F_t)_{t\in[0,1]}$ be an identity isotopy on $\mathbb{C}$. For any two different fixed points z_1 and z_2 of F_1, there exists another identity isotopy I' from $\mathrm{Id}_{\mathbb{C}}$ to F_1 such that I' fixes z_1 and z_2.*

Remark 2.4. Let $z_i \in \mathbf{S}^2$ ($i = 1, 2, 3$) and $\mathrm{Homeo}_*(\mathbf{S}^2, z_1, z_2, z_3)$ be the identity component of the space of all homeomorphisms of $\mathbf{S}^2$ leaving z_i ($i = 1, 2, 3$) pointwise fixed (for the compact-open topology). It is well known that $\pi_1(\mathrm{Homeo}_*(\mathbf{S}^2, z_1, z_2, z_3)) = 0$ (see [Ham66, Han92]). It implies that any two identity isotopies $I, I' \subset \mathrm{Homeo}_*(\mathbf{S}^2, z_1, z_2, z_3)$ with fixed endpoints are homotopic. As a consequence, let $\mathrm{Homeo}_*(\mathbb{C}, z_1, z_2)$ be the identity component of the space of all homeomorphisms of $\mathbb{C}$ leaving two different points z_1 and z_2 of $\mathbb{C}$ pointwise fixed, we have $\pi_1(\mathrm{Homeo}_*(\mathbb{C}, z_1, z_2)) = 0$.

2.4.2. *Linking number.* Let M be a surface that is homeomorphic to the complex plane $\mathbb{C}$ and $I = (F_t)_{t\in[0,1]}$ be an identity isotopy on M. Let us define the *linking number* $i_I(z, z') \in \mathbb{Z}$ for every two different fixed points z and z' of F_1. It is the degree of the map $\xi : \mathbf{S}^1 \to \mathbf{S}^1$ defined by

$$\xi(e^{2i\pi t}) = \frac{h\circ F_t(z') - h\circ F_t(z)}{|h\circ F_t(z') - h\circ F_t(z)|},$$

where $h : M \to \mathbb{C}$ is a homeomorphism. The linking number does not depend on the chosen h.

Let F be the time-one map of an identity isotopy $I = (F_t)_{t\in[0,1]}$ on a closed oriented surface M of genus $g \geq 1$ and $\widetilde{F}$ be the time-one map of the lifted identity isotopy $\widetilde{I} = (\widetilde{F}_t)_{t\in[0,1]}$ on the universal cover $\widetilde{M}$ of M. Let $\pi : \widetilde{M} \to M$ be the covering map and G be the covering transformation group. Denote respectively by Δ and $\widetilde{\Delta}$ the diagonal of $\mathrm{Fix}_{\mathrm{Cont},I}(F) \times \mathrm{Fix}_{\mathrm{Cont},I}(F)$ and the diagonal of $\mathrm{Fix}(\widetilde{F}) \times \mathrm{Fix}(\widetilde{F})$. Endow the surface M with a Riemannian metric and denote by d the distance induced by the metric. Lift the Riemannian metric to $\widetilde{M}$ and write $\widetilde{d}$ for the distance induced by the metric.

When $g > 1$, it is well known that $\pi_1(\mathrm{Homeo}_*(M)) \simeq 0$ (see [Ham66]). It implies that any two identity isotopies $I, I' \subset \mathrm{Homeo}_*(M)$ with fixed endpoints are homotopic. Hence, I is unique up to homotopy, it implies that $\widetilde{F}$ is uniquely defined and does not depend on the choice of the isotopy from Id_M to F. When $g = 1$, $\pi_1(\mathrm{Homeo}_*(M)) \simeq \mathbb{Z}^2$ (see [Ham65]), $\widetilde{F}$ depends on the isotopy I. The universal cover $\widetilde{M}$ is homeomorphic to $\mathbb{C}$.

We define the *linking number* $i(\widetilde{F}; \widetilde{z}, \widetilde{z}')$ for every pair $(\widetilde{z}, \widetilde{z}') \in (\mathrm{Fix}(\widetilde{F}) \times \mathrm{Fix}(\widetilde{F})) \setminus \widetilde{\Delta}$ as

$$i(\widetilde{F}; \widetilde{z}, \widetilde{z}') = i_{\widetilde{I}}(\widetilde{z}, \widetilde{z}'). \tag{2.2}$$

2.4.3. *WB-property and B-property.* In the rest of the article, when we take two distinct fixed points $\widetilde{a}$ and $\widetilde{b}$ of $\widetilde{F}$, it does not mean that $\pi(\widetilde{a})$ and $\pi(\widetilde{b})$ are distinct.

Definition 2.5. We say that I satisfies *the weak boundedness property at* $\widetilde{a} \in \mathrm{Fix}(\widetilde{F})$ *(WB-property at $\widetilde{a}$)* if there exists a positive number $N_{\widetilde{a}}$ such that $|i(\widetilde{F}; \widetilde{a}, \widetilde{b})| \leq N_{\widetilde{a}}$ for all $\widetilde{b} \in \mathrm{Fix}(\widetilde{F}) \setminus \{\widetilde{a}\}$. We say that I satisfies *the weak boundedness property (WB-property)* if it satisfies the weak boundedness property at every $\widetilde{a} \in \mathrm{Fix}(\widetilde{F})$. Let $\widetilde{X} \subseteq \mathrm{Fix}(\widetilde{F})$. We say that I satisfies *the boundedness property on* $\widetilde{X}$ *(B-property on* $\widetilde{X}$*)* if there exists a positive number $N_{\widetilde{X}}$ such that $|i(\widetilde{F};\widetilde{a},\widetilde{b})| \leq N_{\widetilde{X}}$ for all $(\widetilde{a},\widetilde{b}) \in \widetilde{X} \times \mathrm{Fix}(\widetilde{F})$ with $\widetilde{a} \neq \widetilde{b}$. We say that I satisfies *the boundedness property (B-property)* if $\widetilde{X} = \mathrm{Fix}(\widetilde{F})$.

In [Wang11a], we have proved that the WB-property is satisfied if $F \in \mathrm{Diff}(M)$ and the B-property is satisfied if $F \in \mathrm{Diff}^1(M)$. Le Roux [Ler14] has proved that the set of all WB-property points of I is dense in $\mathrm{Fix}(\widetilde{F})$.

Let X be a connected component of $\mathrm{Fix}_{\mathrm{Cont},I}(F)$. Either X is *contractible*, that means it is included in an open disk. In this case, the preimage of X in the universal covering space is a disjoint union of sets $\widetilde{X}$ such that the projection induces a homeomorphim from $\widetilde{X}$ to X. Or X is not contractible, and in this case every connected component of the preimage of X is unbounded.

To prove our main results, we need the following three lemmas whose proofs are provided in Appendix.

Lemma 2.6. *If* $\widetilde{X}$ *is a connected subset of* $\mathrm{Fix}(\widetilde{F})$ *and* $\widetilde{z} \in \mathrm{Fix}(\widetilde{F})$*, then* $i(\widetilde{F};\widetilde{z},\widetilde{z}')$ *(*$\widetilde{z}'$ *as variable,* $\widetilde{z}' \neq \widetilde{z}$*) is a constant on* $\widetilde{X}$*. Furthermore, if* $\widetilde{X}$ *is not reduced to a singleton,* $i(\widetilde{F};\cdot,\cdot)$ *is a constant on* $(\widetilde{X} \times \widetilde{X}) \setminus \widetilde{\Delta}$*.*

Lemma 2.7. *If* $\widetilde{X}$ *is a connected unbounded subset of* $\mathrm{Fix}(\widetilde{F})$*, then* $i(\widetilde{F};\widetilde{z},\widetilde{z}') = 0$ *for all* $(\widetilde{z},\widetilde{z}') \in \mathrm{Fix}(\widetilde{F}) \times \widetilde{X}$ *with* $\widetilde{z} \neq \widetilde{z}'$*. Consequently, if* X *is a connected component of* $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ *and* X *is not contractible,* $i(\widetilde{F};\widetilde{z},\widetilde{z}') = 0$ *for all* $(\widetilde{z},\widetilde{z}') \in \mathrm{Fix}(\widetilde{F}) \times \pi^{-1}(X)$ *with* $\widetilde{z} \neq \widetilde{z}'$*.*

Lemma 2.8. *If* X *is a connected subset of* $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ *and* X *is not reduced to a singleton,* I *satisfies the B-property on* $\pi^{-1}(X)$*. As a conclusion, if the set* $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ *is connected,* I *satisfies the B-property.*

3. The generalized action function revisited

In this section, we explain the approach to defining the generalized action function and study the continuity of this function.

3.1. **The linking number of positively recurrent points.**

Recall that F is the time-one map of an identity isotopy $I = (F_t)_{t\in[0,1]}$ on a closed oriented surface M of genus $g \geq 1$ and $\widetilde{F}$ is the time-one map of the lifted identity isotopy $\widetilde{I} = (\widetilde{F}_t)_{t\in[0,1]}$ on the universal cover $\widetilde{M}$ of M. We can compactify $\widetilde{M}$ into a sphere by adding a point ∞ at infinity and the lift $\widetilde{F}$ may be extended by fixing this point.

For every distinct fixed points $\widetilde{a}$ and $\widetilde{b}$ of $\widetilde{F}$, by Lemma 2.2, we can choose an isotopy $\widetilde{I}_1$ from $\mathrm{Id}_{\widetilde{M}}$ to $\widetilde{F}$ that fixes $\widetilde{a}$ and $\widetilde{b}$.

Fix $z \in \mathrm{Rec}^+(F) \setminus \pi(\{\widetilde{a},\widetilde{b}\})$ and consider an open disk $U \subset M \setminus \pi(\{\widetilde{a},\widetilde{b}\})$ containing z. define the first return map $\Phi : \mathrm{Rec}^+(F) \cap U \to \mathrm{Rec}^+(F) \cap U$ and write $\Phi(z) = F^{\tau(z)}(z)$, where $\tau(z)$ is the first return time, that is, the least number $n \geq 1$ such that $F^n(z) \in U$. For every pair $(z',z'') \in U^2$, choose an oriented simple path $\gamma_{z',z''}$ in U from z' to z''. Denote by $\widetilde{\Phi}$ the lift of the first return map Φ:

$$\begin{array}{rcl} \widetilde{\Phi} : \pi^{-1}(\mathrm{Rec}^+(F)) \cap \pi^{-1}(U) & \to & \pi^{-1}(\mathrm{Rec}^+(F)) \cap \pi^{-1}(U) \\ \widetilde{z} & \mapsto & \widetilde{F}^{\tau(\pi(\widetilde{z}))}(\widetilde{z}). \end{array}$$

For any $\widetilde{z} \in \pi^{-1}(U)$, write $U_{\widetilde{z}}$ the connected component of $\pi^{-1}(U)$ that contains $\widetilde{z}$. For every $j \geq 1$, let $\tau_j(z) = \sum\limits_{i=0}^{j-1} \tau(\Phi^i(z))$. For every $n \geq 1$, consider the following curves in $\widetilde{M}$:

$$\widetilde{\Gamma}^n_{\widetilde{I}_1,\widetilde{z}} = \widetilde{I}_1^{\tau_n(z)}(\widetilde{z})\widetilde{\gamma}_{\widetilde{\Phi}^n(\widetilde{z}),\widetilde{z}_n},$$

where $\widetilde{z}_n \in \pi^{-1}(\{z\}) \cap \widetilde{U}_{\widetilde{\Phi}^n(\widetilde{z})}$, and $\widetilde{\gamma}_{\widetilde{\Phi}^n(\widetilde{z}),\widetilde{z}_n}$ is the lift of $\gamma_{\Phi^n(z),z}$ that is contained in $\widetilde{U}_{\widetilde{\Phi}^n(\widetilde{z})}$. We define the following infinite product (see Section 2.1):

$$\widetilde{\Gamma}^n_{\widetilde{I}_1,z} = \prod_{\pi(\widetilde{z})=z} \widetilde{\Gamma}^n_{\widetilde{I}_1,\widetilde{z}}.$$

In particular, when $z \in \mathrm{Fix}(F)$, $\widetilde{\Gamma}^1_{\widetilde{I}_1,z} = \prod\limits_{\pi(\widetilde{z})=z} \widetilde{I}_1(\widetilde{z})$.

When $\widetilde{U}_{\widetilde{\Phi}^n(\widetilde{z})} = \widetilde{U}_{\widetilde{z}}$, the curve $\widetilde{\Gamma}^n_{\widetilde{I}_1,\widetilde{z}}$ is a loop and hence $\widetilde{\Gamma}^n_{\widetilde{I}_1,z}$ is an infinite family of loops, that will be called a *multi-loop*. When $\widetilde{U}_{\widetilde{\Phi}^n(\widetilde{z})} \neq \widetilde{U}_{\widetilde{z}}$, the curve $\widetilde{\Gamma}^n_{\widetilde{I}_1,\widetilde{z}}$ is a compact path and hence $\widetilde{\Gamma}^n_{\widetilde{I}_1,z}$ is an infinite family of paths (it can be seen as a family of proper paths, that means all of two ends of these paths going to ∞), that will be called a *multi-path*.

In the both cases, for every neighborhood $\widetilde{V}$ of ∞, there are finitely many loops or paths $\widetilde{\Gamma}^n_{\widetilde{I}_1,\widetilde{z}}$ that are not included in $\widetilde{V}$. By adding the point ∞ at infinity, we get a multi-loop on the sphere $\mathbf{S} = \widetilde{M} \sqcup \{\infty\}$.

Hence $\widetilde{\Gamma}^n_{\widetilde{I}_1,z}$ can be seen as a multi-loop in the annulus $A_{\widetilde{a},\widetilde{b}} = \mathbf{S} \setminus \{\widetilde{a},\widetilde{b}\}$ with a finite homology. As a consequence, if $\widetilde{\gamma}$ is a path from $\widetilde{a}$ to $\widetilde{b}$, the intersection number $\widetilde{\gamma} \wedge \widetilde{\Gamma}^n_{\widetilde{I}_1,z}$ is well defined and does not depend on $\widetilde{\gamma}$. By Remark 2.4 and the properties of intersection number, the intersection number is also independent of the choice of the identity isotopy $\widetilde{I}_1$ but depends on U. Moreover, observe that the path $(\prod_{i=0}^{n-1} \gamma_{\Phi^{n-i}(z),\Phi^{n-i-1}(z)})(\gamma_{\Phi^n(z),z})^{-1}$ is a loop in U, we have

$$\widetilde{\gamma} \wedge \widetilde{\Gamma}^n_{\widetilde{I}_1,z} = \widetilde{\gamma} \wedge \prod_{j=0}^{n-1} \widetilde{\Gamma}^1_{\widetilde{I}_1,\Phi^j(z)} = \sum_{j=0}^{n-1} \widetilde{\gamma} \wedge \widetilde{\Gamma}^1_{\widetilde{I}_1,\Phi^j(z)}. \tag{3.1}$$

For $n \geq 1$, we can define the function

$$L_n : ((\mathrm{Fix}(\widetilde{F}) \times \mathrm{Fix}(\widetilde{F})) \setminus \widetilde{\Delta}) \times (\mathrm{Rec}^+(F) \cap U) \to \mathbb{Z},$$

$$L_n(\widetilde{F};\widetilde{a},\widetilde{b},z) = \widetilde{\gamma} \wedge \widetilde{\Gamma}^n_{\widetilde{I}_1,z} = \sum_{j=0}^{n-1} L_1(\widetilde{F};\widetilde{a},\widetilde{b},\Phi^j(z)) \tag{3.2}$$

where $U \subset M \setminus \pi(\{\widetilde{a},\widetilde{b}\})$. The last equation follows from Equation 3.1. The function L_n depends on U but not on the choice of $\gamma_{\Phi^n(z),z}$.

Definition 3.1. Fix $z \in \mathrm{Rec}^+(F) \setminus \pi(\{\widetilde{a},\widetilde{b}\})$. Let us say that the linking number $i(\widetilde{F};\widetilde{a},\widetilde{b},z) \in \mathbb{R}$ is defined, if

$$\lim_{k\to+\infty} \frac{L_{n_k}(\widetilde{F};\widetilde{a},\widetilde{b},z)}{\tau_{n_k}(z)} = i(\widetilde{F};\widetilde{a},\widetilde{b},z)$$

for any subsequence $\{\Phi^{n_k}(z)\}_{k\geq 1}$ of $\{\Phi^n(z)\}_{n\geq 1}$ which converges to z.

Note here that the linking number $i(\widetilde{F};\widetilde{a},\widetilde{b},z)$ does not depend on U since if U and U' are open disks containing z, there exists a disk containing z that is contained in $U\cap U'$. In particular, when $z\in \mathrm{Fix}(F)\setminus\pi(\{\widetilde{a},\widetilde{b}\})$, the linking number $i(\widetilde{F};\widetilde{a},\widetilde{b},z)$ always exists and is equal to $L_1(\widetilde{F};\widetilde{a},\widetilde{b},z)$. Moreover, if $z\in \mathrm{Fix}_{\mathrm{Cont},I}(F)$, we have [Wang11a, page 57]

$$i(\widetilde{F};\widetilde{a},\widetilde{b},z)=\sum_{\pi(\widetilde{z})=z}\left(i(\widetilde{F};\widetilde{a},\widetilde{z})-i(\widetilde{F};\widetilde{b},\widetilde{z})\right). \tag{3.3}$$

3.2. **Some elementary properties of the linking number** [Wang11a, page 71-73]**.**

Proposition 3.2. *For every $\alpha\in G$, every distinct fixed points $\widetilde{a}$ and $\widetilde{b}$ of $\widetilde{F}$, and every $z\in \mathrm{Rec}^+(F)\setminus\pi(\{\widetilde{a},\widetilde{b}\})$, we have $L_n(\widetilde{F};\alpha(\widetilde{a}),\alpha(\widetilde{b}),z)=L_n(\widetilde{F};\widetilde{a},\widetilde{b},z)$ for every n. If $i(\widetilde{F};\widetilde{a},\widetilde{b},z)$ exists, then $i(\widetilde{F};\alpha(\widetilde{a}),\alpha(\widetilde{b}),z)$ also exists and $i(\widetilde{F};\alpha(\widetilde{a}),\alpha(\widetilde{b}),z)=i(\widetilde{F};\widetilde{a},\widetilde{b},z)$.*

Proposition 3.3. *For every distinct fixed points $\widetilde{a}$, $\widetilde{b}$ and $\widetilde{c}$ of $\widetilde{F}$, and every $z\in\mathrm{Rec}^+(F)\setminus\pi(\{\widetilde{a},\widetilde{b},\widetilde{c}\})$, we have $L_n(\widetilde{F};\widetilde{a},\widetilde{b},z)+L_n(\widetilde{F};\widetilde{b},\widetilde{c},z)+L_n(\widetilde{F};\widetilde{c},\widetilde{a},z)=0$ for all n. Moreover, if two among the three linking numbers $i(\widetilde{F};\widetilde{a},\widetilde{b},z)$, $i(\widetilde{F};\widetilde{b},\widetilde{c},z)$ and $i(\widetilde{F};\widetilde{c},\widetilde{a},z)$ exist, then the last one also exists and we have*

$$i(\widetilde{F};\widetilde{a},\widetilde{b},z)+i(\widetilde{F};\widetilde{b},\widetilde{c},z)+i(\widetilde{F};\widetilde{c},\widetilde{a},z)=0.$$

The following lemma gives the continuity property of the function L_k whose proof details will be used in the proof of Proposition 1.3.

Lemma 3.4. *Suppose that $\widetilde{a}\in\mathrm{Fix}(\widetilde{F})$ and $\{\widetilde{a}_n\}_{n\geq 1}\subset \mathrm{Fix}(\widetilde{F})\setminus\{\widetilde{a}\}$ satisfying $\widetilde{a}_n\to\widetilde{a}$ as $n\to+\infty$. Then*

$$\lim_{n\to+\infty} i(\widetilde{F};\widetilde{a}_n,\widetilde{a},z)=0$$

when $z\in\mathrm{Fix}(F)\setminus\{\pi(\widetilde{a})\}$, while

$$\lim_{n\to+\infty} L_k(\widetilde{F};\widetilde{a}_n,\widetilde{a},z)=0$$

for every $k\geq 1$, when $z\in\mathrm{Rec}^+(F)\cap U$ where U is an open disk of $M\setminus\{\pi(\widetilde{a})\}$.

Proof. Let $C_z^k=\pi^{-1}(\{z,F(z),\cdots,F^{\tau_k(z)-1}(z)\})$ where $\tau_k(z)=\sum_{i=0}^{k-1}\tau(\Phi^i(z))$.

For every n, let $\widetilde{I}_n$ be the isotopy that fixes $\widetilde{a}$, $\widetilde{a}_n$ and ∞, which is constructed in Lemma 2.2. Up to conjugacy by a homeomorphism $h:\widetilde{M}\to\mathbb{C}$, we can identify $\widetilde{M}$ with the complex plane $\mathbb{C}$ (refer to Remark 2.4 and Definition 3.1 for the reasons). Through a simple computation (see the proof of Lemma 2.2 [[Wang11a], page 54]), we can get the formula of $\widetilde{I}_n$ as follows

$$\widetilde{I}_n(\widetilde{z})(t)=\frac{\widetilde{a}_n-\widetilde{a}}{\widetilde{F}_t(\widetilde{a}_n)-\widetilde{F}_t(\widetilde{a})}\cdot(\widetilde{F}_t(\widetilde{z})-\widetilde{F}_t(\widetilde{a}))+\widetilde{a}. \tag{3.4}$$

Let $\widetilde{V}_n$ be a disk whose center is $\widetilde{a}$ and radius is $2|\widetilde{a}_n-\widetilde{a}|$. As the functions $L_k(\widetilde{F};\widetilde{a}_n,\widetilde{a},z)$ do not depend on the path from $\widetilde{a}$ to $\widetilde{a}_n$ (see Section 3.1), we can suppose that the path $\widetilde{\gamma}$ from $\widetilde{a}_n$ to $\widetilde{a}$ is always in $\widetilde{V}_n$. As $z\neq\pi(\widetilde{a})$, the value

$$c=\liminf_{n\geq 1}\min_{t\in[0,1],\widetilde{z}\in C_z^k}|\widetilde{F}_t(\widetilde{z})-\widetilde{F}_t(\widetilde{a}_n)| \tag{3.5}$$

is positive which only depends on z and k. For the constant c, we can find $N>0$ large enough such that $\max_{t\in[0,1]}|\widetilde{F}_t(\widetilde{a}_n)-\widetilde{F}_t(\widetilde{a})|<c/3$ when $n\geq N$. This implies that, for every $\widetilde{z}\in C_z^k$ and $t\in[0,1]$,

$$|\widetilde{I}_n(\widetilde{z})(t)-\widetilde{a}|>2|\widetilde{a}_n-\widetilde{a}|$$

when $n \geq N$. As a consequence, we have

$$\lim_{n\to+\infty} i(\widetilde{F};\widetilde{a}_n,\widetilde{a},z)=0$$

in the case where $z \in \mathrm{Fix}(F)\setminus\{\pi(\widetilde{a})\}$, and

$$\lim_{n\to+\infty} L_k(\widetilde{F};\widetilde{a}_n,\widetilde{a},z)=0$$

in the case where $z \in \mathrm{Rec}^+(F)\cap U$. □

3.3. **Definition of the generalized action function.**

Suppose now the function $i(\widetilde{F};\widetilde{a},\widetilde{b},z)$ is μ-integrable. We define the *action difference of* $\widetilde{a}$ *and* $\widetilde{b}$ as follows

$$i_\mu(\widetilde{F};\widetilde{a},\widetilde{b}) = \int_{M\setminus\pi(\{\widetilde{a},\widetilde{b}\})} i(\widetilde{F};\widetilde{a},\widetilde{b},z)\,\mathrm{d}\mu. \tag{3.6}$$

As an immediate consequence of Proposition 3.2, we have:

Corollary 3.5. $i_\mu(\widetilde{F};\alpha(\widetilde{a}),\alpha(\widetilde{b})) = i_\mu(\widetilde{F};\widetilde{a},\widetilde{b})$ *for any* $\alpha\in G$.

Let F be the time-one map of an identity isotopy $I=(F_t)_{t\in[0,1]}$ on M. We suppose now that the action difference $i_\mu(\widetilde{F};\widetilde{a},\widetilde{b})$ is well defined for every two distinct fixed points $\widetilde{a}$ and $\widetilde{b}$ of $\widetilde{F}$. We define the *action difference* as follows:

$$\begin{array}{rcl} i_\mu : (\mathrm{Fix}(\widetilde{F})\times\mathrm{Fix}(\widetilde{F}))\setminus\widetilde{\Delta} & \to & \mathbb{R} \\ (\widetilde{a},\widetilde{b}) & \mapsto & i_\mu(\widetilde{F};\widetilde{a},\widetilde{b}). \end{array}$$

Note that for each of the following cases, the action difference can be defined [Wang11a, page 86] for every pair $(\widetilde{a},\widetilde{b})\in(\mathrm{Fix}(\widetilde{F})\times\mathrm{Fix}(\widetilde{F}))\setminus\widetilde{\Delta}$:

- $F\in\mathrm{Diff}(M)$;
- I satisfies the WB-property and μ has full support;
- I satisfies the WB-property and μ is ergodic.

The following corollary is an immediate conclusion of Proposition 3.3:

Corollary 3.6. *For any distinct fixed points* $\widetilde{a}$, $\widetilde{b}$ *and* $\widetilde{c}$ *of* $\widetilde{F}$, *we have*

$$i_\mu(\widetilde{F};\widetilde{a},\widetilde{b})+i_\mu(\widetilde{F};\widetilde{b},\widetilde{c})+i_\mu(\widetilde{F};\widetilde{c},\widetilde{a})=0.$$

That is, i_μ *is a coboundary on* $\mathrm{Fix}(\widetilde{F})$. *So there is a function* $l_\mu:\mathrm{Fix}(\widetilde{F})\to\mathbb{R}$, *defined up to an additive constant, such that*

$$i_\mu(\widetilde{F};\widetilde{a},\widetilde{b}) = l_\mu(\widetilde{F};\widetilde{b}) - l_\mu(\widetilde{F};\widetilde{a}).$$

We call the function l_μ the *action function* on $\mathrm{Fix}(\widetilde{F})$ defined by the measure μ.

Proposition 3.7 ([Wang11a], page 87). *If* $\rho_{M,I}(\mu)=0$, *then* $i_\mu(\widetilde{F};\widetilde{a},\alpha(\widetilde{a}))=0$ *for every* $\widetilde{a}\in\mathrm{Fix}(\widetilde{F})$ *and every* $\alpha\in G\setminus\{e\}$ *where* e *is the unit element of* G. *As a consequence, there exists a function* L_μ *defined on* $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ *such that for every two distinct fixed points* $\widetilde{a}$ *and* $\widetilde{b}$ *of* $\widetilde{F}$, *we have*

$$i_\mu(\widetilde{F};\widetilde{a},\widetilde{b}) = L_\mu(\widetilde{F};\pi(\widetilde{b})) - L_\mu(\widetilde{F};\pi(\widetilde{a})).$$

We call the function L_μ the *action* on $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ defined by the measure μ. We proved that the function L_μ is a generalization of the classical case (Theorem 1.2, [Wang11a, Theorem 4.3.2]).

3.4. **The continuity of the generalized action function.**

We have the following continuity property of the generalized action function whose proof details will be used in the proof of Theorem 1.4.

Proposition 3.8. *Suppose that F is the time-one map of an identity isotopy I on M and that $\mu \in \mathcal{M}(F)$. Let $\widetilde{X} \subseteq \mathrm{Fix}(\widetilde{F})$. If one of the following three cases is satisfied:*

- *I satisfies the B-property on $\widetilde{X}$ and $F \in \mathrm{Diff}(M)$;*
- *I satisfies the B-property on $\widetilde{X}$ and $\mathrm{Supp}(\mu) = M$;*
- *I satisfies the B-property on $\widetilde{X}$ and μ is ergodic,*

then for any $\widetilde{a} \in \widetilde{X}$ and $\{\widetilde{a}_n\}_{n\geq 1} \subset \widetilde{X} \setminus \{\widetilde{a}\}$ satisfying $\widetilde{a}_n \to \widetilde{a}$ as $n \to +\infty$, we have

$$\lim_{n\to+\infty} i_\mu(\widetilde{F}; \widetilde{a}_n, \widetilde{a}) = 0.$$

As a conclusion, if I satisfies the B-property on $\widetilde{X}$ and the WB-property, the action l_μ is continuous on $\widetilde{X}$. Moreover, if $\rho_{M,I}(\mu) = 0$, the action L_μ is continuous on $\pi(\widetilde{X})$.

Proof. There exists a triangulation $\{U_i\}_{i=1}^{+\infty}$ of $M \setminus \mathrm{Fix}(F)$ such that, for every i, the interior of U_i is an open free disk for F (i.e., $F(U_i) \cap U_i = \emptyset$) and satisfies $\mu(\partial U_i) = 0$. By a slight abuse of notations we will also write U_i for its interior.

According to Lemma 3.4, we have that $\lim_{n\to+\infty} i(\widetilde{F}; \widetilde{a}_n, \widetilde{a}, z) = 0$ for $z \in \mathrm{Fix}(F) \setminus \{\pi(\widetilde{a})\}$, and that $\lim_{n\to+\infty} L_1(\widetilde{F}; \widetilde{a}_n, \widetilde{a}, z) = 0$ for $z \in \mathrm{Rec}^+(F) \cap U_i$, for every i.

Choose a compact set $\widetilde{P} \subset \widetilde{M}$ such that $\widetilde{a} \in \mathrm{Int}(\widetilde{P})$ and $\{\widetilde{a}_n\}_{n\geq 1} \subset \widetilde{P}$. As before, when $\widetilde{a}'$ and $\widetilde{b}'$ are two distinct fixed points of $\widetilde{F}$ in $\widetilde{P}$, we can always suppose that the path $\widetilde{\gamma}$ that joins $\widetilde{a}'$ and $\widetilde{b}'$ is in $\widetilde{P}$. By the definition of B-property, we may suppose that there exists a number $N > 0$ such that

$$N > \operatorname*{ess\,sup}_{n\geq 1} \left\{ \left| i(\widetilde{F}; \widetilde{a}_n, \widetilde{a}, z) \right| \right\},$$

where "ess sup" is the essential supremum[1].

By Lebesgue's dominating convergence theorem (the dominated function is N), we get

$$\lim_{n\to+\infty} \int_{\mathrm{Fix}(F)} \left| (i(\widetilde{F}; \widetilde{a}_n, \widetilde{a}, z) \right| \, \mathrm{d}\mu = 0.$$

It is then sufficient to prove that

$$\lim_{n\to+\infty} \int_{M\setminus\mathrm{Fix}(F)} \left| (i(\widetilde{F}; \widetilde{a}_n, \widetilde{a}, z) \right| \, \mathrm{d}\mu = 0.$$

Fix any $\epsilon > 0$. Since $\mu(\bigcup_{i=1}^{+\infty} U_i) = \mu(M \setminus \mathrm{Fix}(F)) < +\infty$, there exists a positive integer N' such that

$$\mu(\bigcup_{N'+1}^{+\infty} U_i) < \frac{\epsilon}{2N}.$$

[1] Refer to Proposition 4.6.9 in [Wang11a, page 82] for the first case, Proposition 4.6.11 in [Wang11a, page 85] for the second case and Lemma 6.6 in [Wang11b, Proposition 6.8, page 46] for the third case.

For every pair $(\widetilde{a},\widetilde{b})\in(\mathrm{Fix}(\widetilde{F})\times\mathrm{Fix}(\widetilde{F}))\setminus\widetilde{\Delta}$, each i and μ-a.e. $z\in U_i$, we have the following facts

- $\int_{U_i}\tau\,\mathrm{d}\mu=\mu(\bigcup_{k\geq 0}F^k(U_i))$ (by Kac Lemma, see [Kac47]);
- $i(\widetilde{F};\widetilde{a},\widetilde{b},z)$ is the action of F, i.e., $i(\widetilde{F};\widetilde{a},\widetilde{b},F(z))=i(\widetilde{F};\widetilde{a},\widetilde{b},z)$.

Therefore,

$$\int_{\bigcup_{k\geq 0}F^k(U_i)}\left|i(\widetilde{F};\widetilde{a},\widetilde{b},z)\right|\,\mathrm{d}\mu=\int_{U_i}\tau(z)\left|i(\widetilde{F};\widetilde{a},\widetilde{b},z)\right|\,\mathrm{d}\mu. \tag{3.7}$$

As $L_1(\widetilde{F};\widetilde{a},\widetilde{b},z)\in L^1(U_i,\mathbb{R},\mu)$[2], the following limit exists

$$L^*(\widetilde{F};\widetilde{a},\widetilde{b},z)=\lim_{m\to+\infty}\frac{L_m(\widetilde{F};\widetilde{a},\widetilde{b},z)}{m}=\lim_{m\to+\infty}\frac{1}{m}\sum_{j=1}^{m-1}L_1(\widetilde{F};\widetilde{a},\widetilde{b},\Phi^j(z)).$$

Moreover, we also have the following inequality (modulo subsets of measure zero of U_i)

$$\begin{aligned}\left|L^*(\widetilde{F};\widetilde{a},\widetilde{b},z)\right| &= \lim_{m\to+\infty}\frac{1}{m}\left|\sum_{j=0}^{m-1}(L_1(\widetilde{F};\widetilde{a},\widetilde{b},\Phi^j(z))\right|\\ &\leq \lim_{m\to+\infty}\frac{1}{m}\sum_{j=0}^{m-1}\left|L_1(\widetilde{F};\widetilde{a},\widetilde{b},\Phi^j(z))\right|\\ &= \left|L_1(\widetilde{F};\widetilde{a},\widetilde{b},z)\right|^*,\end{aligned} \tag{3.8}$$

where the last equation holds due to Birkhoff Ergodic theorem.

Applying Birkhoff Ergodic theorem again, we get

$$\tau^*(\Phi(z))=\tau^*(z)\quad\text{and}\quad L^*(\widetilde{F};\widetilde{a},\widetilde{b},\Phi(z))=L^*(\widetilde{F};\widetilde{a},\widetilde{b},z),$$

where $\tau^*(z)$ is the limit of the sequence $\{\tau_n(z)/n\}_{n\geq 1}$, Φ is the first return map on U_i.

For μ-a.e. $z\in U_i$, the following limits hold

$$i(\widetilde{F};\widetilde{a},\widetilde{b},z)=\lim_{m\to+\infty}\frac{L_m(\widetilde{F};\widetilde{a},\widetilde{b},z)}{\tau_m(z)}=\lim_{m\to+\infty}\frac{L_m(\widetilde{F};\widetilde{a},\widetilde{b},z)/m}{\tau_m(z)/m}=\frac{L^*(\widetilde{F};\widetilde{a},\widetilde{b},z)}{\tau^*(z)}. \tag{3.9}$$

Therefore, $i(\widetilde{F};\widetilde{a},\widetilde{b},\Phi(z))=i(\widetilde{F};\widetilde{a},\widetilde{b},z)$. Moreover, observing that $\tau(z)|i(\widetilde{F};\widetilde{a},\widetilde{b},z)|\in L^1(U_i,\mathbb{R},\mu)$, we obtain

$$\begin{aligned}&\lim_{m\to+\infty}\frac{1}{m}\sum_{j=0}^{m-1}\left(\tau(\Phi^j(z))\left|i(\widetilde{F};\widetilde{a},\widetilde{b},\Phi^j(z))\right|\right)\\ =\ &\lim_{m\to+\infty}\left(\frac{1}{m}\sum_{j=0}^{m-1}\tau(\Phi^j(z))\right)\cdot\left|i(\widetilde{F};\widetilde{a},\widetilde{b},z)\right|\\ =\ &\tau^*(z)\left|i(\widetilde{F};\widetilde{a},\widetilde{b},z)\right|\end{aligned}$$

for μ-a.e. $z\in U_i$. This implies that

$$\int_{U_i}\tau(z)\left|i(\widetilde{F};\widetilde{a},\widetilde{b},z)\right|\,\mathrm{d}\mu=\int_{U_i}\tau^*(z)\left|i(\widetilde{F};\widetilde{a},\widetilde{b},z)\right|\,\mathrm{d}\mu. \tag{3.10}$$

[2] Refer to Proposition 4.6.10 in [Wang11a, page 81] for the proof.

From the equalities 3.7, 3.9, 3.10 and Inequality 3.8 above, we obtain

$$\begin{aligned}
\int_{\bigcup_{i=1}^{N'} U_i} \left|i(\widetilde{F};\widetilde{a}_n,\widetilde{a},z)\right| \,\mathrm{d}\mu &\leq \sum_{i=1}^{N'} \int_{\bigcup_{k\geq 0} F^k(U_i)} \left|i(\widetilde{F};\widetilde{a}_n,\widetilde{a},z)\right| \,\mathrm{d}\mu \\
&= \sum_{i=1}^{N'} \int_{U_i} \tau(z) \left|i(\widetilde{F};\widetilde{a}_n,\widetilde{a},z)\right| \,\mathrm{d}\mu \\
&= \sum_{i=1}^{N'} \int_{U_i} \tau^*(z) \left|i(\widetilde{F};\widetilde{a}_n,\widetilde{a},z)\right| \,\mathrm{d}\mu \\
&= \sum_{i=1}^{N'} \int_{U_i} \left|L^*(\widetilde{F};\widetilde{a}_n,\widetilde{a},z)\right| \,\mathrm{d}\mu \\
&\leq \sum_{i=1}^{N'} \int_{U_i} \left|L_1(\widetilde{F};\widetilde{a}_n,\widetilde{a},z)\right|^* \,\mathrm{d}\mu \\
&= \sum_{i=1}^{N'} \int_{U_i} \left|L_1(\widetilde{F};\widetilde{a}_n,\widetilde{a},z)\right| \,\mathrm{d}\mu
\end{aligned}$$

As N' is finite, according to Lebesgue's dominating convergence theorem (the dominated function is $N\tau(z)$) and Lemma 3.4, we have

$$\lim_{n\to+\infty} \sum_{i=1}^{N'} \int_{U_i} \left|L_1(\widetilde{F};\widetilde{a}_n,\widetilde{a},z)\right| \,\mathrm{d}\mu = 0.$$

Therefore, there exists a positive number N'' such that when $n \geq N''$,

$$\int_{\bigcup_{i=1}^{N'} U_i} \left|i(\widetilde{F};\widetilde{a}_n,\widetilde{a},z)\right| \,\mathrm{d}\mu < \frac{\epsilon}{2}.$$

Finally, when $n \geq N''$, we obtain

$$\begin{aligned}
& \int_{M\setminus \mathrm{Fix}(F)} \left|(i(\widetilde{F};\widetilde{a}_n,\widetilde{a},z)\right| \,\mathrm{d}\mu \\
= \; & \int_{\bigcup_{i=1}^{N'} U_i} \left|(i(\widetilde{F};\widetilde{a}_n,\widetilde{a},z)\right| \,\mathrm{d}\mu + \int_{\bigcup_{N'+1}^{+\infty} U_i} \left|(i(\widetilde{F};\widetilde{a}_n,\widetilde{a},z)\right| \,\mathrm{d}\mu \\
< \; & \frac{\epsilon}{2} + \frac{\epsilon}{2N} \cdot N \\
= \; & \epsilon.
\end{aligned}$$

Hence, the first statement holds.

Now we turn to prove the second statement. Let $a \in \pi(\widetilde{X})$ and $\{a_n\}_{n\geq 1} \subset \pi(\widetilde{X}) \setminus \{a\}$ that converges to a. By Proposition 3.7, we only need to consider a lift $\widetilde{a} \in \widetilde{X}$ of a and a lifted sequence $\{\widetilde{a}_n\}_{n\geq 1} \subset \widetilde{X}$ of $\{a_n\}_{n\geq 1}$ that converges to $\widetilde{a}$. Then it follows from the first statement. $\square$

4. The proof of Proposition 1.3

Suppose that $X \subseteq \mathrm{Fix}_{\mathrm{Cont},I}(F)$ is connected and not reduced to a singleton. By Lemma 2.8, I satisfies the B-property on $\pi^{-1}(X)$. If I satisfies the hypotheses of Theorem 1.2, according to Proposition 3.8, the action function L_μ is continuous on X. Moreover, we have the following stronger result in this case.

Proposition 1.3 *Under the hypotheses of Theorem 1.2, for every two distinct contractible fixed points a and b of F which belong to a same connected component of* $\mathrm{Fix}_{\mathrm{Cont},I}(F)$*, we have $I_\mu(\widetilde{F};a,b)=0$. As a conclusion, the action function L_μ is a constant on each connected component of* $\mathrm{Fix}_{\mathrm{Cont},I}(F)$.

Given $Y\subset M$ and $\epsilon>0$, let $Y_\epsilon=\{z\in M \mid d(z,y)<\epsilon, y\in Y\}$ be the ϵ-neighborhood of Y. If N is a submanifold of M, the inclusion $i: N\hookrightarrow M$ naturally induces a homomorphism: $i_*:\pi_1(N,p)\to\pi_1(M,p)$, where $p\in N$.

To prove Proposition 1.3, we need the following topological lemma proved in Appendix.

Lemma 4.1. *If Z is a compact subset of M and $z\in Z$, the following direct limits exist:*

$$\varinjlim_{\epsilon\to 0^+} i_*(H_1(Z_\epsilon,\mathbb{Z})) \quad\text{and}\quad \varinjlim_{\epsilon\to 0^+} i_*(\pi_1(Z_\epsilon,z)).$$

Proof. Let X be a connected component of $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ that is not a singleton.

Let us first consider the linking number $i(\widetilde{F};\widetilde{a},\widetilde{b},z)$, where $z\in \mathrm{Rec}^+(F)\setminus \mathrm{Fix}_{\mathrm{Cont},I}(F)$ and $(\widetilde{a},\widetilde{b})\in(\pi^{-1}(X)\times\pi^{-1}(X))\setminus\widetilde{\Delta}$.

Recall the following functions defined in Section 3.1:

$$L_k: ((\mathrm{Fix}(\widetilde{F})\times \mathrm{Fix}(\widetilde{F}))\setminus\widetilde{\Delta})\times(\mathrm{Rec}^+(F)\cap U)\to\mathbb{Z},$$

$$L_k(\widetilde{F};\widetilde{c}_1,\widetilde{c}_2,z)=\widetilde{\gamma}\wedge\widetilde{\Gamma}^k_{\widetilde{I}_1,z}=\sum_{j=0}^{k-1}L_1(\widetilde{F};\widetilde{c}_1,\widetilde{c}_2,\Phi^j(z)),$$

where $z\in\mathrm{Rec}^+(F)$, $U\subset M\setminus\pi(\{\widetilde{c}_1,\widetilde{c}_2\})$ is an open disk containing z, and $\widetilde{I}_1$ is an isotopy from $\mathrm{Id}_{\widetilde{M}}$ to $\widetilde{F}$ that fixes $\widetilde{c}_1$ and $\widetilde{c}_2$.

We claim that, for every $z\in\mathrm{Rec}^+(F)\setminus\mathrm{Fix}_{\mathrm{Cont},I}(F)$ and $k\geq 1$, there exists $\epsilon>0$ which merely depends on z and k such that $L_k(\widetilde{F};\widetilde{a},\widetilde{b},z)=0$ when $\widetilde{d}(\widetilde{a},\widetilde{b})<\epsilon$.

Indeed, since X is compact, $z\notin X$, and $\widetilde{F}_t\circ T=T\circ\widetilde{F}_t$ for any $T\in G$, the value

$$c'=\min_{t\in[0,1],\ \widetilde{z}\in C^k_z,\ \widetilde{z}'\in\pi^{-1}(X)}|\widetilde{F}_t(\widetilde{z})-\widetilde{F}_t(\widetilde{z}')|$$

is positive and only depends on z and k, where $C^k_z=\pi^{-1}(\{z,F(z),\cdots,F^{\tau_k(z)-1}(z)\})$.

Recall that the isotopy

$$\widetilde{I}'(\widetilde{z})(t)=\frac{\widetilde{b}-\widetilde{a}}{\widetilde{F}_t(\widetilde{b})-\widetilde{F}_t(\widetilde{a})}\cdot(\widetilde{F}_t(\widetilde{z})-\widetilde{F}_t(\widetilde{a}))+\widetilde{a}$$

fixes $\widetilde{a}$, $\widetilde{b}$ and ∞. Let $\epsilon>0$ be small enough such that $\max_{t\in[0,1]}|\widetilde{F}_t(\widetilde{a})-\widetilde{F}_t(\widetilde{b})|<c'/3$ when $\widetilde{d}(\widetilde{a},\widetilde{b})<\epsilon$ and let $\widetilde{V}'$ be a disk whose center is $\widetilde{a}$ and radius is $2|\widetilde{b}-\widetilde{a}|$. The claim follows from the proof of Lemma 3.4 if one replaces $\widetilde{I}_n$ in Formula 3.4 by $\widetilde{I}'$, $\widetilde{V}_n$ by $\widetilde{V}'$, and c in Formula 3.5 by c'.

Fix $x\in X$ and a lift $\widetilde{x}\in\widetilde{M}$ of x. By Lemma 4.1, there is $\epsilon_0>0$ such that

$$i_*(\pi_1(X_\epsilon,x))=i_*(\pi_1(X_{\epsilon_0},x))$$

for all $0<\epsilon<\epsilon_0$. Let $\widetilde{X}_\epsilon$ be the connected component of $\pi^{-1}(X_\epsilon)$ that contains $\widetilde{x}$. Denote by $G_{\widetilde{X}_\epsilon}$ the subgroup of G that is the stabilizer of $\widetilde{X}_\epsilon$, i.e., $G_{\widetilde{X}_\epsilon}=\{T\in G \mid T(\widetilde{X}_\epsilon)=\widetilde{X}_\epsilon\}$. It is clear that $i_*(\pi_1(X_\epsilon,x))\simeq G_{\widetilde{X}_\epsilon}$. Hence $G_{\widetilde{X}_{\epsilon_1}}=G_{\widetilde{X}_{\epsilon_2}}$ for all $0<\epsilon_2<\epsilon_1\le\epsilon_0$. Let $\widetilde{Y}_\epsilon=\widetilde{X}_\epsilon\cap\pi^{-1}(X)$. Recall that X is connected. We have $\pi(\widetilde{Y}_\epsilon)=X$ for all $0<\epsilon\le\epsilon_0$ since X_ϵ is path connected. Note that $\widetilde{Y}_\epsilon$ is 4ϵ-*chain connected*, i.e., for any $\widetilde{y},\widetilde{y}'\in\widetilde{Y}_\epsilon$ there exists a sequence $\{\widetilde{y}_i\}_{i=1}^n\subset\widetilde{Y}_\epsilon$ such that $\widetilde{y}_1=\widetilde{y}$, $\widetilde{y}_n=\widetilde{y}'$, and $\widetilde{d}(\widetilde{y}_i,\widetilde{y}_{i+1})<4\epsilon$. Indeed, we can find a path γ in X_ϵ from $\pi(\widetilde{y})$ to $\pi(\widetilde{y}')$ and a lift $\widetilde{\gamma}$ of γ in $\widetilde{X}_\epsilon$ from $\widetilde{y}$ to $\widetilde{y}'$. On the path $\widetilde{\gamma}$, we choose a sequence $\{\widetilde{x}_i\}_{i=1}^n\subset\widetilde{\gamma}$ such that $\widetilde{x}_1=\widetilde{y}$, $\widetilde{x}_n=\widetilde{y}'$, and the disks $\{\widetilde{D}(\widetilde{x}_i,\epsilon)\}_{i=1}^n$ cover $\widetilde{\gamma}$ with $\widetilde{D}(\widetilde{x}_i,\epsilon)\cap\widetilde{D}(\widetilde{x}_{i+1},\epsilon)\ne\emptyset$ for all $i=1,\dots,n-1$, where $\widetilde{D}(\widetilde{x}_i,\epsilon)$ is a disk on $\widetilde{M}$ whose center is $\widetilde{x}_i$ and radius is ϵ. Choose a sequence $\{\widetilde{y}_i\}_{i=1}^n\subset\widetilde{Y}_\epsilon$ such that $\widetilde{y}_1=\widetilde{y}$, $\widetilde{y}_n=\widetilde{y}'$, and $\widetilde{y}_i\in\widetilde{D}(\widetilde{x}_i,\epsilon)\cap\widetilde{Y}_\epsilon$ for $2\le i\le n-1$. Obviously, $\{\widetilde{y}_i\}_{i=1}^n$ is a 4ϵ-chain in $\widetilde{Y}_\epsilon$ from $\widetilde{y}$ to $\widetilde{y}'$ by the triangle inequality.

For any $y\in X$, we claim that $\widetilde{y}\in\widetilde{Y}_\epsilon$ for all $\widetilde{y}\in\pi^{-1}(y)\cap\widetilde{Y}_{\epsilon_0}$ and all $0<\epsilon\le\epsilon_0$. Otherwise, there is $0<\epsilon_1<\epsilon_0$ and $\widetilde{y}\in\widetilde{Y}_{\epsilon_0}\subset\widetilde{X}_{\epsilon_0}$ such that $\widetilde{y}\notin\widetilde{Y}_{\epsilon_1}$, and hence $\widetilde{y}\notin\widetilde{X}_{\epsilon_1}$. However, there is a lift $\widetilde{y}'$ of y such that $\widetilde{y}'\in\widetilde{Y}_{\epsilon_1}\subset\widetilde{X}_{\epsilon_1}\subset\widetilde{X}_{\epsilon_0}$. On the one hand, $T\in G_{\widetilde{X}_{\epsilon_0}}$ since $\widetilde{y},\widetilde{y}'\in\widetilde{X}_{\epsilon_0}$, where $\widetilde{y}=T(\widetilde{y}')$. On the other hand, $T\notin G_{\widetilde{X}_{\epsilon_1}}$ since $\widetilde{y}\notin\widetilde{X}_{\epsilon_1}$. This is impossible because $G_{\widetilde{X}_{\epsilon_1}}=G_{\widetilde{X}_{\epsilon_0}}$ and hence the claim holds. This implies that $\widetilde{Y}_\epsilon=\widetilde{Y}_{\epsilon_0}$ for all $0<\epsilon<\epsilon_0$, and hence $\widetilde{Y}_{\epsilon_0}$ is ϵ-chain connected for all $0<\epsilon\le\epsilon_0/4$.

Recall the equality in Proposition 3.3 for any distinct points $\widetilde{c}_1,\widetilde{c}_2$ and $\widetilde{c}_3$ of $\mathrm{Fix}(\widetilde{F})$:

$$L_k(\widetilde{F};\widetilde{c}_1,\widetilde{c}_2,z)+L_k(\widetilde{F};\widetilde{c}_2,\widetilde{c}_3,z)+L_k(\widetilde{F};\widetilde{c}_3,\widetilde{c}_1,z)=0. \tag{4.1}$$

Applying Equality 4.1, we get that, for all distinct $\widetilde{a},\widetilde{b}\in\widetilde{Y}_{\epsilon_0}$, $L_k(\widetilde{F};\widetilde{a},\widetilde{b},z)=0$ for all k and $z\in\mathrm{Rec}^+(F)\setminus\mathrm{Fix}_{\mathrm{Cont},I}(F)$. This implies that $i(\widetilde{F};\widetilde{a},\widetilde{b},z)=0$ for all $(\widetilde{a},\widetilde{b})\in(\widetilde{Y}_{\epsilon_0}\times\widetilde{Y}_{\epsilon_0})\setminus\widetilde{\Delta}$ and $z\in\mathrm{Rec}^+(F)\setminus\mathrm{Fix}_{\mathrm{Cont},I}(F)$.

Let us now consider the case of $z\in\mathrm{Fix}_{\mathrm{Cont},I}(F)$ to finish our proof, which is divided into two cases:

(1) There is a set $\widetilde{X}$ on $\widetilde{M}$ which is a connected component of $\pi^{-1}(X)$ and satisfies that the covering map $\pi:\widetilde{X}\to X$ is surjective (this case contains the case where X is path connected);
(2) There is no such set satisfying Item 1.

Recall the linking number of z for $\widetilde{a}$ and $\widetilde{b}$ (see Formula 3.3):

$$i(\widetilde{F};\widetilde{a},\widetilde{b},z)=\sum_{\pi(\widetilde{z})=z}\left(i(\widetilde{F};\widetilde{a},\widetilde{z})-i(\widetilde{F};\widetilde{b},\widetilde{z})\right),$$

where $(\widetilde{a},\widetilde{b})\in(\mathrm{Fix}(\widetilde{F})\times\mathrm{Fix}(\widetilde{F}))\setminus\widetilde{\Delta}$ and $i(\widetilde{F};\widetilde{c},\widetilde{z})=i_{\widetilde{I}}(\widetilde{c},\widetilde{z})$ (see Formula 2.2).

In the first case, for any $\widetilde{z}\in\pi^{-1}(z)$, by Lemma 2.6, $i(\widetilde{F};\widetilde{z}',\widetilde{z})$ is a constant (which depends on $\widetilde{z}$) for all $\widetilde{z}'\in\widetilde{X}\setminus\{\widetilde{z}\}$. We get that $i(\widetilde{F};\widetilde{a},\widetilde{b},z)=0$ for any $(\widetilde{a},\widetilde{b})\in(\widetilde{X}\times\widetilde{X})\setminus\widetilde{\Delta}$ and $z\in\mathrm{Fix}_{\mathrm{Cont},I}(F)\setminus\pi(\{\widetilde{a},\widetilde{b}\})$.

Note that $\widetilde{Y}_{\epsilon_0}=\widetilde{X}$ in this case. Therefore, by the definition of the action function, we get that $i_\mu(\widetilde{F};\widetilde{a},\widetilde{b})=0$ for all $(\widetilde{a},\widetilde{b})\in(\widetilde{X}\times\widetilde{X})\setminus\widetilde{\Delta}$. The conclusion follows from the fact that $\pi(\widetilde{X})=X$ and the hypothesis that $\rho_{M,I}(\mu)=0$ in this case.

In the second case, write $\pi^{-1}(X)$ as $\bigsqcup_{\alpha\in\widetilde{\Lambda}}\widetilde{X}_\alpha$ where $\widetilde{X}_\alpha$ is a connected component of $\pi^{-1}(X)$ on $\widetilde{M}$. Note that $2\leq \sharp\widetilde{\Lambda}\leq +\infty$. It is easy to see that every such $\widetilde{X}_\alpha$ is unbounded on $\widetilde{M}$ by the hypotheses and the connectedness of X.

Similar to the proof of the first case, for every $\alpha\in\widetilde{\Lambda}$ and $\widetilde{c}\in\widetilde{X}_\beta$ with $\alpha\neq\beta$, we have the following fact: when $z\in \mathrm{Fix}_{\mathrm{Cont},I}(F)$, the linking number $i(\widetilde{F};\cdot,\widetilde{c},z)\in\mathbb{Z}$ is a constant on $\widetilde{X}_\alpha$, and hence $i(\widetilde{F};\widetilde{a},\widetilde{b},z)=0$ for all $(\widetilde{a},\widetilde{b})\in(\widetilde{X}_\alpha\times\widetilde{X}_\alpha)\setminus\widetilde{\Delta}$. Observing that every $\widetilde{X}_\alpha$ is unbounded, the constant is zero by Formula 3.3 and Lemma 2.7. Therefore, $i(\widetilde{F};\widetilde{a},\widetilde{b},z)=0$ for all $(\widetilde{a},\widetilde{b})\in(\pi^{-1}(X)\times\pi^{-1}(X))\setminus\widetilde{\Delta}$.

Finally, by the definition of the action function, we get that $i_\mu(\widetilde{F};\widetilde{a},\widetilde{b})=0$ for all $(\widetilde{a},\widetilde{b})\in(\widetilde{Y}_{\epsilon_0}\times\widetilde{Y}_{\epsilon_0})\setminus\widetilde{\Delta}$. The conclusion the holds based on the facts that $\pi(\widetilde{Y}_{\epsilon_0})=X$ and that $\rho_{M,I}(\mu)=0$ in the second case. □

5. The proof of Theorem 1.4

To prove Theorem 1.4, we need the following theorem [Mat00, Lec05]:

Theorem 5.1. *Let M be a closed oriented surface with genus $g\geq 1$. If F is the time-one map of a μ-Hamiltonian isotopy I on M, then there exist at least three contractible fixed points of F.*

Remark that Theorem 5.1 is not valid when the measure has no full support (see Example 6.3 and Example 6.4 below).

Theorem 1.4 *Let F be the time-one map of a μ-Hamiltonian isotopy I on a closed oriented surface M with $g\geq 1$. If I satisfies the WB-property and F is not* Id_M*, the action function L_μ is not constant.*

Theorem 1.4 is proved in two cases: $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ is finite and it is infinite.

Proof of Theorem 1.4 for the case $\sharp\mathrm{Fix}_{\mathrm{Cont},I}(F)<+\infty$.

We say that $X\subseteq\mathrm{Fix}_{\mathrm{Cont},I}(F)$ is *unlinked* if there exists an isotopy $I'=(F'_t)_{t\in[0,1]}$ homotopic to I which fixes every point of X. Moreover, we say that X is a *maximal unlinked set* if any set $X'\subseteq\mathrm{Fix}_{\mathrm{Cont},I}(F)$ that strictly contains X is not unlinked.

In the proof of Theorem 5.1 ([Lec05, Theorem 10.1]), Le Calvez has proved that there exists a maximal unlinked set $X\subseteq\mathrm{Fix}_{\mathrm{Cont},I}(F)$ with $\sharp X\geq 3$ if $\sharp\mathrm{Fix}_{\mathrm{Cont},I}(F)<+\infty$.

There exists an oriented topological foliation $\mathcal{F}$ on $M\setminus X$ (or, equivalently, a singular oriented foliation $\mathcal{F}$ on M with X equal to the singular set) such that, for all $z\in M\setminus X$, the trajectory $I(z)$ is homotopic to an arc γ joining z and $F(z)$ in $M\setminus X$ which is positively transverse to $\mathcal{F}$. It means that for every $t_0\in[0,1]$ there exists an open neighborhood $V\subset M\setminus X$ of $\gamma(t_0)$ and an orientation preserving homeomorphism $h:V\to(-1,1)^2$ which sends the foliation $\mathcal{F}$ on the horizontal foliation (oriented with x_1 increasing) such that the map $t\mapsto p_2(h(\gamma(t)))$ defined in a neighborhood of t_0 is strictly increasing, where $p_2(x_1,x_2)=x_2$.

We can choose a point $z\in\mathrm{Rec}^+(F)\setminus\mathrm{Fix}(F)$ and a leaf λ containing z. Proposition 10.4 in [Lec05] states that the ω-limit set $\omega(\lambda)\in X$, the α-limit set $\alpha(\lambda)\in X$, and $\omega(\lambda)\neq\alpha(\lambda)$. Fix an isotopy I' homotopic to I that fixes $\omega(\lambda)$ and $\alpha(\lambda)$ and a lift $\widetilde{\lambda}$ of λ that joins $\widetilde{\omega(\lambda)}$ and $\widetilde{\alpha(\lambda)}$. Let us now study the linking number $i(\widetilde{F};\widetilde{\omega(\lambda)},\widetilde{\alpha(\lambda)},z')$ for $z'\in\mathrm{Rec}^+(F)\setminus X$ when it exists. Observing that for all $z'\in M\setminus X$, the trajectory $I'(z')$ is still homotopic to an arc that is positively transverse to $\mathcal{F}$. Hence, for all $z' \in \mathrm{Rec}^+(F) \setminus X$, without loss of generality, we can choose an open disk U containing z' such that $U \cap \lambda = \emptyset$ by shrinking U and perturbing λ if necessary. Then we have

$$L_n(\widetilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z') = \widetilde{\lambda} \wedge \widetilde{\Gamma}^n_{\widetilde{I}',z'} = \lambda \wedge \Gamma^n_{I',z'} \geq 0$$

for every $n \geq 1$, where $\widetilde{I}'$ is the lift of I' to $\widetilde{M}$ and $\Gamma^n_{I',z'} = \pi(\widetilde{\Gamma}^n_{\widetilde{I}',z'})$.

According to Definition 3.1, we have

$$i(\widetilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z') \geq 0$$

for μ-a.e. $z' \in \mathrm{Rec}^+(F) \setminus \{\omega(\lambda), \alpha(\lambda)\}$.

By the continuity of I' and the hypothesis on μ, there exists an open free disk U containing z such that $\mu(U) > 0$ and $L_1(\widetilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z') > 0$ when $z' \in U \cap \mathrm{Rec}^+(F)$.

Similarly to the proof of Proposition 3.8, we have

$$\begin{aligned}
I_\mu(\widetilde{F}; \omega(\lambda), \alpha(\lambda)) &\geq \int_{\bigcup_{k\geq 0} F^k(U)} i(\widetilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z)\, \mathrm{d}\mu \\
&= \int_U \tau(z) i(\widetilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z)\, \mathrm{d}\mu \\
&= \int_U \tau^*(z) i(\widetilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z)\, \mathrm{d}\mu \\
&= \int_U L^*(\widetilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z)\, \mathrm{d}\mu \\
&= \int_U L_1(\widetilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z)\, \mathrm{d}\mu \\
&> 0.
\end{aligned}$$

□

Before proving the case where the set $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ is infinite, let us recall two results:

Proposition 5.2 (Franks' Lemma [Fra88])**.** *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving homeomorphism. If F possesses a periodic free disk chain, that means a family $(U_r)_{r\in\mathbf{Z}/n\mathbf{Z}}$ of pairwise disjoint free topological open disks such that for every $r \in \mathbf{Z}/n\mathbf{Z}$, one of the positive iterates of U_r meets U_{r+1}, then F has at least one fixed point.*

Theorem 5.3 ([Jau14])**.** *Let M be an oriented surface and F be the time-one map of an identity isotopy I on M. There exists a closed subset $X \subset \mathrm{Fix}(F)$ and an isotopy I' joining $\mathrm{Id}_{M\setminus X}$ to $F|_{M\setminus X}$ in* Homeo(M \ X) *such that*

1. *For all $z \in X$, the loop $I(z)$ is homotopic to zero in M.*
2. *For all $z \in \mathrm{Fix}(F) \setminus X$, the loop $I'(z)$ is not homotopic to zero in $M \setminus X$.*
3. *For all $z \in M\setminus X$, the trajectories $I(z)$ and $I'(z)$ are homotopic with fixed endpoints in M.*
4. *There exists an oriented topological foliation $\mathcal{F}$ on $M \setminus X$ such that, for all $z \in M \setminus X$, the trajectory $I'(z)$ is homotopic to an arc γ joining z and $F(z)$ in $M \setminus X$ which is positively to $\mathcal{F}$.*

Moreover, the isotopy I' satisfies the following property:

(5) *For all finite $Y \subset X$, there exists an isotopy I'_Y joining* Id_M *and F in* $\mathrm{Homeo}(M)$ *which fixes Y such that, if $z \in M \setminus X$, the arc $I'(z)$ and $I'_Y(z)$ are homotopic in $M \setminus Y$. And if $z \in X \setminus Y$, the loop $I'_Y(z)$ is contractible in $M \setminus Y$.*

Proof of Theorem 1.4 for the case $\sharp\mathrm{Fix}_{\mathrm{Cont},I}(F) = +\infty$.

Suppose that X, I' and $\mathcal{F}$ are respectively the closed contractible fixed points set, the isotopy, and the foliation, as stated in Theorem 5.3. Obviously, $X \neq \emptyset$ (see Remark 2.1) and $\mu(M \setminus X) > 0$. Assume that X' is the union of the connected components of X that separate M. Write $M \setminus X' = \sqcup_i S_i$ where S_i is an open F-invariant subsurface of M for every i. For every i, we regard the restriction of I' on S_i, say I'_i, as an identity isotopy on S_i. That means we extend I'_i to the set $S_i \cap X$ by $\mathrm{Id}_{S_i \cap X}$. Let us consider the closure of S_i. Regarding every connected component of ∂S_i as one point, we get a closed surface S'_i and an identity isotopy induced by I'_i, written still I'_i. By the definition of S_i, S'_i and I'_i, the following properties hold

(A1): if S_i is a disk, $X \cap S_i \neq \emptyset$ (by Proposition 5.2 and Item 2 of Theorem 5.3);

(A2): $\rho_{S'_i, I'_i}(\mu) = 0 \in H_1(S'_i, \mathbb{R})$ if S_i is not a subsurface of sphere (by the items 1 and 3 of Theorem 5.3). Thanks to Theorem 5.1, we have $\sharp\mathrm{Fix}_{\mathrm{Cont},I'_i}(F|_{S'_i}) \geq 3$.

It implies that $\sharp\{$the connected components of $\partial S_i \cup (X \cap S_i)\} \geq 2$ for every i.

Fix one subsurface S_i. Similarly to the finite case, we choose a point $z \in (\mathrm{Rec}^+(F) \setminus \mathrm{Fix}(F)) \cap S_i$ and a leaf $\lambda \in \mathcal{F}$ containing z. In [Lec06], the proofs of Proposition 4.1 (page 150, for S_i being a subsurface of sphere) and Proposition 6.1 (page 166, for S_i being not a subsurface of sphere) imply that $\omega(\lambda)$ (resp. $\alpha(\lambda)$) is connected and is contained in a connected component of $\partial S_i \cup (X \cap S_i)$. We write the connected component as $X_+(\lambda)$ (resp. $X_-(\lambda)$). Moreover, $X_+(\lambda) \neq X_-(\lambda)$. Choose a lift $\widetilde{\lambda}$ of λ. We need to consider the following four cases: the set $\omega(\widetilde{\lambda})$ or $\alpha(\widetilde{\lambda})$ contains ∞ or not.

Take two points $a \in \alpha(\lambda)$ and $b \in \omega(\lambda)$. Let $Y = \{a, b\}$ and I'_Y be the isotopy as in Theorem 5.3. Suppose that $\widetilde{I}'_Y$ is the identity lift of I'_Y to $\widetilde{M}$. Notice that

(B1): if $z \in M \setminus X$, the arcs $I'(z)$ and $I'_Y(z)$ are homotopic in $M \setminus Y$ (Item 5, Theorem 5.3), and $I'_Y(z)$ is homotopic to an arc γ joining z and $F(z)$ in $M \setminus Y$ and positively transverse to $\mathcal{F}$ (Item 4, Theorem 5.3);

(B2): if $z \in X \setminus Y$, $\gamma \wedge I'_Y(z) = 0$ where γ is any path from a to b (Item 5, Theorem 5.3).

If both $\alpha(\widetilde{\lambda})$ and $\omega(\widetilde{\lambda})$ do not contain ∞, replacing $\alpha(\lambda)$ by a, $\omega(\lambda)$ by b, and I' by I'_Y in the proof of the finite case, we can get $I_\mu(\widetilde{F}; a, b) > 0$.

We suppose now that either $\alpha(\widetilde{\lambda})$ or $\omega(\widetilde{\lambda})$ contains ∞. Recall that $\widetilde{d}$ is the distance on $\widetilde{M}$ induced by a distance d on M which is further induced by a Riemannian metric on M. Define $\widetilde{d}(\widetilde{z}, \widetilde{C}) = \inf_{\widetilde{c} \in \widetilde{C}} \widetilde{d}(\widetilde{z}, \widetilde{c})$ if $\widetilde{z} \in \widetilde{M}$ and $\widetilde{C} \subset \widetilde{M}$. Take a sequence $\{(\widetilde{a}_m, \widetilde{b}_m)\}_{m \geq 1}$ such that

- $\pi(\widetilde{a}_m) = a$ and $\pi(\widetilde{b}_m) = b$;
- If $\alpha(\widetilde{\lambda})$ (resp. $\omega(\widetilde{\lambda})$) does not contain ∞, we set $\widetilde{a}_m = \widetilde{a}$ (resp. $\widetilde{b}_m = \widetilde{b}$) for every m where $\widetilde{a} \in \pi^{-1}(a) \cap \alpha(\widetilde{\lambda})$ (resp. $\widetilde{b} \in \pi^{-1}(b) \cap \omega(\widetilde{\lambda})$);
- $\lim_{m \to +\infty} \widetilde{d}(\widetilde{a}_m, \widetilde{\lambda}) = 0$ and $\lim_{m \to +\infty} \widetilde{d}(\widetilde{b}_m, \widetilde{\lambda}) = 0$.

For every m, suppose that $\widetilde{c}_m$ (resp. $\widetilde{c}'_m$) is a point of $\widetilde{\lambda}$ such that $\widetilde{d}(\widetilde{a}_m, \widetilde{c}_m) = \widetilde{d}(\widetilde{a}_m, \widetilde{\lambda})$ (resp. $\widetilde{d}(\widetilde{b}_m, \widetilde{c}'_m) = \widetilde{d}(\widetilde{b}_m, \widetilde{\lambda})$). Note that $\widetilde{c}_m = \widetilde{a}_m = \widetilde{a}$ (resp. $\widetilde{c}'_m = \widetilde{b}_m = \widetilde{b}$) and

$\widetilde{d}(\widetilde{a}_m,\widetilde{\lambda}) = 0$ (resp. $\widetilde{d}(\widetilde{b}_m,\widetilde{\lambda}) = 0$) if $\alpha(\widetilde{\lambda})$ (resp. $\omega(\widetilde{\lambda})$) does not contain ∞. Choose a simple path $\widetilde{l}_m$ (resp. $\widetilde{l}'_m$) from $\widetilde{a}_m$ (resp. $\widetilde{c}'_m$) to $\widetilde{c}_m$ (resp. $\widetilde{b}_m$) such that the length of $\widetilde{l}_m$ (resp. $\widetilde{l}'_m$) is $\widetilde{d}(\widetilde{a}_m,\widetilde{\lambda})$ (resp. $\widetilde{d}(\widetilde{b}_m,\widetilde{\lambda})$). Here, we assume that the simple path is empty if its length is 0. Without loss of generality, we may suppose that $\pi(\widetilde{l}_{m+1})\subset\pi(\widetilde{l}_m)$ (resp. $\pi(\widetilde{l}'_{m+1})\subset\pi(\widetilde{l}'_m)$). Let $\widetilde{\gamma}_m=\widetilde{l}_m\widetilde{\lambda}_m\widetilde{l}'_m$ where $\widetilde{\lambda}_m$ is the sub-path of $\widetilde{\lambda}$ from $\widetilde{c}_m$ to $\widetilde{c}'_m$. Then $\widetilde{\gamma}_m$ is a path from $\widetilde{a}_m$ to $\widetilde{b}_m$.

We know that, for every $m\geq 1$, the linking number $i(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m,z')$ exists for μ-a.e. $z'\in M\setminus\{a,b\}$. Hence, the linking number $i(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m,z')$ exists on a full measure subset of $M\setminus\{a,b\}$ for all m.

According to B2 above, we have $i(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m,z')=0$ if $z'\in X\setminus\{a,b\}$. We now claim that

$$\liminf_{m\to+\infty} i(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m,z')\geq 0 \quad \text{for} \quad \mu\text{-a.e.} \quad z'\in \mathrm{Rec}^+(F)\setminus X.$$

Fix one point $z'\in\mathrm{Rec}^+(F)\setminus X$ and choose a disk U containing z' (here again, we suppose that $U\cap\lambda=\emptyset$). By B1 and the construction of $\widetilde{\gamma}_m$, for every $n\geq 1$, there exists $m(z',n)\in\mathbb{N}$ such that when $m\geq m(z',n)$, the value

$$L_n(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m,z')=\widetilde{\gamma}_m\wedge\widetilde{\Gamma}^n_{\widetilde{I}'_Y,z'}=\pi(\widetilde{\gamma}_m)\wedge\Gamma^n_{I'_Y,z'}\geq 0 \tag{5.1}$$

is constant with regard to m.

We now suppose that

$$\mu\{z'\in\mathrm{Rec}^+(F)\setminus X\mid \liminf_{m\to+\infty} i(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m,z')<0\}>0.$$

There exists a small number $c>0$ such that

$$\mu\{z'\in\mathrm{Rec}^+(F)\setminus X\mid \liminf_{m\to+\infty} i(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m,z')<-c\}>c. \tag{5.2}$$

Write $E=\{z'\in\mathrm{Rec}^+(F)\setminus X\mid \liminf_{m\to+\infty} i(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m,z')<-c\}$. Fix a point $z'\in E$ and a disk U containing z' as before. By taking subsequence if necessary, we may suppose that

$$-\infty\leq\lim_{m\to+\infty} i(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m,z')<-c.$$

Then there exists $N(z')$ such that when $m\geq N(z')$, we have

$$i(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m,z')=\lim_{n\to+\infty}\frac{L_n(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m,z')}{\tau_n(z')}<-c.$$

Fix $m_0\geq N(z')$. There exists $n(z',m_0)\in\mathbb{N}$ such that when $n\geq n(z',m_0)$, we have

$$\frac{L_n(\widetilde{F};\widetilde{a}_{m_0},\widetilde{b}_{m_0},z')}{\tau_n(z')}<-c.$$

Then we can choose $n_0\geq n(z',m_0)$ such that

$$L_{n_0}(\widetilde{F};\widetilde{a}_{m_0},\widetilde{b}_{m_0},z')<-c\tau_{n_0}(z').$$

By Inequality (5.1), there exists $m(z',n_0)>m_0$ such that, when $m\geq m(z',n_0)$, we have

$$L_{n_0}(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m,z')\geq 0.$$

Now fix $m_1 \geq m(z', n_0)$. There exists $n(z', m_1) > n_0$ such that when $n \geq n(z', m_1)$, it holds

$$\frac{L_n(\widetilde{F}; \widetilde{a}_{m_1}, \widetilde{b}_{m_1}, z')}{\tau_n(z')} < -c.$$

Then we can choose $n_1 \geq n(z', m_1)$ such that

$$L_{n_1}(\widetilde{F}; \widetilde{a}_{m_1}, \widetilde{b}_{m_1}, z') < -c\tau_{n_1}(z').$$

By induction, we can construct a sequence $\{(m_i, n_i)\}_{i\geq 0} \subset \mathbb{N} \times \mathbb{N}$ satisfying that

(C1): $\{m_i\}_{i\geq 0}$ and $\{n_i\}_{i\geq 0}$ are strictly increasing sequences;

(C2): for all $i \geq 0$, we have

$$L_{n_i}(\widetilde{F}; \widetilde{a}_{m_i}, \widetilde{b}_{m_i}, z') < -c\tau_{n_i}(z') \quad \text{and} \quad L_{n_i}(\widetilde{F}; \widetilde{a}_{m_{i+1}}, \widetilde{b}_{m_{i+1}}, z') \geq 0.$$

By the positively transverse property of $\mathcal{F}$, it is easy to see that the negative part of $L_{n_i}(\widetilde{F}; \widetilde{a}_{m_i}, \widetilde{b}_{m_i}, z')$ only comes from the intersection number of the curve $\widetilde{\Gamma}^n_{\widetilde{I}'_Y, z'}$ with either $\widetilde{l}_{m_i}$ or $\widetilde{l}'_{m_i}$ in the case where either $\alpha(\widetilde{\lambda})$ or $\omega(\widetilde{\lambda})$ contains ∞.

We deal with the case where both $\alpha(\widetilde{\lambda})$ and $\omega(\widetilde{\lambda})$ contain ∞, and other cases follow similarly. In this case, the both sets $\alpha(\lambda), \omega(\lambda) \subset X$ are not contractible. According to Item 5 of Theorem 5.3, for any $z'' \in M \setminus Y$, the loop $I'^{-1}_Y I'(z'')$ is contractible in $M \setminus Y$ (see Section 2.1 for the definition of I'^{-1}_Y). This implies that $\widetilde{\gamma}_m \wedge I'^{-1}_Y I'(z'') = 0$ for all m and $z'' \in \mathrm{Rec}^+(F) \setminus X$. Note that I'_Y fixes a and b. By the facts that the loop $I'_Y(z'')$ is contractible in $M \setminus Y$ for any $z'' \in X \setminus Y$ and that $\alpha(\lambda)$ and $\omega(\lambda)$ are not contractible, and the continuity of I'_Y, we get $|\pi(\widetilde{l}_{m_i}) \wedge I'_Y(x)| \leq 1$ (resp. $|\pi(\widetilde{l}'_{m_i}) \wedge I'_Y(x)| \leq 1$) if the algebraic intersection number is defined and x is close to a (resp. b). By the construction of $\widetilde{\lambda}_m$ and C2, there must be an open sequence of disks $\{U^a_i\}_{i\geq 0}$ containing the set $(I'_Y)^{-1}(\pi(\widetilde{l}_{m_i})) = \cup_{y\in\pi(\widetilde{l}_{m_i})}(I'_Y)^{-1}(y)$ (resp. $\{U^b_i\}_{i\geq 0}$ containing the set $(I'_Y)^{-1}(\pi(\widetilde{l}'_{m_i})) = \cup_{y\in\pi(\widetilde{l}'_{m_i})}(I'_Y)^{-1}(y))$ that satisfies

(D1): $U^a_{i+1} \subset U^a_i$ (resp. $U^b_{i+1} \subset U^b_i$) and $\mu(U^a_i) \to 0$ (resp. $\mu(U^b_i) \to 0$) as $i \to +\infty$ (since the measure μ has no atoms on $\mathrm{Fix}_{\mathrm{Cont},I}(F)$);

(D2): for every $i \geq 0$,

$$\frac{1}{\tau_{n_i}(z')} \sum_{j=0}^{\tau_{n_i}(z')-1} \chi_{U^a_i} \circ F^j(z') > \frac{c}{2} \quad \text{or} \quad \frac{1}{\tau_{n_i}(z')} \sum_{j=0}^{\tau_{n_i}(z')-1} \chi_{U^b_i} \circ F^j(z') > \frac{c}{2},$$

where χ_U is the characteristic function of $U \subset M$.

Denote by $\chi^*_U(x)$ the limit of $\frac{1}{n}\sum_{j=0}^{n-1} \chi_U \circ F^j(x)$ as $n \to +\infty$ for μ-a.e. $x \in M$ (due to Birkhoff Ergodic theorem). By D2 and Inequality 5.2, we have

$$\mu(\{x \in \mathrm{Rec}^+(F) \setminus X \mid \chi^*_{U^a_i}(x) \geq \frac{c}{2} \quad \text{or} \quad \chi^*_{U^b_i}(x) \geq \frac{c}{2}\}) > c$$

for each i. This implies that $\int_M (\chi^*_{U^a_i}(x) + \chi^*_{U^b_i}(x))d\mu \geq \frac{c^2}{2} > 0$ for every i. On the other hand, thanks to Birkhoff Ergodic theorem and D1, we have

$$\int_M (\chi^*_{U^a_i}(x) + \chi^*_{U^b_i}(x))d\mu = \int_M (\chi_{U^a_i}(x) + \chi_{U^b_i}(x))d\mu = \mu(U^a_i) + \mu(U^b_i) \to 0$$

as $i \to +\infty$, which is impossible.

Finally, we get

$$\liminf_{m\to+\infty} i(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m,z') \geq 0 \tag{5.3}$$

for μ-a.e. $z' \in \mathrm{Rec}^+(F)\setminus\{a,b\}$.

From the continuity of I'_Y and the hypothesis on μ, there exists an open free disk U containing z such that $\mu(U)>0$ and for $z'\in U\cap \mathrm{Rec}^+(F)$,

$$\lim_{m\to+\infty} L_1(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m,z') > 0. \tag{5.4}$$

As $\rho_{M,I}(\mu)=0$, by Proposition 3.7, the inequalities 5.3, 5.4 and Fatou Lemma, we have

$$\begin{aligned}
I_\mu(\widetilde{F};a,b) &= \lim_{m\to+\infty} i_\mu(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m)\\
&= \lim_{m\to+\infty}\int_{M\setminus\{a,b\}} i(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m,z)\,\mathrm{d}\mu\\
&\geq \int_{M\setminus\{a,b\}} \liminf_{m\to+\infty} i(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m,z)\,\mathrm{d}\mu\\
&\geq \int_{\bigcup_{k\geq 0}F^k(U)} \liminf_{m\to+\infty} i(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m,z)\,\mathrm{d}\mu\\
&= \int_U \liminf_{m\to+\infty} \tau(z) i(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m,z)\,\mathrm{d}\mu\\
&= \int_U \liminf_{m\to+\infty} \tau^*(z) i(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m,z)\,\mathrm{d}\mu\\
&= \int_U \liminf_{m\to+\infty} L^*(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m,z)\,\mathrm{d}\mu\\
&= \int_U \liminf_{m\to+\infty} L_1(\widetilde{F};\widetilde{a}_m,\widetilde{b}_m,z)\,\mathrm{d}\mu\\
&> 0.
\end{aligned}$$

□

6. Appendix

6.1. Proofs of Lemma 2.6, Lemma 2.7 and Lemma 2.8.

We first recall some properties of $i(\widetilde{F};\widetilde{z},\widetilde{z}')$ defined in Formula 2.2, whose proofs can be found in [Wang11a, page 56]):

(P1): $i(\widetilde{F};\widetilde{z},\widetilde{z}')$ is locally constant on $(\mathrm{Fix}(\widetilde{F})\times\mathrm{Fix}(\widetilde{F}))\setminus\widetilde{\Delta}$;
(P2): $i(\widetilde{F};\widetilde{z},\widetilde{z}')$ is invariant by covering transformation, that is,

$$i(\widetilde{F};\alpha(\widetilde{z}),\alpha(\widetilde{z}')) = i(\widetilde{F};\widetilde{z},\widetilde{z}') \quad \text{for every} \quad \alpha\in G;$$

(P3): $i(\widetilde{F};\widetilde{z},\widetilde{z}')=0$ if $\pi(\widetilde{z})=\pi(\widetilde{z}')$;
(P4): there exists K such that $i(\widetilde{F};\widetilde{z},\widetilde{z}')=0$ if $\widetilde{d}(\widetilde{z},\widetilde{z}')\geq K$.

Lemma 2.7 immediately follows from Lemma 2.6 and P4. Hence we only need to prove Lemma 2.6 and Lemma 2.8.

Proof of Lemma 2.6. If $\widetilde{z} \notin \widetilde{X}$, it is obvious by the continuity (see P1 above) and connectedness. Suppose that $\widetilde{z} \in \widetilde{X}$. Fix a point $\widetilde{a} \in \widetilde{X}$. The linking number $i(\widetilde{F};\widetilde{a},\cdot)$ will be a constant on each connected component of $\widetilde{X} \setminus \widetilde{a}$. Let $\widetilde{b}$ and $\widetilde{b}'$, $\widetilde{c}$ and $\widetilde{c}'$ lie on different components, respectively. Then $i(\widetilde{F};\widetilde{a},\widetilde{b}) = i(\widetilde{F};\widetilde{a},\widetilde{b}')$ and $i(\widetilde{F};\widetilde{a},\widetilde{c}) = i(\widetilde{F};a,\widetilde{c}')$. We have to prove that $i(\widetilde{F};\widetilde{a},\widetilde{b}) = i(\widetilde{F};\widetilde{a},\widetilde{c})$. Now fix $\widetilde{b}$. Let $\widetilde{Y}$ be the connected component of $\widetilde{X} \setminus \{\widetilde{a}\}$ that contains $\widetilde{c}$. Then $\widetilde{a}$ belongs to the closure of $\widetilde{Y}$ and hence $\widetilde{Y} \cup \widetilde{a}$ is connected. Let $\widetilde{Z}$ be the connected component of $\widetilde{X} \setminus \{\widetilde{b}\}$ that contains $\widetilde{a}$. So $(\widetilde{Y} \cup \widetilde{a}) \cap \widetilde{Z} \neq \emptyset$. Hence $\widetilde{c} \in \widetilde{Y} \subset \widetilde{Z}$. We get $i(\widetilde{F};\widetilde{b},\widetilde{a}) = i(\widetilde{F};\widetilde{b},\widetilde{c})$ since $\widetilde{a}$ and $\widetilde{c}$ lie on the same connected component of $X \setminus \widetilde{b}$. Now fix $\widetilde{c}$. Similarly, we have $i(\widetilde{F};\widetilde{c},\widetilde{b}) = i(\widetilde{F};\widetilde{c},\widetilde{a})$ since $\widetilde{b}$ and $\widetilde{a}$ lie on the same connected component of $X \setminus \widetilde{c}$. Obviously, $i(\widetilde{F};\widetilde{z},\widetilde{z}\,')$ is symmetrical on $(\mathrm{Fix}(\widetilde{F}) \times \mathrm{Fix}(\widetilde{F})) \setminus \widetilde{\Delta}$ by the definition of $i_{\widetilde{I}}(\widetilde{z},\widetilde{z}\,')$. Therefore, we obtain

$$i(\widetilde{F};\widetilde{a},\widetilde{b}) = i(\widetilde{F};\widetilde{b},\widetilde{a}) = i(\widetilde{F};\widetilde{b},\widetilde{c}) = i(\widetilde{F};\widetilde{c},\widetilde{b}) = i(\widetilde{F};\widetilde{c},\widetilde{a}) = i(\widetilde{F};\widetilde{a},\widetilde{c}).$$

□

To prove Lemma 2.8, we need the following lemma:

Lemma 6.1. *If $\widetilde{X}$ is a connected subset of* $\mathrm{Fix}(\widetilde{F})$ *and $\widetilde{X}$ is not reduced to a singleton, I satisfies the B-property on $\widetilde{X}$.*

Proof. Let $\widetilde{X}'$ be a connected component of $\mathrm{Fix}(\widetilde{F})$ that contains $\widetilde{X}$. By Lemma 2.7, it is obvious if $\widetilde{X}'$ is unbounded. Suppose now that $\widetilde{X}'$ is bounded. Then $\widetilde{X}'$ is compact. Let us consider the value $i(\widetilde{F},\widetilde{z},\widetilde{z}')$ where $\widetilde{z} \in \mathrm{Fix}(\widetilde{F})$ and $\widetilde{z}' \in \widetilde{X}'$. By the second statement of Lemma 2.6, we only need to consider the case where $\widetilde{z} \in \mathrm{Fix}(\widetilde{F}) \setminus \widetilde{X}'$. If there exists a sequence $\{\widetilde{z}_n\}_{n=1}^{+\infty} \subset \mathrm{Fix}(\widetilde{F}) \setminus \widetilde{X}'$ such that $|i(\widetilde{F},\widetilde{z}_n,\widetilde{z}')| \to +\infty$ as $n \to +\infty$. By P4, the sequence $\{\widetilde{z}_n\}$ must have a convergence subsequence. W.l.o.g, we suppose that $\lim_{n\to+\infty} \widetilde{z}_n = \widetilde{z}_0$. Obviously, $\widetilde{z}_0 \notin \widetilde{X}'$ by the second statement of Lemma 2.6. Finally, it is also impossible in this case since $\widetilde{d}(\widetilde{z}_0,\widetilde{X}') > 0$ and the first statement of Lemma 2.6. □

Proof of Lemma 2.8. If X is not contractible, it follows from Lemma 2.7. Otherwise, it follows from Lemma 6.1 and the properties P2-P4 of $i(\widetilde{F};\widetilde{z},\widetilde{z}\,')$. Furthermore, if $\mathrm{Fix}_{\mathrm{Cont},I}(F)$ is a singleton, it follows from P3. Otherwise, it follows from the first statement of this lemma. □

6.2. **Proof of Lemma 4.1.**

To prove Lemma 4.1, we need the following lemma:

Lemma 6.2. *Let S and S' be two sub-surfaces of an orientable closed surface M, with $S' \subset \mathrm{Int}(S)$ and $z \in S'$. If $i_*(H_1(S,\mathbb{Z}))$ and $i_*(H_1(S',\mathbb{Z}))$ (resp. $i_*(H_1(\mathrm{Cl}(M\setminus S),\mathbb{Z}))$ and $i_*(H_1(\mathrm{Cl}(M\setminus S'),\mathbb{Z}))$) have the same image in $H_1(M,\mathbb{Z})$, then $i_*(\pi_1(S,z))$ and $i_*(\pi_1(S',z))$ have the same image in $\pi_1(M,z)$.*

Proof. Let C be a component of the boundary of S'. It belongs to the boundary of $\mathrm{Cl}(S) \setminus S'$, more precisely, is in a connected component S'' of $\mathrm{Cl}(S) \setminus S'$. The genus of S'' is zero because $i_*(H_1(S,\mathbb{Z}))$ and $i_*(H_1(S',\mathbb{Z}))$ have the same image in $H_1(M,\mathbb{Z})$. We claim that C is the unique component of the boundary of S' and is one of the boundaries of S''. Otherwise, one can find a cycle in S that intersects transversally C, which contradicts the fact that $i_*(H_1(S,\mathbb{Z}))$ and $i_*(H_1(S',\mathbb{Z}))$ have the same image in $H_1(M,\mathbb{Z})$. Secondly, we note that S'' (in fact, every connected component of $\mathrm{Cl}(S)\setminus S'$) is homeomorphic to an annulus because $i_*(H_1(\mathrm{Cl}(M\setminus S),\mathbb{Z}))$ and $i_*(H_1(\mathrm{Cl}(M\setminus S'),\mathbb{Z}))$ have the same image in $H_1(M,\mathbb{Z})$. One deduces that every path in S'' whose endpoints are on C is homotopic in S'' to a path on C. □

Proof of Lemma 4.1. Let us show that $\varinjlim_{\epsilon\to 0} i_*(H_1(Z_\epsilon,\mathbb{Z}))$ exists. Otherwise, there is a positive decrease sequence $\{\epsilon_i\}_{i=1}^{+\infty}$ such that

- $\lim_{i\to+\infty}\epsilon_i = 0$;
- $i_*(H_1(Z_{\epsilon_{i+1}},\mathbb{Z})) \subsetneq i_*(H_1(Z_{\epsilon_i},\mathbb{Z}))$ for every i.

This implies that there is a sequence of simple loops $\{\Gamma_i\}_{i=1}^{+\infty}$ on M which satisfies

- $\Gamma_i \subset Z_{\epsilon_i}$ for every i;
- $\Gamma_i \cap \Gamma_j = \emptyset$ for all $i \neq j$;
- $[\Gamma_i] \neq [\Gamma_j]$ for all $i\neq j$, where $[\Gamma_i]\in H_1(M,\mathbb{Z})$ is the homology class of Γ_i.

This is impossible since $H_1(M,\mathbb{Z})$ is finitely generated.

We now prove that $\varinjlim_{\epsilon\to 0^+} i_*(\pi_1(Z_\epsilon,z))$ exists. By the existence of $\varinjlim_{\epsilon\to 0} i_*(H_1(Z_\epsilon,\mathbb{Z}))$, we can choose a positive number ϵ_0 that is small enough such that

$$i_*(H_1(Z_{\epsilon_1},\mathbb{Z})) = i_*(H_1(Z_{\epsilon_2},\mathbb{Z})) \quad \text{and} \quad i_*(H_1(\mathrm{Cl}(M\setminus Z_{\epsilon_1}),\mathbb{Z})) = i_*(H_1(\mathrm{Cl}(M\setminus Z_{\epsilon_2}),\mathbb{Z}))$$

for all $0<\epsilon_2<\epsilon_1\le\epsilon_0$. The conclusion then follows from Lemma 6.2. □

6.3. **Examples.** Let us give two examples to see what will happen when $\mathrm{Supp}(\mu)\neq M$.

Example 6.3. Consider the following smooth identity isotopy on $\mathbb{R}^2$: $\widetilde{I} = (\widetilde{F}_t)_{t\in[0,1]} : (x,y)\mapsto (x+\frac{t}{2\pi}\cos(2\pi y), y+\frac{t}{2\pi}\sin(2\pi y))$. It induces an identity smooth isotopy $I = (F_t)_{t\in[0,1]}$ on $\mathbb{T}^2$. Let μ have constant density on $\{(x,y)\in\mathbb{T}^2 \mid y=0 \text{ or } y=\frac{1}{2}\}$ and vanish on elsewhere. Obviously, $\rho_{\mathbb{T}^2,I}(\mu)=0$ but $\mathrm{Fix}_{\mathrm{Cont},I}(F_1)=\emptyset$.

Example 6.3 tells us that there is no sense to talk about the action function when $g=1$ and $\mathrm{Supp}(\mu)\neq M$.

The following example belongs to Le Calvez who mentioned me that this example implies that Theorem 1.4 is not true anymore in the case where $g>1$ and $\mathrm{Supp}(\mu)\neq M$.

Example 6.4 ([Lec05], page 73). Let M be the closed orientated surface with $g=2$. He constructed an identity isotopy $I=(F_t)_{t\in[0,1]}$, a point $z'_{1,3}$ which is a periodic point of F_1 with periodic 20 and the arc $\prod_{0\le i\le 19} I(F_1^i(z'_{1,3}))$ is homologic to zero, and two points z_3 and z_4 which are the only two contractible fixed points of F_1.

Let now the measure $\mu = \frac{1}{20}\sum_{i=0}^{19}\delta_{F_1^i(z'_{1,3})}$, where δ_z is the Dirac measure. It is easy to check that $\rho_{M,I}(\mu)=0$ and that the action function is constant.

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