

Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Uniform Entanglement Frames

by

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Preprint no.: 19

2016



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We present several criteria for genuine multipartite entanglement from universal uncertainty relations based on majorization theory. Under non-negative Schur-concave functions, the vector-type uncertainty relation generates a family of infinitely many detectors to check genuine multipartite entanglement. We also introduce the concept of k -separable circles via geometric distance for probability vectors, which include at most $(k - 1)$ -separable states. The entanglement witness is also generalized to a universal entanglement witness which is suitable to detect the k -separable states more accurately.

PACS numbers: 03.65.Ta, 03.67.-a, 42.50.Lc

I. INTRODUCTION

Entanglement is one of the key characteristics of quantum theory and has been a source for new computational methods and algorithms in quantum information theory. Numerous criteria of entanglement have been discovered for pure and mixed quantum states, and many of them are devoted to entanglement of bipartite states. For a multipartite system, separability can be classified into k -separability [1] and the quantum state is called genuinely entangled if it is not separable with respect to any tensor bipartition of its space (see Sect. III C for detailed definition). Due to the importance of genuine entanglement, numerous approaches have been devoted to detecting genuinely multipartite entanglement. Among the influential ones, the methods using entanglement witness [2–9], generalized concurrence [10–14], and Bell-inequality [15] are very useful. Nevertheless, the problem of detecting genuinely entanglement is far from being solved.

The principle behind the entanglement witnesses is that entanglement of multipartite quantum states gives rise to nonlocal correlations of measurement observers, whose measurement outcomes obey certain bounds. In particular the lower bound of the uncertainty relation is expected to provide better entanglement witnesses. In this paper, we use the universal uncertainty relation and majorization theory to derive further criteria for k -separability and genuine entanglement.

We take mixed three-qubit states as an example to explain our approach to entanglement using majorization. In [16], mixed three-qubit states have been classified into three classes of genuine three-qubit entanglement (W-type, GHZ-type), biseparable and fully separable states. Our approach not only gives a new method to detect entanglement, but also provides a fine tuned device to further classify biseparable states into three subclasses of $AB - C$, $A - BC$, and $AC - B$ types. Furthermore, we introduce the notion of a *universal entanglement witness* and derive its canonical form, which can be used to picture layers of k -separability [1] as concentric circles.

The principle behind our majorization method is to use uncertainty relations to detect entanglement. In the original Heisenberg-Robertson [17, 18] uncertainty principle, the lower bound of the product of standard deviations of two incompatible observables A and B is given by

$$\Delta A \cdot \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle|, \quad (1)$$

where ΔA is the standard deviation of A respect to the quantum state φ . Usually the uncertainty relations in terms of standard deviations is state-dependent (but see [19]).

Entropy has been used as a measurement [20] for the uncertainty principle. Deutsch [21] showed that the lower bound of the entropy uncertainty relation (EUR) in a finite dimensional Hilbert space is

$$H(p) + H(q) \geq -2 \ln C, \quad (2)$$

where $C = (1 + \sqrt{c_1})/2$ and $c_1 = \max_{i,j} |\langle a_i | b_j \rangle|$ is the maximal overlap between the bases $|a_i\rangle$ and $|b_j\rangle$, p and q are the probability distributions in the usual manner. In this form the lower bound is dependent on the state.

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Other uncertainty relations and improvements have been given in [22] and [23], where the lower bound are mostly state-dependent and computable from two probability vectors. Good surveys for these bounds can be found in [24–30].

The universality of the information-based uncertainty principle gives a unbreakable lower bound that holds in general. In fact, the uncertainty relations can be quantified by majorization [31], which has numerous advantages over previous formulations. The *universal uncertainty relation* (UUR) based on majorization was first found in [32]

$$p(\rho) \otimes q(\rho) \prec \omega, \quad \forall \rho, \quad (3)$$

where ρ is any mixed state on a finite dimensional Hilbert space and ω is a certain probability vector independent of ρ . Here the *majorization* ' \prec ' is certain partial order among real vectors [33] (see section 2 for definition). In the following we will also generalize the UUR principle to other situations and use them to give new entanglement tests for k -separability.

The paper is roughly organized as follows. First we review some basic backgrounds of majorization theory and the UUR in Sect. II. Then we prove a universal lower bound for partial separable states and use the lower bound to give criteria for genuine entanglement and k -separability. Finally we discuss the matrix form of majorization and how it is used to provide better tests for entanglement of multipartite mixed states.

II. BACKGROUND MATERIALS

We review some basic materials of majorization theory [33], the UUR, and the elementary classification of mixed three-qubit states following mostly [16].

A real vector $x \in \mathbb{R}^d$ is *majorized by* (denoted as \prec) another real vector $y \in \mathbb{R}^d$ provided that $\sum_{j=1}^k x_j^\downarrow \leq \sum_{j=1}^k y_j^\downarrow$ for all $1 \leq k \leq d-1$ and $\sum_{j=1}^d x_j^\downarrow = \sum_{j=1}^d y_j^\downarrow$. The down-arrow means to rearrange the components of the vector in the decreasing order: $x_1^\downarrow \geq x_2^\downarrow \geq \dots \geq x_d^\downarrow$. If $x \prec y$ but $x \neq y$, then x is said to be strictly majorized by y and written as $x \prec\prec y$. In general, any probability vectors $x \in \mathbb{R}^d$ satisfies that $(\frac{1}{d}, \frac{1}{d}, \dots, \frac{1}{d}) \prec x \prec (1, 0, \dots, 0)$.

Consider an n -partite density matrix ρ ($n \geq 2$) with positive operator valued measures (POVMs). Let $\{X_l, E_{\alpha_l}^{X_l}\}_{\alpha_l=1}^{N_l}$ be the l -th POVM, where $1 \leq l \leq n$ and N_l is the number of elements of the l -th POVM. A measurement of ρ with the l -th POVM X_l induces a probability distribution vector $p^l(\rho) = (p_1^l(\rho), p_2^l(\rho), \dots, p_{N_l}^l(\rho))$, where $p_j^l(\rho) = \text{tr}(\rho E_{\alpha_j}^{X_l})$. Then a uncertainty of the form

$$\bigotimes_{l=1}^n p^l(\rho) \prec \omega, \quad \forall \rho, \quad (4)$$

holds, where the LHS represents the joint probability distribution induced by measuring ρ with each POVM X_l . The multitensor product is defined by associativity as follows. Suppose $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$, then the tensor product $a \otimes b$ is the vector $(a_1 b_1, \dots, a_1 b_n, \dots, a_m b_1, \dots, a_m b_n)$ in \mathbb{R}^{mn} .

The vector ω is independent of ρ . If the measurement elements X_l do not have a common eigenstate, then $\omega \prec\prec (1, 0, \dots, 0)$. Moreover for any uncertainty measure Φ , a nonnegative Schur-concave function, one has that

$$\Phi[\bigotimes_{l=1}^n p^l(\rho)] \geq \Phi(\omega), \quad \forall \rho. \quad (5)$$

For example, let ρ be any bipartite density matrix and $|a_m\rangle, |b_m\rangle$ ($m = 1, \dots, d$) be two orthonormal bases of the underlying Hilbert space. Let $p_m(\rho) = \langle a_m | \rho | a_m \rangle$ and $q_n(\rho) = \langle b_n | \rho | b_n \rangle$ be the measurements of ρ given by the basis elements, then they form two probability distribution vectors $\mathbf{p}(\rho)$ and $\mathbf{q}(\rho)$ respectively. It can be shown that the realignment vector of the Kronecker tensor product of the two probability vectors $\mathbf{p}(\rho)$ and $\mathbf{q}(\rho)$ is majored by a vector ω independent of ρ :

$$\mathbf{p}(\rho) \otimes \mathbf{q}(\rho) \prec \omega, \quad \forall \rho, \quad (6)$$

where $\omega \in \mathbb{R}^{d^2}$ is given by

$$\omega = (\Omega_1, \Omega_2 - \Omega_1, \dots, \Omega_d - \Omega_{d-1}, 0, \dots, 0), \quad (7)$$

and

$$\Omega_k = \max_{I_k} \max_{\rho} \sum_{(m,n) \in I_k} p_m(\rho) q_n(\rho). \quad (8)$$

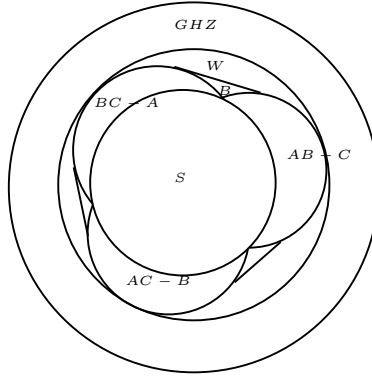


FIG. 1: Schematic picture of mixed states for three qubits. S : Fully separable class; B : Biseparable class (convex hull of biseparable states under three different partitions); W and GHZ : genuine tripartite entangled.

Here I_k are k -element subsets of $[d] \times [d]$ and $[d] = \{1, 2, \dots, d\}$. The outer maximum is over all subsets I_k and the inner maximum runs over all density matrices.

Mixed states of three-qubit systems can be classified by the following: (1) the convex hull S of separable states; (2) the convex hull B of biseparable states (AB-C, AC-B and BC-A); (3) the convex hull of all states, including S , B and genuinely entangled states (W -states and GHZ -states). All these sets are compact and convex, satisfy $S \subset B \subset W \subset GHZ$ (see FIG 1.). Examples of GHZ witness and W witness had been given in [16].

III. ENTANGLEMENT DETECTION

As shown in [31], the universal uncertainty relation can reach its bound on pure states. One can use this to establish an entanglement detector to check whether a density matrix is separable by proving a condition satisfied by all separable states. The method of using universal uncertainty relations, however, has a limit to detect genuine entanglement. In this section, we utilize the Lagrange multiplier to show that the universal uncertainty bound of $\rho = \sum p_i \rho_{AB}^i \otimes \rho_C^i$ can also be reached by the pure state of the form $\rho_{AB} \otimes \rho_C$, which provides a AB-C biseparable detector. This establishes a biseparable detector of all tripartite states in AB-C, AC-B or BC-A, and more generally, it can be generalized to give a k -separable detector for all n -partite states.

A. Universal Uncertainty Bounds on Pure States

For simplicity we limit ourselves to three measurements ($X, \{E_\alpha^X\}$), ($Y, \{E_\beta^Y\}$), and ($Z, \{E_\gamma^Z\}$) that have no common eigenstate. The measurements are performed on a given state $\rho \in \mathbb{H}_A \otimes \mathbb{H}_B \otimes \mathbb{H}_C$. Let $\mathcal{P}^X(\rho) = (\text{tr}(\rho E_\alpha^X))$, $\mathcal{P}^Y(\rho) = (\text{tr}(\rho E_\beta^Y))$, $\mathcal{P}^Z(\rho) = (\text{tr}(\rho E_\gamma^Z))$ and $\mathcal{P}^{X \oplus Y \oplus Z}(\rho) = \mathcal{P}^X(\rho) \otimes \mathcal{P}^Y(\rho) \otimes \mathcal{P}^Z(\rho)$.

We recall the uncertainty bound given in [34]. The tensor product of the propability distribution vectors satisfies that

$$\mathcal{P}^X(\rho) \otimes \mathcal{P}^Y(\rho) \otimes \mathcal{P}^Z(\rho) \prec \omega, \quad (9)$$

where the bound is reached on pure states ρ_0 :

$$\omega = \sup_{\rho_0} [\mathcal{P}^{X \oplus Y \oplus Z} \rho_0] \prec \prec (1, 0, \dots, 0). \quad (10)$$

If $\rho = \sum p_i \rho_A^i \otimes \rho_B^i \otimes \rho_C^i$, then

$$\mathcal{P}^X(\rho) \otimes \mathcal{P}^Y(\rho) \otimes \mathcal{P}^Z(\rho) \prec \omega_{A,B,C} = \sup_{\rho_A \otimes \rho_B \otimes \rho_C} [\mathcal{P}^{X \oplus Y \oplus Z}(\rho_A \otimes \rho_B \otimes \rho_C)], \quad (11)$$

where $\rho_A \otimes \rho_B \otimes \rho_C$ are fully separable pure states. Now we want to prove that if the density matrix of a tripartite states has the form of $\rho = \sum p_i \rho_{AB}^i \otimes \rho_C^i$, then

$$\mathcal{P}^X(\rho) \otimes \mathcal{P}^Y(\rho) \otimes \mathcal{P}^Z(\rho) \prec \omega_{AB,C} = \sup_{\rho_{AB} \otimes \rho_C} [\mathcal{P}^{X \oplus Y \oplus Z}(\rho_{AB} \otimes \rho_C)], \quad (12)$$

where $\rho_{AB} \otimes \rho_C$ is biseparable pure state of type AB-C. To prove this, it is sufficient to show that the maximum value of the sum of i different components of $\mathcal{P}^{X \oplus Y \oplus Z}(\sum p_i \rho_{AB}^i \otimes \rho_C^i)$ is realized on pure states of same type.

Suppose $\rho = \sum_a q_a |\phi_a^{AB}\rangle \langle \phi_a^{AB}| \otimes |\phi_a^C\rangle \langle \phi_a^C|$. To maximize $\mathcal{P}^{X \oplus Y \oplus Z}(\rho)$ we apply the method of the Lagrange multipliers:

$$\begin{aligned} & \sum_{a,b,c} q_a q_b q_c [\langle \phi_a^{AB} | \langle \phi_a^C | E_{\alpha_1}^X | \phi_a^C \rangle | \phi_a^{AB} \rangle \langle \phi_b^{AB} | \langle \phi_b^C | E_{\beta_1}^Y | \phi_b^C \rangle | \phi_b^{AB} \rangle \langle \phi_c^{AB} | \langle \phi_c^C | E_{\gamma_1}^Z | \phi_c^C \rangle | \phi_c^{AB} \rangle + \\ & \langle \phi_a^{AB} | \langle \phi_a^C | E_{\alpha_2}^X | \phi_a^C \rangle | \phi_a^{AB} \rangle \langle \phi_b^{AB} | \langle \phi_b^C | E_{\beta_2}^Y | \phi_b^C \rangle | \phi_b^{AB} \rangle \langle \phi_c^{AB} | \langle \phi_c^C | E_{\gamma_2}^Z | \phi_c^C \rangle | \phi_c^{AB} \rangle + \dots + \\ & \langle \phi_a^{AB} | \langle \phi_a^C | E_{\alpha_i}^X | \phi_a^C \rangle | \phi_a^{AB} \rangle \langle \phi_b^{AB} | \langle \phi_b^C | E_{\beta_i}^Y | \phi_b^C \rangle | \phi_b^{AB} \rangle \langle \phi_c^{AB} | \langle \phi_c^C | E_{\gamma_i}^Z | \phi_c^C \rangle | \phi_c^{AB} \rangle] + \\ & \sum_a \eta_{i,a}^{AB} (1 - \langle \phi_a^{AB} | \phi_a^{AB} \rangle) + \sum_a \eta_{i,a}^C (1 - \langle \phi_a^C | \phi_a^C \rangle) + \xi_i (1 - \sum_a q_a). \end{aligned} \quad (13)$$

where $\eta_{i,a}^{AB}, \eta_{i,a}^C, \xi_i$ are the Lagrange multipliers.

Denote that

$$\begin{aligned} \varepsilon_i^X &= \sum_{k=1}^i \mathcal{P}_{\beta_k}^Y(\rho) \mathcal{P}_{\gamma_k}^Z(\rho) E_{\alpha_k}^X, \\ \varepsilon_i^Y &= \sum_{k=1}^i \mathcal{P}_{\alpha_k}^X(\rho) \mathcal{P}_{\gamma_k}^Z(\rho) E_{\beta_k}^Y, \\ \varepsilon_i^Z &= \sum_{k=1}^i \mathcal{P}_{\alpha_k}^X(\rho) \mathcal{P}_{\beta_k}^Y(\rho) E_{\gamma_k}^Z, \\ \varepsilon_i &= \varepsilon_i^X + \varepsilon_i^Y + \varepsilon_i^Z. \end{aligned} \quad (14)$$

Variations with respect to $\langle \phi_s^{AB} |$ give

$$q_s [\langle \phi_s^C | \varepsilon_i | \phi_s^C \rangle | \phi_s^{AB} \rangle = \eta_{i,s}^{AB} | \phi_s^{AB} \rangle. \quad (15)$$

Similarly with respect to $\langle \phi_s^C |$ we have that

$$q_s [\langle \phi_s^{AB} | \varepsilon_i | \phi_s^{AB} \rangle | \phi_s^C \rangle = \eta_{i,s}^C | \phi_s^C \rangle, \quad (16)$$

which implies $\eta_{i,s}^{AB} = \eta_{i,s}^C$. Denote $\Omega_{AB,C;i}$ as the sum of i different components, and $\langle \phi_s^{AB} | \langle \phi_s^C | \varepsilon_i | \phi_s^C \rangle | \phi_s^{AB} \rangle = \varepsilon_{i,ss}$, then $\Omega_{AB,C;i} = \frac{1}{3} \sum_s q_s \varepsilon_{i,ss} = \frac{1}{3} \text{tr}(\varepsilon_i \rho)$. Finally, variation with respect to q_s leads to $\langle \phi_s^{AB} | \langle \phi_s^C | \varepsilon_i | \phi_s^C \rangle | \phi_s^{AB} \rangle = \xi_i$

which means that $\varepsilon_{i,ss}$ do not depend on s , and $\omega_{AB,C}$ can be reached by pure biseparable states of form $\rho_{AB} \otimes \rho_C$.

It is clear that the above argument works for any multipartite state, and the bound for the tensor product of multipartite state can be reached by pure states with the same type. This implies that majorization uncertainty relations can be used to detect genuine entanglement in multipartite and distinguish different types at the same level of entanglement (for example, triseparable ABC-D vs. quartistate system).

B. Entanglement Detection in Tripartite States

Let $\rho \in \mathbb{H}_A \otimes \mathbb{H}_B \otimes \mathbb{H}_C$ be a tripartite state with a measurement $\{X, E_\alpha^X\}$. Then the probability vector $\mathcal{P}^X(\rho)$ is majorized by a bound vector ω independent of ρ . The following results are clear from our discussion.

[Lemma 1]. Let $\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i \otimes \rho_C^i$ be a mixed tripartite state. The probability density vector $\mathcal{P}^X(\rho)$ associated with the measurement X is majorized by a bound vector $\omega_{A,B,C}$ independent from ρ and reachable by a pure tripartite state:

$$\mathcal{P}^X(\rho) \prec \omega_{A,B,C}. \quad (17)$$

If the probability vector of ρ violates the majorization relation, then ρ is entangled.

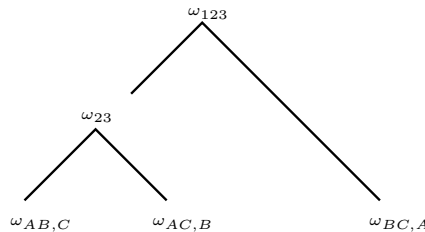


FIG. 2: Preordering of $\omega_{AB,C}$, $\omega_{AC,B}$ and $\omega_{BC,A}$.

[Lemma 2]. If $\rho = \sum_i p_i \rho_{AB}^i \otimes \rho_C^i$, then the probability vector $\mathcal{P}^X(\rho)$ resulting from the measurement X on ρ is majorized by $\omega_{AB,C}$ which is state-independent and reachable by some pure biseparable state of type AB-C:

$$\mathcal{P}^X(\rho) \prec \omega_{AB,C}. \quad (18)$$

If the probability vector of ρ violates the majorization relation, then ρ is entangled but not biseparable of type AB-C.

If the dimensions of \mathbb{H}_A , \mathbb{H}_B and \mathbb{H}_C are different, then the majorization uncertainty bounds $\omega_{AB,C}$, $\omega_{AC,B}$ and $\omega_{BC,A}$ are all different under a suitable measurement $\{X, E_\alpha^X\}$. These probability vectors and that of the whole space form a lattice (see FIG 2), which leads to a majorization uncertainty bound to control any two of $\omega_{AB,C}$, $\omega_{AC,B}$ and $\omega_{BC,A}$.

[Proposition 1]. If $\omega_{AB,C} \neq \omega_{AC,B}$ and both $\prec (1, 0, \dots, 0)$, then there exists a unique probability vector ω_{23} such that

$$\begin{aligned} \omega_{AB,C} &\prec \omega_{23}, \\ \omega_{AC,B} &\prec \omega_{23}, \\ \omega_{23} &\prec (1, 0, \dots, 0), \end{aligned} \quad (19)$$

and majorized by any other vector with property Eq. (19). This vector is also denoted by $\max\{\omega_{AB,C}, \omega_{AC,B}\}$.

With these results in hand, we can give an entanglement criterion to analyze biseparable states.

[Theorem 1]. If the probability vector of tripartite state ρ under measurement $\{X, E_\alpha^X\}$ is not majorized by ω_{23} :

$$\mathcal{P}^X(\rho) \not\prec \max\{\omega_{AB,C}, \omega_{AC,B}\}, \quad (20)$$

then ρ can not be of the form $\rho = \sum_i p_i \rho_{AB}^i \otimes \rho_C^i + \sum_j q_j \rho_{AC}^j \otimes \rho_B^j$.

To study genuine entanglement, we consider both ω_{23} and $\omega_{BC,A}$ and denote the unique probability vector $\max\{\omega_{23}, \omega_{BC,A}\}$ by ω_{123} . The following result gives a criterion for genuine entanglement.

[Theorem 2]. If

$$\mathcal{P}^X(\rho) \not\prec \omega_{123}, \quad (21)$$

then ρ can not be biseparable, and ρ is genuinely entangled.

For any measurement Φ of uncertainty (non-negative Schur-concave function), it follows from $\mathcal{P}^X(\rho) \prec \omega_{123}$ that

$$\Phi(\mathcal{P}^X(\rho)) \geq \Phi(\omega_{123}). \quad (22)$$

One can use uncertainty measurements to give numerical criteria for genuine entanglement.

[Corollary 1]. If there is an uncertainty measurement Φ such that

$$\Phi(\mathcal{P}^X(\rho)) < \Phi(\omega_{123}), \quad (23)$$

then ρ is not a biseparable state, and ρ is genuinely entangled.

If \mathbb{H}_A , \mathbb{H}_B and \mathbb{H}_C have the same dimension, then $\omega_{AB,C} = \omega_{AC,B} = \omega_{BC,A} = \omega_{123}$ by the symmetry property of the superior over pure biseparable states. Although one can not use the above result to detect if a tripartite state has the form $\rho = \sum_i p_i \rho_{AB}^i \otimes \rho_C^i + \sum_j q_j \rho_{AC}^j \otimes \rho_B^j$, one still has the following result for genuine entanglement.

[Theorem 3]. For $\dim \mathbb{H}_A = \dim \mathbb{H}_B = \dim \mathbb{H}_C$, if

$$\mathcal{P}^X(\rho) \not\prec \omega_{AB,C}, \quad (24)$$

then ρ is not a biseparable state, and ρ is genuinely entangled.

Combining with non-negative Schur-concave functions, one immediately gets the following criterion of genuine entanglement.

[Corollary 2]. If there is a non-negative Schur-concave function Φ such that

$$\Phi(\mathcal{P}^X(\rho)) < \Phi(\omega_{AB,C}), \quad (25)$$

then ρ is not biseparable state, and the state ρ is genuinely entangled.

C. Entanglement Detection in Multipartite States

The majorization method can be generalized to characterize entanglement for multipartite quantum states. In this section, we present a general and systematic scheme to study k -separable and n -partite genuinely entangled states.

Following [3], we say a pure n -partite quantum state $|\Psi\rangle$ is k -separable if it can be written as

$$|\Psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_k\rangle, \quad (26)$$

where $|\phi_i\rangle$ is a single subsystem or a group of subsystems. If such a decomposition is not possible, $|\Psi\rangle$ is called genuinely n -partite entangled. For a mixed state ρ : a state is genuinely k -partite entangled if any decomposition into pure states

$$\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|, \quad (27)$$

with probabilities $p_i > 0$ contains at least one genuinely k -partite entangled component. It is called k -separable if any decomposition into pure states $|\psi_i\rangle$ is at least k -separable. All $(k+1)$ -separable states form a compact and convex subset of the set of k -separable states. This hierarchical structure is usually referred to as a *uniform entanglement frame*. Our main idea is to construct criteria to detect k -separability by majorization uncertainty relations and find the principle behind these criteria.

Statistical mixture of biseparable states for tripartite states has been considered, we now focus on k -separable states for multipartite states. For any k -separable state $\rho_{kse} = \sum_i p_i \rho_{kse}^i$, if all ρ_{kse}^i have the same form (for example, systems 1 to $n-k+1$ are entangled and the rest are separable), then by the majorization method there exists a bound vector $\omega_{k,l}$ reachable by pure states with the same form as ρ_{kse}^i for each i . Moreover,

$$\mathcal{P}^X(\rho_{kse}) \prec \omega_{k,l}, \quad (28)$$

where there are \mathbf{C}_n^k possibilities for l . If $\dim \mathbb{H}_1 = \dim \mathbb{H}_2 = \cdots = \dim \mathbb{H}_n$, then

$$\omega_{k,1} = \omega_{k,2} = \cdots = \omega_{k,\mathbf{C}_n^k} := \omega_k.$$

Under this assumption we have

[Theorem 4]. For any k -separable state ρ_{kse} and measurement $\{X, E_{\alpha_i}^X\}$, the following majorization uncertainty relation holds:

$$\mathcal{P}^X(\rho_{kse}) \prec \omega_k, \quad (29)$$

where ω_k can be reached by pure k -separable state.

[Corollary 3]. For any n -partite state ρ , if there exists a measurement $\{X, E_{\alpha_i}^X\}$ such that the probability vector is not majorized by ω_k :

$$\mathcal{P}^X(\rho) \not\prec \omega_k, \quad (30)$$

then ρ is at most $(k-1)$ -separable.

One can generalize Corollary 3 to derive a criterion for genuinely n -partite entangled states.

[Corollary 4]. For any n -partite state ρ , if there exists a probability vector under measurement $\{X, E_{\alpha_i}^X\}$ such that

$$\mathcal{P}^X(\rho) \not\prec \omega_2, \quad (31)$$

then ρ is genuinely n -partite entangled.

Under non-negative Schur-concave functions Φ , the majorization uncertainty bound ω_2 becomes a real number $\Phi(\omega_2)$, then Eq. (31) generates in fact infinite family of genuinely n -partite entangled criteria. [Corollary 5]. For any n -partite state ρ and non-negative Schur-concave functions Φ , if the following equality holds under measurement $\{X, E_{\alpha_i}^X\}$:

$$\Phi(\mathcal{P}^X(\rho)) < \Phi(\omega_2), \quad (32)$$

then ρ is genuinely n -partite entangled.

Entangled states are characterized well under the majorization uncertainty relations as the universal uncertainty bounds are independent from the state ρ . This is particularly so when one applies the uncertainty relations given in Eq. (2). We discuss some examples to show how these are applied.

[Example 1]. Consider the Werner state $\rho_d^{wer}(q)$ [34] defined on the tensor product of two d -dimensional Hilbert spaces:

$$\rho_d^{wer}(q) = \frac{1}{d^2}(1-q)I + q|\mathfrak{B}_1\rangle\langle\mathfrak{B}_1|, \quad (33)$$

where I is the identity matrix in $\mathbb{C}^{d \times d}$ and

$$|\mathfrak{B}_1\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j_A\rangle \otimes |j_B\rangle, \quad (34)$$

is the first of the generalized Bell states $\{|\mathfrak{B}_\alpha\rangle\}_{\alpha=1}^{d^2}$, which form a basis of orthonormal eigenstates for $\rho_d^{wer}(q)$. Let the measurement $\{X, E_\alpha^X\}$ be $E_\alpha^X = |\mathfrak{B}_\alpha\rangle\langle\mathfrak{B}_\alpha|$, $\alpha = 1, 2, \dots, d^2$. The probability vector is then

$$\mathcal{P}^X(\rho_d^{wer}(q)) = (q + d^{-2}(1-q), d^{-2}(1-q), d^{-2}(1-q), \dots, d^{-2}(1-q)). \quad (35)$$

As shown in [35], the maximum overlap of every generalized Bell states with the set of pure, product states is given by $1/\sqrt{d}$, we get that

$$\omega_{A,B} = (1/d, 1/d, \dots, 1/d, 0, 0, \dots, 0). \quad (36)$$

Using our Theorem 1, the Werner state is separable if $\mathcal{P}^X(\rho_d^{wer}(q))$ is majorized by $\omega_{A,B}$, i.e., $q \leq (1+d)^{-1}$. This inequality agrees with the well-known separability condition for the Werner state.

Clearly, the efficiency of uncertainty bounds in judging entanglement depends on the choice of the measurement $\{X, E_\alpha^X\}$. The calculation of ω will be easy if the measurement is optimal in some sense.

[Example 2]. Next we will take the quantum state to be $\rho_d^{wer}(q) \otimes \frac{1}{d}I$ and the measurement to be $\{X, E_{\alpha,j}^X\}$, where $E_{\alpha,j}^X = |\mathfrak{B}_\alpha\rangle \otimes |j\rangle$, $\alpha = 1, 2, \dots, d^2$; $j = 0, 1, \dots, d-1$. It is easy to see that

$$\begin{aligned} & \mathcal{P}^X(\rho_d^{wer}(q) \otimes \frac{1}{d}I) \\ &= \underbrace{(q + d^{-2}(1-q), \dots, q + d^{-2}(1-q), d^{-2}(1-q), d^{-2}(1-q), \dots, d^{-2}(1-q))}_{d\text{-times}}. \end{aligned}$$

The maximum overlap of every measurement element $E_{\alpha,j}^X = |\mathfrak{B}_\alpha\rangle \otimes |j\rangle$ with the set of pure, biseparable state is given by 1. Thus we conclude that

$$\omega_{AB,C} = (1, 0, \dots, 0). \quad (37)$$

It is easy to see that $\mathcal{P}^X(\rho_d^{wer}(q) \otimes \frac{1}{d}I) \prec \omega_{AB,C}$ is always true, and no violation happens, which means that $\rho_d^{wer}(q) \otimes \frac{1}{d}I$ is always biseparable of form AB-C. This coincides with the fact that for any value q , $\rho_d^{wer}(q) \otimes \frac{1}{d}I$ is always biseparable and can only be entangled in the AB system.

IV. MATRIX FORMS OF MAJORIZATION

Majorization was first studied by Schur [36] in relation with Hadamard inequalities and it was proved that two probability vectors $x \prec y$ if and only if $y = Q(x)$ for a bistochastic matrix Q . A matrix $Q = (q_{ij}) \in \mathbb{R}_+^{n \times n}$ is bistochastic if

$$\sum_i q_{ij} = \sum_j q_{ij} = 1.$$

Birkhoff proved that the set of all bistochastic matrices is the convex hull of permutation matrices [37]. The geometric and combinatoric principle behind majorization and bistochastic matrix is that the relation fully characterizes the convexity of the set of all k -separable states. It is a convex and compact subset of $(k-1)$ -separable states, which enables one to construct criteria to detect k -separable states. The majorization method to detect states is relied on the ‘‘interior’’ of the convex set of k -separable states based on Birkhoff’s theorem.

We list the equivalent forms of majorization as follows.

- (1) $x \prec y$;
- (2) $y = Q(x)$ for some bistochastic matrix Q ;
- (3) $\Phi(x) \geq \Phi(y)$ for any non-negative Schur-concave functions Φ ;
- (4) x can be derived from y by successive applications of finitely many T -transformations:

$$T = \lambda I + (1 - \lambda)P, \quad (38)$$

where $0 \leq \lambda \leq 1$ and P is a permutation matrix that interchanges two coordinates;

- (5) x is in the convex hull of the $n!$ permutations of y .

We now give the matrix form of the k -separable criterion.

[Theorem 5]. Let ρ be an n -partite quantum state on $\mathbb{H}^{\otimes n}$ ($\dim(\mathbb{H}) = d$), and let $\{X, E_\alpha^X\}$ be a measurement with the probability vector $\mathcal{P}^X(\rho) = (p_1, p_2, \dots, p_m)$, where $m = d^n$. Suppose the k -separable uncertainty bound is $\omega_k = (\Omega_1, \Omega_2 - \Omega_1, \dots, \Omega_m - \Omega_{m-1})$. If there does not exist a bistochastic matrix $Q = (q_{ij})$ such that

$$\Omega_j - \Omega_{j-1} = \sum_{i=1}^m q_{ij} p_i, \quad j = 1, 2, \dots, m.$$

then ρ is at most $(k-1)$ -separable.

This result follows from the fact that violation of $\mathcal{P}^X(\rho) \prec \omega_k$ is equivalent to there does not exist a bistochastic matrix Q such that $\omega_k = Q(\mathcal{P}^X(\rho))$.

We now consider an information quantity called f -relative entropy to express the closeness between two probability vectors and helps describe k -separability. The f -relative entropy is also called f -divergence in information theory [38]. [Lemma 3]. Let f be a convex function. Then the information quantity $D_f(x \parallel y) = \sum_i x_i f(\frac{y_i}{x_i})$ for probability vectors $x = (x_i)$ and $y = (y_i)$ is monotonic:

$$D_f(x \parallel y) \geq D_f(Q(x) \parallel Q(y)), \quad (39)$$

where Q is stochastic matrix.

It follows from Lemma 3 that $D_f(x \parallel y)$ is an f -relative entropy. This monotonicity condition (39) also holds if Q is replaced by a bistochastic matrix.

Note that $Q(1/m, 1/m, \dots, 1/m) = (1/m, 1/m, \dots, 1/m)$ for any bistochastic matrix. Plugging $y = \mathfrak{K} = (1/m, 1/m, \dots, 1/m)$ and $x = \mathcal{P}^X(\rho)$ associated with a measurement $\{X, E_\alpha^X\}$ in Eq. (39), one immediately gets that

$$D_f(\mathcal{P}^X(\rho) \parallel \mathfrak{K}) \geq D_f(\omega_k \parallel \mathfrak{K}), \quad (40)$$

This gives the following criterion of relevant entanglement.

[Theorem 6]. If there exists a convex function and a measurement $\{X, E_\alpha^X\}$ such that

$$D_f(\mathcal{P}^X(\rho) \parallel \mathfrak{K}) < D_f(\omega_k \parallel \mathfrak{K}), \quad (41)$$

then the state ρ is at most $(k-1)$ -separable.

The f -relative entropy $D_f(\mathcal{P}^X(\rho) \parallel \mathfrak{K})$ expresses the closeness between $\mathcal{P}^X(\rho)$ and a fixed point \mathfrak{K} , but it does not satisfy the axioms of a distance. To get a ‘distance’ to characterize k -separability one needs special convex functions such as $f(x) = 1 - \sqrt{x}$, its square root is called Hellinger distance and denoted by $d_2(x, y)$. It is known that $d_2(x, y)$ satisfies the axioms of a distance. With \mathfrak{K} as the center of the circle, $d_2(\omega_k, \mathfrak{K})$ can be viewed as the radius of a k -separable circle, or simply the k -circle (see FIG 3.).

[Corollary 5]. If ρ is at most $(k-1)$ -separable, then

$$d_2(\mathcal{P}^X(\rho), \mathfrak{K}) < d_2(\omega_k, \mathfrak{K}). \quad (42)$$

The vector $\mathcal{P}^X(\rho) \in \mathbb{R}^m$ is a point inside the k -separable circle.

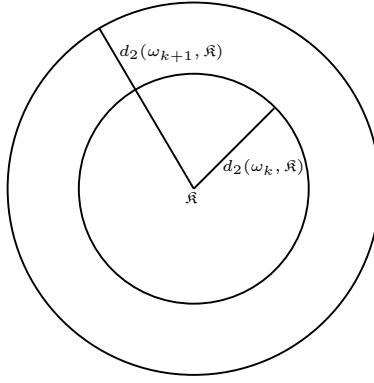


FIG. 3: k -separable radius corresponds to ' k -circle '.

[Example 3]. Consider again the Werner state $\rho_d^{wer}(q)$. Both the probability $\mathcal{P}^X(\rho_d^{wer}(q))$ under measurement $E_\alpha^X = |\mathfrak{B}_\alpha\rangle\langle\mathfrak{B}_\alpha|$ and the uncertainty bound $\omega_{A,B}$ are given as follows.

$$\mathcal{P}^X(\rho_d^{wer}(q)) = (q + d^{-2}(1-q), d^{-2}(1-q), d^{-2}(1-q), \dots, d^{-2}(1-q)), \quad (43)$$

while

$$\omega_{A,B} = (1/d, 1/d, \dots, 1/d, 0, 0, \dots, 0), \quad (44)$$

It is easy to calculate that $d_2^2(\omega_{A,B}, \mathfrak{R}) = 1 - \sqrt{\frac{1}{d}}$. Then $d_2(\mathcal{P}^X(\rho_d^{wer}(q)), \mathfrak{R}) < d_2(\omega_{A,B}, \mathfrak{R})$ implies $q > (1 + d)^{-1}$, which coincides with the necessary and sufficient condition for inseparability of the Werner state.

Hellinger distance is useful to detect entanglement based on the monotonicity condition. However, only the monotonicity property is not good enough for entanglement detection, which is shown by the following example.

[Example 4]. Consider the variational distance given by

$$d_1(x, y) = \frac{1}{2} \sum_i |x_i - y_i|. \quad (45)$$

It is not an f -relative entropy, but it satisfies the monotonicity property

$$d_1(x, y) \geq d_1(Q(x), Q(y)). \quad (46)$$

If we take y as $\mathfrak{R} = (1/m, \dots, 1/m)$ then $d_1(x, y) = d_1(Q(x), Q(y))$. Thus d_1 fails to detect entanglement.

Finally we would like to discuss how entanglement witness is used to detect entanglement. A *witness for genuine k -partite entanglement* is an observable that has a positive expectation value on states with $(k-1)$ -partite entanglement and a negative expectation value on some k -partite entangled states. Entanglement witness is a useful tool for analyzing entanglement in experiments [35]. Usual entanglement witness does have its limit in dealing with the type of k -partite separable state. We now give a universal entanglement witness which can detect the type. We take tripartite states to illustrate the idea. Suppose $\rho \in \mathbb{H}_A \otimes \mathbb{H}_B \otimes \mathbb{H}_C$ and $\dim \mathbb{H}_A \neq \dim \mathbb{H}_B \neq \dim \mathbb{H}_C$. By choosing a suitable measurement $\{X, E_\alpha^X\}$, the bounds $\omega_{AB,C}$, $\omega_{AC,B}$ and $\omega_{BC,A}$ are all different. Write $\omega_{AB,C}$ as $(\Omega_1, \Omega_2 - \Omega_1, \dots, \Omega_m - \Omega_{m-1})$, $m = \dim \mathbb{H}_A \cdot \dim \mathbb{H}_B \cdot \dim \mathbb{H}_C$.

A universal witness operator can be written in the canonical form

$$\mathcal{W}_k = \Omega_k I - \sum_{i=1}^k E_{\alpha_i}^X, k = 1, 2, \dots, m, \quad (47)$$

where I is the identity operator and $E_{\alpha_i}^X$ are from the measurement $\{X, E_\alpha^X\}$. Then for any biseparable state $\rho = \sum_i p_i \rho_{AB}^i \otimes \rho_C^i$ in type AB-C, we have

$$\text{Tr}(\mathcal{W}_k \rho) \geq 0, k = 1, 2, \dots, m. \quad (48)$$

If there is

$$\text{Tr}(\mathcal{W}_k \rho) < 0, \quad (49)$$

for some k , then ρ can not be of type AB-C. We remark that the universal entanglement witness can be used to analyze the complement of the compact convex subset inside another one.

V. CONCLUSIONS AND DISCUSSIONS

In this paper, we have developed entanglement criteria based on both majorization and universal uncertainty relations. These criteria have advantage over the scalar detecting algorithms as they are often stronger and tighter due to specific formulas for the bound ω of the uncertainty relation. They are tight enough to detect k -separable n -partite states and their types by choosing a suitable measurement $\{X, E_\alpha^X\}$ when the underlying particles have different dimensions.

We have also presented the matrix form of majorization. One feature in this approach is by choosing a suitable monotonic function one can define a distance in the set of entangled states, where k -separable states form concentric circles. In this regard, viable functions are non-negative Schur-concave functions such as Shannon, Rényi and Tsallis entropies [39, 40]. We have also generalized the entanglement witness to detect the type of a k -separable state, and indicate how they can be implemented in experiments.

Acknowledgments The work is supported in part by National Natural Science Foundation of China (grant Nrs. 11271138, 11531004), China Scholarship Council and Simons Foundation grant No. 198129.

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