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by

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Abstract. The Euclidean mixed isoperimetric-isodiametric inequality states that the round ball maximizes the volume under constraint on the product between boundary area and radius. The goal of the paper is to investigate such mixed isoperimetric-isodiametric inequalities in Riemannian manifolds. We first prove that the same inequality, with the sharp Euclidean constants, holds on Cartan-Hadamard spaces as well as on minimal submanifolds of $\mathbb{R}^n$. The equality cases are also studied and completely characterized; in particular, the latter gives a new link with free boundary minimal submanifolds in a Euclidean ball. We also consider the case of manifolds with non-negative Ricci curvature and prove a new comparison result stating that metric balls in the manifold have product of boundary area and radius bounded by the Euclidean counterpart and equality holds if and only if the ball is actually Euclidean.

We then pass to consider the problem of the existence of optimal shapes (i.e. regions minimizing the product of boundary area and radius under the constraint of having fixed enclosed volume), called here isoperimetric-isodiametric regions. While it is not difficult to show existence if the ambient manifold is compact, the situation changes dramatically if the manifold is not compact: indeed we give examples of spaces where there exists no isoperimetric-isodiametric region (e.g. minimal surfaces with planar ends and more generally C^0-locally-asymptotic Euclidean Cartan-Hadamard manifolds), and we prove that on the other hand on C^0-locally-asymptotic Euclidean manifolds with non-negative Ricci curvature there exists an isoperimetric-isodiametric region for every positive volume (this class of spaces includes a large family of metrics playing a key role in general relativity and Ricci flow: the so called Hawking gravitational instantons and the Bryant-type Ricci solitons).

Finally we pass to prove the optimal regularity of the boundary of isoperimetric-isodiametric regions: in the part which does not touch a minimal enclosing ball the boundary is a smooth hypersurface outside of a closed subset of Hausdorff co-dimension 8, and in a neighborhood of the contact region the boundary is a $C^{1,1}$-hypersurface with explicit estimates on the L^∞-norm of the mean curvature.

Contents

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1. Introduction

One of the oldest questions of mathematics is the *isoperimetric* problem: *What is the largest amount of volume that can be enclosed by a given amount of area?* A related classical question is the *isodiametric* problem: *What is the largest amount of volume that can be enclosed by a domain having a fixed diameter?*

In this paper we address a mix of the previous two questions, namely we investigate the following mixed isoperimetric-isodiametric problem: *What is the largest amount of volume that can be enclosed by a domain having a fixed product of diameter and boundary area?*

Of course, if we ask the three above questions in the Euclidean space, the answer is given by the round balls of the suitable radius; but, of course, the situation in non-flat geometries is much more subtle. We start by recalling classical material on the isoperimetric problem which motivated our investigation on the mixed isoperimetric-isodiametric one.

The solution of the isoperimetric problem in the Euclidean space $\mathbb{R}^n$ can be summarized by the classical isoperimetric inequality

$$n\,\omega_n^{\frac{1}{n}}\,\mathrm{Vol}(\Omega)^{\frac{n-1}{n}} \leq \mathcal{A}(\partial\Omega)\,, \quad \text{for every } \Omega \subset \mathbb{R}^n \text{ open subset with smooth boundary,} \tag{1.1}$$

where $\mathrm{Vol}(\Omega)$ is the n-dimensional Hausdorff measure of Ω (i.e. the "volume" of Ω), $\mathcal{A}(\partial\Omega)$ is the $(n-1)$-dimensional Hausdorff measure of $\partial\Omega$ (i.e. the "area" of $\partial\Omega$), and $\omega_n := \mathrm{Vol}(B^n)$ is the volume of the unit ball in $\mathbb{R}^n$. As it is well known, the regularity assumption on Ω can be relaxed a lot (for instance (1.1) holds for every set Ω of finite perimeter) but let us not enter in technicalities here since we are just motivating our problem.

As anticipated above, in the present paper we will not deal with the isoperimetric problem itself but we will focus on a mixed isoperimetric-isodiametric problem. Let us start by stating the Euclidean mixed isoperimetric-isodiametric inequality which will act as model for this paper. Given a bounded open subset $\Omega \subset \mathbb{R}^n$ with smooth boundary, by the divergence theorem in $\mathbb{R}^n$ (see Section 2 for the easy proof), we have

$$n\,\mathrm{Vol}(\Omega) \leq \mathrm{rad}(\Omega)\,\mathcal{A}(\partial\Omega), \tag{1.2}$$

where $\mathrm{rad}(\Omega)$ is the radius of the smallest ball of $\mathbb{R}^n$ containing Ω (see (2.1) for the precise definition). As observed in Remark 2.1, inequality (1.2) is *sharp* and *rigid*; indeed, equality occurs if and only if Ω is a round ball in $\mathbb{R}^n$.

In sharp contrast with the classical isoperimetric problem, where both problems are still open in the general case, it is not difficult to show that the inequality (1.2) holds in Cartan-Hadamard spaces (i.e. simply connected Riemannian manifolds with non-positive sectional curvature) and on minimal submanifolds of $\mathbb{R}^n$, see Proposition 3.1, Proposition 3.3 and Proposition 3.7. Even if the validity of inequality (1.2) in such spaces is probably known to experts, we included it here in order to motivate the reader and also because the equality case for minimal submanifolds present an interesting link with free-boundary minimal surfaces: equality is attained in (1.2) if and only if the minimal submanifold is a free boundary minimal surface in a Euclidean ball (see Proposition 3.3 for the precise statement and Remarks 3.5-3.6 for more information about free boundary minimal surfaces).

If on one hand the negative curvature gives a stronger isoperimetric-isodiametric inequality, on the the other hand we show that non-negative Ricci curvature forces metric balls to satisfy a weaker isoperimetric-isodiametric inequality. The precise statement is the following.

Theorem 1.1 (Theorem 4.1)**.** *Let (M^n, g) be a complete (possibly non compact) Riemannian n-manifold. Let $B_r \subset M$ be a metric ball of volume $V = \mathrm{Vol}_g(B_r)$, and denote with $B^{\mathbb{R}^n}(V)$ the round ball in $\mathbb{R}^n$ having volume V. Then*

$$\mathrm{rad}(B_r)\ \mathcal{A}(\partial B_r) = r\ \mathcal{A}(\partial B_r) \leq n\,\mathrm{Vol}_g(B_r) = \mathrm{rad}_{\mathbb{R}^n}(B^{\mathbb{R}^n}(V))\ \mathcal{A}_{\mathbb{R}^n}(\partial B^{\mathbb{R}^n}(V)). \tag{1.3}$$

Moreover equality holds if and only if B_r is isometric *to a round ball in the Euclidean space $\mathbb{R}^n$. In particular, for every $V \in (0, \mathrm{Vol}_g(M))$ it holds*

$$\inf\{\mathrm{rad}(\Omega)\mathcal{P}(\Omega)\,:\,\Omega \subset M,\,\mathrm{Vol}_g(\Omega) = V\} \leq nV = \inf\{\mathrm{rad}(\Omega)\mathcal{P}(\Omega)\,:\,\Omega \subset \mathbb{R}^n,\,\mathrm{Vol}_{\mathbb{R}^n}(\Omega) = V\}, \tag{1.4}$$

with equality for some $V \in (0, \mathrm{Vol}_g(M))$ if and only if every metric ball in M of volume V is isometric to a round ball in $\mathbb{R}^n$. In particular if equality occurs for some $V \in (0, \mathrm{Vol}_g(M))$ then (M, g) is flat, i.e. it has identically zero sectional curvature.

Remark 1.2. Since by Bishop-Gromov volume comparison we know that if $Ric_g \geq 0$ then for every metric ball $B_r(x_0) \subset M$ it holds $\mathrm{Vol}_g(B_r(x_0)) \leq \omega_n r^n = \mathrm{Vol}_{\mathbb{R}^n}(B_r^{\mathbb{R}^n})$, it follows that

$$\mathrm{rad}(B_r(x_0)) \geq \mathrm{rad}_{\mathbb{R}^n}(B^{\mathbb{R}^n}(V)),$$

where $B^{\mathbb{R}^n}(V)$ is a Euclidean ball of volume $V = \mathrm{Vol}_g(B_r(x_0))$. Therefore Theorem 1.1 in particular implies that $\mathcal{P}(B_r(x_0)) \leq \mathcal{P}_{\mathbb{R}^n}(B^{\mathbb{R}^n}(V))$, but is a strictly stronger statement which at best of our knowledge is original. The aforementioned counterpart of Theorem 1.1 for the isoperimetric problem was proved instead by Morgan-Johnson [42, Theorem 3.5] for compact manifolds and extended to non-compact manifolds in [41, Proposition 3.2]. $\square$

In Section 5 we investigate the existence of optimal shapes in a general Riemannian manifold (M, g). More precisely, given a measurable subset $E \subset M$ we denote with $\mathcal{P}(E)$ its perimeter and define its extrinsic radius as

$$\mathrm{rad}(E) := \inf \{r > 0 \ : \ \mathrm{Vol}_g(E \setminus B_r(z_0)) = 0 \text{ for some } z_0 \in M\},$$

where $B_r(z_0)$ denotes the open metric ball with center z_0 and radius $r > 0$. We consider the following minimization problem: for every fixed $V \in (0, \mathrm{Vol}_g(M))$,

$$\min \Big\{ \mathrm{rad}(E)\, \mathcal{P}(E) \ : \ E \subset M,\ \mathrm{Vol}_g(E) = V \Big\}, \tag{1.5}$$

and call the minimizers of (1.5) isoperimetric-isodiametric sets (or regions). To best of our knowledge this is first time such a problem is considered in literature.
As it happens also for the isoperimetric problem, we will find that if the ambient manifold is compact then for every volume there exists an isoperimetric-isodiametric region (see Theorem 5.2 and Corollary 5.3) but if the ambient space is non-compact the situation changes dramatically. Indeed in Examples 5.6-5.7 we show that in complete minimal submanifolds with planar ends (like the helicoid) and in asymptotically locally Euclidean Cartan-Hadamard manifolds there exists no isoperimetric-isodiametric region of positive volume. On the other hand, we show that in C^0-locally asymptotically Euclidean manifolds (see Definition 5.4 for the precise notion) with non negative Ricci curvature for every volume there exists an isoperimetric-isodiametric region:

Theorem 1.3 (Theorem 5.5)**.** *Let (M, g) be a complete Riemannian n-manifold with non-negative Ricci curvature and fix any reference point $\bar{x} \in M$. Assume that for any diverging sequence of points $(x_k)_{k\in N} \subset M$, i.e. $\mathsf{d}(x_k, \bar{x}) \to \infty$, the sequence of pointed manifolds (M, g, x_k) converges in the pointed C^0-topology to the Euclidean space $(\mathbb{R}^n, g_{\mathbb{R}^n}, 0)$.*
Then for every $V \in (0, \mathrm{Vol}_g(M))$ there exists a minimizer of the problem (1.5)*, in other words there exists an isoperimetric-isodiametric region of volume V.*

Let us mention that the counterpart of Theorem 1.3 for the isoperimetric problem was proved in [41] capitalizing on the work by Nardulli [43].

Remark 1.4. It is well known that the only manifold with non-negative Ricci curvature and C^0-*globally* asymptotic to $\mathbb{R}^n$ is $\mathbb{R}^n$ itself. Indeed if M is C^0-globally asymptotic to $\mathbb{R}^n$ then

$$\lim_{R\to\infty} \frac{\mathrm{Vol}_g(B_R(\bar{x}))}{\omega_n R^n} = 1,$$

which by the rigidity statement associated to the Bishop-Gromov inequality implies that (M, g) is globally isometric to $\mathbb{R}^n$. On the other hand, the assumption of Theorem 1.3 is much weaker as it ask (M, g) to be just *locally* asymptotic to $\mathbb{R}^n$ in C^0 topology and many important examples enter in this framework as explained in next Example 1.5. $\square$

Example 1.5. The class of manifolds satisfying the assumptions of Theorem 1.3 contains many geometrically and physically relevant examples.

- *Eguchi-Hanson and more generally ALE gravitational instantons.* These are 4-manifolds, solutions of the Einstein vacuum equations with null cosmological constant (i.e. they are Ricci flat, $Ric_g \equiv 0$), they are non-compact with just one end which is topologically a quotient of $\mathbb{R}^4$ by a finite subgroup of $O(4)$, and the Riemannian metric g on this end is asymptotic to the Euclidean metric up to terms of order $O(r^{-4})$,

$$g_{ij} = \delta_{ij} + O(r^{-4}),$$

  with appropriate decay in the derivatives of g_{ij} (in particular, such metrics are C^0-locally asymptotic, in the sense of Definition 5.4, to the Euclidean 4-dimensional space). The first example of such manifolds was discovered by Eguchi and Hanson in [19]; the authors, inspired by the discovery of self-dual instantons in Yang-Mills Theory, found a self-dual ALE instanton metric. The Eguchi-Hanson example was then generalized by Gibbons and Hawking [23], see also the work by Hitchin [29]. These metrics constitute the building blocks of the Euclidean quantum gravity theory of Hawking (see [27, 28]). The ALE Gravitational Instantons were classified in 1989 by Kronheimer (see [34, 35]).
- *Bryant-type solitons.* The Bryant solitons, discovered by R. Bryant [10], are special but fundamental solutions to the Ricci flow (see for instance the work of Brendle [8, 9] for higher dimension). Such metrics are complete, have non-negative Ricci curvature (they actually satisfy the stronger condition of having nonnegative curvature operator) and are locally C^0-asymptotically Euclidean. Other soliton examples fitting our assumptions are given by Catino-Mazzieri in [15]. □

The last Section 6 is then devoted to establish the optimal regularity for isoperimetric-isodiametric regions under suitable assumptions on regularity of the enclosing ball. We first observe that outside of the contact region with the minimal enclosing ball B, such sets are locally minimizers of the perimeter under volume constraint. Therefore by classical results (see, for example, [40, Corollary 3.8]) in the interior of B the boundary of the region is a smooth hypersurface outside a singular set of Hausdorff co-dimension at least 8.

The rest of the paper is devoted to prove the optimal regularity at the contact region. We first show in Section 6.1 that isoperimetric-isodiametric regions are almost-minimizers for the perimeter (see Lemma 6.3) and therefore, by a result of Tamanini [47] their boundaries are $C^{1,1/2}$ regular (see Proposition 6.1). In Section 6.2, by means of geometric comparisons and sharp first variation arguments, we show that the mean curvature of the boundary of an isoperimetric-isodiametric region is in L^∞ with explicit estimates. Finally in Section 6.3 we establish the optimal $C^{1,1}$ regularity. We mention that, strictly speaking, Section 6.2 is not needed to prove the optimal regularity; in any case we included such section since provides an explicit sharp L^∞-estimate on the mean curvature and is of independent interest. Now the let us state the main regularity result.

Theorem 1.6 (Theorem 6.11)**.** *Let $E \subset M$ be an isoperimetric-isodiametric set and $x_0 \in M$ be such that $\mathrm{Vol}_g(E \setminus B_{\mathrm{rad}(E)}(x_0)) = 0$. Assume that $B := B_{\mathrm{rad}(E)}(x_0))$ has smooth boundary. Then, there exists $\delta > 0$ such that $\partial E \setminus B_{\mathrm{rad}(E)-\delta}(x_0)$ is $C^{1,1}$ regular.*

An essential ingredient in the proof of Theorem 1.6 is Proposition 6.12, which roughly tells that the boundary of E leaves the obstacle at most quadratically. Then the conclusion will follow by combining Schauder estimates outside of the contact region (see Lemma 6.13) with the general fact that functions which leave the first order approximation quadratically are $C^{1,1}$ – see Lemma 6.14. Although the techniques exploited for this part of the paper are inspired by the ones introduced in the study of the classical obstacle problem (cf., for example, [12]), here we treat the geometric case of the area functional in a Riemannian manifold with volume constraints and we take several short-cuts by thanks to some specifically geometric arguments, such as the theory of almost minimizers.

Remark 1.7. Note that the $C^{1,1}$ regularity is optimal, because in general one cannot expect to have continuity of the second fundamental form of ∂E across the free boundary of ∂E, i.e. the points on the relative (with respect to ∂B) boundary of $\partial E \cap \partial B$. The same is indeed true for the simplest case of the classical obstacle problem.

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2. Notation, Preliminaries and the Euclidean case

Let (Z, d) be a metric space. Given an open subset $\Omega \subset Z$, we define its *extrinsic radius* as

$$\mathrm{rad}(\Omega) := \inf\{r > 0 : \ \Omega \subset B_r(z_0) \text{ for some } z_0 \in Z\} \quad , \tag{2.1}$$

where $B_r(z_0)$ denotes the open metric ball of center z_0 and radius $r > 0$.

The model inequality for the first part of the paper is the Euclidean mixed isoperimetric-isodiametric inequality obtained by the following integration by parts. Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset with smooth boundary and let $x_0 \in \mathbb{R}^n$ be a point such that

$$\max_{x \in \bar{\Omega}} |x - x_0| = \mathrm{rad}(\Omega). \tag{2.2}$$

Denoted with X the vector field $X(x) := x - x_0$, by the divergence theorem in $\mathbb{R}^n$ we then get

$$n \,\mathrm{Vol}(\Omega) = \int_\Omega \mathrm{div}\, X \, d\,\mathcal{H}^n = -\int_{\partial\Omega} X \cdot \nu \, d\mathcal{H}^{n-1} \leq \mathrm{rad}(\Omega)\, \mathcal{A}(\partial\Omega), \tag{2.3}$$

where $\mathrm{Vol}(\Omega)$ denotes the Euclidean n-dimensional volume of Ω, ν is the inward pointing unit normal vector and $\mathcal{A}(\partial\Omega)$ is the Euclidean $(n-1)$-dimensional area of $\partial\Omega$, which here is assumed to be smooth. Notice that, analogously, if $\Omega \subset \mathbb{R}^n$ is a finite perimeter set one gets the inequality

$$\mathrm{Vol}(\Omega) \leq \frac{\mathrm{rad}(\Omega)}{n}\, \mathcal{P}(\Omega), \tag{2.4}$$

where, of course, $\mathcal{P}(\Omega)$ denotes the perimeter of Ω (see § 5.1 for the definitions of $\mathcal{P}(\Omega)$ and $\mathrm{rad}(\Omega)$ for finite perimeter sets).

Remark 2.1. The inequalities (2.3) and (2.4) are *sharp* and *rigid*: indeed equality occurs if and only if Ω is a round ball. □

3. Euclidean isoperimetric-isodiametric inequality in Cartan-Hadamard manifolds and minimal submanifolds

In order to motivate and gently introduce the reader to the topic, in this section we will prove that the Euclidean isoperimetric-isodiametric inequality holds with the same constant in Cartan-Hadamard spaces and in minimal submanifolds. Possibly apart from the rigidity statements, here we do not claim originality since such inequalities are probably well known to experts (cf. [11], [30], [39]). However we included this section for the following reasons:

- While for the isoperimetric-isodiametric inequality the proofs are a consequence of a non difficult integration by parts argument, the corresponding statements for the classical isoperimetric inequality are still open problems (see Remark 3.2 and Remark 3.4). This suggest that possibly also in other situations isoperimetric-isodiametric inequalities may behave better than the classical isoperimetric ones.
- The rigidity statements, in case of minimal submanifolds, show interesting connections between the isoperimetric-isodiametric inequality and free boundary minimal surfaces, a topic which recently has received a lot of attention (for more details see Remark 3.5 and Remark 3.6).

3.1. **The case of Cartan-Hadamard manifolds.** Recall that a Cartan-Hadamard n-manifold is a complete simply connected Riemannian n-dimensional manifold with non-positive sectional curvature. By a classical theorem of Cartan and Hadamard (see for instance [18]) such manifolds are diffeomorphic to $\mathbb{R}^n$ via the exponential map. The next result is a sharp and rigid mixed isoperimetric-isodiametric inequality in such spaces. For this section, without loosing much, the non-expert reader may assume the region $\Omega \subset M$ to have smooth boundary, in this case the perimeter is just the standard $(n-1)$-volume of the boundary (the perimeter will instead play a role in the next sections about existence and regularity of optimal sets).

Proposition 3.1. *Let (M^n, g) be a Cartan-Hadamard manifold. Then for every smooth open subset (or more generally for every finite perimeter set) $\Omega \subset M^n$ it holds*

$$n\,\mathrm{Vol}(\Omega) \leq \mathrm{rad}(\Omega)\ \mathcal{A}(\partial\Omega) \tag{3.1}$$

where $\mathrm{Vol}(\Omega)$ *denotes the n-dimensional Riemannian volume of Ω and $\mathcal{A}(\partial\Omega)$ the $(n-1)$-dimensional area of the smooth boundary $\partial\Omega$ (in case Ω is a finite perimeter set, just replace $\mathcal{A}(\partial\Omega)$ with $\mathcal{P}(\Omega)$, the perimeter of Ω in the right hand side, and* $\mathrm{rad}(\Omega)$ *is as in § 5.1*[1]*). Moreover if for some Ω the equality is achieved, then Ω is isometric to* an Euclidean ball.

Proof. Let $\Omega \subset M^n$ be a subset with finite perimeter; without loss of generality we can assume that Ω is bounded (otherwise $\mathrm{rad}(\Omega) = +\infty$ and the inequality is trivial). Let $x_0 \in M^n$ be such

[1]For readers' convenience we recall here the definition of $\mathrm{rad}(\Omega)$ for a finite perimeter set $\Omega \subset M$: $\mathrm{rad}(\Omega) := \inf\{r > 0 \,:\, \mathrm{Vol}(\Omega \setminus B_r) = 0,\ B_r \subset M \text{ metric ball}\}$.

that

$$\max_{x\in\bar\Omega} \mathsf{d}(x,x_0) = \mathrm{rad}(\Omega),$$

where d is the Riemannian distance on (M^n, g), for convenience we will also denote $\mathsf{d}_{x_0}(\cdot) := \mathsf{d}(x_0, \cdot)$. Let $u := \frac{1}{2}\mathsf{d}^2_{x_0}$; by the aforementioned Cartan-Hadamard Theorem (see for instance [18]) we know that $u : M^n \to \mathbb{R}^+$ is smooth and by the Hessian comparison Theorem one has $(D^2u)_{ij} \geq g_{ij}$; in particular, by tracing, we get $\Delta u \geq n$. Therefore, by the divergence theorem, we infer

$$\begin{aligned} n\,\mathrm{Vol}(\Omega) &\leq \int_\Omega \Delta u\, d\mu_g = -\int_{\partial^*\Omega} g(\nabla u, \nu)\, d\,\mathcal{H}^{n-1} = -\int_{\partial^*\Omega} \mathsf{d}(x,x_0)\; g(\nabla \mathsf{d}_{x_0}, \nu)\, d\,\mathcal{H}^{n-1} \\ &\leq \mathrm{rad}(\Omega)\, \mathcal{H}^{n-1}(\partial^*\Omega) = \mathrm{rad}(\Omega)\, \mathcal{P}(\Omega), \end{aligned} \tag{3.2}$$

where μ_g is the measure associated to the Riemannian volume form, $\partial^*\Omega$ is the reduced boundary of Ω (of course, in case Ω is a smooth open subset one has $\partial^*\Omega = \partial\Omega$), ν is the inward pointing unit normal vector (recall that it is $\mathcal{H}^{n-1}$-a.e. well defined on $\partial^*\Omega$), and we used that d_{x_0} is 1-Lipschitz. Of course (3.2) implies (3.1). Notice that if equality holds in the second line, then Ω is a metric ball of center x_0 and radius $\mathrm{rad}(\Omega)$. Moreover if equality occurs in the first inequality of the first line then we must have $(D^2\mathsf{d}^2_{x_0})_{ij} \equiv 2g_{ij}$ on Ω, and by standard comparison (see for instance [45, Section 4.1]) it follows that Ω is flat. But since the exponential map in M is a global diffeomorphism it follows that Ω is isometric to an Euclidean ball. □

Remark 3.2 (Euclidean isoperimetric inequality on Cartan-Hadamard spaces)**.** The statement corresponding to Proposition 3.1 for the isoperimetric problem is the following celebrated conjecture: *Let (M^n, g) be a Cartan-Hadamard space, i.e. a complete simply connected Riemannian n-manifold with non-positive sectional curvature. Then every smooth open subset $\Omega \subset M^n$ satisfies the Euclidean isoperimetric inequality.*

This conjecture is generally attributed to Aubin [4, Conj. 1] but has its roots in earlier work by Weil [48], as we are going to explain. The problem has been solved affirmatively in the following cases: in dimension 2 by Weil [48] in 1926 (Beckenbach and Radó [6] gave an independent proof in 1933, capitalizing on a result of Carleman [14] for minimal surfaces), in dimension 3 by Kleiner [33] in 1992 (see also the survey paper by Ritoré [45] for a variant of Kleiner's arguments), and in dimension 4 by Croke [17] in 1984. An interesting feature of this problem is that the above proofs have nothing to do one with the other and that they work only for one specific dimension; probably also for this reason such a problem is still open in the general case. □

3.2. **The case of minimal submanifolds.** Given a smoothly immersed submanifold $M^n \hookrightarrow \mathbb{R}^{n+k}$, by the first variation formula for the area functional we know that for every $\Omega \subset M^n$ open bounded subset with smooth boundary and every smooth vector field X along Ω it holds

$$\int_\Omega \mathrm{div}_M X\, d\,\mathcal{H}^n = -\int_\Omega H\cdot X\, d\,\mathcal{H}^n - \int_{\partial\Omega} X\cdot\nu\, d\,\mathcal{H}^{n-1}, \tag{3.3}$$

where H is the mean curvature vector of M and ν is the inward pointing conormal to Ω (i.e. ν is the unit vector tangent to M, normal to $\partial\Omega$ and pointing inside Ω).

We are interested in the case $M^n \hookrightarrow \mathbb{R}^{n+k}$ is a minimal submanifold, i.e. $H \equiv 0$, and $\Omega \subset M^n$ is a bounded open subset with smooth boundary $\partial\Omega$. Let $x_0 \in \mathbb{R}^{n+k}$ be such that

$$\max_{x \in \bar{\Omega}} |x - x_0|_{\mathbb{R}^{n+k}} = \mathrm{rad}_{\mathbb{R}^{n+k}}(\Omega),$$

and observe that, called $X(x) := x - x_0$, one has $\mathrm{div}_M X \equiv n$. By applying (3.3), we then infer

$$n\,\mathcal{H}^n(\Omega) = \int_\Omega \mathrm{div}_M X \, d\,\mathcal{H}^n = -\int_{\partial\Omega} X \cdot \nu \, d\,\mathcal{H}^{n-1} \le \mathrm{rad}_{\mathbb{R}^{n+k}}(\Omega)\,\mathcal{H}^{n-1}(\partial\Omega). \tag{3.4}$$

Notice that equality is achieved if and only if Ω is the intersection of M with a round ball in $\mathbb{R}^{n+k}$ centered at x_0 and $\nu(x)$ is parallel to $x - x_0$, or in other words if and only if Ω is a free boundary minimal n-submanifold in a ball of $\mathbb{R}^{n+k}$. So we have just proved the following result.

Proposition 3.3. *Let $M^n \hookrightarrow \mathbb{R}^{n+k}$ be a minimal submanifold and $\Omega \subset M^n$ a bounded open subset with smooth boundary $\partial\Omega$. Then*

$$n\,\mathcal{H}^n(\Omega) \le \mathrm{rad}_{\mathbb{R}^{n+k}}(\Omega)\ \mathcal{H}^{n-1}(\partial\Omega)$$

with equality if and only if Ω is a free boundary minimal n-submanifold in a ball of $\mathbb{R}^{n+k}$.

Remark 3.4 (Euclidean isoperimetric inequality on minimal submanifolds)**.** The statement corresponding to Proposition 3.3 for the isoperimetric problem is the following celebrated conjecture: *Let $M^n \subset \mathbb{R}^m$ be a minimal n-dimensional submanifold and let $\Omega \subset M^n$ be a smooth open subset. Then Ω satisfies the Euclidean isoperimetric inequality* (1.1)*, and equality holds if and only if Ω is a ball in an affine n-plane of $\mathbb{R}^m$.*

To our knowledge the only two solved cases are $i)$ when $\partial\Omega$ lies on an $(m-1)$-dimensional Euclidean sphere centered at a point of Ω (the argument is by monotonicity, see for instance [16, Section 8.1]) and $ii)$ when Ω is area minimizing with respect to its boundary $\partial\Omega$ by Almgren [2]. Let us mention that a complete solution of the above conjecture is still not available even for minimal surfaces in $\mathbb{R}^m$, i.e. for $n = 2$; however, in the latter situation, the statement is known to be true in many cases (let us just mention that in case Ω is a topological disk the problem was solved by Carleman [14] in 1921, and the case $m = 3$ and $\partial\Omega$ has two connected components was settled much later by Li-Schoen-Yau [36]; for more results in this direction and for a comprehensive overview see the beautiful survey paper [16] by Choe). Let us finally observe that, when $n = 2$ and $m = 3$, the above c onjecture is a special case of the Aubin Conjecture recalled in Remark 3.2, since of course the induced metric on a immersed minimal surface in $\mathbb{R}^3$ has non-positive Gauss curvature; this case was settled in the pioneering work by Weil [48]. □

Remark 3.5 (Free boundary minimal submanifolds and critical metrics)**.** After a classical work of Nitsche [44] in the 80'ies, the last years have seen an increasing interest on free boundary submanifolds also thanks to recent works of Fraser and Schoen [21, 22] on the topic. By definition, a *free boundary submanifold* M^n of the unit ball B^{n+k}, is a proper submanifold which is critical for the area functional with respect to variations of M^n that are allowed to move also the boundary ∂M^n,but under the constraint $\partial M^n \subset \partial B^{n+k}$. As a consequence of the 1^{st} variational formula, such definition forces on one hand the mean curvature to vanish on $M^n \cap B^{n+k}$ and on the other hand the submanifold to the meet the ambient boundary ∂B^{n+k} orthogonally. These are characterized by the condition that the coordinate functions are Steklov eigenfunctions with eigenvalue 1 [21, Lemma 2.2]; that is,

$$\Delta x_i = 0 \text{ on } M \text{ and } \nabla_\nu x_i = -x_i \text{ on } \partial M.$$

It turns out that surfaces of this type arise naturally as extremal metrics for the Steklov eigenvalues (see [22] for more details); Steklov eigenvalues are eigenvalues of the Dirichlet-to-Neumann map, which sends a given smooth function on the boundary to the normal derivative of its harmonic extension to the interior. □

Remark 3.6 (Examples of free boundary minimal submanifolds)**.** Let us recall here some well known examples of free boundary minimal submanifolds in the unit ball $B^{n+k} \subset \mathbb{R}^{n+k}$, for a deeper discussion on the examples below see [22].

- *Equatorial Disk.* Equatorial n-disks $D^n \subset B^{n+k}$ are the simplest examples of free boundary minimal submanifolds. By a result of Nitsche [44] any simply connected free boundary minimal surface in B^3 must be a flat equatorial disk. However, if we admit minimal surfaces of a different topological type, there are other examples, as the critical catenoid described below.
- *Critical Catenoid.* Consider the catenoid parametrized on $\mathbb{R} \times S^1$ by the function
$$\varphi(t,\theta) = (\cosh t \cos\theta, \cosh t \sin\theta, t)\,.$$
For a unique choice of $T_0 > 0$, the restriction of φ to $[-T_0, T_0] \times S^1$ defines a minimal embedding into a ball meeting the boundary of the ball orthogonally. By rescaling the radius of the ball to 1 we get the critical catenoid in B^3. Explicitly, T_0 is the unique positive solution of $t = \coth t$.
- *Critical Möbius band.* We think of the Möbius band M^2 as $\mathbb{R} \times S^1$ with the identification $(t,\theta) \sim (-t, \theta + \pi)$. There is a minimal embedding of M^2 into $\mathbb{R}^4$ given by
$$\varphi(t,\theta) = (2\sinh t \cos\theta, 2\sinh t \sin\theta, \cosh 2t \cos 2\theta, \cosh 2t \sin 2\theta)\,.$$
For a unique choice of $T_0 > 0$, the restriction of φ to $[-T_0, T_0] \times S^1$ defines a minimal embedding into a ball meeting the boundary of the ball orthogonally. By rescaling the radius of the ball to 1 we get the critical Möbis band in B^4. Explicitly T_0 is the unique positive solution of $\coth t = 2\tanh 2t$.
- A consequence of the results of [22] is that for every $k \geq 1$ there exists an embedded free boundary minimal surface in B^3 of genus 0 with k boundary components.

□

Since of course $\mathrm{rad}_{\mathbb{R}^{n+k}}(\Omega) \leq \mathrm{rad}_M(\Omega)$, where $\mathrm{rad}_M(\cdot)$ is the extrinsic radius in the metric space (M, d_g), we have a fortiori that

$$n\,\mathcal{H}^n(\Omega) \leq \mathrm{rad}_M(\Omega)\;\mathcal{H}^{n-1}(\partial\Omega). \tag{3.5}$$

But in this case the rigidity statement is much stronger, indeed in case of equality the center of the ball x_0 must be a point of M, moreover for every $x \in \partial\Omega$ the segment $\overline{x, x_0}$ must be contained in

M, therefore M contains a portion of a minimal cone $\mathcal{C}$ centered at x_0. But since by assumption M is a smooth submanifold and since the only cone smooth at its origin is an affine subspace, it must be that M contains a portion of an affine subspace. By the classical weak unique continuation property for solutions to the minimal submanifold system, we conclude that M is an affine subspace of $\mathbb{R}^{n+k}$. Therefore we have just proven the next result.

Proposition 3.7. *Let $M^n \hookrightarrow \mathbb{R}^{n+k}$ be a connected smooth minimal submanifold and $\Omega \subset M^n$ a bounded open subset with smooth boundary $\partial\Omega$. Then*

$$n\ \mathcal{H}^n(\Omega) \le \mathrm{rad}_M(\Omega)\ \mathcal{H}^{n-1}(\partial\Omega) \tag{3.6}$$

with equality if and only if M is an affine subspace and Ω is the intersection of M with a round ball in $\mathbb{R}^{n+k}$ centered at a point of M.

Remark 3.8. If we allow M to have conical singularities, then (3.6) still holds with equality if and only if M is a minimal cone and Ω is the intersection of M with a round ball in $\mathbb{R}^{n+k}$ centered at a point of M.

Concerning this, recall that in case $n = 2$ and $k = 1$ every minimal cone smooth away from the vertex is totally geodesic, indeed one of the principal curvatures is always null for cones and so the mean curvature vanishes if and only if all the second fundamental form is null. Therefore equality in (3.6) is attained if and only if M^2 is an affine plane and Ω is a flat 2-disk. The analogous result for $n = 3$ and $k = 1$ is due to Almgren [1] (see also the work of Calabi [13]).

For the general case of higher dimensions and co-dimensions note that a minimal submanifold Σ^k in S^n is naturally the boundary of a minimal submanifold of the ball, the cone $C(\Sigma)$ over Σ. Using this correspondence it is possible to construct many non trivial minimal cones: Hsiang [31]-[32] gave infinitely many co-dimension 1 examples for $n \ge 4$, the higher co-dimensional problem was investigated in the celebrated paper of Simons [46] and the related work of Bombieri-De Giorgi-Giusti [5]. $\square$

posse concludere). If we assume V to be an integer rectifiable varifold after finitely many steps we have exhausted the varifold.

4. The isoperimetric-isodiametric inequality in manifolds with non-negative Ricci curvature

In this section we show a comparison result for manifolds with non-negative Ricci curvature which will be used in Section 5 to get existence of isoperimetric-isodiametric regions in manifolds which are asymptotically locally Euclidean and have non-negative Ricci (the so called ALE spaces).

Theorem 4.1. *Let (M^n, g) be a complete (possibly non compact) Riemannian n-manifold. Let $B_r \subset M$ be a metric ball of volume $V = \mathrm{Vol}(B_r)$, and denote with $B^{\mathbb{R}^n}(V)$ the round ball in $\mathbb{R}^n$ having volume V. Then*

$$\mathrm{rad}(B_r)\,\mathcal{P}(B_r) = r\,\mathcal{P}(B_r) \le nV = \mathrm{rad}_{\mathbb{R}^n}(B^{\mathbb{R}^n}(V))\,\mathcal{P}_{\mathbb{R}^n}(B^{\mathbb{R}^n}(V)). \tag{4.1}$$

Moreover equality holds if and only if B_r is isometric *to a round ball in the Euclidean space $\mathbb{R}^n$. In particular, for every $V \in (0, \mathrm{Vol}(M))$ it holds*

$$\inf\{\mathrm{rad}(\Omega)\mathcal{P}(\Omega) \,:\, \Omega \subset M,\, \mathrm{Vol}(\Omega) = V\} \leq nV = \inf\{\mathrm{rad}(\Omega)\mathcal{P}(\Omega) \,:\, \Omega \subset \mathbb{R}^n,\, \mathrm{Vol}_{\mathbb{R}^n}(\Omega) = V\}, \tag{4.2}$$

with equality for some $V \in (0, \mathrm{Vol}(M))$ if and only if every metric ball in M of volume V is isometric to a round ball in $\mathbb{R}^n$. In particular if equality occurs for some $V \in (0, \mathrm{Vol}(M))$ then (M, g) is flat, i.e. it has identically zero sectional curvature.

Proof. Let us fix an arbitrary $x_0 \in M$ and let $B_r = B_r(x_0)$ be the metric ball in M centered at x_0 of radius $r > 0$. It is well known that the distance function $\mathsf{d}_{x_0}(\cdot) := \mathsf{d}(x_0, \cdot)$ is smooth outside the cut locus $\mathcal{C}_{x_0}$ of x_0 and that $\mu_g(\mathcal{C}_{x_0}) = 0$. From the co-area formula it follows that for $\mathcal{L}^1$-a.e. $r \geq 0$ one has $\mathcal{H}^{n-1}(\mathcal{C}_{x_0} \cap \partial B_r(x_0)) = 0$ and, since the cut locus is closed by definition, we get that for $\mathcal{L}^1$-a.e. $r \geq 0$ the distance function $\mathsf{d}_{x_0}(\cdot)$ is smooth on an open subset of full $\mathcal{H}^{n-1}$-measure on $\partial B_r(x_0)$.

Let us first assume that $r > 0$ is one of these regular radii, the general case will be settled in the end by an approximation argument. It is immediate to see that on $\partial B_r(x_0) \setminus \mathcal{C}_{x_0}$ we have $|\nabla \mathsf{d}_{x_0}| = 1$ and thus $\partial B_r(x_0) \setminus \mathcal{C}_{x_0}$ is a smooth hypersurface. In particular, since $\mathcal{H}^{n-1}(\partial B_r(x_0) \cap \mathcal{C}_{x_0}) = 0$, we have that $B_r(x_0)$ is a finite perimeter set whose reduced boundary is contained $\partial B_r(x_0) \setminus \mathcal{C}_{x_0}$. Called ν the inward pointing unit normal to $\partial B_r(x_0)$ on the regular part $\partial B_r(x_0) \setminus \mathcal{C}_{x_0}$, from the Gauss Lemma we have

$$\nu = -\nabla \mathsf{d}_{x_0}, \quad \text{on } \partial B_r(x_0) \setminus \mathcal{C}_{x_0}. \tag{4.3}$$

Therefore, called $u := \frac{1}{2}\mathsf{d}_{x_0}^2$, we infer

$$\begin{aligned} r\mathcal{P}(B_r(x_0)) &= -\int_{\partial B_r(x_0)\setminus \mathcal{C}_{x_0}} \mathsf{d}_{x_0}(x)\, g(\nabla \mathsf{d}_{x_0}(x), \nu(x))\, d\mathcal{H}^{n-1}(x) = -\int_{\partial B_r(x_0)\setminus \mathcal{C}_{x_0}} g(\nabla u, \nu)\, d\mathcal{H}^{n-1} \\ &= -\lim_{\varepsilon \downarrow 0} \int_{\partial B_r(x_0)\setminus \mathcal{C}_{x_0}} g(\nabla u_\varepsilon, \nu)\, d\mathcal{H}^{n-1} \end{aligned}$$

where $u_\varepsilon \in C^2(M)$ is a approximation by convolution of u such that $\|\nabla u_\varepsilon - \nabla u\|_{L^\infty(\partial B_r(x_0), \mathcal{H}^{n-1})} \to 0$, $\Delta u_\varepsilon \to \Delta u$ in $C^0_{loc}(M \setminus \mathcal{C}_{x_0})$ and $\Delta u_\varepsilon \leq n$ where in the last estimate we used the global Laplacian comparison stating that Δu is a Radon measure with $\Delta u \leq n\, \mu_g$. More precisely, one has that $\Delta u \llcorner M \setminus \mathcal{C}_{x_0}$ is given by μ_g multiplied by a smooth function bounded above by n, and the singular part $(\Delta u)^s$ of Δu is a non-positive measure concentrated on $\mathcal{C}_{x_0}$. Now ∇u_ε is a C^1 vector field and we can apply the Gauss-Green formula for finite perimeter sets [3, Theorem 3.36] to infer

$$\begin{aligned} r\mathcal{P}(B_r(x_0)) &= \lim_{\varepsilon \downarrow 0} \int_{B_r(x_0)} \Delta u_\varepsilon \, d\mu_g = \lim_{\varepsilon \downarrow 0} \int_{B_r(x_0)\setminus \mathcal{C}_{x_0}} \Delta u_\varepsilon \, d\mu_g \leq \int_{B_r(x_0)\setminus \mathcal{C}_{x_0}} \limsup_{\varepsilon \downarrow 0} \Delta u_\varepsilon \, d\mu_g \\ &= \int_{B_r(x_0)\setminus \mathcal{C}_{x_0}} \Delta u \, d\mu_g \leq n \, \mathrm{Vol}(B_r), \end{aligned} \tag{4.4}$$

where the first inequality we used Fatou's Lemma combined with the upper bound $\Delta u_\varepsilon \leq n$ and the last inequality is ensured by the local Laplacian Comparison Theorem. Notice that if equality occurs then $\Delta u = n\, \mu_g$ on $B_r(x_0) \setminus \mathcal{C}_{x_0}$ and, by analyzing the equality in Riccati equations, it is well known that this implies $B_r(x_0)$ to be isometric to the round ball in $\mathbb{R}^n$.

If now $r > 0$ is a singular radius, in the sense that $\mathcal{H}^{n-1}(\partial B_r(x_0) \cap \mathcal{C}_{x_0}) > 0$, then by the above discussion we can find a sequence of regular radii $r_n \to r$ and, by the lower semicontinuity of the perimeter under L^1_{loc} convergence [3, Proposition 3.38] combined with (4.4) which is valid for $B_{r_n}(x_0)$, we infer

$$\begin{aligned} r\mathcal{P}(B_r(x_0)) &\leq \liminf_{n\to\infty} r_n \mathcal{P}(B_{r_n}(x_0)) \leq \liminf_{n\to\infty} \int_{B_{r_n}(x_0)\setminus\mathcal{C}_{x_0}} \Delta u \, d\mu_g \leq \limsup_{n\to\infty} \int_{M\setminus\mathcal{C}_{x_0}} \chi_{B_{r_n}(x_0)} \Delta u \, d\mu_g \\ &\leq \int_{M\setminus\mathcal{C}_{x_0}} \limsup_{n\to\infty} \chi_{B_{r_n}(x_0)} \Delta u \, d\mu_g = \int_{B_r(x_0)\setminus\mathcal{C}_{x_0}} \Delta u \, d\mu_g \leq n \operatorname{Vol}(B_r), \end{aligned} \tag{4.5}$$

where in the first inequality of the second line we used Fatou's Lemma (we are allowed since $\chi_{B_{r_n}(x_0)} \Delta u \leq n$ on $M \setminus \mathcal{C}_{x_0}$), and the last inequality follows again by local Laplacian comparison. Notice that, as before, equality in (4.5) forces $\Delta u = n \, \mu_g$ on $B_r(x_0) \setminus \mathcal{C}_{x_0}$ and then $B_r(x_0)$ is isometric to a Euclidean ball.

The second part of the statement clearly follows from the first part combined with the Euclidean isoperimetric-isodiametric inequality (2.3). $\square$

5. Existence of isoperimetric-isodiametric regions

In Section 3 we have seen explicit isoperimetric-inequalities in some special situations: Cartan-Hadamard spaces and minimal submanifolds. In the present section we investigate the existence of optimal shapes: as it happens also for the isoperimetric problem, we will find that if the ambient manifold is compact an optimal set always exists but if the ambient space is non-compact the situation changes dramatically. The subsequent sections will be devoted to establish the sharp regularity for the optimal sets.

5.1. **Notation.** Let (M^n, g) be a complete Riemannian manifold and denote by d_g the geodesic distance, by μ_g the measure associated to the Riemannian volume form and by $\mathfrak{X}(M)$ the smooth vector fields. Given a measurable subset $E \subset M$, the perimeter of E is denoted by $\mathcal{P}(E)$ and is given by the following formula

$$\mathcal{P}(E) := \sup \left\{ \int_E \operatorname{div} X \, \mathrm{d}\mu_g \; : \; X \in \mathfrak{X}(M), \; \operatorname{spt}(X) \subset\subset M, \; \|X\|_{L^\infty(M,g)} \leq 1 \right\},$$

and, for any open subset $\Omega \subset M$, we write $\mathcal{P}(E, \Omega)$ when the fields X are restricted to have compact support in Ω. It is out of the scope of this paper to discuss the theory of finite perimeter sets; standard references are [3], [20] and [37].

Since from now on we will work with sets of finite perimeter, which are well defined up to subsets of measure zero, we will adopt the following definition of extrinsic radius of a measurable subset $E \subset M$:

$$\operatorname{rad}(E) := \inf \left\{ r > 0 \; : \; \mu_g(E \setminus B_r(z_0)) = 0 \text{ for some } z_0 \in M \right\},$$

where $B_r(z_0)$ denotes the open metric ball with center z_0 and radius $r > 0$. A metric ball $B_r(z_0)$ satisfying $\mu_g(E \setminus B_r(z_0)) = 0$, is called an *enclosing ball* for E.

We consider the following minimization problem: for every fixed $V \in (0, \mu_g(M))$,

$$\min \Big\{ \operatorname{rad}(E)\, \mathcal{P}(E) \; : \; E \subset M, \; \mu_g(E) = V \Big\}, \tag{5.1}$$

and call the minimizers of (5.1) isoperimetric-isodiametric sets (or regions).

5.2. **Existence of isoperimetric-isodiametric regions in compact manifolds.** Let us start with the following lemma, stating the lower semi continuity of the extrinsic radius under L^1_{loc} convergence.

Lemma 5.1 (Lower semi-continuity of extrinsic radius under L^1_{loc} convergence)**.** *Let (M,g) be a (non necessarily compact) Riemannian manifold and let $(E_k)_{k\in\mathbb{N}\cup\{\infty\}}$ be a sequence of measurable subsets such that $\chi_{E_k}\to\chi_{E_\infty}$ in $L^1_{loc}(M,\mu_g)$. Then*

$$\mathrm{rad}(E_\infty)\le\liminf_{k\in\mathbb{N}}\mathrm{rad}(E_k).$$

Proof. Without loss of generality we can assume $\liminf_{k\in\mathbb{N}}\mathrm{rad}(E_k)<\infty$ so, up to selecting a subsequence, we can assume $\chi_{E_k}\to\chi_{E_\infty}$ a.e. and $\lim_{k\uparrow+\infty}\mathrm{rad}(E_k)=\ell<\infty$. Let $B_k:=B_{\mathrm{rad}(E_k)}(x_k)$ be enclosing balls for E_k. Then two cases can occur. Either x_k is unbounded, i.e. $\sup_k d_g(x_k,\bar{x})=\infty$ for any $\bar{x}\in M$, in which case it follows that $E_\infty=\emptyset$ and the conclusion of the lemma is proved. Or there exists $x_\infty\in M$ such that, up to passing to a subsequence, $x_k\to x_\infty$. In this case it is readily verified that

$$\mu_g\big(E_k\setminus B_{\mathrm{rad}(E_k)+|x_k-x_\infty|}(x_\infty)\big)=0$$

from which it follows, by taking the limit as $k\to+\infty$, that $\mu_g\big(E_\infty\setminus B_\ell(x_\infty)\big)=0$, which by definition implies that $\mathrm{rad}(E_\infty)\le\ell$. □

The next theorem is a general existence result for minimizers of the problem (5.1), as special cases it will be applied in Corollary 5.3 to compact manifolds and in Theorem 5.5 for asymptotically locally Euclidean manifolds (ALE for short) having non-negative Ricci curvature. Let us observe that the existence of a minimizer in a non-compact manifold for the classical isoperimetric problem is much harder due to the possibility of "small tentacles" going to infinity in a minimizing sequence; this difficulty is simply not there in the isoperimetric-isoperimetric problem we are considering, since it would imply the radius to go to infinity. We believe that this simplification, together with sharp inequalities obtained in the previous section, is another motivation to look at the isoperimetric-isoperimetric inequality since it appears more manageable in many situations than the classical isoperimetric one.

Theorem 5.2 (Sufficient conditions for existence of isoperimetric-isodiametric regions)**.** *Let (M^n,g) be a possibly non compact Riemannian n-manifold satisfying the following two conditions:*

1. $\liminf_{r\to0^+}\sup_{x\in M}\mu_g(B_r(x))=0$.
2. *There exists $\varepsilon_0>0$ and a function*

$$\Phi_{Isop}:[0,\varepsilon_0)\to\mathbb{R}^+\quad\textit{with}\quad\lim_{t\downarrow0}\Phi_{Isop}(t)=0,$$

such that for every finite perimeter set $E\subset M$ with $\mathcal{P}(E)<\varepsilon_0$ the weak isoperimetric inequality $\mu_g(E)\le\Phi_{Isop}(\mathcal{P}(E))$ holds.

Let $V\in(0,\mu_g(M))$ be fixed and let $(E_k)_{k\in\mathbb{N}}\subset M$ be a sequence of finite perimeter sets satisfying

$$\mu_g(E_k)=V,\,\forall k\in\mathbb{N},\quad\textit{and}\quad\sup_{k\in\mathbb{N}}\Big(\mathrm{rad}(E_k)\,\mathcal{P}(E_k)\Big)<\infty.\tag{5.2}$$

Then there exist $R > 0$ and a sequence $(x_k)_{k\in N}$ of points in M such that $\mu_g(E_k \setminus B_R(x_k)) = 0$, i.e. $B_R(x_k)$ are inclosing balls for E_k.

In particular, if there exists a minimizing sequence $(E_k)_{k\in\mathbb{N}}$ for the problem (5.1) *relative to some fixed $V \in (0, \mu_g(M))$ such that $\mu_g(E_k \cap K) > 0$ for infinitely many k and a fixed compact subset $K \subset M$, then there exists an isoperimetric-isodiametric region of volume V.*

Proof. We start the proof by the following two claims.

Claim 1: $\inf_k \mathrm{rad}(E_k) > 0$.

Otherwise, up subsequences in k, there exist $r_k \downarrow 0$ and $x_k \in M$ such that $\mu_g(E_k \setminus B_{r_k}(x_k)) = 0$. But then the assumption (1) implies $\mu_g(E_k) \le \mu_g(B_{r_k}(x_k)) = 0$, contradicting (5.2).

Claim 2: $\inf_k \mathcal{P}(E_k) > 0$.

Otherwise, by the assumption (2) we get $\mu_g(E_k) \le \Phi_{Isop}(\mathcal{P}(E_k)) \to 0$, contradicting again (5.2).

Combining the two claims with (5.2), we infer that there exists $C > 1$ such that

$$\frac{1}{C} \le \mathcal{P}(E_k) \le C \quad \text{and} \quad \frac{1}{C} \le \mathrm{rad}(E_k) \le C, \tag{5.3}$$

so that the first part of the proposition is proved.

If now there exists a compact subset $K \subset M$ such that $\mu_g(E_k \cap K) > 0$ for infinitely many k then by (5.3), up to enlarging K and selecting a subsequence in k, we can assume $\mu_g(E_k \setminus K) = 0$. But then the characteristic functions $(\chi_{E_k})_{k\in\mathbb{N}}$ are pre-compact in $L^1(K, \mu_g)$ since the total variations of χ_{E_k} are equi-bounded by (5.3) (cf. [3, Theorem 3.23]). The thesis then follows by the lower semicontinuity of the perimeter under L^1_{loc} convergence (cf. [3, Proposition 3.38]) combined with Lemma 5.1. □

Clearly if the manifold is compact all the assumptions of Theorem 5.2 are satisfied and we can state the following corollary.

Corollary 5.3 (Existence of isoperimetric-isodiametric regions in compact manifolds)**.** *Let (M^n, g) be a compact Riemannian manifold. Then for every $V \in (0, \mu_g(M))$ there exists a minimizer of the problem* (5.1)*, in other words there exists an isoperimetric-isodiametric region of volume V.*

5.3. Existence of isoperimetric-isodiametric regions in non-compact ALE spaces with non-negative Ricci curvature.

Let us start by recalling the notion of pointed C^0-convergence of metrics.

Definition 5.4. *Let (M^n, g) be a smooth complete Riemannian manifold and fix $\bar{x} \in M$. A sequence of pointed smooth complete Riemannian n-manifolds (M_k, g_k, x_k) is said to converge in the pointed C^0-topology to the manifold $(M, g, \bar{x})$, and we write $(M_k, g_k, x_k) \to (M, g, \bar{x})$, if for every $R > 0$ we can find a domain Ω_R with $B_R(\bar{x}) \subseteq \Omega_R \subseteq M$, a natural number $N_R \in \mathbb{N}$, and C^1-embeddings $F_{k,R} : \Omega_R \to M_k$ for large $k \ge N_R$ such that $B_R(x_k) \subseteq F_{k,R}(\Omega_R)$ and $F^*_{k,R}(g_k) \to g$ on Ω_R in the C^0-topology.*

Theorem 5.5. *Let (M, g) be a complete Riemannian n-manifold with non-negative Ricci curvature and fix any reference point $\bar{x} \in M$. Assume that for any diverging sequence of points $(x_k)_{k\in N} \subset M$, i.e. $\mathsf{d}(x_k, \bar{x}) \to \infty$, the sequence of pointed manifolds (M, g, x_k) converges in the pointed C^0-topology*

to the Euclidean space $(\mathbb{R}^n, g_{\mathbb{R}^n}, 0)$.
Then for every $V \in [0, \mu_g(M))$ *there exists a minimizer of the problem* (5.1)*, in other words there exists an isoperimetric-isodiametric region of volume* V.

Proof. Since volume and perimeter involve only the metric tensor g and not its derivatives, the hypothesis on the manifold (M, g) of being C^0-locally asymptotic to $\mathbb{R}^n$ implies directly that assumptions (1) and (2) of Theorem 5.2 are satisfied. Therefore the thesis will be a consequence of Theorem 5.2 once we show the following: given $E_k \subset M$ a minimizing sequence of the problem (5.1) for some fixed volume $V \in [0, \mu_g(M))$, then there exists a compact subset $K \subset M$ such that $\mu_g(E_k \cap K) > 0$ for infinitely many k. We will show that if this last statement is violated then (M, g) is flat and minimizers are metric balls of volume V.

By the first part of Theorem 5.2 we know that there exist $R > 0$ and a sequence $(x_k)_{k\in\mathbb{N}}$ of points in M such that $\mu_g(E_k \setminus B_R(x_k)) = 0$, i.e. $B_R(x_k)$ are inclosing balls for E_k.
Fixed any reference point $\bar{x} \in M$, if $\liminf_k \mathsf{d}(x_k, \bar{x})$ then clearly we can find a compact subset $K \subset M$ such that $\mu_g(E_k \cap K) > 0$ for infinitely many k and the conclusion follows from the last part of Theorem 5.2. So assume that $\mathsf{d}(\bar{x}, x_k) \to \infty$. Since M is C^0-locally asymptotic to $\mathbb{R}^n$, combining Definition 5.4 with the Euclidean isoperimetric-isodiametric inequality (2.3), we get that

$$\liminf_{k\to\infty} \mathrm{rad}(E_k)\, \mathcal{P}(E_k) \geq n\, V. \tag{5.4}$$

But since (M, g) has non-negative Ricci curvature, the comparison estimate (4.2) yields that

$$\lim_{k\to\infty} \mathrm{rad}(E_k)\, \mathcal{P}(E_k) = \inf\{\mathrm{rad}(\Omega)\mathcal{P}(\Omega) \,:\, \Omega \subset M,\, \mathrm{Vol}(\Omega) = V\} \leq nV. \tag{5.5}$$

The combination of (5.4) with (5.5) clearly implies

$$\inf\{\mathrm{rad}(\Omega)\mathcal{P}(\Omega) \,:\, \Omega \subset M,\, \mathrm{Vol}(\Omega) = V\} = nV.$$

The rigidity statement of Theorem 4.1 then gives that any metric ball in (M, g) of volume V is isometric to a round ball in $\mathbb{R}^n$, and therefore in particular is a minimizer of the problem (5.1). □

5.4. **Examples of non-compact spaces where existence of isoperimetric-isodiametric regions fails.**

Example 5.6 (Mimimal surfaces with planar ends)**.** If $M \subset \mathbb{R}^3$ is an helicoid, or more generally a minimal surface with planar ends, then it is in particular C^0-locally asymptotic to $\mathbb{R}^2$ in the sense of Definition 5.4. Then, if we consider a sequence of metric balls $B_{r_k}(x_k) \subset M$ of fixed volume $V > 0$ such that $x_k \to \infty$ we get $\lim_{k\to\infty} \mathrm{rad}(B_{r_k}(x_k))\, \mathrm{Vol}(B_{r_k}(x_k)) = 2\,V$. In particular, for every $V > 0$ we have

$$\inf\{\mathrm{rad}(\Omega)\mathcal{P}(\Omega) \,:\, \Omega \subset M,\, \mathrm{Vol}(\Omega) = V\} \leq 2V.$$

But then Proposition 3.7 implies that the infimum is never achieved, or more precisely it is achieved if and only if M is an affine subspace.
The same argument holds for any minimal n-dimensional sub-manifold in $\mathbb{R}^m$ with ends which are C^0-locally asymptotic to $\mathbb{R}^n$. □

Example 5.7 (ALE spaces of negative sectional curvature)**.** Let (M^n, g) be a simply connected non-compact Riemannian manifold with negative sectional curvature and assume that (M, g) is C^0-locally asymptotic to $\mathbb{R}^n$ in the sense of Definition 5.4. Then, if we consider a sequence of metric balls $B_{r_k}(x_k) \subset M$ of fixed volume $V > 0$ such that $x_k \to \infty$ we get $\lim_{k\to\infty} \mathrm{rad}(B_{r_k}(x_k)) \,\mathrm{Vol}(B_{r_k}(x_k)) = n\,V$. In particular, for every $V > 0$ we have

$$\inf\{\mathrm{rad}(\Omega)\mathcal{P}(\Omega) \,:\, \Omega \subset M,\, \mathrm{Vol}(\Omega) = V\} \leq nV.$$

But then Proposition 3.1 implies that the infimum is never achieved, or more precisely it is achieved by a region Ω if and only if Ω is isometric to a Euclidean region, which is forbidden since M has negative sectional curvature. □

6. Optimal regularity of isoperimetric-isodiametric regions

In this last section we establish the optimal regularity for the isoperimetric-isodiametric regions, i.e. the minimizers of problem (5.1), under the assumption that the enclosing ball is regular.

6.1. $C^{1,\frac{1}{2}}$ regularity.

6.1.1. *First properties.* Let E be a minimizer of the isoperimetric–isodiametric problem in (M, g) with volume $\mu_g(E) = V > 0$. Let $x_0 \in M$ satisfy $\mu_g(E \setminus B_{\mathrm{rad}(E)}(x_0)) = 0$ and, for the sake of simplicity, we fix the notation $B := B_{\mathrm{rad}(E)}(x_0)$ for an enclosing ball. In the sequel, we always assume that B has regular boundary and we assume to be in the non-trivial case $\mu_g(B \setminus E) > 0$.

By the very definition of isoperimetric-isodiametric sets, we have that

$$\mathcal{P}(E) \leq \mathcal{P}(F) \quad \forall\, F \triangle E \subset\subset B \;:\; \mu_g(F) = V. \tag{6.1}$$

In particular, E is a minimizer of the perimeter with constrained volume in B, and therefore we can apply the classical regularity results (see, for example, [40, Corollary 3.8]) in order to deduce that there exists a relatively closed set $\mathrm{Sing}(E) \subset B$ such that $\dim_{\mathcal{H}}(\mathrm{Sing}(E)) \leq n - 8$ and $\partial E \cap B \setminus \mathrm{Sing}(E)$ is a smooth $(n-1)$-dimensional hypersurface.

Moreover, by the first variations of the area functional under volume constraint, one deduces that the mean curvature is constant on the regular part of the boundary: i.e. there exits $H_0 \in \mathbb{R}$ such that

$$\vec{H}_E(x) = H_0\,\nu_E \quad \forall\, x \in \partial E \cap B \setminus \mathrm{Sing}(E), \tag{6.2}$$

where

$$\vec{H}_E(x) := \sum_{i=1}^{n-1} \nabla_{\tau_i}\tau_i,$$

for $\{\tau_1, \ldots, \tau_{n-1}\}$ a local orthonormal frame of ∂E around $x \in \partial E \cap B \setminus \mathrm{Sing}(E)$, ν_E the interior normal to E and ∇ the Riemannian connection on (M, g).

In this section we prove the following.

Proposition 6.1. *Let $E \subset M$ be an isoperimetric-isodiametric set and $x_0 \in M$ be such that $\mu_g(E \setminus B_{\mathrm{rad}(E)}(x_0)) = 0$. Assume that $B := B_{\mathrm{rad}(E)}(x_0))$ has smooth boundary. Then, there exists $\delta > 0$ such that $\partial E \setminus B_{\mathrm{rad}(E)-\delta}(x_0)$ is $C^{1,\frac{1}{2}}$ regular.*

Remark 6.2. In particular, given the partial regularity in B as explained in § 6.1.1, we conclude that E is a closed set whose boundary is $C^{1,\frac{1}{2}}$ regular except at most a closed singular set $\mathrm{Sing}(E)$ of dimension less or equal to $n-8$. □

6.1.2. *Almost minimizing property.* The main ingredient of the proof of Proposition 6.1 is the following almost-minimizing property.

Lemma 6.3. *Let E be an isoperimetric-isodiametric set in M and let B denote an enclosing ball as above. There exist constants $C, r_0 > 0$ such that, for every $x \in B$ and for every $0 < r < r_0$, the following holds*

$$\mathcal{P}(E) \le \mathcal{P}(F) + C\, r^n \quad \forall\, F \triangle E \subset\subset B_r(x). \tag{6.3}$$

Remark 6.4. Note that $B_r(x)$ is not necessarily contained in B. □

Proof. We start fixing parameters $\eta, c_1 > 0$ and two points $y_1, y_2 \in B$ such that $d_g(y_1, y_2) > 4\,\eta$, $B_{4\eta}(y_1) \subset B$, $B_{4\eta}(y_2) \subset B$ and

$$\mathcal{P}(E, B_\eta(y_i)) > c_1 \quad i = 1, 2. \tag{6.4}$$

Note that the possibility of such a choice is easily deduced from the regularity of the previous subsection, or more elementary from the density estimates for sets of finite perimeter in points of the reduced boundary. Set for simplicity of notation $D_i := B_\eta(y_i)$. By a result by Giusti [26, Lemma 2.1], there exist $v_0, C_1 > 0$ such that, for every $v \in \mathbb{R}$ with $|v| < v_0$ and for every $i = 1, 2$, there exists F_i which satisfies the following

$$\begin{cases} F_i \triangle E \subset D_i, \\ \mu_g(F_i) = \mu_g(E) + v, \\ \mathcal{P}(F_i) \le \mathcal{P}(E) + C_1\, v. \end{cases} \tag{6.5}$$

Note that in [26, Lemma 2.1] the property (6.5) is proven in the Euclidean space with the flat metric, but the proof remains unchanged in a Riemannian manifold (up to a suitable choice of the constants v_0, C_1).

Next, let $r_0 > 0$ be a constant to be fixed momentarily such that $r_0 < \eta$ and

$$\sup_{x \in B} \mu_g(B_r(x)) \le C_2\, r^n < v_0, \quad \forall r \in [0, r_0] \tag{6.6}$$

for some $C_2 > 0$ depending just on B and r_0. Since $d_g(y_1, y_2) > 4\,\eta$, for every $x \in B$, $B_{r_0}(x)$ cannot intersect both D_1 and D_2: therefore, without loss of generality, we can assume $B_{r_0}(x) \cap D_1 = \emptyset$. If $r < r_0$ and $F \subset M$ is any set such that $F \triangle E \subset\subset B_r(x)$, we consider $F' := F \cap B$. Note that $F' \subset B$ and moreover

$$|\mu_g(F') - \mu_g(E)| \le \mu_g(B_r(x)) \le C_2\, r^n < v_0.$$

According to (6.5) we can then find $F'' \subset B$ such that $\mu_g(F'') = \mu_g(E)$, $F'' \triangle F' \subset\subset D_1$ and

$$\mathcal{P}(F'') \le \mathcal{P}(F') + C_1 |\mu_g(F') - \mu_g(E)|. \tag{6.7}$$

Using the fact that E minimizes the perimeter among compactly supported perturbation in $\bar{B}$, we deduce that

$$\begin{aligned}\mathcal{P}(E) \leq \mathcal{P}(F'') &\overset{(6.7)}{\leq} \mathcal{P}(F') + C_1|\mu_g(F') - \mu_g(E)| \\ &\leq \mathcal{P}(F) + \mathcal{P}(B) - \mathcal{P}(F \cup B) + C_2\, r^n. \end{aligned} \tag{6.8}$$

Next note that, if ∂B is $C^{1,1}$ regular, then one can choose $r_0 > 0$ such that the following holds: there exists a constant $C_3 > 0$ such that, for every $x \in B$ and for every $r \in (0, r_0)$,

$$\mathcal{P}(B) \leq \mathcal{P}(G) + C_3\, r^n \quad \forall\, G \triangle B \subset\subset B_r(x). \tag{6.9}$$

In order to show this claim, it it enough to take r_0 small enough (in particular smaller than half the injectivity radius) in such a way that, for every $p \in \partial B$, there exists a co-ordinate chart $\phi : B_{2r_0}(p) \to \mathbb{R}^n$ such that $\phi(\partial B) \subset \{x_n = 0\}$ and ϕ is a $C^{1,1}$ diffeomorphism with $\mathrm{d}\phi(p) \in SO(n), \phi(p) = 0$ and $g(0) = \mathrm{Id}$, g being the metric tensor in the coordinates induced by ϕ . Indeed, in this case we have that $\mathcal{P}(B, B_r(p)) \leq (1 + Cr)\omega_{n-1} r^{n-1}$ for every $r < r_0$ and, for every G such that $G \triangle B \subset\subset B_r(p)$,

$$\mathcal{P}(G, B_r(p)) \geq (1 - Cr)\mathcal{P}(\mathrm{proj}(\phi(G)), \phi(B_r(p))) \geq (1 - Cr)\omega_{n-1} r^{n-1},$$

where proj denotes the orthogonal Euclidean projection on $\{x_n = 0\}$ and we have used the regularity of ϕ.

Applying (6.9) to $G = F \cup B$ and using (6.8), we conclude the proof. $\square$

6.1.3. *Proof of Proposition 6.1.* Now we are in the position to apply a result by Tamanini [47, Theorem 1] (the result is proved in $\mathbb{R}^n$ with a flat metric, but the proof is unchanged in a Riemannian manifold) in order to give a proof of the above proposition.

To this aim, we start considering any point $p \in \partial B \cap \partial E$; we denote with $Exp_p : T_pM \to M$ the exponential map and we let $r_0 > 0$ be less then the injectivity radius. Since by Lemma 6.3 the set E is an almost minimizer of the perimeter, the rescaled sets

$$E_{p,r} := \frac{Exp_p^{-1}(E \cap B_{r_0}(p))}{r} \subset T_pM \simeq \mathbb{R}^n \tag{6.10}$$

converge up to passing to a suitable subsequence to a minimizing cone C_∞ in the Euclidean space (see [37, Theorem 28.6]). Moreover, since E is enclosed by B and ∂B is $C^{1,1}$, it is immediate to check that if $r_0 > 0$ is chosen small enough in (6.10), then $C_\infty \subset \{x : g(\nu_B(p), x) \geq 0\}$, we deduce that every tangent cone to E at p needs to be contained in a half-space, and therefore by the Bernstein theorem is flat (cf. [25, Theorem 17.4]). This implies that every such point p is a point of the reduced boundary of the set (see [3, Definition 3.54]) and therefore we can apply the aforementioned result by Tamanini to conclude that ∂E is a $C^{1,\frac{1}{2}}$ regular hypersurface in $B_r(p)$ for every $p \in \partial B \cap \partial E$ and for every $r < \frac{r_0}{2}$. By a simple covering argument, the conclusion of the corollary follows.

6.2. **L^∞ estimates on the mean curvature of the minimizer.** In this section we prove that the boundary of E has generalized mean curvature in the sense of varifolds which is bounded in L^∞. To this aim, we compute the first variations of the perimeter of E along suitable diffeomorphisms.

6.2.1. *First variations.* We start fixing two points $y_1, y_2 \in \partial E \cap B \setminus \mathrm{Sing}(E)$ and a real number $\eta > 0$ such that $B_{4\eta}(y_1) \subset B$, $B_{4\eta}(y_2) \subset B$ and

$$B_{4\eta}(y_1) \cap B_{4\eta}(y_2) = B_{4\eta}(y_1) \cap \mathrm{Sing}(E) = B_{4\eta}(y_2) \cap \mathrm{Sing}(E) = \emptyset.$$

Note that such a choice is possible in the hypothesis that $\mu_g(B \setminus E) > 0$ because of the partial regularity in § 6.1.1. Let $X \in \mathfrak{X}(M)$ be a vector field with support contained in a metric ball $B_\eta(y)$ for some $y \in M$. Clearly, $B_\eta(y)$ cannot intersect both $B_{2\eta}(y_1)$ and $B_{2\eta}(y_2)$, because $d_g(y_1, y_2) \geq 8\eta$; therefore, without loss of generality let us assume that $B_\eta(y) \cap B_{2\eta}(y_1) = \emptyset$. It is not difficult to construct a smooth vector field Y supported in $B_\eta(y_1)$ such that the generated flow $\{\Phi_t^Y\}$ satisfies the following properties for small $|t|$:

$$\mu_g(\Phi_t^Y \circ \Phi_t^X(E)) = \mu_g(E). \tag{6.11}$$

Note that the generated flows $\{\Phi_t^X\}_{t\in\mathbb{R}}$ and $\{\Phi_t^Y\}_{t\in\mathbb{R}}$ are well-defined and for $|t|$ sufficiently small are diffeomorphisms of M. Moreover, $\Phi_t^Y \circ \Phi_t^X(E) \subset B_{\mathrm{rad}(E)+|t|\|X\|_\infty}$. We can then deduce that

$$\begin{aligned} \mathrm{rad}(E)\mathcal{P}(E) &\leq \mathrm{rad}\left(\Phi_t^Y \circ \Phi_t^X(E)\right)\mathcal{P}\big(\Phi_t^Y \circ \Phi_t^X(E)\big) \\ &\leq \big(\mathrm{rad}(E) + |t|\|X\|_\infty\big)\mathcal{P}\big(\Phi_t^Y \circ \Phi_t^X(E)\big) =: f(t). \end{aligned} \tag{6.12}$$

Taking the derivative of the last functional as $t \downarrow 0^+$ and as $t \uparrow 0^-$, by the well-known computation of the first variations of the area we infer that

$$\begin{aligned} 0 &\leq \lim_{t\downarrow 0^+} \frac{f(t) - f(0)}{t} \\ &= \|X\|_\infty \mathcal{P}(E) + \mathrm{rad}(E) \int_{\partial E} \mathrm{div}_{\partial E} X \, \mathrm{d}\mathcal{H}^{n-1} - \int_{\partial E} g\big(\vec{H}_E, Y\big) \, \mathrm{d}\mathcal{H}^{n-1} \end{aligned} \tag{6.13}$$

$$\begin{aligned} 0 &\geq \lim_{t\uparrow 0^-} \frac{f(t) - f(0)}{t} \\ &= -\|X\|_\infty \mathcal{P}(E) + \mathrm{rad}(E) \int_{\partial E} \mathrm{div}_{\partial E} X \, \mathrm{d}\mathcal{H}^{n-1} - \int_{\partial E} g\big(\vec{H}_E, Y\big) \, \mathrm{d}\mathcal{H}^{n-1}, \end{aligned} \tag{6.14}$$

where $\mathrm{div}_{\partial E} X := \sum_{i=1}^{n-1} g(\nabla_{\tau_i} X, \tau_i)$ for a (measurable) local orthonormal frame $\{\tau_1, \ldots, \tau_{n-1}\}$ of ∂E. (Note that in writing (6.13) and (6.14) we have used that ∂E is a $C^{1,\frac{1}{2}}$ regular submanifold up to singular set of dimension at most $n-8$ and that Y is supported in $B_\eta(y)$ where ∂E is smooth in order to make the integration by parts.) In the case $V \in (0, \mu_g(M))$, we have $\mathrm{rad}(E) > 0$ and thus $\mathcal{P}(E) < \infty$. Moreover, from (6.11) we deduce that

$$0 = \frac{d}{dt}_{|t=0} \mu_g\left(\Phi_t^Y \circ \Phi_t^X(E)\right) = -\int_{\partial E} g(X, \nu_E) \, \mathrm{d}\mathcal{H}^{n-1} - \int_{\partial E} g(Y, \nu_E) \, \mathrm{d}\mathcal{H}^{n-1}. \tag{6.15}$$

Therefore, from (6.2), (6.13), (6.14) and (6.15) we conclude that

$$\begin{aligned}\left|\int_{\partial E}\mathrm{div}_{\partial E}X\,\mathrm{d}\mathcal{H}^{n-1}\right| &\leq \frac{1}{\mathrm{rad}(E)}\left(\mathcal{P}(E)\|X\|_\infty+\left|\int_{\partial E}g\big(\vec{H}_E,Y\big)\,\mathrm{d}\mathcal{H}^{n-1}\right|\right)\\ &\leq \frac{1}{\mathrm{rad}(E)}\left(\mathcal{P}(E)\|X\|_\infty+|H_0|\left|\int_{\partial E}g(Y,\nu_E)\,\mathrm{d}\mathcal{H}^{n-1}\right|\right)\\ &= \frac{1}{\mathrm{rad}(E)}\left(\mathcal{P}(E)\|X\|_\infty+|H_0|\left|\int_{\partial E}g(X,\nu_E)\,\mathrm{d}\mathcal{H}^{n-1}\right|\right)\\ &\leq C\|X\|_\infty \end{aligned}\tag{6.16}$$

for some $C=C(\mathrm{rad}(E),\mathcal{P}(E),|H_0|)>0$, for every vector field X with support contained in a metric ball $B_\eta(y)$ for some $y\in M$. By a simple partition of unity argument, (6.16) holds for every $X\in\mathfrak{X}(M)$. In particular, by the use of Riesz representation theorem we have proved the following lemma. To this regard we denote with $\mathcal{M}(M,TM)$ the vectorial Radon measures $\vec{\mu}$ on M with values in the tangent bundle TM.

Lemma 6.5 (The mean curvature is represented by a vectorial Radon measure)**.** *Let $E\subset M$ be an isoperimetric-isodiametric region for some $V\in(0,\mu_g(M))$ and denote by B an enclosing ball. If ∂B is smooth, then there exists a vectorial radon measure $\vec{\mathbf{H}}_E\in\mathcal{M}(M,TM)$ concentrated on ∂E such that for every C^1 vector field X on M with compact support, called $\Phi_t^X:M\to M$ the corresponding one-parameter family of diffeomorphisms for $t\in\mathbb{R}$, it holds*

$$\delta E(X):=\frac{d}{dt}_{|t=0}\mathcal{P}(\Phi_t^X(E))=-\int_M g(X,\vec{\mathbf{H}}_E).\tag{6.17}$$

Moreover the total variation of $\vec{\mathbf{H}}_E$ is finite, i.e.

$$|\vec{\mathbf{H}}_E|(M)\leq C=C\big(\mathcal{P}(E),\mathrm{rad}(E),|H_0|\big)\in[0,\infty).$$

Remark 6.6. *Note that*

$$\vec{\mathbf{H}}_E\llcorner B:=\vec{H}_E\,\mathcal{H}^{n-1}\llcorner(\partial E\cap B),\tag{6.18}$$

where $\vec{H}_E$ is the mean curvature vector on the smooth part of ∂E as defined in (6.2).

We close this subsection by noting that if

$$g\big(X(x),\nu_B(x)\big)\geq 0\quad\forall\,x\in\partial B\cap B_\eta(y),\tag{6.19}$$

where ν_B is the interior normal to ∂B (note that $\partial B\cap B_\eta(y)$ can also be empty), then $\Phi_t^Y\circ\Phi_t^X(E)\subset B$ for $t\geq 0$. In particular, the minimizing property of E gives

$$\mathcal{P}\big(\Phi_t^Y\circ\Phi_t^X(E)\big)\geq\mathcal{P}(E)\quad\forall\,t\geq 0,\tag{6.20}$$

which combined with (6.2) and (6.15) implies

$$\begin{aligned}0\leq\frac{d}{dt}\Big|_{t=0^+}\mathcal{P}\big(\Phi_t^Y\circ\Phi_t^X(E)\big)&=\int_{\partial E}\mathrm{div}_{\partial E}X\,\mathrm{d}\mathcal{H}^{n-1}-\int_{\partial E}g\big(\vec{H}_E,Y\big)\\ &=\int_{\partial E}\mathrm{div}_{\partial E}X\,\mathrm{d}\mathcal{H}^{n-1}+H_0\int_{\partial E}g\big(\nu_E,X\big),\end{aligned}\tag{6.21}$$

which in view of (6.17) gives

$$g\big(\nu_B, \vec{\mathbf{H}}_E\big) \llcorner (\partial E \cap \partial B) \leq H_0\, \mathcal{H}^{n-1} \llcorner (\partial E \cap \partial B), \tag{6.22}$$

where the inequality is intended in the sense of measures, i.e. $\int_A g(\nu_B, \vec{\mathbf{H}}_E) \leq H_0 \mathcal{H}^{n-1}(A)$ for every measurable set $A \subset \partial E \cap \partial B$.

6.2.2. *Orthogonality of* $\vec{\mathbf{H}}_E$*.* We have seen in the previous section that $\vec{\mathbf{H}}_E$ is well-defined as a measure on all ∂E. Translated into the language of varifolds, we have shown that the integral varifold associated to ∂E has finite first variation. A classical result due to Brakke [7, Section 5.8] (see also [38] for an alternative proof and for fine structural properties of varifolds with locally finite first variation) implies that for $\mathcal{H}^{n-1}$-a.e. $x \in \partial E$ it holds $\vec{\mathbf{H}}_E(x) \in (T_x \partial E)^\perp$. This is not quite enough to our purposes, indeed in the next lemma we will show that $\vec{\mathbf{H}}_E$ is *normal to* ∂E *as measure*, which is a strictly stronger statement. Note that the proof is based on the fact that E is a minimizer for the problem (5.1), and will not make use of the aforementioned structural result by Brakke.

Lemma 6.7 (The mean curvature measure is orthogonal to ∂E)**.** *Let* $E, B, M, V, \vec{\mathbf{H}}_E$ *be as in Lemma 6.5. Then* $\vec{\mathbf{H}}_E(x) \in (T_x \partial E)^\perp$ *for* $|\vec{\mathbf{H}}_E|$*-a.e.* $x \in \partial E$*, i.e. the mean curvature is orthogonal to* ∂E *as a measure.*

Remark 6.8. *In other words there exists an* $\mathbb{R}$*-valued finite radon measure* $\mathbf{H}_E$ *on* M *concentrated on* ∂E *such that* $\vec{\mathbf{H}}_E = \mathbf{H}_E\, \nu_E$*; moreover, by* (6.2)*,* $\mathbf{H}_E \llcorner (B \cap \partial E) = H_0\, \mathcal{H}^{n-1} \llcorner (\partial E \cap B)$*.*

Proof. In view of (6.2) we only need to prove the claim for $\vec{\mathbf{H}}_E \llcorner \partial B$. Assume by contradiction that there exists a compact subset $K \subset \partial B \cap \partial E$ such that

$$|\vec{\mathbf{H}}_E^T|(K) > 0, \tag{6.23}$$

where $\vec{\mathbf{H}}_E^T := P_{T\partial E}(\vec{\mathbf{H}}_E)$ is the projection of $\vec{\mathbf{H}}_E$ onto the tangent space of ∂E (or, equivalently, onto $T\partial B$, because ∂E and ∂B are C^1 and $T_x \partial E = T_x \partial B$ for every $x \in \partial B \cap \partial E$).

The geometric idea of the proof is very neat: if the mean curvature along $K \subset \partial E \cap \partial B$ has a non trivial tangential part, then deforming infinitesimally E along this tangential direction will not increase the extrinsic radius (since the deformation of E will stay in the ball B), will not increase the volume (because the deformation is tangential to ∂E) but will strictly decrease the perimeter; so, after adjusting the volume in a smooth portion of ∂E, this procedure builds an infinitesimal deformation of E which preserves the volume, does not increase the extrinsic radius but strictly decreases the perimeter, contradicting that E is a minimizer of the problem (5.1). The rest of the proof is a technical implementation of this neat geometric idea.

For every $\varepsilon > 0$ we construct a suitable C^1 regular tangential vector field. To this aim, we consider the polar decomposition of the measure $\vec{\mathbf{H}}_E^T = v\, |\vec{\mathbf{H}}_E^T|$ where v is a Borel vector field such that $v(x) \in T\partial B$ and $g(v(x), v(x)) = 1$ for $|\vec{\mathbf{H}}_E^T|$-a.e. $x \in M$. By the Lusin theorem we can find a continuous vector field w such that $|\vec{\mathbf{H}}_E^T|(\{v \neq w\}) \leq \varepsilon$ and $\mathrm{spt}(w) \subset K_\varepsilon := \{x \in \partial E \cap \partial B : d_g(x, K) < \varepsilon\}$. Moreover, by a standard regularization procedure via mollification and projection on $T\partial B$, we find a vector field X_ε such that $X_\varepsilon(x) \in T\partial B$ for every $x \in \partial B \cap K_{2\varepsilon}$, $\|X_\varepsilon - w\|_\infty \leq \varepsilon$ and $\mathrm{spt}(X_\varepsilon) \subset K_{2\varepsilon}$. Note that

$$\int_M g\big(X_\varepsilon, \vec{\mathbf{H}}_E\big) = \int_M g\big(X_\varepsilon - w, \vec{\mathbf{H}}_E\big) + \int_{\{w=v\}} g\big(v, \vec{\mathbf{H}}_E\big) + \int_{\{w\neq v\}} g\big(w, \vec{\mathbf{H}}_E\big)$$
$$\to |\vec{\mathbf{H}}_E^T|(K) \quad \text{as } \varepsilon \to 0. \tag{6.24}$$

Since X_ε is a smooth vector field compactly supported in M and tangent to ∂B, the generated flow $\Phi_t^{X_\varepsilon}$ is well defined and maps B into B for every $t \in \mathbb{R}$ and by (6.24)

$$\frac{d}{dt}_{|t=0} \mathcal{P}\big(\Phi_t^{X_\varepsilon}(E)\big) = -\int_{\partial E} g(X_\varepsilon, \vec{\mathbf{H}}_E) \leq -\frac{|\vec{\mathbf{H}}_E^T|(K)}{2} < 0, \tag{6.25}$$

for $\varepsilon > 0$ small enough. Moreover, since X_ε is supported in $K_{2\varepsilon}$ and $K \subset \partial B$ and X_ε is tangent to $\partial B = \partial E$ in K, we have that

$$\frac{d}{dt}_{|t=0} \mu_g\big(\Phi_t^{X_\varepsilon}(E)\big) = -\int_{\partial E} g(\nu_E, X_\varepsilon)\, d\mathcal{H}^{n-1} \to 0 \quad \text{as } \varepsilon \to 0. \tag{6.26}$$

Up to choosing a smaller compact set, we can suppose that K is contained in a small ball $B_{r_0}(x)$ with $x \in \partial E \cap \partial B$ such that $(\partial E \setminus \partial B) \cap (M \setminus B_{4r_0}(x)) \neq \emptyset$. Now fix $y \in \partial E \setminus (\partial B \cup B_{4r_0}(x) \cup \mathrm{Sing}(E))$ and let $r \in (0, r_0)$ be such that $B_{2r}(y) \cap (\partial B \cup B_{4r_0}(x) \cup \mathrm{Sing}(E)) = \emptyset$. For $\varepsilon > 0$ small enough it is not difficult to construct a smooth vector field Y_ε supported in $B_r(y)$ such that the generated flow $\Phi_t^{Y_\varepsilon}$ satisfies the following properties ((6.28) is intended for small t):

$$\frac{d}{dt}_{|t=0} \mu_g(\Phi_t^{Y_\varepsilon} \circ \Phi_t^{X_\varepsilon}(E)) \quad = \quad 0 \tag{6.27}$$

$$|\mathcal{P}(\Phi_t^{Y_\varepsilon}(E), B_{2r}(y)) - \mathcal{P}(E, B_{2r}(y))| \quad \leq \quad C\mu_g(\Phi_t^{Y_\varepsilon}(E) \Delta E). \tag{6.28}$$

Notice that the combination of (6.26), (6.27) and (6.28) gives

$$\left| \frac{d}{dt}_{|t=0} \mathcal{P}(\Phi_t^{Y_\varepsilon}(E)) \right| \leq C \left| \frac{d}{dt}_{|t=0} \mu_g(\Phi_t^{Y_\varepsilon}(E)) \right| = C \left| \frac{d}{dt}_{|t=0} \mu_g(\Phi_t^{X_\varepsilon}(E)) \right| \to 0, \text{ as } \varepsilon \to 0. \tag{6.29}$$

Moreover, since for small $t > 0$ we have $\Phi_t^{Y_\varepsilon}(E) \Delta E \subset B_{2r}(y)$ which is disjoint from ∂B, and since by construction $\Phi_t^{X_\varepsilon}$ maps B into B, it is clear that

$$\Phi_t^{Y_\varepsilon} \circ \Phi_t^{X_\varepsilon}(E) \subset B, \quad \text{for } t > 0 \text{ sufficiently small.}$$

Therefore, since by assumption E is a minimizer for the problem (5.1), we infer

$$\frac{d}{dt}_{|t=0} \mathcal{P}(\Phi_t^{Y_\varepsilon} \circ \Phi_t^{X_\varepsilon}(E)) \geq 0. \tag{6.30}$$

But on the other hand, combining (6.25) and (6.29) we get

$$\begin{aligned} \frac{d}{dt}_{|t=0} \mathcal{P}(\Phi_t^{Y_\varepsilon} \circ \Phi_t^{X_\varepsilon}(E)) \quad &= \quad \frac{d}{dt}_{|t=0} \mathcal{P}(\Phi_t^{Y_\varepsilon}(E)) + \frac{d}{dt}_{|t=0} \mathcal{P}(\Phi_t^{X_\varepsilon}(E)) \\ &\leq \quad -\frac{|\vec{\mathbf{H}}_E^T|(K)}{4} < 0, \text{ for } \varepsilon > 0 \text{ small enough.} \end{aligned}$$

Clearly the last inequality contradicts (6.30). We conclude that it is not possible to find a compact subset $K \subset \partial B \cap \partial E$ satisfying (6.23); therefore the measure $|\vec{\mathbf{H}}_E^T|$ vanishes identically and the proof is complete. $\square$

6.2.3. *L^∞ estimate.* The next step is to show that the signed measure $\mathbf{H}_E$ is actually absolutely continuous with respect to $\mathcal{H}^{n-1} \llcorner \partial E$ with L^∞ bounds on the density. The upper bound follows from (6.22). For the lower bound we use the following lemma which is an adaptation of [49, Theorem 2] to our setting (notice that the statement of [49, Theorem 2] is more general as includes higher co-dimensions and arbitrary varifolds, but let us state below just the result we will use in the sequel).

Lemma 6.9. *Let $N^n \subset M^n$ be an n-dimensional submanifold with C^2-boundary ∂N and denote with ν_N the inward pointing unit normal to ∂N. Fix a compact subset $K \subset \partial N$ and assume that, denoted with $\vec{H}_N$ the mean curvature of ∂N, it holds*

$$g(\vec{H}_N, \nu_N) \geq \eta, \quad \text{on } K.$$

Then, for every $\varepsilon > 0$ there exists a C^1-vector field X_ε on M with the following properties:

$$
\begin{aligned}
X_\varepsilon(x) &= \nu_N, \quad \forall x \in K && (6.31)\\
|X_\varepsilon|(x) &\leq 1, \quad \forall x \in M && (6.32)\\
\operatorname{spt}(X_\varepsilon) &\subset K_\varepsilon := \{x \in M \,:\, d(x, K) \leq \varepsilon\} && (6.33)\\
g(X_\varepsilon, \nu_N)(x) &\geq 0, \quad \forall x \in \partial N, && (6.34)\\
\frac{d}{dt}_{|t=0} \mathcal{P}(\Phi_t^{X_\varepsilon}(E)) &\leq -\eta \int_{\partial E} |X_\varepsilon| \, d\mathcal{H}^{n-1}, && (6.35)
\end{aligned}
$$

for every subset $E \subset N$ with C^1 boundary ∂E, where $\Phi_t^{X_\varepsilon}$ denotes the flow generated by the vector field X_ε.

Lemma 6.9 will be used to prove the following lower bound on the mean curvature measure $\mathbf{H}_E$ of ∂E.

Lemma 6.10 (Lower bound on $\mathbf{H}_E$)**.** *Let $E, B, M, V, \vec{\mathbf{H}}_E, \mathbf{H}_E$ be as in Lemma 6.7. Assume $\eta := \inf_{\partial B} H_B > -\infty$, where $H_B := g(\vec{H}_B, \nu_B)$ and $\vec{H}_B$ is the mean curvature vector of ∂B. Then*

$$\mathbf{H}_E \llcorner (\partial E \cap \partial B) \geq \eta \, \mathcal{H}^{n-1} \llcorner (\partial E \cap \partial B). \tag{6.36}$$

Proof. Fix any $K \subset \partial E \cap \partial B$. For every $\varepsilon \in (0, 1)$ let X_ε be the C^1 vector field obtained by applying Lemma 6.9 with $N = B$, then by (6.35) and (6.33) we get

$$
\begin{aligned}
-\eta \int_{\partial E} |X_\varepsilon| \, d\mathcal{H}^{n-1} &\geq \frac{d}{dt}_{|t=0} \mathcal{P}(\Phi_t^{X_\varepsilon}(E)) = -\int_{K_\varepsilon} g(X_\varepsilon, \nu_E) \, d\mathbf{H}_E \\
&= -\int_K g(X_\varepsilon, \nu_B) \, d\mathbf{H}_E - \int_{K_\varepsilon \setminus K} g(X_\varepsilon, \nu_E) \, d\mathbf{H}_E \\
&\quad \to -\mathbf{H}_E(K), \quad \text{as } \varepsilon \to 0, && (6.37)
\end{aligned}
$$

where in the second identity we used that $\nu_B = \nu_E$ on $K \subset \partial E \cap \partial B$. Using (6.31) and (6.32), we have

$$-\eta \int_{\partial E} |X_\varepsilon| \, d\mathcal{H}^{n-1} = -\eta \int_K |X_\varepsilon| \, d\mathcal{H}^{n-1} - \eta \int_{\partial E \cap (K_\varepsilon \setminus K)} |X_\varepsilon| \, d\mathcal{H}^{n-1} \to -\eta \, \mathcal{H}^{n-1}(K) \quad \text{as } \varepsilon \to 0. \tag{6.38}$$

In particular, in the limit as $\varepsilon \to 0$ we deduce from (6.37) that

$$\eta \, \mathcal{H}^{n-1}(K) \le \mathbf{H}_E(K). \tag{6.39}$$

Since this holds for every $K \subset \partial E \cap \partial B$, it is easily recognized that (6.36) follows. □

6.3. **Optimal regularity.** In this section we prove that the boundary of an isoperimetric-isodiametric set E is $C^{1,1}$ regular away from the singular set.

Theorem 6.11. *Let $E \subset M$ be an isoperimetric-isodiametric set and $x_0 \in M$ be such that $\mu_g(E \setminus B_{\mathrm{rad}(E)}(x_0)) = 0$. Assume that $B := B_{\mathrm{rad}(E)}(x_0))$ has smooth boundary. Then, there exists $\delta > 0$ such that $\partial E \setminus B_{\mathrm{rad}(E)-\delta}(x_0)$ is $C^{1,1}$ regular.*

Note that the $C^{1,1}$ regularity is optimal, because in general one cannot expect to have continuity of the second fundamental form of ∂E across the free boundary of ∂E, i.e. the points on the relative (with respect to ∂B) boundary of $\partial E \cap \partial B$.

6.3.1. *Co-ordinate charts.* We start fixing suitable co-ordinate charts. Since E is bounded, there exists $r_0 > 0$ such that for every $x_0 \in \partial E$ there is a normal co-ordinate chart (Ω, φ) with $x_0 \in \Omega$ and

$$\varphi : \Omega \subset M \to B^{n-1}_{r_0} \times (-r_0, r_0) \subset \mathbb{R}^{n-1} \times \mathbb{R}$$

such that $\varphi(x_0) = 0$, $g(0) = \mathrm{Id}$ and $\nabla g(0) = 0$, where g denotes the metric tensor in these co-ordinates. Moreover, by the $C^{1,\frac{1}{2}}$ regularity of ∂E established in § 6.1, up to rotating these co-ordinate chart and eventually changing r_0, we can also assume that for every point $x_0 \in \partial B \cap \partial E$ also the following holds:

- ∂E and ∂B are, respectively, $C^{1,\frac{1}{2}}$ and C^∞ regular submanifolds, given in this chart as graphs of functions $u, \psi : B^{n-1}_{r_0} \to \left(-\frac{r_0}{2}, \frac{r_0}{2}\right)$ with $u \in C^{1,\frac{1}{2}}$ and $\psi \in C^\infty$;
- the functions u and ψ satisfy $\psi(x) \le u(x)$ for every $x \in B^{n-1}_{r_0}$,
$$u(0) = \psi(0) = |\nabla u(0)| = |\nabla \psi(0)| = 0,$$
and $\|u\|_{C^1} \le \delta_0$ and $\|\psi\|_{C^1} \le \delta_0$ for a fixed $\delta_0 > 0$ which will be later assumed to be suitably small.

On every such a chart, the $C^{1,\frac{1}{2}}$ regular submanifold $\partial E \cap \Omega$ is given as the set $\{(x, u(x)) : x \in B^{n-1}_{r_0}\}$. We can consider the natural co-ordinate chart on it given by $(x, u(x)) \mapsto x \in B^{n-1}_r$ with induced metric tensor given by $h_{ij} := g(E_i, E_j)$, where $E_i := e_i + \partial_i u \, e_n$ for $i = 1, \dots, n-1$. In particular,

$$h_{ij} = g_{ij} + \partial_i u \, g_{nj} + \partial_j u \, g_{ni} + \partial_i u \, \partial_j u \, g_{nn}, \tag{6.40}$$

where $\partial_i u = \partial_i u(x)$ and $g_{ij} = g_{ij}(x, u(x))$. We will use the notation $\tilde h$ for the function $\tilde h : B^{n-1}_{r_0} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$

$$\tilde h_{ij}(x,z,p) = g_{ij}(x,z) + p_i g_{jn}(x,z) + p_j g_{ni}(x,z) + p_i\, p_j g_{nn}(x,z)$$

with the obvious relation that $h_{ij} = \tilde h_{ij}\big(x, u(x), \nabla u(x)\big)$. Note that, as a function in (x,z,p), $\tilde h$ is smooth.

6.3.2. *First variation formula in local co-ordinates.* We consider next functions $\phi \in C^\infty_c(B^{n-1}_{r_0})$ and $\chi \in C^\infty_c(-r_0, r_0)$, and we assume that $\chi|_{(-\frac{r_0}{2},\frac{r_0}{2})} \equiv 1$, in such a way to assure that $\chi \circ u(x) = 1$ for every $x \in B^{n-1}_{r_0}$ (by the assumptions made on u). Consider the associated vector field $X(x,y) := \phi(x)\,\chi(y)\,e_n$ and note that $X \in C^\infty_c(\Omega, \mathbb{R}^n)$ and $X|_{\partial E} = \phi(x)\,e_n$. Called $F(t,p) := p + t\,X(p)$, there exists $\epsilon_0 > 0$ such that $F_t := F(t,\cdot)$ is a diffeomorphism of Ω into itself for every $|t| \le \epsilon_0$.

Consider now the variations of the area along these one-parameter family of diffeomorphisms under the assumption $\phi \ge 0$ on $\Lambda(u) := \{x \in B^{n-1}_{r_0} \,:\, u(x) = \psi(x)\}$. Arguing as in (6.21), we get that

$$\begin{aligned} 0 &\le \int_{\partial E} \mathrm{div}_{\partial E} X \,\mathrm{d}\mathcal{H}^{n-1} - H_0 \int_{\partial E} g(X, \nu_E)\,\mathrm{d}\mathcal{H}^{n-1} \\ &= \int_\Sigma h^{ij} g(\nabla_{E_i} X, E_j)\,\mathrm{d}\mathcal{H}^{n-1} - H_0 \int g(X,\nu_E)\,\mathrm{d}\mathcal{H}^{n-1}, \end{aligned} \tag{6.41}$$

where in the second line we have used a simple computation for the tangential divergence of X. Noting that

$$\begin{aligned} \nabla_{E_i} X &= \nabla_{e_i + \partial_i u\, e_n} X = \nabla_{e_i} X + \partial_i u\,\nabla_{e_n} X \\ &= \partial_i\phi\, e_n + \phi\,\nabla_{e_i} e_n + \partial_i u\,\phi\,\nabla_{e_n} e_n \\ &= \partial_i \phi\, e_n + \phi\,\Gamma^k_{in}\, e_k + \partial_i u\,\phi\,\Gamma^k_{nn}\, e_k, \end{aligned}$$

we get that

$$\begin{aligned} h^{ij}\, g(\nabla_{E_i} X, E_j) &= h^{ij}\,\big(\partial_i\phi\, g_{jn} + \phi\,\Gamma^k_{in}\, g_{jk} + \partial_i u\,\phi\,\Gamma^k_{nn}\, g_{jk}\big) \\ &\quad + h^{ij}\,\big(\partial_j u\,\partial_i\phi\, g_{nn} + \phi\,\partial_j u\,\Gamma^k_{in}\, g_{kn} + \partial_j u\,\partial_i u\,\phi\,\Gamma^k_{nn}\, g_{kn}\big) \\ &= \partial_i\phi\,\big(h^{ij}\, g_{jn} + h^{ij}\,\partial_j u\, g_{nn}\big) \\ &\quad + \phi\,\big(h^{ij}\,\partial_i u\,\Gamma^k_{nn}\, g_{jk} + h^{ij}\,\partial_j u\,\partial_i u\,\Gamma^k_{nn}\, g_{kn}\big) \\ &\quad + \phi\,\big(h^{ij}\,\Gamma^k_{in}\, g_{jk} + h^{ij}\,\partial_j u\,\Gamma^k_{in}\, g_{kn}\big). \end{aligned} \tag{6.42}$$

In particular, by a simple integration by parts, (6.41) reads as

$$\int_{B^{n-1}_r} \phi\, Lu\,\sqrt{\det(h_{ij})}\,\mathrm{d}x \le 0 \quad \forall\, \phi \in C^1_c(B^{n-1}_r),\ \phi|_{\Lambda(u)} \ge 0, \tag{6.43}$$

where $\Lambda(u) := \{x \in B^{n-1}_r : u(x) = \psi(x)\}$ and

$$Lu(x) := \mathrm{div}\big(A(x, u(x), \nabla u(x))\nabla u(x) + b(x, u(x), \nabla u(x))\big) - f(x) \tag{6.44}$$

with

- $A = (a^{ij})_{i,j=1,\dots,n-1} : B_r^{n-1} \times (-r, r) \times \mathbb{R}^{n-1} \to \mathbb{R}^{(n-1)\times(n-1)}$ is a smooth function given by
$$a^{ij}(x,z,p) := g_{nn}(x,z)\,\tilde{h}^{ij}(x,z,p);$$
- $b : B_r^{n-1} \times (-r, r) \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is a smooth regular function given by
$$b^i(x,z,p) := \tilde{h}^{ij}(x,z,p)\,g_{jn}(x,z);$$
- $f : B_r^{n-1} \to \mathbb{R}$ is a $C^{0,\alpha}$ regular function given by
$$\begin{aligned} f(x) &:= h^{ij}\,\partial_i u\,\Gamma^k_{nn}\,g_{jk} + h^{ij}\,\partial_j u\,\partial_i u\,\Gamma^k_{nn}\,g_{kn} \\ &\quad + h^{ij}\,\Gamma^k_{in}\,g_{jk} + h^{ij}\,\partial_j u\,\Gamma^k_{in}\,g_{kn} - H_0\,g\big(e_n, \nu_E\big), \end{aligned}$$
where $h^{ij} = \tilde{h}^{ij}\big(x, u(x), \nabla u(x)\big)$, $g_{ij} = g_{ij}\big(x, u(x)\big)$, $\Gamma^k_{ij} = \Gamma^k_{ij}\big(x, u(x)\big)$ and $\nu_E = \nu_E(x, u(x))$.

Explicitly expanding the divergence term in Lu we deduce that
$$Lu(x) = c^{ij}\partial_{ij}u + d, \tag{6.45}$$
where
$$c^{ij} = a^{ij} + g_{nn}\,\partial_l u\,\partial_{p^j} h^{il} + g_{ln}\,\partial_{p_j} h^{il}, \tag{6.46}$$
with $\partial_{p^j} h^{il} = \partial_{p^j}\tilde{h}^{il}\big(x, u(x), \nabla u(x)\big)$, $g_{ij} = g_{ij}\big(x, u(x)\big)$ and $d \in C^{0,\alpha}(B_r^{n-1})$ is given by
$$\begin{aligned} d &= g_{nn}\partial_i h^{ij}\partial_j u + g_{nn}\partial_z h^{ij}\partial_i u\,\partial_j u + \partial_i g_{nn} h^{ij}\partial_j u + \partial_n g_{nn} h^{ij}\partial_i u\,\partial_j u \\ &\quad + g_{jn}\partial_i h^{ij} + g_{jn}\partial_z h^{ij}\partial_i u + \partial_i g_{jn} h^{ij} + \partial_n g_{jn} h^{ij}\partial_i u - f \end{aligned} \tag{6.47}$$
with the entries of h and of its derivatives are computed in $\big(x, u(x), \nabla u(x)\big)$, while those of g and the derivatives of the metric are computed in $\big(x, u(x)\big)$.

Note that (6.43) is equivalent to the following couple of differential relations:
$$\begin{cases} Lu \le 0 & \text{in } B_r^{n-1}, \\ Lu = 0 & \text{in } B_r^{n-1} \setminus \Lambda(u), \end{cases} \tag{6.48}$$
where the first inequality is meant in the sense of distribution, while the second equation is pointwise (also recalling that u is smooth outside the contact set $\Lambda(u)$).

6.3.3. *Quadratic growth.* Note that by the explicit expressions of the previous subsection it turns out that $c^{ij}, d \in C^{0,\alpha}(B_{r_0}^{n-1})$ with uniform estimates (by the assumptions in § 6.3.1)
$$\|c^{ij}\|_{C^{0,\alpha}(B_{r_0}^{n-1})} + \|d\|_{C^{0,\alpha}(B_{r_0}^{n-1})} \le C. \tag{6.49}$$
Since $c(0) = \mathrm{Id}$ and c^{ij} are Hölder continuous, up to choosing a smaller $\delta_0 > 0$ (and consistently a smaller $r_0 > 0$) we can also ensure that c^{ij} is uniformly elliptic with bounds
$$\frac{\mathrm{Id}}{2} \le c \le 2\,\mathrm{Id}.$$

The next lemma shows that u leaves the obstacle ψ at most as a quadratic function of the distance to the free-boundary point.

Proposition 6.12. *Let $E \subset M$ be an isoperimetric-isodiametric set. Then, there exists a constant $C > 0$ such that, for every $x_0 \in \partial E \cap \partial B$, setting co-ordinates as in § 6.3.1, we have that*

$$u(x) - \psi(x) \le C\,|x|^2 \quad \forall\, x \in B^{n-1}_{\frac{r_0}{2}}. \tag{6.50}$$

Proof. Let us consider the homogeneous part of the operator L, i.e. $\mathcal{L}w := c^{ij}\partial_{ij}w$. Since $\mathcal{L}(u-\psi) = Lu - \mathcal{L}\psi - d$, for every $r \le r_0$ we can write $(u-\psi)|_{B^{n-1}_r} = w_1 + w_2$ with

$$\begin{cases} \mathcal{L}w_1 = 0 & \text{in } B^{n-1}_r, \\ w_1 = u - \psi & \text{on } \partial B^{n-1}_r, \end{cases} \tag{6.51}$$

and

$$\begin{cases} \mathcal{L}w_2 = Lu - \mathcal{L}\psi - d & \text{in } B^{n-1}_r, \\ w_2 = 0 & \text{on } \partial B^{n-1}_r. \end{cases} \tag{6.52}$$

We start estimating w_2 from below. Considering that $\mathcal{L}w_2 + \mathcal{L}\psi + d = Lu \le 0$, we can apply the L^∞-estimate for elliptic equations [24, Theorem 8.16]. In order to understand the dependence of the constant on the domain, we can rescale the variables in this way: $v : B^{n-1}_1 \to \mathbb{R}$ given by $v(y) := r^{-2}w_2(r\,y)$. Then, the equation satisfied by v is

$$\mathcal{L}v(y) + \mathcal{L}\psi(ry) + d(ry) = Lu(ry) \le 0.$$

We can then conclude using [24, (8.39)] that

$$\sup_{B^{n-1}_1}(-v) \le C\,\|\mathcal{L}\psi(ry) + d(ry)\|_{L^{\frac{q}{2}}(B^{n-1}_1)} \le C,$$

where now C is a dimensional constant (only depending on $q > n-1$, which for us is any fixed exponent – note that the hypothesis (8.8) in [24, Theorem 8.16] is satisfied because we are considering the operator $\mathcal{L}$ which has no lower order terms). In particular, scaling back to w_2 we deduce that

$$w_2(x) \ge -C\,r^2, \quad \forall\, x \in B^{n-1}_r. \tag{6.53}$$

This clearly implies that $w_1(0) = u(0) - \psi(0) - w_2(0) \le C\,r^2$. We can then use Harnack inequality for w_1 (cf. [24, Theorem 8.20]) and conclude that

$$w_1(x) \le C \inf_{B^{n-1}_{\frac{r}{2}}} w_1 \le C\,w_1(0) \le C\,r^2, \quad \forall\, x \in B^{n-1}_{\frac{r}{2}}. \tag{6.54}$$

Finally note that in $B^{n-1}_r \setminus \Lambda(u)$ we have the equality $\mathcal{L}w_2 = -\mathcal{L}\psi - d$. Therefore, the function $z := w_2 + C\,|x|^2$ satisfies $\mathcal{L}z \ge 0$ for a suitably chosen constant $C = C(\|\mathcal{L}\psi\|_{L^\infty}, \|d\|_{L^\infty})$. By the strong maximum principle [24, Theorem 8.19] we deduce that

$$\max_{B^{n-1}_r \setminus \Lambda(u)} z \le \max_{\partial(B^{n-1}_r \setminus \Lambda(u))} z \le C\,r^2,$$

where we used that $z|_{\partial B^{n-1}_r} = C\,r^2$ and that for every $x \in \Lambda(u) \cap B^{n-1}_r$ we have $z(x) = -w_1(x) + C\,|x|^2 \le C\,r^2$ by the positivity of w_1. In conclusion, we have that $u(x) - \psi(x) \le |w_1(x)| + |w_2(x)| \le C\,r^2$ for every $x \in B^{n-1}_{\frac{r}{2}}$. Since $r \le r_0$ is arbitrary, by eventually changing the constant C we conclude the proof of the proposition. □

6.3.4. *Curvature bounds away from the contact set.* Next we analyze the points $p \in \partial E \setminus \partial B$ which are close to ∂B. To this aim we fix a constant $s_0 > 0$ such that the following holds: if $\mathrm{dist}(p, \partial E \cap \partial B) = \mathrm{dist}(p, x_0) < s_0$, then p belongs to the co-ordinate chart Ω around x_0 as fixed in § 6.3.1 and moreover, in these co-ordinates, $p = (x, z) \in B^{n-1}_{r_0} \times (-r_0, r_0)$ (necessarily with $x \notin \Lambda(u)$) satisfies

$$B^{n-1}_{4\delta}(x) \subset B^{n-1}_{r_0} \quad \text{with} \quad \delta := \frac{\mathrm{dist}(x, \Lambda(u))}{2}.$$

Note that the existence of such a constant $s_0 > 0$ is ensured by a simple compactness argument. Recall also that by the quadratic growth proved in the previous section we now that

$$\|u\|_{L^\infty(B^{n-1}_{2\delta}(x))} \le C\,\delta^2.$$

The following lemma gives a curvature bound for ∂E in points p as above.

Lemma 6.13. *Let $p \in \partial E \setminus \partial B$ satisfy $\mathrm{dist}(p, \partial E \cap \partial B) < s_0$. Fixing $x_0 \in \partial E \cap \partial B$ and the corresponding co-ordinate chart as in § 6.3.1 with the notation fixed above, we then conclude that*

$$\|D^2 u\|_{L^\infty(B^{n-1}_{\delta}(x))} \le C, \tag{6.55}$$

where $C > 0$ is a dimensional constant.

Proof. Since on $B^{n-1}_{4\delta} \subset B^{n-1}_{r_0} \setminus \Lambda(u)$ the *equation* $Lu = 0$ is satisfied, the proof is a consequence of the basic interior Schauder estimates for second order elliptic equations (cp. [24, Theorem 6.2]). More precisely we write the equation as $\mathcal{L}u = -d$ where $d \in C^{0,\alpha}$ was defined is (6.47) and satisfies (6.49), and we apply [24, Theorem 6.2]) to such an equation. Indeed, by simply recalling the definition of the norms in [24, Theorem 6.2] we have that, setting $\mathsf{d}_y := \mathrm{dist}(y, \partial B^{n-1}_{2\delta}(x))$

$$\begin{aligned}
\delta^2\,\|D^2 u\|_{L^\infty(B^{n-1}_{\delta}(x))} &\le C\,\Bigg(\|u\|_{L^\infty(B^{n-1}_{2\delta}(x))} + \sup_{y \in B^{n-1}_{2\delta}(x)} \mathsf{d}_y^2 |d(y)|\Bigg) \\
&\quad + C \sup_{y,z \in B^{n-1}_{2\delta}(x)} \min\{\mathsf{d}_y, \mathsf{d}_z\}^{2+\alpha}\, \frac{|d(y) - d(z)|}{|y - z|^\alpha} \\
&\le C\,\Big(\|u\|_{L^\infty(B^{n-1}_{2\delta}(x))} + \delta^2 \|d\|_{L^\infty(B^{n-1}_{2\delta}(x))}\Big) \\
&\quad + C\,\delta^{2+\alpha} [d]_{C^{0,\alpha}(B^{n-1}_{2\delta}(x))} \le C\,\delta^2.
\end{aligned}$$

□

6.3.5. $C^{1,1}$*-regularity.* In this section we finally prove Theorem 6.11. The proof is based on the following property: by Proposition 6.12 and Lemma 6.13, there exists $\delta > 0$ such that for every $x_0 \in \partial B \cap \partial E$ there exists $r_0 > 0$ satisfying the following: fixing co-ordinates as in § 6.3.1,

$$|u(y) - u(x) - \nabla u(x) \cdot (y - x)| \le \frac{\bar{C}}{2}\,|x - y|^2, \quad \forall\, x, y \in B_{r_0}(x_0). \tag{6.56}$$

Indeed, if $x \in \partial E \cap \partial B$, then centering the co-ordinates at x we have $0 = u(0) = |\nabla u(0)|$, and (6.56) is a direct consequence of (6.50). On the other hand, if $x \notin \partial E \cap \partial B$, then setting the co-ordinates as in Lemma 6.13, we deduce (6.56) from (6.55).

The conclusion of Theorem 6.11 is then a direct consequence of the following lemma combined with a standard partition of unity argument.

Lemma 6.14. *Let $\Omega \subset \mathbb{R}^n$ be an open subset and let $u : \Omega \to \mathbb{R}$ be a C^1-function. Assume there exist $\bar{C} > 0$ and a countable covering $\{B_i\}_{i\in\mathbb{N}}$ of Ω made by open balls $B_i \subset \Omega$ such that for every $x, y \in B_i$ it holds*

$$|u(y) - u(x) - \nabla u(x) \cdot (y - x)| \leq \frac{\bar{C}}{2} \, |x - y|^2. \tag{6.57}$$

Then the distribution $\partial^2_{ij} u \in \mathcal{D}'(\Omega)$ is represented by an $L^\infty(\Omega)$ function, and

$$\|\partial^2_{ij} u\|_{L^\infty(\Omega)} \leq \bar{C}.$$

Proof. By a standard partition of unity argument it is enough to prove that for every ball B_i the restriction of the distribution $\partial^2_{ij} u \llcorner B_i$ is represented by an $L^\infty(B_i)$ function, and $\|\partial^2_{ij} u\|_{L^\infty(B_i)} \leq \bar{C}$. In order to simplify the notation let us fix $i \in \mathbb{N}$ and denote $B := B_i$. For every fixed $\varphi \in C^\infty_c(B)$ let $Q^\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$Q^\varphi(v_1, v_2) := \int_B u \, \frac{\partial^2 \varphi}{\partial v_1 \partial v_2} \, . \tag{6.58}$$

We first claim that

$$|Q^\varphi(v, v)| \leq \bar{C} |v|^2 \|\varphi\|_{L^1(B)}, \quad \forall \varphi \in C^\infty_c(B), \; \forall v \in \mathbb{R}^n, \tag{6.59}$$

where $\bar{C}$ is given is (6.57). In order to prove (6.59), we write (6.57) exchanging x and y and sum up to get

$$|(\nabla u(x) - \nabla u(y)) \cdot (x - y)| \leq \bar{C} \, |x - y|^2.$$

Choosing $y = x + tv$ in the last estimate, we get

$$\frac{|(\nabla u(x + tv) - \nabla u(x)) \cdot v|}{t} \leq \bar{C}, \quad \forall v \in S^{n-1}, \; \forall t \in (0, 1 - |x|). \tag{6.60}$$

Now using that u is C^1 and $\varphi \in C^\infty_c(B)$, we can integrate by parts to get

$$\begin{aligned}
\left| \int_B u \, \frac{\partial^2 \varphi}{\partial v \partial v} \right| &= \left| \int_B \frac{\partial u}{\partial v} \, \frac{\partial \varphi}{\partial v} \right| = \left| \int_B (\nabla u(x) \cdot v) \, \lim_{t \downarrow 0} \frac{\varphi(x + tv) - \varphi(x)}{t} \, dx \right| \\
&= \left| \lim_{t \downarrow 0} \int_B \left(\frac{\nabla u(x - tv) - \nabla u(x)}{t} \cdot v \right) \, \varphi(x) \, dx \right| \\
&\leq \bar{C} \, \|\varphi\|_{L^1(B)}, \quad \forall v \in S^{n-1},
\end{aligned} \tag{6.61}$$

where in the second line we used the change of variable $x \mapsto x + tv$, and the last inequality follows from (6.60). The inequality (6.61) proves our claim (6.59).

We now show that (6.59) implies that the distribution $\partial^2_{ij} u$ is represented by an $L^\infty(B)$ function and $\|\partial^2_{ij} u\|_{L^\infty(B)} \leq \bar{C}$. To this aim observe that for every $\varphi \in C^\infty_c(B)$, by the Schwartz lemma, the map $Q^\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined in (6.58) is a symmetric bilinear form. Using (6.59), by polarization of Q^φ we infer

$$|Q^\varphi(\partial_i, \partial_j)| = \frac{1}{4} |Q^\varphi(\partial_i + \partial_j, \partial_i + \partial_j) - Q^\varphi(\partial_i - \partial_j, \partial_i - \partial_j)| \leq \bar{C} \, \|\varphi\|_{L^1(B)}, \tag{6.62}$$

for every $i, j = 1, \ldots, n$. But now

$$Q^\varphi(\partial_i, \partial_j) = < \partial^2_{ij} u, \varphi >_{\mathcal{D}', \mathcal{D}} \quad ,$$

where $< \cdot, \cdot >_{\mathcal{D}',\mathcal{D}}$ denotes the pairing between distributions and C_c^∞-test functions. Therefore (6.62) combined with Riesz representation Theorem concludes the proof. □

6.4. **Further comments.** We have proven the above regularity of the isoperimetric-isodiametric sets $E \subset M$ under the assumptions that the enclosing ball $B = B_{\mathrm{rad}(E)}(x_0)$ has smooth boundary. Actually, the following is true and is a direct consequence of the argument used above.

(A) If $\partial B \in C^{1,\alpha}$ for some $\alpha \in (0, 1]$, then in a neighbourhood of ∂B the isoperimetric-isodiametric sets have the boundary ∂E which are $C^{1,\alpha}$ regular.

Indeed, under the assumption in (A), the arguments in Lemma 6.3 show that ∂E is $C^{1,\kappa}$ regular in a neighbourhood of ∂B for $k = \min\{\alpha, \frac{1}{2}\}$. Moreover, a careful inspection of the proof of the optimal regularity in Theorem 6.11 shows that the conclusion of (A) holds true with the right Hölder exponent (in the case $\alpha = 1$ the proof is a straightforward generalization; for $\alpha \in (\frac{1}{2}, 1)$ more details need to be checked). Nevertheless, we do not do it here.

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