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Differentiable stratified groupoids and a de Rham theorem for inertia spaces

by

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DIFFERENTIABLE STRATIFIED GROUPOIDS AND A DE RHAM THEOREM FOR INERTIA SPACES

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Abstract. We introduce the notions of a differentiable groupoid and a differentiable stratified groupoid, generalizations of Lie groupoids in which the spaces of objects and arrows have the structures of differentiable spaces, respectively differentiable stratified spaces, compatible with the groupoid structure. After studying basic properties of these groupoids including Morita equivalence, we prove a de Rham theorem for locally contractible differentiable stratified groupoids. We then focus on the study of the inertia groupoid associated to a proper Lie groupoid. We show that the loop and the inertia space of a proper Lie groupoid can be endowed with a natural Whitney B stratification, which we call the orbit Cartan type stratification. Endowed with this stratification, the inertia groupoid of a proper Lie groupoid becomes a locally contractible differentiable stratified groupoid.

Contents

1. Introduction

The theory of Lie group actions on smooth manifolds is fundamental for several areas in mathematics and has a long tradition. But examples of natural Lie group actions do not only comprise actions on smooth manifolds but also on singular spaces such as orbifolds [LT97, Sch01, Ver96] or manifolds with boundary or corners [MROD93, Mel91]. Lie group actions on singular spaces arise also in singular symplectic reduction [SL91] and the transverse cotangent bundle [DCPV13, PV09]. In these cases, the corresponding translation groupoid, while not a Lie groupoid, has significantly more structure than merely that of a topological groupoid. In particular, the spaces of objects and arrows inherit differentiable stratified space structures compatible with the groupoid structure maps. One of the main goals of this paper is to define and study the category of such groupoids and their singular structures.

In this paper, we propose in Definition 2.10 the notion of a *differentiable stratified groupoid*, a class of groupoids within the category of differentiable stratified spaces. That definition is designed so that, under mild hypotheses, the restriction of a differentiable stratified groupoid to a stratum of the orbit space is a Lie groupoid; cf. Proposition 2.19.

One major motivating example is that of a compact Lie group acting differentiably on a differentiable stratified space in such a way that the strata are permuted by the action. We will see in Proposition 3.1 that the corresponding translation groupoid will always satisfy our definition, from which it follows that our definition includes many of the examples described above. Indeed, most examples are *locally translation differentiable stratified groupoids*, a particular subclass of differentiable stratified groupoid which can be locally described as a translation groupoid in a compatible way; see Definitions 2.6 and 2.28. We study the properties of these groupoids and their object spaces, including the appropriate notion of Morita equivalence in these categories, and prove a de Rham theorem that relates the singular cohomology of the orbit space of a locally translation differentiable stratified groupoid fulfilling a local contractibility hypotheses to the cohomology of basic differential forms on the object space; see Theorem 5.9. To prove this theorem, we first localize to groupoid charts and then prove a Poincaré lemma for basic forms in the context of locally translation differentiable stratified groupoids.

A particularly important example which we consider here is that of the *inertia groupoid* of a proper Lie groupoid G. The inertia groupoid is presented as the translation groupoid of G acting on the so-called *loop space* $\Lambda_0\mathsf{G} \subset \mathsf{G}$ which consist of all arrows having the same source and target. When G is an orbifold groupoid, the loop space $\Lambda_0\mathsf{G}$ is a smooth manifold so that the inertia groupoid presents an orbifold, the *inertia orbifold*. The inertia orbifold plays an import role in the geometry and index theory of orbifolds; see [ALR07, PPT07]. However, when G is not an orbifold groupoid, the loop space $\Lambda_0\mathsf{G}$ becomes singular, and very little is known about its singularity structure in general. One of the goals of this paper is to deepen the understanding of this groupoid by exhibiting an explicit stratification. The inertia groupoid represents the *inertia stack* $\Lambda\mathfrak{X}$ of the differentiable stack $\mathfrak{X}$ represented by G, which plays an important role in the string topology of $\mathfrak{X}$; see [BX11]. The inertia stack can be interpreted as the collection of *hidden loops*, elements of the free loop stack $\mathrm{L}\,\mathfrak{X}$ of $\mathfrak{X}$ that vanish on the course moduli space $|\mathfrak{X}|$; see [BGNX12] for more details. Note that when the adjoint action of G on itself is considered, the loop space corresponds to the set of pairs of commuting elements in G, which has been considered in [AC07, PY07, Ric79]. Similarly, one may consider the space

of (conjugacy classes of) commuting m-tuples of elements of G [AG12, GPS12, TGS08], as well as the spaces of conjugacy classes of homomorphisms $\pi \to G$ where π is a finitely generated discrete group [AC07, FS14].

When G is the translation groupoid $G \ltimes M$ associated to the smooth action of a compact Lie group G on a manifold M, the orbit space of the G-action on the corrresponding loop space - here this orbit space is called the *inertia space* - was first considered in [Bry87]. In the previous paper [FPS15], we introduced an explicit Whitney B stratification of the loop space and its associated orbit space. The stratification given there does in general not yield a well-defined global stratification of the loop space of a proper Lie groupoid, though. Specifically, the local stratifications from that paper may not coincide on intersections of charts. Here, we will give a modification of the stratification in [FPS15] that yields a well-defined stratification of the loop and inertia spaces of an arbitrary proper Lie groupoid. We call this stratification the *orbit Cartan type stratification*. For the translation groupoid of a compact group action the orbit Cartan type stratification presented here is in general coarser than the one considered in our previous paper [FPS15]. Because the construction is local, we carry it out on a single groupoid chart, which, up to Morita equivalence, is given by the translation groupoid associated to a finite-dimensional G-representation V where G is a proper Lie groupoid, see Theorem 6.5. For this case, the main ideas of our construction rely on providing a stratification of slices, see Section 6.2 for details. To achieve this, we first decompose the Cartan subgroups of the stabilizers into equivalence classes determined by their fixed sets in V, which motivates the name of the stratification. We then use these decompositions together with the fixed sets of the stabilizers to stratify the slice and take saturations to stratify the loop and orbit spaces. In Section 6.3 we demonstrate that our orbit Cartan type decomposition indeed satisfies all the properties of a stratification and turns the loop and the inertia space into differentiable stratified spaces. Moreover, in Section 6.4, we show that the orbit Cartan type stratification is Whitney B regular. That this stratification is given explicitly allows us finally to verify in Theorem 6.19 that the local contractibility hypotheses of Definition 5.7 is fulfilled for the inertia space, hence the de Rham Theorem 5.9 holds in this case.

Let us mention that the latter result provides the main tool for the proof of Brylinski's claim [Bry87, Prop., p. 25], which is not fully proven in his paper, that the complex of basic differential forms on the loop space of a transformation groupoid $G \ltimes M$ is acyclic. Since this complex of basic differential forms naturally coincides with the Hochschild homology of the convolution algebra on $G \ltimes M$, the E^2-term of the spectral sequence from Hochschild to cyclic homology is given by the cohomology of the sheaf which the sheaf complex of basic differential forms resolves. The results of our paper will be crucial to extend Brylinski's observations to the proper Lie groupoid case. Work on this is in progress.

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2. Fundamentals

In this section, we give the definitions and basic properties of the groupoids under consideration. See Appendices A.1 and A.2 for a review of the notions of differentiable and differentiable stratified spaces used in this paper.

2.1. **Topological groupoids.** Recall that by a groupoid G one understands a small category with object set G_0 and arrow set G_1 such that all arrows are invertible. We use $s, t\colon \mathsf{G}_1 \to \mathsf{G}_0$ to denote the source and target maps, respectively, write $u\colon \mathsf{G}_0 \to \mathsf{G}_1$ for the unit map, $i\colon \mathsf{G}_1 \to \mathsf{G}_1$ for the inverse map, and finally denote by $m\colon \mathsf{G}_1 {}_s\!\times_t \mathsf{G}_1 \to \mathsf{G}_1$ the multiplication or composition map. The maps s, t, u, i, and m are collectively referred to as the *structure maps* of the groupoid. If G and H are groupoids, a *morphism* $f\colon \mathsf{G} \to \mathsf{H}$ is a functor, i.e. a pair of functions $f_0\colon \mathsf{G}_0 \to \mathsf{H}_0$ and $f_1\colon \mathsf{G}_1 \to \mathsf{H}_1$ that commute with each of the structure maps.

The *orbit* through a point $x \in \mathsf{G}_0$ is defined as the set of all $y \in \mathsf{G}_0$ for which there exists a $g \in \mathsf{G}_1$ such that $s(g) = x$ and $t(g) = y$. It is denoted by $\mathsf{G}x$. Obviously, the object space is partitioned into orbits. We denote the set of orbits of a groupoid G by $|\mathsf{G}|$, and the canonical projection from the object to the orbit space by $\pi\colon \mathsf{G}_0 \to |\mathsf{G}|$.

A groupoid G is called a *topological groupoid* if G_0 and G_1 are topological spaces and each of the structure maps are continuous. This implies in particularly that the unit map is a homeomorphism onto its image. If the topological groupoid G is *Hausdorff*, which means that G_1 is Hausdorff, the image $u(\mathsf{G}_0)$ is closed in G_1, cf. [Ren80, Chap. 1 Sec. 2]. The orbit space of a topological groupoid G is always assumed to carry the quotient topology with respect to the canonical projection $\pi\colon \mathsf{G}_0 \to |\mathsf{G}|$. If G and H are topological groupoids and $f\colon \mathsf{G} \to \mathsf{H}$ a morphism of groupoids, then f is a *morphism of topological groupoids*, if in addition f_0 and f_1 are continuous functions.

If the source map of a topological groupoid G is an open map one says that G is an *open topological groupoid*. Note that as i is a homeomorphism and $t = s \circ i$, the target map of an open topological groupoid is open as well. A topological groupoid G for which the source map s is a local homeomorphism is called an *étale groupoid*. G is called a *quasi-proper groupoid* if $s \times t\colon \mathsf{G}_1 \times \mathsf{G}_1 \to \mathsf{G}_0$ is a proper map, and a *proper groupoid* if it is Hausdorff and quasi-proper. Finally, we say that a topological groupoid G is *compact*, respectively *locally compact*, if G_1 as a topological space has the respective property. Note that G_0 is then as well compact, respectively locally compact, by the proof of [Tu04, Prop. 2.5].

Remark 2.1. In this paper we follow the definitions of quasi-compact, compact, and locally compact spaces by Bourbaki [Bou98]. That means, a topological space X is called *quasi-compact* if every open cover of X admits a finite subcover, *compact* if X is both Hausdorff and quasi-compact, and finally *locally compact* if it is Hausdorff and every point $x \in X$ has a compact neighborhood. If each point of a topological space X possesses a Hausdorff (respectively quasi-compact) neighborhood, we say that X is *locally Hausdorff* (respectively *locally quasi-compact*). A continuous map $f\colon X \to Y$ between not necessarily Hausdorff spaces X and Y is called *proper* if $f \times \mathrm{id}_Z\colon X \times Z \to Y \times Z$ is a closed map for every topological space Z, cf. [Bou98, I.10.1. Def. 1] or [Tu04, Sec. 1.3].

By [Bou98, I.10.2. Thm. 1], properness of f is equivalent to the property that f is closed and $f^{-1}(y)$ is quasi-compact for each $y \in Y$.

Proposition 2.2. *Let G be a topological groupoid. Then the following holds true:*

(1) *If G is proper and the object space G_0 is locally compact, then G is a locally compact groupoid.*
(2) *The quotient map $\pi\colon \mathsf{G}_0 \to |\mathsf{G}|$ is an open map of topological spaces if G is an open groupoid.*
(3) *The orbit space of G is Hausdorff if G is open, G_0 is locally compact, and $(s,t)(\mathsf{G}_1)$ is closed in $\mathsf{G}_0 \times \mathsf{G}_0$. In particular, this is the case if G is a proper open groupoid.*
(4) *The orbit space of G is locally compact if G is a locally compact open groupoid and $(s,t)(\mathsf{G}_1)$ is locally closed in $\mathsf{G}_0 \times \mathsf{G}_0$.*

Proof. Let $g \in \mathsf{G}_1$, K be a compact neighborhood of $s(g)$, and L be a compact neighborhood of $t(g)$. Then $(s,t)^{-1}(K \times L)$ is a compact neighborhood of g. Since G_1 is Hausdorff, the first claim is proved. Point (2) is an immediate consequence of [Tu04, Prop. 2.11]. The claims (3) and (4) follow from [Tu04, Prop. 2.12]. □

2.2. **Differentiable groupoids.**

Definition 2.3. Let G be an open topological groupoid. We say that G is a *differentiable groupoid* if G_0 and G_1 are differentiable spaces and the structure maps s, t, i, u, and m are morphisms of differentiable spaces. We say that G is a *reduced differentiable groupoid* if G_0 and G_1 are reduced differentiable spaces. A morphism of topological groupoids $f\colon \mathsf{G} \to \mathsf{H}$ between differentiable groupoids G and H is called a *morphism of differentiable groupoids* if the functions f_0 and f_1 are both morphisms of differentiable spaces.

The requirement that the structure maps $s\colon \mathsf{G}_1 \to \mathsf{G}_0$ and $t\colon \mathsf{G}_1 \to \mathsf{G}_0$ are morphisms of the differentiable spaces G_0 and G_1 implies by [NGSdS03, Thm. 7.6] that the fibred product $\mathsf{G}_1 {}_s\!\times_t \mathsf{G}_1$ inherits the structure of a differentiable space. It is with respect to this structure that we require that $m\colon \mathsf{G}_1 {}_s\!\times_t \mathsf{G}_1 \to \mathsf{G}_1$ is a morphism of differentiable spaces. As the inverse map $i\colon \mathsf{G}_1 \to \mathsf{G}_1$ is clearly invertible with $i^{-1} = i$, it follows that i is an isomorphism of differentiable spaces. Similarly, $u\colon \mathsf{G}_0 \to \mathsf{G}_1$ is an embedding of differentiable spaces.

Remark 2.4. If G is a differentiable groupoid, the underlying topological spaces G_0 and G_1 need not be Hausdorff. However, because both carry the structure of a differentiable space, G_0 and G_1 are locally Hausdorff and locally quasi-compact, see [NGSdS03, Chap. 4].

Example 2.5. (a) Let G be a Lie group and let X be a Hausdorff differentiable space with a differentiable G-action, cf. [NGSdS03, Sec. 11.2]. Then $G \ltimes X$ is a differentiable groupoid with space of objects X and space of arrows $G \times X$, the latter being a differentiable space by [NGSdS03, Thm. 7.6]. The source map $s\colon G \times X \to X$ is the projection, hence smooth and open, and the target map $t\colon G \times X \to X$ is given by $(g,x) \mapsto gx$, which is differentiable by the definition of a differentiable action. The unit map $u\colon x \mapsto (e,x)$ is easily seen to be a closed embedding since $\{e\}$ is closed in G. The domain of the product map m is a differentiable subspace of $G \times X \times G \times X$ by [NGSdS03, Thm. 7.6] and the proof of [NGSdS03, Rem. 7.5]. Moreover, the map $G \times G \times X \to (G \times X) {}_s\!\times_t (G \times X)$, $(g,h,x) \mapsto \big((g,hx),(h,x)\big)$ is an isomorphism of

differentiable spaces onto the domain of m. Since m pulled back by this isomorphism is the smooth map $G \times G \times X \to G \times X$, $(g, h, x) \mapsto (gh, x)$, multiplication on $G \ltimes X$ is smooth. Similarly one shows that the inverse map $i\colon G \times X \to G \times X$, $(g, x) \mapsto (g^{-1}, gx)$ is smooth.

By definition, the quotient space X/G of the G-space X coincides with the orbit space of the groupoid $|G \ltimes X|$. Hence, if X is Hausdorff, the orbit space becomes a Hausdorff space by Prop. 2.2 and even is a differentiable space by [NGSdS03, Thm. 11.17] when equipped with the sheaf of G-invariant functions on X as the structure sheaf.

(b) In general, the orbit space $|\mathsf{G}|$ of a differentiable groupoid G need not admit a differentiable structure with respect to which the quotient map $\mathsf{G}_0 \to |\mathsf{G}|$ is a smooth map. As a well-known example, consider the action of $\mathbb{Z}$ on the circle S^1 by an irrational rotation. Then the translation groupoid $\mathbb{Z} \ltimes S^1$ is a (non-quasi-proper) differentiable groupoid whose orbit space $|\mathbb{Z} \ltimes S^1|$ is not locally Hausdorff and hence can not carry the structure of a differentiable space.

Let G be a reduced differentiable groupoid and Y a differentiable subspace of G_0. A *bisection of* G *over* Y then is a smooth map $\sigma\colon Y \to \mathsf{G}_1$ such that $s \circ \sigma = \mathrm{id}_Y$, and such that $t \circ \sigma$ is an isomorphism of Y onto the differentiable subspace $t \circ \sigma(Y)$ of G_0, cf. [Mac05, Def. 1.4.8]. Note that we will consider bisections over sets Y which may not be open in G_0.

In the case that G is proper, the question of whether $|\mathsf{G}|$ admits a differentiable structure remains open in general. However, from Example 2.5 (a), it is clear that the quotient map of the action groupoid of a differentiable action on a Hausdorff differentiable space is a morphism of differentiable spaces. Many of the examples we consider will be of this form, at least locally, which motivates the following.

Definition 2.6. We say that a reduced differentiable groupoid G is *locally translation* if the following conditions are satisfied for every $x \in \mathsf{G}_0$.

(LT0) The isotropy group G_x becomes a Lie group with the induced topological and differentiable structures.

(LT1) There is an open Hausdorff neighborhood U_x of x in G_0 and a relatively closed connected reduced differentiable subspace $Y_x \subset U_x$ containing x together with a smooth G_x-action on Y_x such that x is a fixed point of the G_x-action and such that the restriction $\mathsf{G}_{|U_x}$ is isomorphic as a differentiable groupoid to the product of the translation groupoid $G_x \ltimes Y_x$ and the pair groupoid $O_x \times O_x$, where O_x is an open neighborhood of x in its orbit.

(LT2) For each $z \in Y_x$, we may choose U_z and Y_z with $U_z \subset U_x$ and $Y_z \subset Y_x$.

(LT3) For each arrow $g \in \mathsf{G}$ with $s(g) = x$ there exist open neigborhoods U_x of x and U_y of $y := t(g)$ as in (LT1) and a bisection σ over U_x such that $\sigma(x) = g$ and such that the morphism of differentiable groupoids $f\colon \mathsf{G}_{|U_x} \to \mathsf{G}_{|U_y}$ with components $f_0 := t \circ \sigma\colon U_x \to U_y$ and $f_1\colon \mathsf{G}_{|U_x,1} \to \mathsf{G}_{|U_y,1}$, $h \mapsto \sigma(t(h))h\big(\sigma(s(h))\big)^{-1}$ is an isomorphism.

A neighborhood U_x as in (LT1) is called a *trivializing neighborhood* of x, a differentiable subspace Y_x as in (LT1) a G-*slice* or *groupoid-slice* of x.

Remark 2.7. By the following observation one can assume, possibly after shrinking, that a G-slice Y_x of x possesses a G_x-equivariant singular chart $\iota\colon Y_x \hookrightarrow T_xY_x \cong \mathbb{R}^{\operatorname{rk} x}$ with $\iota(x) = 0$; see Appendix A.1 for the definition of a singular chart.

Proposition 2.8. *Let G be a compact Lie group, Y a differentiable space carrying a differentiable G-action, and $x \in Y$ a fixed point. Then the Zariski tangent space T_xY inherits a natural G-action from the G-action on Y. Moreover, there exists an open G-invariant neighborhood W of x in Y and a G-equivariant singular chart $\iota : W \hookrightarrow T_xY$ mapping x to the origin.*

Proof. The proof is literally identical to the proof of [NW14, Lem. 5.2.6] when replacing "holomorphic" with "smooth". □

Proposition 2.9. *Let G be a locally translation differentiable groupoid. Then the orbit space $|\mathsf{G}|$ inherits the structure of a differentiable space with respect to which the quotient map $\pi\colon \mathsf{G}_0 \to |\mathsf{G}|$ is a smooth map. Specifically, the structure sheaf of $|\mathsf{G}|$ is given by the differentiable functions on G_0 that are constant on orbits.*

Proof. We define the structure sheaf of $|\mathsf{G}|$ to be the sheaf of continuous functions on $|\mathsf{G}|$ which pull back under the projection π to G-invariant smooth functions on G_0. For each orbit $\mathsf{G}x \in |\mathsf{G}|$, we may use condition (LT1) in Definition 2.6 to identify a neighborhood of $\mathsf{G}x$ with $|G_x \ltimes Y_x|$, a Hausdorff differentiable space as explained in Example 2.5 (a). That these local identifications are well defined and isomorphisms of Hausdorff differentiable spaces are consequences of (LT2) and (LT3). □

2.3. **Differentiable stratified groupoids.** Recall that by a *stratified submersion* (respectively *stratified immersion*), one understands a morphism $f\colon X \to Y$ of reduced differentiable stratified spaces such that the restriction of f to a connected component of a stratum of the maximal decomposition of X is a submersion (respectively immersion), cf. [Pfl01, 1.2.10]. A *stratified surjective submersion* is a stratified submersion that maps connected components of strata onto connected components of strata, and a *stratified embedding* is a stratied immersion that is injective on connected components of strata.

Definition 2.10. A *differentiable stratified groupoid* is a reduced differentiable groupoid G such that the following properties hold true.

(DSG1) G_0 and G_1 are differentiable stratified spaces with respective stratifications $\mathcal{S}^0$ and $\mathcal{S}^1$.
(DSG2) The structure maps s, t, i, u, and m are stratified mappings.
(DSG3) The maps s and t are stratified surjective submersions, and u is a stratified embedding.
(DSG4) For every $x \in \mathsf{G}_0$ and arrow $g \in s^{-1}(x)$ the germ $[s^{-1}(\mathcal{S}^0_x)]_g$ is a subgerm of $\mathcal{S}^1_g$.
(DSG5) Let $x \in \mathsf{G}_0$, $g \in \mathsf{G}_1$ with $t(g) = x$, and U be an open connected neighborhood of x within the stratum of G_0 containing x. Assume that $\sigma\colon U \to \mathsf{G}_1$ is a bisection of G. Then the map $L_\sigma\colon t^{-1}(U) \to \mathsf{G}_1$ defined by $h \mapsto \sigma(t(h))h$ satisfies $L_\sigma(\mathcal{S}^1_g) = \mathcal{S}^1_{L_\sigma(g)}$.

If G and H are differentiable stratified groupoids, a *morphism of differentiable stratified groupoids* is a morphism of differentiable groupoids $f\colon \mathsf{G} \to \mathsf{H}$ such that f_0 and f_1 are in addition stratified mappings.

A differentiable stratified groupoid G is called a *Lie groupoid* if in addition the following axiom holds true.

(DSG6) The stratifications $\mathcal{S}^0$ and $\mathcal{S}^1$ are induced by G_0 and G_1, respectively, which in other words means that both G_0 and G_1 are smooth manifolds and their stratifications have only one stratum.

We say that G is a *differentiable Whitney stratified groupoid* if the stratifications of G_0 and G_1, as well as the induced stratification of $\mathsf{G}_1 {}_s\!\times_t \mathsf{G}_1$, are Whitney (B)-regular. Similarly, G is *topologically locally trivial* if the stratifications of G_0, G_1, and the induced stratification of $\mathsf{G}_1 {}_s\!\times_t \mathsf{G}_1$ are topologically locally trivial (cf. [Pfl01, Sec. 1.4]).

Remark 2.11. (a) In the definition, $[s^{-1}(\mathcal{S}^0_x)]_g$ of course means the germ of $s^{-1}(S)$ at g, where S is a set defining the germ $\mathcal{S}^0_x$ at x. Condition (DSG4) is a kind of equivariance condition for the stratification. It entails that every set germ of a source fiber is contained in a stratum of the arrow space, so that the stratification of the arrow space is not unnecessarily fine. This will be used to show that G-orbits are locally contained in strata; see Lemma 2.14 below. Likewise, (DSG5) requires that bisections defined on open subsets of strata act on G_1 in a way compatibe with its stratification. Note that a bisection $\sigma\colon S \to \mathsf{G}_1$ as in (DSG5) has image in the stratum of G_1 through $g = \sigma(x)$ by (DSG2). The existence of such bisections σ is demonstrated by Lemma 2.15 and Corollary 2.16 below.

(b) Though G_0 and G_1 are locally Hausdorff spaces, we do not require that they are Hausdorff. See Appendix A for stratifications of locally Hausdorff spaces.

(c) One readily checks that Definition 2.10 reduces to the standard definition of a Lie groupoid if (DSG6) is fulfilled. Observe that in the case of a Lie groupoid, conditions (DSG4) and (DSG5) become trivial.

(d) In order for the requirement that $m\colon \mathsf{G}_1 {}_s\!\times_t \mathsf{G}_1 \to \mathsf{G}_1$ is a stratified mapping to make sense, it must be that $\mathsf{G}_1 {}_s\!\times_t \mathsf{G}_1$ is a stratified space. We will always let $\mathsf{G}_1 {}_s\!\times_t \mathsf{G}_1$ carry the stratification induced by the stratification of G_1, the existance of which is guaranteed by Lemma A.4.

Our definition of a differentiable Whitney stratified groupoid is stronger than the one of a stratified Lie groupoid given in [FOR09, Def. 4.16] in that we require conditions (DSG4) and (DSG5). The following examples illustrate the kinds of behavior we preclude by requiring these conditions.

Example 2.12. (a) Consider the translation groupoid $\mathsf{G} = \mathsf{S}^1 \ltimes \mathsf{S}^1$ where the action is by left-translation. We decompose $\mathsf{G}_0 = \mathsf{S}^1$ into pieces $\{1\}$ and $\mathsf{S}^1 \smallsetminus \{1\}$, and $\mathsf{G}_1 = \mathsf{S}^1 \times \mathsf{S}^1$ into pieces $\{(1,1)\}$, $\{(a,1) \mid a \neq 1\}$, $\{(a,a^{-1}) \mid a \neq 1\}$, and $\{(a,b) \mid b \neq 1,\ a \neq b^{-1}\}$. Then it is immediate to check that (DSG1), (DSG2), and (DSG3) are satisfied. However, condition (DSG4) fails, as the germ of $s^{-1}(1) = \{(a,1) \mid a \in \mathsf{S}^1\}$ is not contained in the stratum $\{(1,1)\}$. Note that G_0 consists of a single connected orbit that is given an "artificial" stratification that is too fine. Of course, G is a Lie groupoid when given the trivial stratifications of G_0 and G_1.

(b) Let G be a compact Lie group of positive dimension and let $\mathsf{G} = G \ltimes \{p\}$ be the translation groupoid associated to the trivial action of G on a one-point space. Define a stratification of $\mathsf{G}_1 = G \times \{p\}$ by the decomposition into $\{(1,p)\}$ and $\{(g,p) \mid g \neq 1\}$. Obviously, the source and target maps are stratified submersions, the unit map is a stratified embedding, and the inverse map is a stratified mapping. Though the multiplication map $\mathsf{G}_1 {}_s\!\times_t \mathsf{G}_1 = \mathsf{G}_1 \times \mathsf{G}_1 \to \mathsf{G}_1$ given by $((g,p),(h,p)) \mapsto (gh,p)$ is not a stratified mapping with respect to the induced stratification, the space $\mathsf{G}_1 {}_s\!\times_t \mathsf{G}_1$ does admit a stratification with respect to which the multiplication map m

is stratified. Specifically, we may decompose $\mathsf{G}_1 {}_s\!\times_t \mathsf{G}_1$ into the pieces

$$\{((g^{-1},p),(g,p)) \mid g \in G\} \quad \text{and} \quad \{((g,p),(h,p)) \mid g \neq h^{-1}\},$$

and then m becomes stratified. In this case, while G_0 consists of a single stratum, the stratification of G_1 is "too fine" so that (DSG4) again fails; the preimage $s^{-1}(p) = \mathsf{G}_1$ is not contained in the stratum $\{(1,p)\}$ through $(1,p)$.

(c) Let G be the pair groupoid on the topological disjoint union $\mathbb{R} \sqcup \{p\}$. Give G_0 and G_1 the stratifications by connected components and natural differentiable structures. One checks that (DSG1), (DSG2), (DSG3), and (DSG4) are satisfied. However, let $\sigma\colon \{p\} \to \mathsf{G}_1$ be the bisection with the single value $\sigma(p) = (0,p)$. For any arrow of the form (p,y) one has $\sigma(t(p,y))(p,y) = (0,y)$. Hence the mapping $h \mapsto \sigma(t(h))h$ maps the stratum $\{(p,y) \mid y \in \mathbb{R}\}$ of G_1 into the stratum $\mathbb{R}^2$ as the y-axis. Thus (DSG5) fails.

We next collect some useful consequences of Definition 2.10. We hereby assume for the remainder of this section that G denotes a differentiable stratified groupoid.

Lemma 2.13. *Let $g \in \mathsf{G}_1$ with $s(g) = x$ and $t(g) = y$. Then one has*

$$\mathcal{S}^1_g = [s^{-1}(\mathcal{S}^0_x)]_g, \text{ i.e. } \mathcal{S}^1 \text{ is the pullback of } \mathcal{S}^0 \text{ via } s, \text{ [Mat73, (2.3)]}, \tag{2.1}$$

$$\mathcal{S}^0_x = [s(\mathcal{S}^1_g)]_x, \tag{2.2}$$

$$\mathcal{S}^1_g = [t^{-1}(\mathcal{S}^0_y)]_g, \tag{2.3}$$

$$\mathcal{S}^0_y = [t(\mathcal{S}^1_g)]_y, \text{ and}, \tag{2.4}$$

$$\mathcal{S}^0_y = [t(s^{-1}(\mathcal{S}^0_x))]_y. \tag{2.5}$$

Proof. Let R_g be the connected component of the stratum of G_1 containing g, and let S_x and S_y be the connected components of the strata of G_0 containing x and y, respectively. Then $s_{|R_g}$ and $t_{|R_g}$ are, respectively, surjective submersions onto S_x and S_y. By (DSG4), there exists a relatively open neighborhood U_x of x in S_x and an open neighborhood V_g of g in G_1 such that $V_g \cap s^{-1}(U_x) \subset R_g$. As $s_{|R_g}$ is a smooth map onto S_x, $s^{-1}(U_x)$ is a relatively open neighborhood of g in R_g, proving (2.1). Since $s_{|R_g}$ is a submersion, hence an open map, $s(V_g \cap s^{-1}(U_x))$ is an open neighborhood of x in S_x, which gives (2.2). Then (2.3) and (2.4) follow from the fact that $t = s \circ i$ and that i is a stratified mapping with $i^2 = \mathrm{id}_{\mathsf{G}_1}$. Finally, (2.5) is a consequence of (2.1) and (2.4). □

Lemma 2.14. *Let G be a differentiable stratified groupoid and let $x \in \mathsf{G}_0$ be a point. Then the connected component of x in the orbit $\mathsf{G}x$ through x is contained in the stratum of G_0 containing x.*

Proof. Let S_x be the connected component of the stratum of G_0 containing x. Suppose for contradiction that the germ of $\mathsf{G}x$ at x is not a subgerm of $\mathcal{S}^1_x$. Then one may construct a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $\mathsf{G}x$ such that $\lim_{n\to\infty} x_n = x$, yet each x_n is not contained in S_x. Since the set $K = \{x_n\}_{n\in\mathbb{N}} \cup \{x\}$ is compact and G is proper, the preimage $(s,t)^{-1}(K)$ is a quasi-compact subset of $\mathsf{G}_1 \times \mathsf{G}_1$. Hence projecting onto the first factor yields a quasi-compact subset $C := \mathrm{pr}_1((s,t)^{-1}(S))$ of G_1. Note that C is the set of elements of G_1 with source in K.

For each n, choose an arrow $g_n \in \mathsf{G}_1$ with source x and target x_n. As each g_n is in the quasi-compact set C, there is a subsequence $(g_{n_k})_{k\in\mathbb{N}}$ with limit $g \in \mathsf{G}_1$. However, (DSG4) entails the existence of an open neighborhood U_x of x in S_x and a neighborhood

V_g of g in G_1 such that $V_g \cap s^{-1}(U_x)$ is contained in the connected component R_g of the stratum of G_1 containing g. Infinitely many of the g_{n_k} must be contained in V_g and each g_{n_k} is an element of $s^{-1}(U_x)$ by construction, so infinitely many of the g_{n_k} are contained in R_g. But since t is a stratified mapping, infinitely many of the x_{n_k} are contained in S_x, which is a contradiction. It follows that the germ of $\mathsf{G}x$ at x is a subgerm of $\mathcal{S}_x$, hence that the connected component of $\mathsf{G}x$ containing x is a subset of S_x. □

The following property is important in realizing the consequences of (DSG5) and is proven as in the case of a Lie groupoid; cf. [Mac05, Prop. 1.4.9].

Lemma 2.15. *Let x, y be two points of the differentiable stratified groupoid G and g an arrow with $s(g) = x$ and $t(g) = y$. Denote by S_x and S_y the connected components of the strata of G_0 containing x and y, respectively. If $\dim S_x \leq \dim S_y$, then there exists a relatively open neighborhood U_x of x in S_x and a bisection σ of G on U_x such that σ is a stratified mapping and $\sigma(x) = g$.*

Of course, since σ is only defined on a subset of the stratum through x, σ being a stratified mapping means that its image is contained in the stratum containing g. The hypothesis that $\dim S_x \leq \dim S_y$ will be seen to be unnecessary below.

Proof. Let R_g be the connected component of the stratum of G containing g. Then $s_{|R_g}$ and $t_{|R_g}$ are surjective submersions onto S_x and S_y, respectively. Moreover, there are relatively open neighborhoods of g in $s^{-1}(x)$ and $t^{-1}(y)$ contained in R_g by (DSG4) and Lemma 2.13 (2.3). Then there are subspaces $E \subset F$ of the tangent space T_gR_g such that $T_gR_g = T_g(s^{-1}(x)) \times E = T_g(t^{-1}(y)) \times F$. Choose a local section $\sigma\colon U_x \to R_g$ such that $\sigma(x) = g$ and such that the image of $T_x\sigma$ is E. Then $T_x(t \circ \sigma)$ is injective so that we can shrink U_x in a way that $t \circ \sigma$ is a diffeomorphism onto its image. □

As an important consequence, we may now conclude that the strata of G_0 that meet the orbit $\mathsf{G}x$ of a point $x \in \mathsf{G}_0$ must all have the same dimension. The proof follows [Mac05, Corollaries 1.4.11 & 1.4.12].

Corollary 2.16. *Let G be a differentiable stratified groupoid, and let S be a connected component of a stratum of G_0. Then the following holds true.*

(1) *Each stratum of G_1 contained in $s^{-1}(S)$ has the same dimension.*
(2) *The rank of t on $s^{-1}(S)$ is constant.*
(3) *Each connected component S' of a stratum of G_0 such that $s^{-1}(S) \cap t^{-1}(S') \neq \emptyset$ has the same dimension as S.*

Proof. Let S_1 and S_2 be (not necessarily distinct) connected components of strata of G_0 such that $s^{-1}(S_1) \cap t^{-1}(S_2) \neq \emptyset$. We assume $\dim S_1 \leq \dim S_2$. Otherwise, we may switch roles and apply the inverse map, so this hypothesis introduces no loss of generality. As s and t are stratified surjective submersions, $s^{-1}(S_1)$ and $t^{-1}(S_2)$ are both unions of connected components of strata of G_1. Hence their intersection is a union of connected components of strata. Let $g \in \mathsf{G}_1$ with $s(g) = x \in S_1$ and $t(g) = y \in S_2$. Then there exists by Lemma 2.15 a bisection σ of G on a relatively open neighborhood U_x of x in S_1 such that $\sigma(x) = g$. Let $R_{u(x)}$ and R_g denote the connected components of the strata of G_1 containing $u(x)$ and g, respectively. By (DSG5), there is a relatively open neighborhood $V_{u(x)}$ of $u(x)$ in $R_{u(x)}$ such that $L_\sigma(V_{u(x)})$ is a relatively open neighborhood of g in R_g. The requirement that $t \circ \sigma$ is injective implies that L_σ is injective, hence that $R_{u(x)}$ and R_g have the same dimension. Since S_2 and g were arbitrary, (1) follows.

Moreover, from the definition of L_σ, we have $t_{|L_\sigma(V_{u(x)})} \circ L_\sigma = (t \circ \sigma) \circ t_{|V_{u(x)}}$. Then, as $L_\sigma(V_{u(x)})$ is an open neighborhood of g in R_g and $t \circ \sigma$ is a diffeomorphism onto its image, the ranks of $t_{|R_{u(x)}}$ at $u(x)$ and $t_{|R_g}$ at g coincide, yielding (2). Since t is a stratified surjective submersion, (3) is immediate. □

Remark 2.17. In particular, the hypothesis $\dim S_x \leq \dim S_y$ in Lemma 2.15 is now seen to be superfluous. Indeed, given the other hypotheses, we always have $\dim S_x = \dim S_y$.

Example 2.18. Note that if S_1 and S_2 are connected components of strata of G_0, even connected components of the same stratum, it need not be the case that the strata of $s^{-1}(S_1)$ and $s^{-1}(S_2)$ have the same dimension. As an example, let G and H be compact Lie groups, and let G be the disjoint union of $G \ltimes \{p\}$ and $H \ltimes \{q\}$. Then G_0 is the discrete set $\{p, q\}$. The maximal stratification of G_0 contains a single stratum with two one-point connected components, yet the space of arrows of $\mathsf{G}_{|\{p\}}$ and $\mathsf{G}_{|\{q\}}$ have dimensions $\dim G$ and $\dim H$, respectively, which clearly need not coincide.

Proposition 2.19. *Assume that G_0 and G_1 are topologically locally trivial, and let $S \subset \mathsf{G}_0$ be a connected component of a stratum of G_0. Let $\mathcal{P}$ be a collection of connected components of strata of G_0 such that $S \in \mathcal{P}$, and such that for each $S' \in \mathcal{P}$ the relation $s^{-1}(S) \cap t^{-1}(S') \neq \emptyset$ is satisfied. Letting $P = \bigcup_{S' \in \mathcal{P}} S'$, the restriction $\mathsf{G}_{|P}$ then is a Lie groupoid.*

Proof. By Corollary 2.16, each element of $\mathcal{P}$ has the same dimension as S, hence each connected component of a stratum of G_1 contained in $s^{-1}(P)$ has the same dimension as well. The hypothesis that G_0 is a topologically locally trivial implies that connected components of strata of the same dimension are separated from one another's closures so that P is a manifold. The same holds for G_1 so that $s^{-1}(P) \cap t^{-1}(P)$ is a union of strata of the same dimension and therefore a manifold. Hence P and $s^{-1}(P) \cap t^{-1}(P)$ are manifolds, which can both be given their trivial stratifications. The claim follows. □

Of course, the hypothesis that G_0 and G_1 are topologically locally trivial is only required so that the strata of G_0 and G_1 of the same dimension are not contained in one another's closures. Any other hypothesis that ensures this is as well sufficient.

We also note the following, which will be useful in the sequel.

Corollary 2.20. *Assume that G is proper and let $x \in \mathsf{G}_0$. Then each connected component of the orbit $\mathsf{G}x$ is a smooth submanifold of G_0. If G_0 and G_1 are in addition topologically locally trivial, then $\mathsf{G}x$ is a smooth submanifold of G_0.*

Proof. Let S_x be the connected component of the stratum containing x. Then the connected component of $\mathsf{G}x$ containing x is contained in S_x by Lemma 2.14. The restricted groupoid $\mathsf{G}_{|S_x}$ is a Lie groupoid by Proposition 2.19. Since $\mathsf{G}x \cap S_x = (\mathsf{G}_{|S_x})x$ this implies that the connected components of $\mathsf{G}x$ contained in S_x are smooth submanifolds of S_x, hence of G_0. If G_0 and G_1 are topologically locally trivial, then the restriction of G to the saturation of S_x is as well a Lie groupoid, and the same argument applies. □

Finally, we include the following example to demonstrate that the hypothesis that G_0 and G_1 are topologically locally trivial (or a similar requirement) is necessary in Proposition 2.19 and Corollary 2.20, cf. [Pfl01, 1.1.12].

Example 2.21. Let $\mathsf{G}_0 = S_1 \cup S_2 \subset \mathbb{R}^2$ where $S_1 = \{0\} \times (-1, 1)$ and $S_2 = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y = \sin 1/x\}$, and let G be the pair groupoid on G_0. Then S_1 and S_2, both of

dimension 1, are the pieces of a decomposition of G_0 with $S_1 \subset \overline{S_2}$. The decomposition of G_1 is into pieces of the form $S_i \times S_j$, each of dimension 2. Then G is a differentiable stratified groupoid. However, the single orbit G_0 is not locally connected and hence not a manifold, and G is not a Lie groupoid. Clearly, however, the restriction of G to either stratum of G_0 is a Lie groupoid.

2.4. **Morita equivalence.** Let G be a topological groupoid, Y a topological space, and $f\colon Y \to \mathsf{G}_0$ a continuous function. Following [Tu04], we denote by $\mathsf{G}[Y]$ the groupoid with object space $\mathsf{G}[Y]_0 := Y$ and arrow space

$$\mathsf{G}[Y]_1 := (Y \times Y)\,{}_{(f,f)}\!\times_{(t,s)} \mathsf{G}_1 = \{(y,z,g) \in Y \times Y \times \mathsf{G}_1 \mid t(g) = f(y),\ s(g) = f(z)\},$$

and structure maps given as follows. The source of $(y,z,g) \in \mathsf{G}[Y]_1$ is z, its target is y. Multiplication maps $(y,w,g),(w,z,h) \in \mathsf{G}[Y]_1$ with $s(g) = t(h)$ to

$$(y,w,g)\cdot(w,z,h) := (y,z,gh).$$

The unit map is $\mathsf{G}[Y]_0 \to \mathsf{G}[Y]_1$, $y \mapsto (y,y,u(f(x)))$, and the inverse map $\mathsf{G}[Y]_1 \to \mathsf{G}[Y]_1$, $(y,z,g) \mapsto (z,y,g^{-1})$. Then $\mathsf{G}[Y]$ is a subgroupoid of $Y \times Y \times \mathsf{G}$ where $Y \times Y$ denotes the pair groupoid and where we identify $\mathsf{G}[Y]_0 = Y$ with $Y\,{}_f\!\times_{\mathrm{id}} \mathsf{G}_0$. By [Tu04, Prop. 2.7], $\mathsf{G}[Y]$ is a locally closed subgroupoid of $Y \times Y \times \mathsf{G}$, if G_0 is locally Hausdorff. Moreover, $\mathsf{G}[Y]$ is locally compact, if G and T are locally compact. Finally, by [Tu04, Prop. 2.22], $\mathsf{G}[Y]$ is proper, if G is proper.

Now suppose that G is a differentiable groupoid, Y a differentiable space, and f a differentiable map. Then it is straightforward to see that $\mathsf{G}[Y]$ is a differentiable subgroupoid of $Y \times Y \times \mathsf{G}$. In particular, both $\mathsf{G}[Y]_0$ and $\mathsf{G}[Y]_1$ are defined as fibered products. Similarly, if G and H are differentiable stratified groupoids, Y is a differentiable stratified space, and f a differentiable stratified surjective submersion, then $\mathsf{G}[Y]$ is a differentiable stratified groupoid as well, where $\mathsf{G}[Y]_0$ and $\mathsf{G}[Y]_1$ are given the induced stratifications.

Definition 2.22. Two open topological groupoids G and H are called *Morita equivalent as topological groupoids*, if there exists a topological space Y together with open surjective continuous functions $f\colon Y \to \mathsf{G}_0$ and $g\colon Y \to \mathsf{H}_0$ such that $\mathsf{G}[Y]$ and $\mathsf{H}[Y]$ are isomorphic as topological groupoids. If G and H are differentiable groupoids, Y is a differentiable space, and f and g are differentiable maps, then G and H are called *Morita equivalent as differentiable groupoids*, if $\mathsf{G}[Y]$ and $\mathsf{H}[Y]$ are isomorphic as differentiable groupoids. Similarly, if G and H are differentiable stratified groupoids, Y is a differentiable stratified space, and f and g are differentiable stratified surjective submersions, then G and H are called *Morita equivalent as differentiable stratified groupoids*, if $\mathsf{G}[Y]$ and $\mathsf{H}[Y]$ are isomorphic as differentiable stratified groupoids.

One verifies immediately that Morita equivalence is transitive. Specifically, if the maps $\mathsf{G}_0 \xleftarrow{f_1} Y \xrightarrow{g_1} \mathsf{H}_0$ realize a Morita equivalence between (topological, differentiable, or differentiable stratified) groupoids G and H, and $\mathsf{H}_0 \xleftarrow{f_2} Y' \xrightarrow{g_2} \mathsf{K}_0$ is a Morita equivalence between H and K, then $\mathsf{G}_0 \xleftarrow{f_1\circ pr_1} Y\,{}_{g_1}\!\times_{f_2} Y' \xrightarrow{g_2\circ pr_2} \mathsf{K}_0$ induces a Morita equivalence between G and K.

Proposition 2.23. *If G and H are Morita equivalent open topological groupoids, then the orbit spaces $|\mathsf{G}|$ and $|\mathsf{H}|$ are homeomorphic. Moreover, if G and H are Morita equivalent differentiable groupoids such that $|\mathsf{G}|$ and $|\mathsf{H}|$ admit the structure of differentiable spaces*

and the quotient maps are differentiable, then $|\mathsf{G}|$ *and* $|\mathsf{H}|$ *are isomorphic as differentiable spaces.*

Proof. Because the orbit spaces $|\mathsf{G}[Y]|$ and $|\mathsf{H}[Y]|$ are clearly homeomorphic, it is sufficient to show that $|\mathsf{G}|$ is homeomorphic to $|\mathsf{G}[Y]|$. To see this, define the map $\mathsf{G}[Y]_0 \to \mathsf{G}_0$ by $(y, x) \mapsto x$. Given an arrow $g \in \mathsf{G}_1$ from x to x', there exists, by the surjectivity of f, a y' such that $f(y') = x'$ and hence an arrow (y, y', g) from (y, x) to (y', x'). Conversely, if (y, y', g) is an arrow from (y, x) to (y', x'), then g is by definition an arrow from x to x'. Hence $(y, x) \mapsto x$ maps orbits to orbits. In the differentiable case, this map is obviously differentiable, so that if the quotient map $\mathsf{G}_0 \to |\mathsf{G}|$ is differentiable, then its composition with $(y, x) \mapsto x$ is also differentiable. □

For Lie groupoids, the notion of Morita equivalence is often defined in terms of morphisms called *weak equivalences*. We introduce a similar notion as follows.

Definition 2.24. A morphism $f\colon \mathsf{G} \to \mathsf{H}$ of differentiable stratified groupoids is called a *weak equivalence* if it is essentially surjective and fully faithful, i.e. if the following two conditions are satisfied.

(ES) The map $t \circ \mathrm{pr}_1\colon \mathsf{H}_1 {}_s\!\times_{f_0} \mathsf{G}_0 \to \mathsf{H}_0$ is a stratified surjective submersion.
(FF) The arrow space G_1 is a fibered product via the diagram

$$\begin{tikzcd} \mathsf{G}_1 \arrow[r, "f_1"] \arrow[d, "(s,t)"'] & \mathsf{H}_1 \arrow[d, "(s,t)"] \\ \mathsf{G}_0 \times \mathsf{G}_0 \arrow[r, "(f_0,f_0)"] & \mathsf{H}_0 \times \mathsf{H}_0. \end{tikzcd}$$

Remark 2.25. There is an analogous notion of a weak equivalence between differentiable groupoids, where axiom (ES) is replaced with the requirement that $t \circ \mathrm{pr}_1$ is an open surjective map that admits local sections.

Note that if the object and arrow spaces of G and H are topologically locally trivial, then the restrictions of G and H to connected components of strata are Lie groupoids by Proposition 2.19. One immediately checks that the restriction of a weak equivalence to strata yields a weak equivalence of Lie groupoids. We have the following.

Proposition 2.26. *Let* G *and* H *be differentiable stratified groupoids. The following are equivalent.*

(1) G *and* H *are Morita equivalent as differentiable stratified groupoids.*
(2) *There is a differentiable stratified groupoid* K *together with weak equivalences* $h\colon \mathsf{K} \to \mathsf{G}$ *and* $k\colon \mathsf{K} \to \mathsf{H}$.

Proof. Assume that (1) holds true. Then there exists a differentiable stratified space Y together with open surjective differentiable stratified submersions $f\colon Y \to \mathsf{G}_0$ and $g\colon Y \to \mathsf{H}_0$ such that $\mathsf{G}[Y]$ and $\mathsf{H}[Y]$ are isomorphic. Define the groupoid K by setting $\mathsf{K}_0 = Y$ and $\mathsf{K}_1 = \mathsf{G}_1 {}_{(f\circ s_{\mathsf{G}}, f\circ t_{\mathsf{G}})}\!\times_{(g\circ s_{\mathsf{H}}, g\circ t_{\mathsf{H}})} \mathsf{H}_1$, and put $s_{\mathsf{K}} := f \circ s_{\mathsf{G}} = g \circ s_{\mathsf{H}}$ and $t_{\mathsf{K}} := f \circ t_{\mathsf{G}} = g \circ t_{\mathsf{H}}$. The unit, inverse, and multiplication maps are defined componentwise. Then it is straightforward to see that K is a differentiable stratified groupoid, where K_1 is given the induced stratification. We define a morphism $h\colon \mathsf{K} \to \mathsf{G}$ by setting $h_0 = f$ and $h_1 = \mathrm{pr}_1$. Then $h_0 = f$ is in fact surjective by hypothesis so that $t_{\mathsf{G}} \circ \mathrm{pr}_1\colon \mathsf{G}_1 {}_s\!\times_{h_0} \mathsf{K}_0 \to \mathsf{G}_0$ is a composition of stratified surjective submersions, and K_1

is a fibered product by construction, so h is a weak equivalence. The weak equivalence $k\colon \mathsf{K} \to \mathsf{H}$ is defined identically.

Now assume that we have a weak equivalence $h\colon \mathsf{K} \to \mathsf{G}$. Let $Y = \mathsf{G}_1 {}_{s_\mathsf{G}}\!\times_{h_0} \mathsf{K}_0$ with the induced stratification, and let $f = \mathrm{pr}_2\colon Y \to \mathsf{K}_0$ and $g = s_\mathsf{G} \circ \mathrm{pr}_1\colon Y \to \mathsf{G}_0$. Then f is obviously an open surjective differentiable stratified submersion, while the same for g follows from condition (ES) in Definition 2.24 (along with applying the inverse map). By definition, $\mathsf{K}[Y]$ and $\mathsf{G}[Y]$ both have Y as object space. The arrow spaces are given by

$$\mathsf{K}[Y]_1 = (\mathsf{G}_1 {}_{s_\mathsf{G}}\!\times_{h_0} \mathsf{K}_0 \times \mathsf{G}_1 {}_{s_\mathsf{G}}\!\times_{h_0} \mathsf{K}_0)_{(\mathrm{pr}_2,\mathrm{pr}_2)}\!\times_{(t_\mathsf{K},s_\mathsf{K})} \mathsf{K}_1,$$

and

$$\mathsf{G}[Y]_1 = (\mathsf{G}_1 {}_{s_\mathsf{G}}\!\times_{h_0} \mathsf{K}_0 \times \mathsf{G}_1 {}_{s_\mathsf{G}}\!\times_{h_0} \mathsf{K}_0)_{(s_\mathsf{G}\circ\mathrm{pr}_1,s_\mathsf{G}\circ\mathrm{pr}_1)}\!\times_{(t_\mathsf{G},s_\mathsf{G})} \mathsf{G}_1.$$

We define an isomorphism $\mathsf{K}[Y] \to \mathsf{G}[Y]$ as the identity on objects and by applying h_1 to the last factor on arrows. This is obviously a differentiable stratified mapping. The fact that it is an isomorphism follows from condition (FF) in Definition 2.24. □

We will need the following, whose proof is that of [PPT14, Prop. 3.7] with minor modifications.

Lemma 2.27. *Let G be a differentiable stratified groupoid. Suppose Y is a locally closed differentiable stratified subspace of G_0 such that $\mathsf{G}_{|Y}$ is a differentiable stratified subgroupoid. Then the inclusion map $\iota\colon \mathsf{G}_{|Y} \to \mathsf{G}_{|\,\mathrm{Sat}(Y)}$ is a weak equivalence, where*

$$\mathrm{Sat}(Y) := \{y \in \mathsf{G}_0 \mid y = t(g) \text{ for some } g \in \mathsf{G}_1 \text{ with } s(g) \in Y\}$$

denotes that saturation of Y.

Proof. Note that $(\mathsf{G}_{|\,\mathrm{Sat}(Y)})_1 {}_s\!\times_{\iota_0} Y = \{g \in \mathsf{G}_1 \mid s(g) \in Y\}$, so that t clearly restricts to this set as a map that is surjective onto Y. Moreover, t restricts to a stratified surjective submersion $(\mathsf{G}_{|Y})_1 \to Y$ by hypothesis so that (ES) is satisfied. The condition (FF) clearly follows from the definition of $\mathrm{Sat}(Y)$. □

With this, we make the following.

Definition 2.28. Let G be a differentiable stratified groupoid. We say that G is a *locally translation differentiable stratified groupoid* if for every $x \in \mathsf{G}_0$ the conditions (LT1) to (LT3) from Definition 2.6 are satisfied and if in addition the following holds true:

(LT4) The slice Y_x can be chosen in such a way that Y_x is a differentiable stratified subspace of U_x, and such that Y_x has the form $Z_x \times R_x$, where Z_x is a G_x-invariant subspace of Y_x and R_x is the stratum through x. Moreover, the isomorphism between $\mathsf{G}_{|U_x}$ and $(O_x \times O_x) \times (\mathsf{G}_x \ltimes Y_x)$ then becomes an isomorphism of differentiable stratified groupoids, where O_x carries the trivial stratification.

We will see that important examples of differentiable stratified groupoids are locally translation differentiable stratified groupoids. Under additional hypotheses, we have the following.

Proposition 2.29. *Let G be a locally translation differentiable stratified groupoid, and suppose that for each $x \in \mathsf{G}_0$, the set Y_x as in* (LT1) *of Definition* 2.6 *can be chosen so that on each stratum of Y_x the G_x-orbit type is constant. Then the assignment*

$$|\mathsf{G}| \ni \mathsf{G}x \mapsto \mathcal{S}_{\mathsf{G}x} = \pi(\mathcal{S}^0_x)$$

defines a stratification of $|\mathsf{G}|$ *with respect to which the orbit map* $\pi\colon \mathsf{G}_0 \to |\mathsf{G}|$ *is a stratified surjective submersion.*

Proof. Note that $|\mathsf{G}|$ is a differentiable space by Proposition 2.9. Choose U_x, Y_x, etc. as in Definitions 2.6 and 2.28, and note that we may shrink U_x if necessary to assume that the stratification of Y_x consists of a finite number of strata. By Lemma 2.27 and Proposition 2.23, the inclusion of Y_x into its saturation in G_0 induces an isomorphism of the differentiable spaces Y_x/G_x and $|\mathsf{G}_{\mathrm{Sat}(Y_x)}|$. As $\mathrm{Sat}(Y_x)$ contains U_x by definition, and as the orbit map is open by Proposition 2.2(2), it follows that Y_x/G_x is isomorphic as a differentiable space to an open neighborhood of $\mathsf{G}x$ in $|\mathsf{G}|$. Moreover, by (LT4), the embedding of Y_x into its saturation in G_0 preserves strata, so it is sufficient to show that the stratification of Y_x induces a stratification of the orbit space Y_x/G_x.

Now, each stratum S of Y_x is a smooth manifold with G_x-action so that S/G_x is stratified by G_x-orbit types. Since each S has a single orbit type by hypothesis, the stratification of S/G_x is trivial, i.e. S/G_x is a smooth submanifold of Y_x/G_x. As Y_x is assumed to have finitely many strata, the resulting stratification of Y_x/G_x is clearly finite. As G_x is compact, the orbit map $Y_x \to Y_x/\mathsf{G}_x$ is closed so that $\overline{(S/\mathsf{G}_x)} = \overline{S}/\mathsf{G}_x$. Therefore, if $S/\mathsf{G}_x \cap \overline{S'/\mathsf{G}_x} \neq \emptyset$ for strata S and S' of Y_x, then $S \cap \overline{S'} \neq \emptyset$, implying $S \subset \overline{S'}$ and hence $S/\mathsf{G}_x \subset \overline{S'/\mathsf{G}_x}$. The pieces S/G_x therefore satisfy the condition of frontier and define a stratification of Y_x/G_x. □

Remark 2.30. Note that the hypotheses of Proposition 2.29 need not be satisfied even in the case of a proper Lie groupoid. For instance, when G is a compact Lie group, a smooth G-manifold equipped with the trivial stratification satisfies these hypotheses if and only if G acts with a single orbit type.

3. Examples of differentiable stratified groupoids

Many examples of differentiable stratified groupoids arise naturally from differentiable actions of Lie groupoids on stratified differentiable spaces. Recall that if G is a topological groupoid and X is a topological space, an *action* of G on X is given by a continuous *anchor map* $\alpha\colon X \to \mathsf{G}_0$ together with a continuous map $\cdot\colon \mathsf{G}_1 {}_s\!\times_\alpha X \to X$ such that for all $x \in X$ and $g, h \in \mathsf{G}_1$ with $s(g) = \alpha(x)$ and $t(g) = s(h)$ the relations

$$\alpha(g \cdot x) = t(g), \quad h \cdot (g \cdot x) = (hg) \cdot x, \quad \text{and} \quad u(\alpha(x)) \cdot x = x$$

hold true. As in the case of group actions, the G-*orbit of* $x \in X$ is defined to be $\{g \cdot x \mid s(g) = x\}$. The *translation groupoid* $\mathsf{G} \ltimes X$ associated to the action has object space $(\mathsf{G} \ltimes X)_0 = X$ and arrow space $(\mathsf{G} \ltimes X)_1 = \mathsf{G}_1 {}_s\!\times_\alpha X$. The structure maps are given by

$$\begin{aligned} &s_{\mathsf{G}\ltimes X}(g,x) = x, \quad t_{\mathsf{G}\ltimes X}(g,x) = g \cdot x, \quad u_{\mathsf{G}\ltimes X}(x) = (u \circ \alpha(x), x), \\ &i_{\mathsf{G}\ltimes X}(g,x) = (g^{-1}, g \cdot x), \quad m_{\mathsf{G}\ltimes X}\big((h, g \cdot x), (g, x)\big) = (hg, x). \end{aligned} \tag{3.1}$$

If G is a differentiable groupoid and X a differentiable space, we say that the action is *differentiable* if α and $\cdot$ are morphisms of differentiable space. In this case, the arrow space $(\mathsf{G} \ltimes X)_1 = \mathsf{G}_1 {}_s\!\times_\alpha X$ inherits a differentiable structure by [NGSdS03, Thm. 7.6]. Then, as each of the structure maps of $\mathsf{G} \ltimes X$ is defined in terms of the structure maps of G and the differentiable maps α and $\cdot$, $\mathsf{G} \ltimes X$ is a differentiable groupoid. For actions of differentiable stratified groupoids, one even has the following.

Proposition 3.1. *Let X be a differentiable stratified space that is topologically locally trivial. Further let G be a differentiable stratified groupoid with G_0 and G_1 topologically locally trivial that acts differentiably on X in such a way that the G-orbit of each $x \in X$ is a subset of the stratum containing x. Then the translation groupoid $\mathsf{G} \ltimes X$ is a differentiable stratified groupoid.*

Proof. Let $\mathcal{Z}$ denote the maximal decomposition of X, and endow $(\mathsf{G} \ltimes X)_1 = \mathsf{G}_1 {}_s\!\times_\alpha X$ with the stratification induced by those of G_1 and X. Then by Lemma A.4, $(\mathsf{G} \ltimes X)_1$ is a differentiable stratified space. By definition of the induced stratification, the germ of the stratum of $(\mathsf{G} \ltimes X)_1$ at the point (g, x) is given by the fibered product over s and α of neighborhoods of strata in G_1 and X. Since i, u, m, and α are stratified mappings, the structure maps $s_{\mathsf{G}\ltimes X}$, $t_{\mathsf{G}\ltimes X}$, $u_{\mathsf{G}\ltimes X}$, $i_{\mathsf{G}\ltimes X}$, and $m_{\mathsf{G}\ltimes X}$ defined by Equation (3.1) are as well stratified mappings, so that (DSG1) and (DSG2) are satisfied. To see that $s_{\mathsf{G}\ltimes X}$ is a stratified surjective submersion, let $(g, x) \in (\mathsf{G} \ltimes X)_1$. Note that as s is a stratified surjective submersion, the restriction of s to the connected component R_g of the stratum of G_1 containing g is a surjective submersion onto the connected component $S_{\alpha(x)}$ of the stratum of G_0 containing $\alpha(x)$. As α is a stratified mapping, it maps the connected component P_x of the stratum of X containing x into $S_{\alpha(x)}$. Hence the restriction of $s_{\mathsf{G}\ltimes X}$ to $R_g {}_s\!\times_\alpha P_x$ is a surjective submersion onto P_x. It follows that $s_{\mathsf{G}\ltimes X}$ is a stratified surjective submersion. The proof for $t_{\mathsf{G}\ltimes X}$ is identical. In a similar fashion one shows that $u_{\mathsf{G}\ltimes X}$ is a stratified embedding, because α is a stratified map and u a stratified embedding, so (DSG3) is satisfied. Property (DSG4) follows from the definition of the induced stratification, as the connected component of the stratum of $(\mathsf{G} \ltimes X)_1$ containing the arrow (g, x) equals the fibered product $R_g {}_s\!\times_\alpha P_x$ with R_g and P_x as above. Therefore, the germ at (g, x) of the set of points $(h, y) \in (\mathsf{G} \ltimes X)_1$ such that $y \in P_x$ is contained in the germ $[R_g {}_s\!\times_\alpha P_x]_{(g,x)}$ of the stratum through (g, x).

To verify (DSG5), consider again $P_x \subset X$, the connected component of the stratum through x. Since by assumption on X the G-orbit of each point in P_x is a subset of the stratum containing P_x, the saturation $P := \operatorname{Sat} P_x$ has to be a union of connected components of strata, which are separated from one another's closures by topological local triviality. Assume that $\sigma\colon P \to (\mathsf{G} \ltimes X)_1$ is a bisection as in (DSG5). Since α is a stratified mapping, $\alpha(P_x)$ is contained in a connected component S of a stratum of G_0. The saturation $\operatorname{Sat} S$ of S by G now is given by $t(s^{-1}(S))$ and hence is a union of connected components of strata of G_0 such that for each such connected component S', we have $s^{-1}(S) \cap t^{-1}(S') \neq \emptyset$ by construction. Then by Proposition 2.19, the restriction $\mathsf{G}_{|\operatorname{Sat} S}$ is a Lie groupoid. Moreover, it follows from (DSG4) that $\sigma(P) \subset (\mathsf{G}_{|\operatorname{Sat} S} \ltimes P)_1$. Therefore, the map L_σ described in (DSG5) is a left translation of the Lie groupoid $\mathsf{G}_{|\operatorname{Sat} S} \ltimes P$, see [Mac05, page 22], implying in particular that it is a diffeomorphism of the stratum $s^{-1}_{\mathsf{G}\ltimes X}(P)$ onto itself. Hence (DSG5) holds, completing the proof. □

A particularly important special case appears when a Lie group G acts differentiably on a differentiable stratified space X in such a way that the action restricts to a smooth action on each stratum. The resulting translation groupoid $G \ltimes X$ then is a differentiable stratified groupoid by Proposition 3.1. Many significant and naturally occuring examples of differentiable stratified groupoids are constructed that way. In the following, we will provide a few.

Example 3.2 (Singular symplectic reduction)**.** Suppose (M, ω) is a symplectic manifold equipped with a Hamiltonian G-action with moment map $J\colon M \to \mathfrak{g}^*$ such that 0 is

a singular value for J. By [SL91, Thm. 2.1], the corresponding *symplectic quotient* $J^{-1}(0)/G$ is stratified by orbit types. It is then easy to see that $J^{-1}(0)$ inherits the structure of a smooth stratified space on which G acts in a way such that the hypotheses of Proposition 3.1 are satisfied. Therefore, the singular symplectic quotient can be realized as the orbit space of the differentiable stratified groupoid $G \ltimes J^{-1}(0)$. Note that since $J^{-1}(0)$ is equivariantly embedded as a differentiable subspace in the smooth G-manifold M the contractibility hypotheses (LC) in Definition 5.7) is satisfied. Hence, Theorem 5.9 below reduces to the de Rham theorem of [Sja05] in this case.

Example 3.3 (Lie Groupoid actions on manifolds with corners)**.** *Manifolds with corners* and *manifolds with boundary* are in a natural way locally trivial differentiable stratified spaces, and even more $\mathcal{C}^\infty$-cone spaces, see [Pfl01, 1.1.19 & 3.10.3] and [SLV15]. Compact Lie group actions on manifolds with corners have been considered, e.g. in [MROD93, Mel91]. By Proposition 3.1, the corresponding translation groupoids are differential stratified groupoids under mild hypotheses.

Example 3.4 (Semialgebraic actions)**.** Recall that a *semialgebraic set* is a locally closed subset of $\mathbb{R}^n$ locally given by the solution of a finite collection of polynomial equations and inequalities, see [Shi97]. Semialgebraic sets are differentiable stratified spaces in a natural way, since they admit a minimal Whitney, and hence topologically locally trivial, stratification into semialgebraic manifolds. If a compact Lie group acts on a semialgebraic set and preserves this stratification, the resulting translation groupoid is again a stratified differentiable groupoid by Proposition 3.1. Lie group actions on semialgebraic sets have been considered in [CPS04, Par13, PS02].

Example 3.5 (Transverse cotangent bundle)**.** Let G be a compact Lie group and M a G-manifold. The *transverse cotangent bundle* $T^*_G M$ is the subspace of the cotangent bundle T^*M consisting of elements that are conormal to the G-orbits in M. The transverse cotangent bundle appears in the study of transversally elliptic operators G-invariant pseudodifferential operators on M, see [DCPV11, Par12, PV09]. The action of G on M induces an action of G on $T^*_G M$. It is not difficult to show that the stratification of M by orbit types induces a stratification of $T^*_G M$ that is compatible with the smooth structure $T^*_G M$ inherits as a subset of T^*M. Hence the corresponding translation groupoid is a differentiable stratified groupoid.

If G is a proper Lie groupoid, the *transverse cotangent bundle* $T^*_{\mathsf{G}}\mathsf{G}_0 \subset T^*\mathsf{G}_0$ can be defined in a similar fashion as the subspace of all $\alpha \in T^*\mathsf{G}_0$ such that $\langle \alpha, v\rangle = 0$ for all $v \in T_x\mathcal{O}_x$, where $x \in \mathsf{G}_0$ is the footpoint of α and $\mathcal{O}_x$ the orbit through x. By the slice theorem for groupoids as stated in [PPT14, Cor. 3.11] recalled in Section 6.2 below, it follows that locally around a point $x \in \mathsf{G}_0$ the transverse cotangent bundle $T^*_{\mathsf{G}}\mathsf{G}_0$ is isomorphic to the exterior tensor product of the transverse cotangent bundle $T^*_{\mathsf{G}_x} Y_x$ of a slice Y_x through x with the cotangent bundle T^*O of an open connected neighborhood O of x in the orbit through x. By the preceeding considerations it follows that the transverse cotangent bundle of a proper Lie groupoid is locally smoothly contractible as well.

Example 3.6 (Singular riemannian foliations)**.** Let M be a smooth, connected manifold and let $\mathcal{F}$ be a singular riemannian foliation of M, see [Mol88, Section 6.1]. That is, $\mathcal{F}$ is a partition of M into connected, immersed submanifolds called *leaves* such that the module of smooth vector fields on M that are tangent to the leaves is transitive on each

leaf, and there is a riemannian metric on M with respect to which every geodesic that is normal to a leaf is normal to every leaf it intersects. A singular riemannian foliation is an example of a singular Stefan–Sussmann foliation, see [Ste74, Sus73]. By [Mol88, Section 6.2], M is stratified by unions of leaves of the same dimension; see also [RPSAW05, Sections 1.2–3] and [BM93].

Suppose $(M, \mathcal{F})$ is *almost regular*, meaning that the union of leaves of maximal dimension k is an open, dense subset of M. Suppose further that the foliation $\mathcal{F}$ can be defined by a Lie algebroid of dimension k. In [Deb01], the holonomy groupoid G of $(M, \mathcal{F})$ is constructed as a Lie groupoid with object space $\mathsf{G}_0 = M$ and such that the orbits of G correspond to the leaves $\mathcal{F}$; see also [Pra85]. Giving M the stratification by leaves of the same dimension described above and G_1 the stratification given by the pullback of this stratification via s, it is easy to see that G has, along with its structure as a Lie groupoid, an alternate structure as a differential stratified groupoid. These two structures coincide if and only if the foliation $\mathcal{F}$ is regular, i.e. all leaves have the same dimension.

Note that the holonomy groupoid of a general singular Stefan–Sussmann foliation was constructed as a topological groupoid in [AS09] and coincides with the holonomy groupoid of [Deb01] when the latter is defined. Other hypotheses under which this holonomy groupoid naturally has the structure of a differentiable stratified groupoid are not yet clear.

4. The algebroid of a differentiable stratified groupoid

Given a reduced differentiable Whitney stratified groupoid G with $\mathcal{S}^i$, $i = 0, 1$ the decomposition of G_i into its strata we obtain the so-called stratified tangent bundles

$$T^{\mathrm{st}}\mathsf{G}_0 := \bigcup_{S \in \mathcal{S}^0} TS \text{ and } T^{\mathrm{st}}\mathsf{G}_1 := \bigcup_{R \in \mathcal{S}^1} TR = \bigcup_{S \in \mathcal{S}^0} TS^1, \text{ where } S^1 := s^{-1}(S) \text{ for } S \in \mathcal{S}^0.$$

Since, by assumption, the spaces G_0 and G_1 satisfy Whitney's condition B, hence A, the stratified tangent bundles $T^{\mathrm{st}}\mathsf{G}_0$ and $T^{\mathrm{st}}\mathsf{G}_1$ inherit the structures of differentiable stratified spaces by [Pfl01, Thm. 2.1.2]. Moreover, we have tangent maps $Ts\colon T^{\mathrm{st}}\mathsf{G}_1 \to T^{\mathrm{st}}\mathsf{G}_0$ and $Tt\colon T^{\mathrm{st}}\mathsf{G}_1 \to T^{\mathrm{st}}\mathsf{G}_0$. Now we can define what we understand by the algebroid of G.

Definition 4.1. Given a differentiable stratified groupoid G, the *differentiable stratified algebroid* of a differentiable Whitney stratified groupoid G is defined as the space

$$\mathsf{A} := \bigcup_{S \in \mathcal{S}^0} \mathsf{A}_S, \tag{4.1}$$

where $\mathsf{A}_S := u^*_{|S} \ker T_{|S_1} s$ denotes the Lie algebroid of the Lie groupoid $\mathsf{G}_{|S}$. In other words, A can be identified with $u^* \ker Ts$, the restriction of $\ker Ts$ to G_0. We define the *anchor map* of the algebroid A as the map

$$\varrho\colon \mathsf{A} \to T^{\mathrm{st}}\mathsf{G}_0, \quad v \mapsto Tt(v).$$

Proposition 4.2. *Let G be a proper reduced differentiable Whitney stratified groupoid. Then the differentiable stratified algebroid A of G is a reduced differentiable stratified space, where the differentiable structure is that inherited from $T^{\mathrm{st}}\mathsf{G}_1$ and the stratification is that induced by the decomposition in Equation* (4.1)*. The anchor map $\varrho\colon \mathsf{A} \to T^{\mathrm{st}}\mathsf{G}_0$ is a differentiable stratified submersion.*

Proof. Since G_1 is locally compact, $u(\mathsf{G}_0)$ is locally closed in G_1 by [Tu04, Prop. 2.5]. For each stratum S of G_0 the source map s restricts to S_1 as a submersion which implies that $\ker T_{|S_1} s$ is a subbundle, hence closed subset, of TS_1. Hence, if U is an open subset of G_1 intersecting finitely many strata, $\mathsf{A} \cap T^{\mathrm{st}}U$ is a finite union of locally closed sets, therefore locally closed itself. Then A inherits the structure of a reduced differentiable space from that of $T^{\mathrm{st}}\mathsf{G}_1$, see [NGSdS03, Ex. 3.21].

Since each A_S is a subbundle of the restriction of TS_1 to $S_1 \cap u(\mathsf{G}_0)$ one concludes that each A_S is a smooth submanifold of A. Note that the projection $\pi\colon T^{\mathrm{st}}\mathsf{G}_1 \to \mathsf{G}_1$ is clearly open as its restriction to an element of $\mathcal{S}^1$ is a bundle map. The fact that $\mathcal{S}^1$ is locally finite implies then that $(\mathsf{A}_S)_{S\in\mathcal{S}^0}$ is a locally finite decomposition of A.

To verify the condition of frontier, suppose $\mathsf{A}_S \cap \overline{\mathsf{A}_{S'}} \neq \emptyset$ for $S, S' \in \mathcal{S}^0$, and let $S_1 = s^{-1}(S)$ and $S_1' = s^{-1}(S')$. By [Pfl01, Thm. 2.1.2], $\pi\colon T^{\mathrm{st}}\mathsf{G}_1 \to \mathsf{G}_1$ is a topological projection, hence $S_1 \subset \overline{S_1'}$ and $TS_1 \subset \overline{TS_1'}$ follow.

Choose a tangent vector $v \in \ker T_{|S_1} s$ and assume for simplicity that v is a unit vector. Let $x \in S$ be the footpoint, i.e. x is the unique object such that $\pi(v) = u(x)$. Choosing a singular chart for G_1 at $u(x)$, we may reduce to the case where S_1 and S_1' are closed subsets of $\mathbb{R}^n$. Let $\widetilde{s}$ denote a smooth function from $\mathbb{R}^n$ into a singular chart of G_0 at x that extends s. Note that $\widetilde{s}$ may not be a submersion. Let $p(t)\colon [-1,1] \to S_1$ be a smooth path in S_1 with $p(0) = u(x)$ and tangent $p'(0) = v$, and put $\alpha_i = p(1/i)$ for $i \in \mathbb{N}$. As $\alpha_1 \in S_1 \subset \overline{S_1'}$, we may choose a sequence $(\beta_{1,j})_{j\in\mathbb{N}} \subset S_1'$ such that $\lim_{j\to\infty} \beta_{1,j} = \alpha_1$. We then set $x_j = s(\beta_{1,j}) \in S'$ for each j. By continuity of u and s, we have $\lim_{j\to\infty} x_j = x$ and $\lim_{j\to\infty} u(x_j) = u(x)$. As $s_{|S_1'}$ is a submersion, each $s^{-1}(x_j)$ is a closed submanifold of $\mathbb{R}^n$. Then the intersection of each $s^{-1}(x_j)$ with a closed ball in $\mathbb{R}^n$ is compact. Hence we may define, for each $i > 1$ and each $j \geq 1$, the element $\beta_{i,j}$ of S_1' to be the point on the connected component of $s^{-1}(x_j)$ containing $u(x_j)$ intersected with a closed annulus of external radius $1/j$ about $u(x_j)$ that minimizes the distance to α_i. In particular, for each j, $\lim_{i\to\infty} \beta_{i,j} = u(x_j)$, and for i sufficiently large, $\lim_{j\to\infty} \beta_{i,j} = \alpha_i$; if necessary, we restrict to a subsequence in the j direction so that this intersection is not empty. With this, we set

$$v_j := \lim_{i\to\infty} \frac{\beta_{i,j} - u(x_j)}{\|\beta_{i,j} - u(x_j)\|},$$

and then $v_j \in T_{x_j} S_1'$ for each j. Moreover, as $\beta_{i,j} \in s^{-1}(x_j)$, it follows that $v_j \in \ker T_{|S_1'} s$. However,

$$\lim_{j\to\infty} v_j = \lim_{i\to\infty} \lim_{j\to\infty} \frac{\beta_{i,j} - u(x_j)}{\|\beta_{i,j} - u(x_j)\|} = \lim_{i\to\infty} \frac{\alpha_i - u(x_j)}{\|\alpha_i - u(x_j)\|} = v$$

so that $v \in \overline{\ker T_{|S_1'} s}$. It follows that the decomposition of A into the A_S satisfies the condition of frontier.

Finally, the anchor map is a differentiable stratified morphism by Eq. (2.5) in Lemma 2.13 and because t is a stratified map. Since the restriction of ϱ to each stratum is a submersion, the anchor map is a stratified submersion. □

5. A de Rham theorem

In this section, we prove a de Rham theorem for proper reduced differentiable stratified groupoids which are locally translation and satisfy a certain local contractibility condition. As we will see below, this hypothesis is satisfied in a number of the examples we have considered.

Throughout this section let G be a proper differentiable Whitney stratified groupoid. Let $\Omega^\bullet$ be the sheaf complex of abstract forms on G_0 as constructed in Section A.5. Recall from [PPT14, Def. 8.1] that a differential form on the object space of a proper Lie groupoid is called *basic* if contraction with any smooth section of the Lie algebroid vanishes and if the form is invariant under the conormal action of the Lie groupoid.

Definition 5.1. Let $U \subset |\mathsf{G}|$ be an open subset of the orbit space of G, and U_0 its preimage under the canonical projection $\pi\colon \mathsf{G}_0 \to |\mathsf{G}|$. One calls an abstract k-form $\omega \in \Omega^k(U_0)$ G-*horizontal* or simply *horizontal*, if for each smooth section $\xi\colon U_0 \to \mathsf{A}$ of the differentiable stratified algebroid of G the stratawise contracted form $(\varrho \circ \xi)\lrcorner\omega$ vanishes.

Next let $S \subset \mathsf{G}_0$ be a (relatively closed and open) component of a stratum of G_0 such that the projection $\pi(S)$ to the orbit space is connected. For $x \in \mathsf{G}_0$ consider the Zariski tangent space $T_x\mathsf{G}_0$; see Appendix A.4. Note that $T_x\mathsf{G}_0$ does in general not coincide with $T_x^{\mathrm{st}}\mathsf{G}_0$, but that the latter is always contained in the former.

Lemma 5.2. *Under the assumption that G is locally translation, the space $T_{|S}\mathsf{G}_0 := \bigcup_{x\in S} T_x\mathsf{G}_0$ naturally carries the structure of a smooth vector bundle over S.*

Proof. Given $x \in S$ choose a trivializing neighborhood U_x and a groupoid slice Y_x with a G_x-action such that there exists an isomorphism of differentiable groupoids $\mathsf{G}_{|U_x} \to (O_x \times O_x) \times (\mathsf{G}_x \ltimes Y_x)$, where O_x is an open contractible neighborhood of x in its orbit. Such U_x, Y_x, and O_x exist according to (LT1). After possibly shrinking these data, we can assume by (LT4) that the slice Y_x is stratified and is isomorphic as a differentiable stratified space to the product space $Z_x \times R_x$, where $Z_x \subset Y_x$ is a G_x-invariant subspace, and $R_x \subset Y_x$ is the stratum through x. Hence $\bigcup_{y\in O_x\times R_x} T_y\mathsf{G}_0$ is a vector bundle isomorphic to $T(O_x \times R_x) \times T_xZ_x$. But the set $O_x \times R_x$ contains x and is relatively open in the stratum S. This means that, locally, $T_{|S}\mathsf{G}_0$ is a vector bundle. By construction, the transition maps arise from the local isomorphisms $\mathsf{G}_{U_x} \to (O_x \times O_x) \times (\mathsf{G}_x \ltimes Y_x)$, and are vector bundle isomorphisms, hence the claim follows. □

We now have the means to define basic forms on G. Note that here we apply the pull-back morphisms constructed in Appendix A.5.

Definition 5.3. Let $U \subset |\mathsf{G}|$ be an open subset of the orbit space, and $U_0 := \pi^{-1}(U)$. One calls an abstract k-form $\omega \in \Omega^k(U_0)$ G-*basic* or simply *basic* if it is horizontal and if for every $x \in U_0$ and every smooth bisection $\sigma : U_x \to \mathsf{G}$ defined on an open neighborhood $U_x \subset U_0$ of x the equality $(t \circ \sigma)^*\omega = \omega_{|U_x}$ holds true.

By definition, $\Omega^0_{\mathrm{basic}}(U)$ coincides with the space of smooth functions on U_0 which are invariant under the G-action, hence $\Omega^0_{\mathrm{basic}}(U)$ can be naturally identified with $\mathcal{C}^\infty(U)$. Moreover, the spaces $\Omega^k_{\mathrm{basic}}(U)$, where U runs through the open sets of $|\mathsf{G}|$, form the space of sections of a sheaf on $|\mathsf{G}|$ which will be denoted by $\Omega^k_{\mathrm{basic}}$. By construction the sheaf $\Omega^k_{\mathrm{basic}}$ is a $\mathcal{C}^\infty_{|\mathsf{G}|}$-module. This entails the first part of the following result.

Proposition 5.4. *The sheaves of basic k-forms $\Omega^k_{\mathrm{basic}}$ are fine. Moreover, the exterior differential on $\Omega^\bullet$ descends to a differential d on $\Omega^\bullet_{\mathrm{basic}}$ turning $\big(\Omega^\bullet_{\mathrm{basic}}, d\big)$ into a sheaf of commutative differential graded algebras over the orbit space $|\mathsf{G}|$.*

Proof. It remains to show that d descends to $\Omega^\bullet_{\text{basic}}$. But that follows from the fact that d commutes with the pull-back morphism $(t\circ\sigma)^*$ for every bisection $\sigma : U_x \to \mathsf{G}$ defined on an open neighborhood U_x of $x \in \mathsf{G}_0$. □

The next result will be needed for a proof of a Poincaré Lemma for basic forms.

Proposition 5.5. *Let V be an open subset of the object space of a locally translation proper differentiable Whitney stratified groupoid, and let $\omega \in \Omega^k(V)$ be an abstract k-form which is $\mathsf{G}_{|V}$-basic. Then there exists a unique form $\widehat{\omega} \in \Omega^k\big(\operatorname{Sat}(V)\big)$ which is G-basic and whose restriction to V coincides with ω. We call $\widehat{\omega}$ the* basic extension *of ω.*

Proof. Let $x \in \operatorname{Sat}(V)$, and choose a bisection $\sigma : U_x \to \mathsf{G}_1$ defined on an open neighborhood of x such that $(t\circ\sigma)(U_x) \subset V$. Then put $\omega_{U_x} := (t\circ\sigma)^*(\omega)$. If $\eta : U_x \to \mathsf{G}_1$ is another bisection with $U_z := (t\circ\eta)(U_x) \subset V$, where $z := (t\circ\eta)(x)$, then the bisection $\mu\colon U_z \to \mathsf{G}_1$ defined by

$$\mu(y) = \sigma\big((t\circ\eta)^{-1}(y)\big)\cdot\eta^{-1}\big((t\circ\eta)(y)\big) \quad \text{for } y \in U_z$$

starts and ends in V. Hence, by assumption, $(t\circ\mu)^*\omega = \omega_{|U_z}$. But $t\circ\mu = (t\circ\sigma)\circ(t\circ\eta)^{-1}$, which implies

$$(t\circ\eta)^*\omega = (t\circ\eta)^*(t\circ\mu)^*\omega = (t\circ\sigma)^*\omega.$$

This shows that ω_{U_x} does not depend on the particular choice of the bisection $\sigma : U_x \to t^{-1}(V)$. An analogous argument proves that for points $x, y \in \operatorname{Sat}(V)$ the forms ω_{U_x} and ω_{U_y} coincide over $U_x \cap U_y$. Let $\widehat{\omega} \in \Omega^k\big(\operatorname{Sat}(V)\big)$ be the abstract k-form such that $\widehat{\omega}_{|U_x} = \omega_{U_x}$ for all $x \in \operatorname{Sat}(V)$. Obviously, $\widehat{\omega}$ is horizontal and basic by construction. Uniqueness of $\widehat{\omega}$ is clear since the form has to be basic. □

Since the kernel of the sheaf morphism $d : \Omega^0_{\text{basic}} \to \Omega^1_{\text{basic}}$ can be naturally identified with $\mathbb{R}_{|\mathsf{G}|}$, the sheaf of locally constant real-valued functions on $|\mathsf{G}|$, we obtain a sheaf complex

$$0 \longrightarrow \mathbb{R}_{|\mathsf{G}|} \longrightarrow \Omega^\bullet_{\text{basic}}$$

which is exact at $\mathbb{R}_{|\mathsf{G}|}$ and at Ω^0_{basic}. Now observe that the orbit space $|\mathsf{G}|$ is paracompact, locally contractible and locally path connected by [PPT14]. Since the sheaves Ω^k_{basic} are fine, the following is a consequence of [God58, Sec. 3.9].

Theorem 5.6. *If the sheaf complex $\big(\Omega^\bullet_{\text{basic}}, d\big)$ of basic forms on a proper differentiable Whitney stratified groupoid G is exact, the* basic cohomology *$H^\bullet_{\text{basic}}(\mathsf{G}) := H^\bullet\big(\Omega^\bullet_{\text{basic}}(|\mathsf{G}|)\big)$ of G coincides naturally with the real singular cohomology of $|\mathsf{G}|$.*

In the remainder of this section we will show that the sheaf complex of basic forms on a groupoid is exact if a certain local contractibility condition is satisfied. Before we come to stating the local contractibility condition recall that by Proposition 2.8 every G-slice can be, possibly after shrinking, equivariantly embedded around the fixed point into the Zariski tangent space.

Definition 5.7. Let G be a proper locally translation differentiable stratified groupoid in the sense of Definition 2.28. We then say that G fulfills the *local contractibility hypothesis* if the following condition holds true.

(LC) For each $x \in \mathsf{G}_0$ there exists a groupoid slice Y_x as in (LT4), a linear G_x-action on some $\mathbb{R}^n$ together with a singular G_x-equivariant chart $\iota : Z_x \to \widetilde{V_x} \subset \mathbb{R}^n$, $y \mapsto \widetilde{y}$, and a smooth homotopy $h\colon \widetilde{V_x} \times [0,1] \to \widetilde{V_x}$ having the following properties:
  (1) The chart ι maps x to 0 and the stratum R_x through x to the subspace of $\mathbb{R}^n$ fixed by G_x. Moreover, $\widetilde{V_x}$ is an open neighborhood of 0 in $\mathbb{R}^n$, and $\widetilde{Y_x} := \iota(Y_x)$ is relatively closed in $\widetilde{V_x}$.
  (2) One has $\operatorname{im} h_0 = \{0\}$ and $h_1 = \mathrm{id}_{\widetilde{V_x}}$.
  (3) The homotopy h is a homotopy along $\widetilde{Y_x}$ which means that $h(\widetilde{y}, t) \in \widetilde{Y_x}$ for all $y \in Z_x$ and $t \in [0,1]$.
  (4) The homotopy h is G_x-equivariant.
  (5) The homotopy h preserves the stratification which means for each $y \in Y_x$ and $t \in (0,1]$ the points $\iota^{-1}h(\widetilde{y}, t)$ and y are in the same stratum.

Example 5.8. Let G be a compact Lie group and M is a G-manifold. The transverse cotangent bundle described in Example 3.5 satisfies the local contractibility hypothesis. To verify this, observe first that $T^*_G M$ is equivariantly embedded in the G-manifold T^*M, and that $\mathbb{R}_{\geq 0}$ acts on T^*M by fiberwise homotheties. Since the transverse cotangent bundle is invariant under these homotheties, it contracts smoothly to the 0-section which is diffeomorphic to M. But M is a smooth manifold, so is locally smoothly contractible for trivial reasons. Hence $T^*_G M$ is locally smoothly contractible as well.

We will illustrate in Section 6 that the inertia groupoid of a proper Lie groupoid fulfills the local contractibility hypothesis.

Now we will prove that the sheaf complex of basic forms on a locally translation proper Lie groupoid fulfilling the contractibility hypothesis is exact, or in other words satisfies Poincaré's Lemma. We first consider the case where the groupoid G is of the form $(O \times O) \times (G \ltimes Y) \rightrightarrows O \times Y$, where O is an open contractible set in some $\mathbb{R}^m$, $O \times O$ denotes the corresponding pair groupoid, G is a Lie group acting linearly on some $\mathbb{R}^n$, and $Y \subset \mathbb{R}^n$ is an affine G-invariant differentiable stratified space on which G acts by strata preserving maps. In addition we assume that $0 \in Y$ and that there exists a smooth homotopy $h : \widetilde{V} \times [0,1] \to \widetilde{V}$ defined on an open G-invariant subset $\widetilde{V} \subset \mathbb{R}^n$ such that the five conditions of the local contractibility hypothesis are satisfied with $\iota : Y \hookrightarrow \mathbb{R}^n$ being the identical embedding. Denote by $I \subset \mathcal{C}^\infty(O \times \widetilde{V})$ the vanishing ideal of $O \times Y$. Observe that by [DJZ04, Sec. 2] and the smooth contractibility of Y the subcomplex $I^\bullet \subset \Omega^\bullet(O \times \widetilde{V})$ defined by

$$I^k := \begin{cases} I, & \text{for } k = 0, \\ I\Omega^k(O \times \widetilde{V}) + dI \wedge \Omega^{k-1}(O \times \widetilde{V}), & \text{for } k = 1, \dots, n+m \end{cases}$$

is contractible. More precisely, an (algebraic) contraction is given by

$$K\omega = \begin{cases} 0, & \text{for } \omega \in \Omega^0(O \times \widetilde{V}) = \mathcal{C}^\infty(O \times \widetilde{V}), \\ \int_0^1 H_t^*(\xi_t \lrcorner \omega)\, dt, & \text{for } \omega \in \Omega^k(O \times \widetilde{V}),\ k \in \mathbb{N}^*, \end{cases} \tag{5.1}$$

where the homotopy h has been extended to a homotopy on $O \times \widetilde{V}$ by putting $H_t(v, x) = (v, h_t(x))$ for $v \in O$, $x \in \widetilde{V}$, $t \in [0,1]$, and $\xi_t : O \times \widetilde{V} \to T\widetilde{V}$ is the vector field defined by $\xi_t := \partial_t H_t$. Cartan's magic formula implies that

$$\omega - H_0^*\omega = dK\omega + Kd\omega, \quad \text{for } \omega \in \Omega^k(O \times \widetilde{V}),\ k \in \mathbb{N}. \tag{5.2}$$

But this entails that the restriction of K to $I^\bullet$ is an algebraic contraction indeed, since every form $\omega \in I^k$ satisfies the relation $H_0^*\omega = 0$.

Now consider the subcomplex $\Omega^\bullet_{\text{r-basic}}(O \times \widetilde{V})$ of *relative basic forms* or more precisely of *basic forms relative* $(O \times Y)$. It consists of all $\omega \in \Omega^k(O \times \widetilde{V})$ which are invariant under the G-action and which have the property that for each stratum S of $O \times Y$ the form $\iota_S^*\omega$ is a basic form for the restricted Lie groupoid $\mathsf{G}_{|S}$. In particular, this implies that $H_0^*\omega = \iota^*_{O\times\{0\}}\omega$ vanishes for each relative basic form ω. Since H_t commutes with the G-action and maps fibers $O \times \{y\}$ to $O \times \{h_t(y)\}$, the algebraic contraction K maps basic forms to basic forms. One concludes that the complex $\Omega^\bullet_{\text{r-basic}}(O \times \widetilde{V})$ is exact, and that the subcomplex $I^\bullet_{\text{r-basic}} := I^\bullet \cap \Omega^\bullet_{\text{r-basic}}(O \times \widetilde{V})$ is contractible. Hence the quotient complex $\Omega^\bullet_{\text{r-basic}}(O \times \widetilde{V})/I^\bullet_{\text{r-basic}}$ is exact. But one has

$$\Omega^\bullet_{\text{basic}}(Y/G) = \Omega^\bullet_{\text{r-basic}}(O \times \widetilde{V})/I^\bullet_{\text{r-basic}}.$$

This can be seen by averaging a representative of an element $[\omega] \in \Omega^k_{\text{basic}}(Y/G)$ over the orbits of the G-action using a bi-invariant Haar measure on G. The resulting new representative ω then is G-invariant. Since $[\omega]$ is basic, the pull-back of such a representative ω to each stratum S of $O \times Y$ has to be basic as well, hence $\omega \in \Omega^\bullet_{\text{r-basic}}(O \times \widetilde{V})$. So we have shown Poincare's Lemma in the special case where the groupoid G is of the form $(O \times O) \times (G \ltimes Y) \rightrightarrows O \times Y$ with Y and O as stated above.

Next let us consider the general case of a proper locally translation differentiable Whitney stratified groupoid G which fulfills the local contractibility hypothesis. Let $x \in \mathsf{G}_0$ be a point in the object space. Since G is locally translation, we can choose a trivializing neighborhood $U_x \subset \mathsf{G}_0$ of x, an open contractible neighborhood O_x in the orbit through x and a groupoid slice $Y_x \subset U_x$ with a G_x-action such that $\mathsf{G}_{|U_x}$ as a differentiable stratified groupoid is isomorphic to the groupoid $(O_x \times O_x) \times (\mathsf{G}_x \ltimes Y_x)$. We will show that $\Omega^\bullet_{\text{basic}}(\pi(U_x))$ is exact. To this end let $\omega \in \Omega^k(\text{Sat}(U_x))$ be a closed basic k-form. The restriction of ω to U_x is a closed and $\mathsf{G}_{|U_x}$-basic form. By the preceeding considerations there exists a $\mathsf{G}_{|U_x}$-basic $(k-1)$-form $\eta \in \Omega^{k-1}(U_x)$ such that $d\eta = \omega_{|U_x}$. By Proposition 5.5, η has a unique extension to a basic form $\widehat{\eta} \in \Omega^{k-1}_{\text{basic}}(\pi(U_x))$. By construction of $\widehat{\eta}$ we have $d\eta = \omega$, since d commutes which each of the isomorphisms $t \circ \sigma$, where $\sigma : U \to \mathsf{G}$ is a bisection. This proves exactness of $\Omega^\bullet_{\text{basic}}(\pi(U_x))$, and entails the following result.

Theorem 5.9. *Let G be a proper locally translation differentiable Whitney stratified groupoid satisfying the local contractibility hypothesis. The complex of sheaves $(\Omega^\bullet_{\text{basic}}, d)$ on $|\mathsf{G}|$ then is a fine resolution of the sheaf of locally constant real-valued functions on $|\mathsf{G}|$. In particular this implies that the cohomology of the complex $(\Omega^\bullet_{\text{basic}}, d)$ of basic differential forms on G is naturally isomorphic to the singular cohomology of $|\mathsf{G}|$ with coefficients in $\mathbb{R}$.*

Remark 5.10. If G is a proper Lie groupoid, then the complex of sheaves $(\Omega^\bullet_{\text{basic}}, d)$ coincides with the sheaf of basic differential forms defined in [PPT14, Def. 8.1] so that Theorem 5.9 reduces to [PPT14, Prop. 8.6 & Cor. 8.7].

If G is a locally translation groupoid, then $(s,t)(\mathsf{G}_1)$ is necessarily locally closed in $\mathsf{G}_0 \times \mathsf{G}_0$. Specifically, using the local model of G given by condition (LT1), the condition that $(s,t)(\mathsf{G}_1)$ is locally closed is equivalent to the requirement that the G_x-action on Y_x is a proper group action, which is automatically satisfied as G_x is compact by [Tu04,

Prop. 2.10(ii)]. Therefore, $|\mathsf{G}|$ is locally compact by Proposition 2.2(4). If we assume further that $|\mathsf{G}|$ carries a stratification according to Proposition 2.29 and that that stratification fulfills Whitney's condition B, then $|\mathsf{G}|$ is a differentiable stratified space with control data in the sense of Mather by [Pfl01, Thm. 3.6.9]. Hence $|\mathsf{G}|$ admits in this case a triangulation subordinate to its stratification, cf. [PPT14, Thm. 7.1]. Therefore, given an open covering of $|\mathsf{G}|$, there exists a subordinate good covering. See [PPT14, Sec. 7] for more details. As in the case of Lie groupoids, cf. [PPT14, Cor. 8.8], one concludes that the singular cohomology of $|\mathsf{G}|$ with real coefficients coincides with the Čech cohomology under these hypotheses, and that it is finite-dimensional if $|\mathsf{G}|$ is compact.

6. The inertia groupoid of a proper Lie groupoid as a differentiable stratified groupoid

The goal of this section is to construct an explicit Whitney stratification of the loop space of a proper Lie groupoid G with respect to which the inertia groupoid $\Lambda\mathsf{G}$ becomes a reduced differentiable stratified groupoid which is locally translation and satisfies the local contractibility hypotheses. As the general strategy we hereby use the slice theorem for proper Lie groupoids to describe the groupoid G locally in terms of translation groupoids by compact Lie groups. This will allow us to describe $\Lambda\mathsf{G}$ locally in terms of the inertia groupoid associated to such a translation groupoid. Using isotropy types and an equivalence relation on the group defined in terms of this action, we will construct stratifications for inertia groupoids of such translation groupoids that patch together to a well-defined stratification of $\Lambda_0\mathsf{G}$.

6.1. The inertia groupoid of a proper Lie groupoid.

Let G be a proper Lie groupoid. Define the loop space of G to be

$$\Lambda_0\mathsf{G} := \{h \in \mathsf{G}_1 \mid s(h) = t(h)\}.$$

Since the loop space is closed in G_1 it inherits the structure of a reduced differentiable space from the ambient manifold G_1. The map $s = t\colon \Lambda_0\mathsf{G} \to \mathsf{G}_0$ serves as an anchor map for the action of G on $\Lambda_0\mathsf{G}$ by conjugation. More precisely, the action of $g \in \mathsf{G}_1$ on $h \in \Lambda_0\mathsf{G}$ with $s(g) = s(h)$ is given by

$$g \cdot h = ghg^{-1}. \tag{6.1}$$

Definition 6.1. The *inertia groupoid* of a proper Lie groupoid G is the action groupoid $\Lambda\mathsf{G} := \mathsf{G} \ltimes \Lambda_0\mathsf{G}$. The space of its objects is the loop space $\Lambda_0\mathsf{G}$, its space of arrows is $\mathsf{G}_1 {}_s\!\times_s \Lambda_0\mathsf{G}$. The *inertia space* of G then is the orbit space $|\Lambda\mathsf{G}|$.

Remark 6.2. Note that, while $\Lambda_0\mathsf{G}$ is a differentiable subspace of the smooth manifold G_1, the action of G on $\Lambda_0\mathsf{G}$ does not necessarily extend to an action on G_1 or an open neighborhood of $\Lambda_0\mathsf{G}$ in G_1.

6.2. The stratification of the loop space.

The compact Lie group action case. Assume that the compact Lie group G acts by diffeomorphisms on the smooth manifold M. The loop space $\Lambda_0(G \ltimes M)$ coincides in this case with the union $\bigcup_{g\in G}\{g\} \times M^g$, where M^g denotes the fixed point space of $g \in G$. To describe our stratification of $\Lambda_0(G \ltimes V)$ recall first [BtD95, IV Def. 4.1] that a closed subgroup T of the Lie group G is called a *Cartan subgroup* if it is closed, topologically cyclic, and of finite index in its normalizer. As in [FPS15], we say T is

associated to an element $h \in G$ if $h \in \mathsf{T}$, and $\mathsf{T}/\mathsf{T}^\circ$ is generated by $h\mathsf{T}^\circ$. Now let $(h,x) \in \Lambda_0(G \ltimes V) \subset G \times V$, and choose a slice Y_x at x for the G-action on M. Then, after possibly shrinking Y_x and the choice of an appropriate G-invariant riemannian metric on M, Y_x is the image under the exponential map of an open ball $B_x \subset N_x$ around the origin of the normal space $N_x := T_xM/T_x(Gx)$ to the tangent space of the orbit through x. Let $H = G_{(h,x)}$, and note that $H = G_x \cap \mathsf{Z}_G(h) = \mathsf{Z}_{G_x}(h)$ is the centralizer of h in G_x. Let $\mathsf{T}_{(h,x)}$ be a Cartan subgroup of H associated to h. Note that if G_x is connected, the relation $h \in (\mathsf{Z}_{G_x}(h))^\circ = H^\circ$ holds true by [DK00, Thm. 3.3.1 (i)], so that $\mathsf{T}_{(h,x)}$ is a maximal torus of H° containing h. Define an equivalence relation $\simeq$ on $\mathsf{T}_{(h,x)}$ by $s \simeq t$ if $N_x^s = N_x^t$, and let $\mathsf{T}^*_{(h,x)}$ denote the connected component of the $\simeq$ class containing h. Note that by construction, $s \simeq t$ if and only if the germs of the sets Y_x^s and Y_x^t at x coincide.

Next choose a slice $V_{(h,x)}$ at (h,x) for the G_x-action on $G_x \times Y_x$ which is given by $g(k,y) = (gkg^{-1}, gy)$, i.e. by the diagonal action with conjugation on the G_x-factor. Then assign to (h,x) the germ

$$\mathcal{S}_{(h,x)} = \left[G\left(V^H_{(h,x)} \cap \left(\mathsf{T}^*_{(h,x)} \times Y_x^{G_x}\right)\right)\right]_{(h,x)}. \tag{6.2}$$

It will be demonstrated below that this yields a stratification of the loop space $\Lambda_0(G \ltimes M)$; see Theorem 6.5. We refer to this as the *orbit Cartan type stratification* of the loop space of the Lie groupoid $G \ltimes M$.

The germ $\mathcal{S}_{(h,x)}$ is obviously G_x-invariant, and hence, if intersected with $\Lambda_0(G_x \ltimes Y_x)$ (i.e. take G_x-orbits rather than G-orbits) defines a germ in the quotient $|\Lambda(G_x \ltimes Y_x)|$. This stratification depends only on the G-orbit of (h,x), and hence defines a germ in the quotient $|\Lambda(G \ltimes M)|$ as well. To see this, note that if $g \in G$, then gY_x is a slice at gx for the G-action on M, and conjugation by g maps G_x onto G_{gx}. Choosing $V_{g(h,x)}$ and $\mathsf{T}_{g(h,x)}$ to be the images of $V_{(h,x)}$ and $\mathsf{T}_{(h,x)}$ under the induced isomorphism $G_x \ltimes Y_x \to G_{gx} \ltimes gY_x$, the germ $\mathcal{S}_{g(h,x)}$ coincides with $\mathcal{S}_{(h,x)}$. Then the *orbit Cartan type stratification* of the inertia space $|\Lambda(G \ltimes M)|$ is given by

$$\mathcal{R}_{(h,x)} = G\backslash\left[G\left(V^H_{(h,x)} \cap \left(\mathsf{T}^*_{(h,x)} \times Y_x^{G_x}\right)\right)\right]_{(h,x)}. \tag{6.3}$$

The proper Lie groupoid case. We now turn to the case of a proper Lie groupoid G, and endow G with a transversally invariant riemannian metric. Recall [PPT14, Cor. 3.11] that for each point $\mathsf{x} \in \mathsf{G}_0$, there is an open neighborhood U of x in G_0 diffeomorphic to $O \times B_{\mathsf{x}}$ such that $\mathsf{G}_{|U}$ is isomorphic to the product of the pair groupoid over O and $\mathsf{G}_{\mathsf{x}} \ltimes B_{\mathsf{x}}$. Hereby, O is an open ball around x in the orbit of x and B_{x} is a G_{x}-invariant open ball around the origin in the normal space $N_{\mathsf{x}} = T_{\mathsf{x}}\mathsf{G}_0/T_{\mathsf{x}}\mathcal{O}_{\mathsf{x}}$ to the tangent space of the orbit through x. According to [PPT14, Thm. 4.1], one can achieve that the corresponding diffeomorphism $O \times B_{\mathsf{x}} \to \mathsf{G}_0$ is given, over the factor B_{x}, by the exponential map with respect to the chosen transversally invariant riemannian metric. We let $Y_{\mathsf{x}} \subset \mathsf{G}_0$ denote the image of $\{\mathsf{x}\} \times B_{\mathsf{x}}$ under this diffeomorphism and call it a *slice for* G *at* x. By [PPT14, Thm. 4.1], $\mathsf{G}_{|Y_{\mathsf{x}}}$ is isomorphic to $\mathsf{G}_{\mathsf{x}} \ltimes B_{\mathsf{x}}$. Since the latter is isomorphic to $\mathsf{G}_{\mathsf{x}} \ltimes Y_{\mathsf{x}}$, we obtain an isomorphism between $\mathsf{G}_{|Y_{\mathsf{x}}}$ and $\mathsf{G}_{\mathsf{x}} \ltimes Y_{\mathsf{x}}$ which is induced by the exponential map and the canonical action of G_{x} on N_{x}. This isomorphism gives rise to an embedding $\mathsf{G}_{\mathsf{x}} \ltimes Y_{\mathsf{x}} \hookrightarrow \mathsf{G}$ of differentiable stratified groupoids.

Remark 6.3. Note that if $\mathsf{G} = G \ltimes M$ is a translation groupoid, then a slice as defined here corresponds to a slice for the G-action on M, so using the same notation for both will cause no confusion.

To define a stratification of $\Lambda_0\mathsf{G}$, we will employ the stratification constructed above of the loop space $\Lambda_0(\mathsf{G}_x \ltimes Y_x)_0$. To this end choose $g \in \Lambda_0\mathsf{G}$ with $s(g) = t(g) = x \in \mathsf{G}_0$. We then define the germ $\mathcal{S}_g^{\mathsf{G}}$ of the stratification of $\Lambda_0\mathsf{G}$ as follows. Let $h \in \mathsf{G}_x$ denote the element such that (h, x) corresponds to the arrow g under the isomorphism between $\mathsf{G}_{|Y_x}$ and $\mathsf{G}_x \ltimes Y_x$. Let $\mathcal{S}_{(h,x)}$ be the germ of the orbit Cartan type stratification of $\Lambda_0(\mathsf{G}_x \ltimes Y_x)$, and let $\mathrm{Sat}(\mathcal{S}_{(h,x)})$ denote its saturation within G, i.e. the germ of the saturation of a defining set for $\mathcal{S}_{(h,x)}$. Putting

$$\mathcal{S}_g^{\mathsf{G}} := \mathrm{Sat}(\mathcal{S}_{(h,x)}) \tag{6.4}$$

then defines the *orbit Cartan type stratification of the loop space* $\Lambda_0\mathsf{G}$. Similarly, we define

$$\mathcal{R}_{\Lambda\pi(g)}^{\mathsf{G}} := \Lambda\pi(\mathrm{Sat}(\mathcal{S}_{(h,x)})), \tag{6.5}$$

where $\Lambda\pi\colon \Lambda_0\mathsf{G} \to |\Lambda\mathsf{G}|$ is the orbit map of the inertia groupoid. That means that the germ $\mathcal{R}_{\Lambda\pi(g)}^{\mathsf{G}}$ in the orbit space $|\Lambda\mathsf{G}|$ is defined to be the projection of $\mathcal{S}_g^{\mathsf{G}}$ to the orbit space.

With these definitions, we have the following.

Theorem 6.4. *Let G be a proper Lie groupoid. Then Equation* (6.4) *defines a Whitney stratification of the loop space $\Lambda_0\mathsf{G}$ with respect to which the inertia groupoid $\Lambda\mathsf{G}$ is a locally translation differentiable stratified groupoid. Moreover, the inertia space $|\Lambda\mathsf{G}|$ inherits a differentiable structure, and Equation* (6.5) *defines a stratification with respect to which the inertia space is a differentiable stratified space and the orbit map $\Lambda_0\mathsf{G} \to |\Lambda\mathsf{G}|$ a differentiable stratified surjective submersion.*

In order to prove Theorem 6.4, the primary focus will be to demonstrate the following.

Theorem 6.5. *Let G be a compact Lie group and M a smooth G-manifold. Then Equation* (6.2) *defines a G-invariant stratification of the loop space $\Lambda_0(G \ltimes M)$ with respect to which $\Lambda_0(G \ltimes M)$ is a differentiable stratified space such that the G-orbits are subsets of strata.*

Before turning to the proof of Theorem 6.5, assume it holds. Assume further that the stratification of $\Lambda_0\mathsf{G}$ is well-defined, i.e. that it does not depend on the choice of a point in an orbit nor a slice at that point, and that the stratification fulfills Whitney's condition B. Recall that Whitney stratified spaces are topologically locally trivial, see [Pfl01, Cor. 3.9.3], and that strata contain orbits because they are defined as saturations. Hence $\Lambda G = \mathsf{G} \ltimes \Lambda_0\mathsf{G}$ is a differentiable stratified groupoid by Proposition 3.1. That G is a locally translation differentiable groupoid follows from [PPT14, Prop. 3.9 & Cor. 3.11]. Similarly, by the definition of the stratification of $\Lambda_0\mathsf{G}$ in terms of stratifications of the inertia spaces of slices $\mathsf{G}_x \ltimes Y_x$, $x \in \mathsf{G}_0$, the inertia groupoid $\Lambda\mathsf{G}$ is even a locally translation differentiable stratified groupoid. Moreover, since the elements of a stratum of a slice Y_x all have the same G_x-orbit type, the inertia space is a differentiable stratified space by Proposition 2.29, and the orbit map is a stratified surjective submersion.

Hence, once we demonstrate that the stratification of $\Lambda_0\mathsf{G}$ is well-defined, prove Theorem 6.5 and verify Whitney's condition B, Theorem 6.4 will follow. We first show here

that the stratification is well-defined, assuming well-definition of the stratification for a translation groupoid $G \ltimes M$. In the following section, we prove Theorem 6.5. Afterwards we verify Whitney's condition B to hold true for the orbit Cartan type stratification of the inertia groupoid and the inertia space of a proper Lie groupoid.

Proposition 6.6. *Let G be a proper Lie groupoid,and let $x, y \in \mathsf{G}_0$ be points in the same orbit. Let $g \in \mathsf{G}_1$ such that $s(g) = t(g) = x$, and let $h \in \mathsf{G}_1$ such that $s(h) = x$ and $t(h) = y$. Put $g' := hgh^{-1}$. Then $\mathcal{S}_g = \mathcal{S}_{g'}$. In particular, $\mathcal{S}_g$ does not depend on the choice of a slice Y_x.*

Proof. Choose slices Y_x and Y_y for G at x and y, respectively. Then there are identifications $\mathsf{G}_{|Y_x} \cong \mathsf{G}_x \ltimes Y_x$ and $\mathsf{G}_{|Y_y} \cong \mathsf{G}_y \ltimes Y_y$ by [PPT14, Thm. 3.3]. Under these identifications let $g = (h, x)$ and $g' = (k, y)$ for some $h \in \mathsf{G}_x$ and $k \in \mathsf{G}_y$. Choose a local bisection $\sigma\colon U \to \mathsf{G}_1$ defined on an open neighborhood of x such that $\sigma(x) = h$ and such that $t \circ \sigma_{|Y_x}$ induces a diffeomorphism from Y_x to Y_y. The existence of such a bisection, after possibly shrinking Y_x and Y_y, is guaranteed by [PPT14, Prop. 3.9 & proofs of Lemmata 5.1 & 5.2]. Then we obtain an isomorphism $\Psi\colon \mathsf{G}_x \ltimes Y_x \to \mathsf{G}_y \ltimes Y_y$ which is given by the composition of the diffeomeorphism

$$\mathsf{G}_{|Y_x} \to \mathsf{G}_{|Y_y}, \quad k \mapsto \big(\sigma(t(k))\big)k\big(\sigma(s(k))\big)^{-1}$$

with the identifications $\mathsf{G}_{|Y_x} \cong \mathsf{G}_x \ltimes Y_x$ and $\mathsf{G}_{|Y_y} \cong \mathsf{G}_y \ltimes Y_y$. Note that Ψ obviously restricts to a diffeomorphism from $\Lambda_0(\mathsf{G}_x \ltimes Y_x)$ onto $\Lambda_0(\mathsf{G}_y \ltimes Y_y)$. Moreover, by construction, $\Psi(h, x)$ is the image of

$$\big(\sigma(x)\big)g\big(\sigma(x)\big)^{-1} = hgh^{-1} = g'$$

under the identification $\mathsf{G}_{|Y_y} \cong \mathsf{G}_y \ltimes Y_y$, hence $\Psi(h, x) = (k, y)$. Now we choose a slice $V_{(h,x)}$ at (h, x) for the G_x-action on $\mathsf{G}_x \ltimes Y_x$ and a Cartan subgroup $\mathsf{T}_{(h,x)}$ of $\mathsf{Z}_{\mathsf{G}_x}(h)$ associated to h. Then $\Psi(V_{(h,x)})$ is a slice at (k, y) for the G_y-action on $\mathsf{G}_y \times Y_y$, and $\Psi(\mathsf{T}_{(h,x)} \times \{x\})$ a Cartan subgroup of $\mathsf{Z}_{\mathsf{G}_y}(k)$ associated to k. Moreover, we claim that $\Psi(\mathsf{T}^*_{(h,x)} \times \{x\}) = \mathsf{T}^*_{(k,y)} \times \{y\}$. To see this, note that $\Psi_0\colon Y_x \to Y_y$ is a diffeomorphism that is $\mathsf{T}_{(h,x)}$-equivariant with respect to the isomorphism $\tau\colon \mathsf{T}_{(h,x)} \to \mathsf{T}_{(k,y)}$ given by $\tau(k) = \pi_1 \circ \Psi(k, x)$, where π_1 denotes the projection $\pi_1\colon \mathsf{G}_y \times Y_y \to \mathsf{G}_y$. Then for $z \in Y_y$ and $s \in \mathsf{T}_{(h,x)}$, $\tau(s)z = \tau(s)(\Psi_0 \circ \Psi_0^{-1}(z)) = \Psi_0(s(\Psi_0^{-1}(z)))$. Hence $\tau(s)z = z$ if and only if $s(\Psi_0^{-1}(z)) = \Psi_0^{-1}(z)$, from which it follows that Ψ_0 maps the set of points of Y_x fixed by $s \in \mathsf{T}_{(h,x)}$ onto the fixed set of $\tau(s)$ in Y_y. The isomorphism Ψ then maps $\mathcal{S}_{(h,x)}$ onto $\mathcal{S}_{(k,y)}$. This proves the claim. If $x = y$, this argument shows that the stratification does not depend on the choice of the slice Y_x. □

6.3. **Proof of Theorem 6.5.** Because the stratification given by Equation (6.2) is a variation on the stratification given by [FPS15, Thm. 4.1], we refer the reader there for arguments that are identical or only slightly modified. Observe though that the stratification given here is coarser than the one from [FPS15, Thm. 4.1], and that the virtue of the new definition is that it is invariant under Morita equivalences, hence can be glued together via chart changes.

We assume $G \times M$ is equipped with a riemannian metric given by the product of a G-invariant metric on M and a bi-invariant metric on G. For points $y \in M$ and $(h, x) \in \Lambda_0(G \ltimes M)$, we will denote by Y_y always a slice at y for the G-action on M and by $V_{(h,x)}$ a slice at (h, x) for the G_x-action on $G_x \times Y_x$. We use H to denote the

isotropy group $G_{(h,x)} = \mathsf{Z}_{G_x}(h)$ of (h,x) and the symbol $N_{(h,x)}$ to denote the normal space $T_{(h,x)}(G_x \times Y_x)/T_{(h,x)}(G_x(h,x)) \cong T_{(h,x)}(G \times M)/T_{(h,x)}(G(h,x))$.

It will be helpful to observe the following, which is a slight strengthening of a special case of [BtD95, Prop. 4.6].

Lemma 6.7. *Let G be a compact Lie group, let h and k be elements of a single connected component of G, and let T_h and T_k be Cartan subgroups of G associated to h and k, respectively. Then there is an element $g \in G^\circ$ such that*

$$g\mathsf{T}_k g^{-1} = \mathsf{T}_h \quad \textit{and} \quad g(k\mathsf{T}_k^\circ)g^{-1} = h\mathsf{T}_h^\circ.$$

Proof. By [BtD95, Prop. 4.6], we know a priori that T_h and T_k are conjugate and hence isomorphic. Let $\mathsf{T}_k \cong \mathbb{T}^\ell \times \mathbb{Z}/r\mathbb{Z}$ for some ℓ and r. Then the topological generators of $\mathbb{T}^\ell \times \mathbb{Z}/r\mathbb{Z}$, i.e. the elements that generate a dense subset of $\mathbb{T}^\ell \times \mathbb{Z}/r\mathbb{Z}$, are given by (s,α) where s is a topological generator of $\mathbb{T}^\ell$ and α is a generator of $\mathbb{Z}/r\mathbb{Z}$; see [BtD95, proof of Prop. 4.6]. Since $k\mathsf{T}_k^\circ$ generates $\mathsf{T}_k/\mathsf{T}_k^\circ \cong \mathbb{Z}/r\mathbb{Z}$, we may therefore choose a topological generator t of T_k such that $t \in k\mathsf{T}_k^\circ$. Note that t, h, and k are all in the same connected component of G. By the same argument a topological generator of T_h can be chosen to be an element of $h\mathsf{T}_h^\circ$. Hence there is an element $g \in G^\circ$ such that $gtg^{-1} \in h\mathsf{T}_h^\circ$. But then gtg^{-1} generates a subgroup of T_h that is dense in $g\mathsf{T}_k g^{-1}$. Therefore $g\mathsf{T}_k g^{-1}$ is contained in T_h. But as $g\mathsf{T}_k g^{-1}$ is isomorphic to T_k, hence to T_h, we have $g\mathsf{T}_k g^{-1} = \mathsf{T}_h$. □

The following is a simple consequence of the definition of a Cartan subgroup.

Lemma 6.8. *Let $(h,x) \in \Lambda_0(G \ltimes M)$ and $H = G_{(h,x)}$. A Cartan subgroup $\mathsf{T}_{(h,x)}$ of G_x associated to h is also a Cartan subgroup of H associated to h.*

Recall that the equivalence relation $\simeq$ on a Cartan subgroup T of the isotropy group G_x is defined by setting $s \simeq t$ if and only if $N_x^s = N_x^t$, where N_x denotes the normal space $T_xM/T_x(Gx)$. We denote by $[s]$ the $\simeq$ class of $s \in \mathsf{T}$. Note that by definition of a slice, $s \simeq t$ holds true if and only $Y_x^s = Y_x^t$ for one, hence all (sufficiently small) slices Y_x at x. We recall the following properties of the relation $\simeq$, whose proofs are analogous to [FPS15, Lemmata 4.7, 4.8, & 4.10] and hence are omitted.

Proposition 6.9. *Let Y be a slice for the G-action on M through a point $x \in M$ and let $\mathsf{T} \subset G_x$ be a Cartan subgroup. Then the following holds true.*

(1) *The group T is partitioned into a finite number of $\simeq$ classes, each with a finite number of connected components. Each $\simeq$ class $[t]$ is an open subset of the closed subgroup $t^\bullet$ of T defined by*

$$t^\bullet := \bigcap_{t \in H_i} H_i = \bigcap_{y \in Y^t} \mathsf{T}_y,$$

where $\{H_0, \dots, H_r\}$ is the finite set of isotropy groups for the T-action on Y, T_y is the isotropy group of y in T, and $\overline{[t]}$ consists of a union of connected components of $t^\bullet$.

(2) *If $s,t \in \mathsf{T}$ with $[s] \cap \overline{[t]} \neq \emptyset$, then for each connected component $[s]^\circ$ of $[s]$ and $[t]^\circ$ of $[t]$ such that $[s]^\circ \cap \overline{[t]}^\circ \neq \emptyset$, we have $[s]^\circ \subset \overline{[t]}^\circ$.*

(3) *If $s,t \in \mathsf{T}$ such that $s \not\simeq t$ and $[s]$ is diffeomorphic to $[t]$, then $[s] \cap \overline{[t]} = \emptyset$.*

Using the observation that equivariant diffeomeorphisms map $\simeq$ classes in a Cartan subgroup T onto $\simeq$ classes in the image Cartan subgroup T', proven as in Proposition 6.6 above, it is straightforward to verify the following; see also [FPS15, Lem. 4.11].

Lemma 6.10. *The normalizer $\mathsf{N}_H(\mathsf{T})$ of T in $H = G_{(h,x)}$ acts on the finite set of $\simeq$ classes in T in such a way that for each $n \in \mathsf{N}_H(\mathsf{T})$ and $t \in \mathsf{T}$, the submanifold $n[t]n^{-1}$ is diffeomorphic to $[t]$. Moreover, either $n[t]n^{-1} = [t]$ or $\overline{n[t]n^{-1}} \cap [t] = \emptyset$.*

The set germ $\mathcal{S}_{(h,x)}$ is contained in the germ at (h,x) of points in $\Lambda_0(G \ltimes M)$ having the same isotropy type as (h,x) with respect to the G-action on $G \times M$. For, if $(k,y) \in V^H_{(h,x)} \cap \big(\mathsf{T}^*_{(h,x)} \times Y_x^{G_x}\big)$, then $G_x = G_y$, so the isotropy group of (k,y) with respect to the G-action on $G \times M$ coincides with the isotropy group of (k,y) with respect to the G_x-action on $G_x \times Y_x$. But this isotropy group is equal to H as $(k,y) \in V^H_{(h,x)}$. Similarly, using the same argument as for [FPS15, Lem. 4.14], one shows that the germ $\mathcal{S}_{(h,x)}$ does not depend on the choice of a Cartan subgroup $\mathsf{T}_{(h,x)}$ of H associated to h.

In order to prove that the stratification does not depend on the choice of a slice $V_{(h,x)}$, we will need the following lemma, which essentially demonstrates that two slices at the same point are related by a local bisection that acts on $\mathsf{T}_{(h,x)}$ by conjugation in a way that fixes the connected component of h.

Proposition 6.11. *Let $V_{(h,x)}$ and $W_{(h,x)}$ be slices at (h,x) for the G_x-action on $G_x \times Y_x$ and let $H = \mathsf{Z}_{G_x}(h)$.*

(1) *Possibly after shrinking the slices, there is a smooth function $\sigma\colon W_{(h,x)} \to G_x$, $(k,y) \to \sigma_{(k,y)}$ such that the map $\tau\colon W_{(h,x)} \to G_x \times Y_x$ given by $\tau(k,y) = \sigma_{(k,y)}(k,y)$ is an H-diffeomorphism of $W_{(h,x)}$ onto $V_{(h,x)}$ which satisfies $\tau(h,x) = (h,x)$, meaning $\sigma_{(h,x)} \in H$. Moreover, one can choose σ in such a way that $\sigma_{(h,x)}$ is any arbitrary element of H.*

(2) *For any σ as in (1), one has $\sigma(W^H_{(h,x)}) \subset \sigma_{(h,x)}\mathsf{N}_{G_x}(H)^\circ$.*

(3) *Fix a $\sigma\colon W_{(h,x)} \to G_x$ as above and suppose $\sigma_{(h,x)} \in \mathsf{N}_{G_x}(H)^\circ$. Let $\mathsf{T}_{(h,x)}$ be a Cartan subgroup of H associated to h and define*

$$K = \mathsf{N}_H(\mathsf{T}_{(h,x)}) \cap \mathsf{N}_H(h\mathsf{T}^\circ_{(h,x)}) \cap H^\circ.$$

Then K is a closed subgroup of H°. There is a continuous function $(W_{(h,x)})^H \to K\backslash H^\circ$, which we denote $(k,y) \mapsto Kg_{(k,y)}$, such that for each $(k,y) \in W^H_{(h,x)}$ the relations

$$g_{(k,y)}\sigma_{(k,y)}\mathsf{T}_{(h,x)}\sigma^{-1}_{(k,y)}g^{-1}_{(k,y)} = \mathsf{T}_{(h,x)} \quad \text{and} \quad g_{(k,y)}\sigma_{(k,y)}\big(h\mathsf{T}^\circ_{(h,x)}\big)\sigma^{-1}_{(k,y)}g^{-1}_{(k,y)} = h\mathsf{T}^\circ_{(h,x)}$$

are fulfilled. In other words, $g_{(k,y)}\sigma_{(k,y)} \in \mathsf{N}_H(\mathsf{T}_{(h,x)})$, and the action of $g_{(k,y)}\sigma_{(k,y)}$ on $\mathsf{T}_{(h,x)}$ by conjugation fixes the connected component containing h.

Proof. Statement (1) is a special case of [PPT14, Prop. 3.9] where it is shown to be true for proper Lie groupoids. The claim that $\sigma_{(h,x)}$ can be an arbitrary element of H can be verified by choosing an arbitrary σ and an $\ell \in H$ and then redefining σ as the map $(k,y) \mapsto \ell\sigma^{-1}_{(h,x)}\sigma_{(k,y)}$.

To prove (2), fix an arbitrary σ as in (1). If $(k,y) \in W^H_{(h,x)}$, then $G_{\sigma_{(k,y)}(k,y)} = \sigma_{(k,y)}G_{(k,y)}\sigma^{-1}_{(k,y)} = \sigma_{(k,y)}H\sigma^{-1}_{(k,y)}$. As $\sigma_{(k,y)}(k,y) \in V_{(h,x)}$ implies $G_{\sigma_{(k,y)}(k,y)} \leq H$, one obtains $G_{\sigma_{(k,y)}(k,y)} = H$. Therefore, σ maps $W^H_{(h,x)}$ into $\mathsf{N}_{G_x}(H)$. By the connectedness of $W^H_{(h,x)}$, the funtion σ therefore maps $(W_{(h,x)})^H$ into $\sigma_{(h,x)}\mathsf{N}_{G_x}(H)^\circ$.

We now turn to (3). For each $(k,y) \in W^H_{(h,x)}$ the action of $\sigma_{(k,y)}$ by conjugation on H fixes the connected components of H. This follows from $\sigma_{(k,y)} \in \mathsf{N}_{G_x}(H)^\circ$ and

the fact that the induced map from $\mathsf{N}_{G_x}(H)$ into the symmetric group of the connected components of H is continuous and hence maps $\mathsf{N}_{G_x}(H)^\circ$ into the identity. Then $\sigma_{(k,y)}\mathsf{T}_{(h,x)}\sigma_{(k,y)}^{-1}$ is a Cartan subgroup of H associated to $\sigma_{(k,y)}h\sigma_{(k,y)}^{-1} \in hH^\circ$. By Lemma 6.7, there is a $g_{(k,y)} \in H^\circ$ such that $g_{(k,y)}\sigma_{(k,y)}\mathsf{T}_{(h,x)}\sigma_{(k,y)}^{-1}g_{(k,y)}^{-1} = \mathsf{T}_{(h,x)}$ and such that $g_{(k,y)}\big(\sigma_{(k,y)}h\mathsf{T}_{(h,x)}^\circ\sigma_{(k,y)}^{-1}\big)g_{(k,y)}^{-1} = h\mathsf{T}_{(h,x)}^\circ$. Then $g_{(k,y)}\sigma_{(k,y)} \in \mathsf{N}_H(\mathsf{T}_{(h,x)})$ is clear.

Of course, $g_{(k,y)}$ is not unique; however, if $g_{(k,y)}$ and $g'_{(k,y)}$ are two such choices, then a routine computation demonstrates that $g'_{(k,y)}g_{(k,y)} \in \mathsf{N}_H(\mathsf{T}_{(h,x)})$, and $g'_{(k,y)}g_{(k,y)} \in \mathsf{N}_H(h\mathsf{T}_{(h,x)}^\circ)$. Similarly, as both $g_{(k,y)}$ and $g'_{(k,y)}$ are elements of H° by construction, $g_{(k,y)}g'_{(k,y)} \in H^\circ$. That is, $g'_{(k,y)}g_{(k,y)}^{-1} \in K$.

Conversely, if $n \in K$ and $g_{(k,y)}$ is given as above, then we have $ng_{(k,y)} \in H^\circ$

$$ng_{(k,y)}\sigma_{(k,y)}\mathsf{T}_{(h,x)}\sigma_{(k,y)}^{-1}g_{(k,y)}^{-1}n^{-1} = n\mathsf{T}_{(h,x)}n^{-1} = \mathsf{T}_{(h,x)},$$

and

$$ng_{(k,y)}\sigma_{(k,y)}\big(h\mathsf{T}_{(h,x)}^\circ\big)\sigma_{(k,y)}^{-1}g_{(k,y)}^{-1}n^{-1} = n\big(h\mathsf{T}_{(h,x)}^\circ\big)n^{-1} = h\mathsf{T}_{(h,x)}^\circ,$$

so that $ng_{(k,y)}$ satisfies the desired properties as well. Therefore, while $g_{(k,y)}$ is not unique, the right coset $Kg_{(k,y)}$ is determined uniquely. Note that K is indeed a closed subgroup of H, as $\mathsf{N}_H(h\mathsf{T}_{(h,x)}^\circ) \cap \mathsf{N}_H(\mathsf{T}_{(h,x)})$ is a union of connected components of $\mathsf{N}_H(\mathsf{T}_{(h,x)})$; this is clear by considering the homomorphism of $\mathsf{N}_H(\mathsf{T}_{(h,x)})$ into the symmetric group on the connected components of $\mathsf{T}_{(h,x)}/\mathsf{T}_{(h,x)}^\circ$. We claim that the assignment $(k,y) \mapsto Kg_{(k,y)}$ is a continuous function $W_{(h,x)}^H \to K\backslash H^\circ$.

Let $(k_i,y_i)_{i\in\mathbb{N}} \subset W_{(h,x)}^H$ be a convergent sequence with $\lim_{i\to\infty}(k_i,y_i) = (k,y)$. Put $g_i := g_{(k_i,y_i)}$ for each $i \in \mathbb{N}$. Without loss of generality, we may assume that $\lim_{i\to\infty} g_i = g \in H^\circ$. Choose a topological generator t of $\mathsf{T}_{(h,x)}$. As $\mathsf{T}_{(h,x)}$ is a Cartan subgroup associated to h, we may assume that t is in the same connected component $h\mathsf{T}_{(h,x)}^\circ$ of $\mathsf{T}_{(h,x)}$ as h; see the proof of Lemma 6.7 above. Then we have $g_i\sigma_{(k_i,y_i)}t\sigma_{(k_i,y_i)}^{-1}g_i^{-1} \in h\mathsf{T}_{(h,x)}^\circ$ for each i. Since σ is a continuous function, we may take the limit to conclude $g\sigma_{(k,y)}t\sigma_{(k,y)}^{-1}g^{-1} \in h\mathsf{T}_{(h,x)}^\circ$. However, as t generates a subgroup that is dense in $\mathsf{T}_{(h,x)}$, we have that $g\sigma_{(k,y)}t\sigma_{(k,y)}^{-1}g^{-1}$ generates a subgroup that is dense in a subgroup of $\mathsf{T}_{(h,x)}$ that is isomorphic to $\mathsf{T}_{(h,x)}$, and then must be equal to $\mathsf{T}_{(h,x)}$. This implies

$$g\sigma_{(k,y)}\mathsf{T}_{(h,x)}\sigma_{(k,y)}^{-1}g^{-1} = \mathsf{T}_{(h,x)},$$

and

$$g_{(k,y)}\big(\sigma_{(k,y)}h\mathsf{T}_{(h,x)}\sigma_{(k,y)}^{-1}\big)^\circ g_{(k,y)}^{-1} = h\mathsf{T}_{(h,x)}^\circ.$$

Hence, we can take $g_{(k,y)} = g$, determining the unique coset of $K\backslash H^\circ$, and hence the map $W_{(h,x)}^H \to K\backslash H^\circ$ given by $(k,y) \mapsto Kg_{(k,y)}$ is continuous.

□

With this, we have the following.

Lemma 6.12. *The germ $\mathcal{S}_{(h,x)}$ is independent of the particular choice of the slice $V_{(h,x)}$ at (h,x). In fact, given slices $V_{(h,x)}$ and $W_{(h,x)}$ at (h,x), the germs of $V_{(h,x)}^H \cap \big(\mathsf{T}_{(h,x)}^* \times Y_x^{G_x}\big)$ and $W_{(h,x)}^H \cap \big(\mathsf{T}_{(h,x)}^* \times Y_x^{G_x}\big)$ coincide at (h,x).*

Proof. Suppose $V_{(h,x)}$ and $W_{(h,x)}$ are two choices of slices at (h,x) for the G_x-action on $G_x \times Y_x$. Let $\mathsf{T}_{(h,x)}$ be a Cartan subgroup of H associated to h. As the germ of the stratification does not depend on the choice of Cartan subgroup, we may assume that the stratum containing (h,x) is defined with respect to each of the two slices using this Cartan subgroup.

By Proposition 6.11 (1) there exists, after shrinking slices if necessary, a function $\sigma\colon W_{(h,x)} \to G_x$ such that $(k,y) \mapsto \sigma_{(k,y)}y$ is a H-diffeomorphism of $W_{(h,x)}$ onto $V_{(h,x)}$. We may assume that $\sigma_{(h,x)} = 1$. Choose a Cartan subgroup $\mathsf{T}_{(h,x)}$ of H associated to h and then, by Proposition 6.11 (3), a continuous function $W^H_{(h,x)} \to K\backslash H^\circ$ denoted $(k,y) \mapsto Kg_{(k,y)}$ such that for each $(k,y) \in W^H_{(h,x)}$, we have $g_{(k,y)}\sigma_{(k,y)}\mathsf{T}_{(h,x)}\sigma^{-1}_{(k,y)}g^{-1}_{(k,y)} = \mathsf{T}_{(h,x)}$, and $g_{(k,y)}\sigma_{(k,y)}\big(h\mathsf{T}^\circ_{(h,x)}\big)\sigma^{-1}_{(k,y)}g^{-1}_{(k,y)} = h\mathsf{T}^\circ_{(h,x)}$. In particular, as $\sigma_{(h,x)} = 1$, it follows that $Kg_{(h,x)} = K$. Choose a local section for the fiber bundle $H^\circ \to K\backslash H^\circ$ near the point K such that $K \mapsto 1$. Shrinking slices if necessary so that each $g_{(k,y)}$ is contained in the domain of this section for $(k,y) \in W^H_{(h,x)}$ (which is possible as $g_{(h,x)} = K$), we have a continuous choice of a specific representative $g_{(k,y)}$ of each coset $Kg_{(k,y)}$. In particular, as $\sigma_{(h,x)} = 1$ (and as the section was chosen such that $K \mapsto 1$, we have $g_{(h,x)} = 1$).

For each $(k,y) \in W_{(h,x)}$, we have that $g_{(k,y)}\sigma_{(k,y)} \in \mathsf{N}_{G_x}(\mathsf{T}_{(h,x)})$, and hence conjugation by $g_{(k,y)}\sigma_{(k,y)}$ yields an element of $\mathrm{Aut}(\mathsf{T}_{(h,x)})$. That is, we have a function $W^H_{(h,x)} \to \mathrm{Aut}(\mathsf{T}_{(h,x)})$ which is continuous by construction. However, the automorphism group of $\mathsf{T}_{(h,x)}$ is discrete, see [BtD95, proof of Proposition 4.2], so that as $W^H_{(h,x)}$ is connected, the map into $\mathrm{Aut}(\mathsf{T}_{(h,x)})$ must be constant. Moreover, as $g_{(h,x)}\sigma_{(h,x)} = 1$, we see that each $g_{(k,y)}\sigma_{(k,y)}$ maps to the trivial element of $\mathrm{Aut}(\mathsf{T}_{(h,x)})$.

We let $U = G_xV_{(h,x)} \cap G_xW_{(h,x)}$ and claim that

$$U \cap \Big(W^H_{(h,x)} \cap \big(\mathsf{T}^*_{(h,x)} \times Y_x^{G_x}\big)\Big) = U \cap \Big(V^H_{(h,x)} \cap \big(\mathsf{T}^*_{(h,x)} \times Y_x^{G_x}\big)\Big)$$

Suppose $(k,y) \in W^H_{(h,x)} \cap \big(\mathsf{T}^*_{(h,x)} \times Y_x^{G_x}\big)$. Using the above construction, we have $\sigma_{(k,y)}(k,y) \in V_{(h,x)}$. As $\sigma_{(k,y)} \in \mathsf{N}_{G_x}(H)$, we have $\sigma_{(k,y)}(k,y) \in V^H_{(h,x)}$. Similarly, $\sigma_{(k,y)} \in G_x$ and $y \in Y_x^{G_x}$, so that $\sigma_{(k,y)}y = y \in Y_x^{G_x}$. Then as H fixes $\sigma_{(k,y)}(k,y)$ and $g_{(k,y)} \in H$, it follows that

$$g_{(k,y)}\sigma_{(k,y)}k\sigma^{-1}_{(k,y)}g^{-1}_{(k,y)} \in g_{(k,y)}\sigma_{(k,y)}\mathsf{T}_{(h,x)}\sigma^{-1}_{(k,y)}g^{-1}_{(k,y)} = \mathsf{T}_{(h,x)}.$$

But then as $g_{(k,y)}\sigma_{(k,y)}$ acts trivially on $\mathsf{T}_{(h,x)}$, we have that $g_{(k,y)}\sigma_{(k,y)}k\sigma^{-1}_{(k,y)}g^{-1}_{(k,y)} = k$, so that the point $(k,y) \in V^H_{(h,x)} \cap \big(\mathsf{T}^*_{(h,x)} \times Y_x^{G_x}\big)$ to begin with. Switching roles completes the proof. □

With this, the proof of the following proposition uses identical techniques to those of [FPS15, Prop. 4.16]. In particular, it is a matter of choosing specific slices in terms of orthogonal complements using the riemannian metric.

Proposition 6.13. *Each $\mathcal{S}_{(h,x)}$ is the germ of a smooth G-submanifold of $G \times M$ that intersects $G_x \times Y_x$ as a smooth submanifold. Each $\mathcal{R}_{(h,x)}$ is the germ of a smooth submanifold of the differentiable space $G\backslash(G \times M)$ that intersects $G_x\backslash(G_x \times Y_x)$ as a smooth submanifold.*

In order for the germs $\mathcal{S}_{(h,x)}$ to define a stratification, one must verify that for each $(h,x) \in \Lambda_0(G \ltimes V)$ there is a neighborhood U in $\Lambda_0(G \ltimes V)$ and a decomposition $\mathcal{Z}$

of U such that for all $(k,y) \in \Lambda_0(G \ltimes V)$, the germ $\mathcal{S}_{(k,y)}$ coincides with the germ of the piece of $\mathcal{Z}$ containing (k,y). For the remainder of this section, we fix (h,x), a slice Y_x at x for the G_x-action on V, and a slice $V_{(h,x)}$ at (h,x) for the G_x-action on $G_x \times Y_x$. Set $U := GV_{(h,x)} \cap \Lambda_0(G \ltimes V)$. Note that U is indeed an open neighborhood of (h,x) in $\Lambda_0(G \ltimes V)$, as $G_xV_{(h,x)}$ is an open G_x-invariant neighborhood of (h,x) in $\Lambda_0(G_x \ltimes Y_x)$ and so the G-saturation is as well by [FPS15, Prop. 3.6]. We now define the decomposition $\mathcal{Z}$ of U.

Given $(\tilde{k},\tilde{y}) \in U$ there is a $\tilde{g} \in G$ such that $\tilde{g}(\tilde{k},\tilde{y}) \in V_{(h,x)}$. Put $(k,y) = \tilde{g}(\tilde{k},\tilde{y})$ and $K = G_{(k,y)} \leq H$, and let $\mathsf{T}_{(k,y)}$ be a Cartan subgroup in K associated to k. Define $\mathcal{U}_{\tilde{g}}^{\mathsf{T}_{(k,y)}}(\tilde{k},\tilde{y})$ to be the G-saturation of the set of points $(l,z) \in (V_{(h,x)})_K \cap (\mathsf{T}_{(k,y)} \times (Y_x)_{G_y})$ such that $\mathsf{T}_{(k,y)}$ is also a Cartan subgroup of K associated to l and such that the $\simeq$ class of l at z in $\mathsf{T}_{(k,y)}$ is diffeomorphic to $\mathsf{T}^*_{(k,y)}$. Define the piece $\mathcal{Z}$ containing $(\tilde{k},\tilde{y})$ to be the connected component of $\mathcal{U}_{\tilde{g}}^{\mathsf{T}_{(k,y)}}(\tilde{k},\tilde{y})$ containing $(\tilde{k},\tilde{y})$.

By [Sch80, Proposition 1.2(3)], the slice representations of points in the same orbit type are isomorphic. Hence, if $z \in (Y_x)_{G_y}$, then the slice representations N_y and N_z for the action of G_x on Y_x at y and z, respectively, are isomorphic as G_y-representations. This isomorphism induces an isomorphism of $\mathsf{T}_{(k,y)}$-representations, which induces a diffeomorphism of $\simeq$ classes at y onto $\simeq$ classes at z. Then the $\simeq$ class of l at y is diffeomorphic to the $\simeq$ class of l at z, and hence to the $\simeq$ class of k at y. Therefore, the set $\mathcal{U}_{\tilde{g}}^{\mathsf{T}_{(k,y)}}(\tilde{k},\tilde{y})$ can be written as

$$\mathcal{U}_{\tilde{g}}^{\mathsf{T}_{(k,y)}}(\tilde{k},\tilde{y}) = G\Big((V_{(h,x)})_K \cap \big(\mathsf{T}^{**}_{(k,y)} \times (Y_x)_{G_y}\big)\Big),$$

where $\mathsf{T}^{**}_{(k,y)}$ denotes the union of $\simeq$ classes in $\mathsf{T}_{(k,y)}$ that are diffeomorphic to $\mathsf{T}^*_{(k,y)}$. Choosing representatives $k_0, \ldots, k_r$ from the collection of such $\simeq$ classes with $k_0 = k$ and noting that this collection is finite by Proposition 6.9, we can express

$$\mathcal{U}_{\tilde{g}}^{\mathsf{T}_{(k,y)}}(\tilde{k},\tilde{y}) = G\Big(\bigcup_{i=0}^{r}(V_{(h,x)})_K \cap \big(\mathsf{T}^*_{(k_i,y)} \times (Y_x)_{G_y}\big)\Big). \tag{6.6}$$

In particular, we will see below that for every $(k,y) \in V_{(h,x)}$ with isotropy group K and every Cartan subgroup $\mathsf{T}_{(k,y)}$ of K associated to k, the connected component of $\mathsf{T}_{(k,y)}$ containing k is the only connected component that intersects the projection of $V_{(h,x)}$ onto G_x. Note that $\mathcal{U}_{\tilde{g}}^{\mathsf{T}_{(k,y)}}(\tilde{k},\tilde{y})$ is clearly a subset of $\Lambda_0(G \ltimes V)$ as $\mathsf{T}_{(k,y)} \leq G_y$.

Using arguments identical to those in [FPS15, Lemmata 4.18 & 4.19], one can demonstrate that $\mathcal{U}_{\tilde{g}}^{\mathsf{T}_{(k,y)}}(\tilde{k},\tilde{y})$ depends only on the orbit $G(k,y)$, and does not depend on the choice of the Cartan subgroup $\mathsf{T}_{(k,y)}$. Hence, we we may denote $\mathcal{U}_{\tilde{g}}^{\mathsf{T}_{(k,y)}}(\tilde{k},\tilde{y})$ simply as $\mathcal{U}(k,y)$ and let $\mathcal{U}_{(\tilde{k},\tilde{y})}$ denote the connected component of $\mathcal{U}(k,y)$ containing $(\tilde{k},\tilde{y})$. The partition $\mathcal{Z}$ of U then can be written as

$$\mathcal{Z} = \big\{\mathcal{U}_{(\tilde{k},\tilde{y})} \subset U \mid (\tilde{k},\tilde{y}) \in U\big\}. \tag{6.7}$$

By [FPS15, Lemma 4.17], using the linearity of the action on slices, one can demonstrate that the exponential map associated to the product metric on $G_x \times Y_x$ maps the subset

$$(N_{(h,x)})_K \cap \big(T_h(k_i^\bullet) \oplus (T_xY_x)_{G_y}\big) \cap B_{(h,x)} \tag{6.8}$$

onto $(V_{(h,x)})_K \cap \Big(\overline{\mathsf{T}^*_{(k_i,y)}} \times (Y_x)_{G_y}\Big)$. Recall that $B_{(h,x)}$ is an H-invariant ball around the origin in the normal space $N_{(h,x)}$ and $k_i^\bullet$ is defined in Proposition 6.9. By construction, (6.8) is a semialgebraic subset of $N_{(h,x)}$ and is invariant under the action of $t \in (0,1]$. Similarly, because there are only finitely many $\simeq$ classes in $\mathsf{T}_{(k,y)}$, there are $l_1, \dots, l_N \in \mathsf{T}_{(k,y)}$ such that each group $l_j^\bullet$, $j = 1, \dots, N$, has dimension less than $\dim k^\bullet$, and

$$\mathsf{T}^*_{(k,y)} = \overline{\mathsf{T}^*_{(k,y)}} \smallsetminus \bigcup_{j=1}^{N} l_j^\bullet \,.$$

Then the exponential function maps the semialgebraic set

$$(N_{(h,x)})_K \cap \Big(\big(T_h(k^\bullet) \smallsetminus \bigcup_{j=1}^{N} T_h(l_j^\bullet)\big) \oplus (T_x Y_x)_{G_y}\Big) \cap B_{(h,x)} \tag{6.9}$$

onto $(V_{(h,x)})_K \cap \big(\mathsf{T}^*_{(k,y)} \times (Y_x)_{G_y}\big)$.

Restricting the inverse of the exponential map from $V_{(h,x)}$ to $V_{(h,x)} \cap \Lambda_0(G_x \ltimes Y_x)$ yields an H-equivariant embedding ι of a neighborhood of (h,x) in $V_{(h,x)} \cap \Lambda_0(G_x \ltimes Y_x)$ into the normal space $N_{(h,x)}$, where the stratum of (h,x) is mapped onto the subspace

$$N^H_{(h,x)} \cap \Big(T_h h^\bullet \times (T_x Y_x)^{G_x}\Big).$$

Using the description of the image of each stratum given in Equation (6.9), one sees immediately that the homotopy defined as multiplication in $N_{(h,x)}$ by scalars $t \in [0,1]$ contracts the image of ι onto the origin preserving the image of ι. The linearity of the H-action on $N_{(h,x)}$ ensures that this homotopy is H-equivariant, and scalars $t \in (0,1]$ preserve the images of strata as demonstrated above. We therefore observe the following.

Proposition 6.14. *The inertia groupoid $\Lambda\mathsf{G}$ of a proper Lie groupoid G satisfies the local contractibility hypothesis of Definition 5.7.*

Following [FPS15, Prop. 4.20], it is easy to see the following.

Proposition 6.15. *The germs of the $\mathcal{U}\big(G(k,y)\big)$ coincide with the stratification. That is, for $(\tilde{k},\tilde{y}) \in U = GV_{(h,x)}$, the germs $[\mathcal{U}\big(G(k,y)\big)]_{(\tilde{k},\tilde{y})}$, $[\mathcal{U}_{(\tilde{k},\tilde{y})}]_{(\tilde{k},\tilde{y})}$ and $\mathcal{S}_{(\tilde{k},\tilde{y})}$ coincide.*

Since the $\mathcal{S}_{(h,x)}$ are germs of smooth G-submanifolds of $G \times M$, and the piece associated to a point $(\tilde{k},\tilde{y}) \in U$ has the same set germ as $\mathcal{S}_{(l,z)}$ at $(l,z) \in \mathcal{U}_{(\tilde{k},\tilde{y})}$, it follows that the pieces of $\mathcal{Z}$ are smooth submanifolds of $G \times M$ invariant under the G-action. The proof of the following is a minor variation of that of [FPS15, Lemma 4.21].

Lemma 6.16. *The partition $\mathcal{Z}$ of $U = GV_{(h,x)}$ given by Equation* (6.6) *is finite.*

We now verify that $\mathcal{Z}$ is a decomposition indeed, cf. [Pfl01, Def. 1.1.1 (DS2)]. The proof is similar to that of [FPS15, Prop. 4.22].

Proposition 6.17. *The pieces of $\mathcal{Z}$ satisfy the condition of frontier.*

Proof. Suppose there are points (h,x) and (k,y) with $\mathcal{U}\big(G(h,x)\big) \cap \overline{\mathcal{U}\big(G(k,y)\big)} \neq \emptyset$. As the pieces of $\mathcal{Z}$ are defined to be connected components, it is sufficient to show that $\mathcal{U}\big(G(h,x)\big) \cap \overline{\mathcal{U}\big(G(k,y)\big)}$, which is obviously closed in $\mathcal{U}\big(G(h,x)\big)$, is also open in $\mathcal{U}\big(G(h,x)\big)$. Note that we can assume with no loss of generality that one of the points

in question is (h,x), the point used to define U, as we may restrict consideration to a neighborhood of that point. Moreover, as the piece $\mathcal{U}\big(G(h,x)\big)$ may be defined in terms of any point it contains, we may assume that $G(h,x) \subset \mathcal{U}\big(G(h,x)\big) \cap \overline{\mathcal{U}\big(G(k,y)\big)}$. Similarly, we assume by choosing another representative of the orbit if necessary that $(k,y) \subset V_{(h,x)}$. By Proposition 6.15, an open neighborhood of (h,x) in $\mathcal{U}\big(G(h,x)\big)$ is given by $G\big(V^H_{(h,x)} \cap (\mathsf{T}^*_{(h,x)} \times Y_x^{G_x})\big)$ for a sufficiently small slice $V_{(h,x)}$ at (h,x). We will show that $G\big(V^H_{(h,x)} \cap (\mathsf{T}^*_{(h,x)} \times Y_x^{G_x})\big)$ is contained in $\overline{\mathcal{U}\big(G(k,y)\big)}$.

Let $K := G_{(k,y)} \leq H$, and then $K^\circ \leq H^\circ$. Then any maximal torus in K° is contained in a maximal torus in H°. Moreover, as $V^K_{(h,x)}$ is connected and closed under multiplication by scalars $t \in (0,1]$, taking the limit of $t(k,y)$ as $t \to 0$, we see that $h \in K$. Similarly, h and k are in the same connected component of K. It follows that we may choose a Cartan subgroup of H associated to h by taking the group generated by a maximal torus in H° that contains a maximal torus in K° and h. That is, we may assume that $h \in \mathsf{T}_{(k,y)} \leq \mathsf{T}_{(h,x)}$ and $\mathsf{T}_{(k,y)} = \mathsf{T}_{(h,x)} \cap K$. Then we have $h \in \overline{\mathsf{T}^*_{(k',y)}}$ for some k' whose $\simeq$ class at y is diffeomorphic to $\mathsf{T}^*_{(k,y)}$. Similarly, as $G_y \leq G_x$, it follows that $Y_x^{G_x} \subset \overline{(Y_x)_{G_y}}$. From these observations, we have

$$(h,x) \in V^H_{(h,x)} \cap \left(\overline{\mathsf{T}^*_{(k',y)}} \times Y_x^{G_x}\right) \subset \overline{(V_{(h,x)})_K} \cap \big(\overline{\mathsf{T}^*_{(k',y)}} \times \overline{(Y_x)_{G_y}}\big).$$

Now, let $l \in \mathsf{T}^*_{(h,x)}$ so that $Y_x^l = Y_x^h$. In particular, as $h \in K \leq G_y$ and $y \in Y_x$, it follows that $l \in G_y$. Similarly, as $l \in \mathsf{T}_{(h,x)}$, as $k \in \mathsf{T}_{(k,y)} \leq \mathsf{T}_{(h,x)}$, and as $\mathsf{T}_{(h,x)}$ is abelian, we have $l(k,y) = (k,y)$ so that $l \in K$. In particular, $l \in \mathsf{T}_{(h,x)} \cap K = \mathsf{T}_{(k,y)}$. This demonstrates $\mathsf{T}^*_{(h,x)} \subset \mathsf{T}_{(k,y)}$. As $Y_y \subset Y_x$, we have that the relation $\simeq$ at x implies $\simeq$ at y, so that the $\simeq$ classes at y are the intersection with $\mathsf{T}_{(k,y)}$ of a (finite) union of $\simeq$ classes at y. That is, using Proposition 6.9 (2), $\mathsf{T}^*_{(h,x)} \subset \overline{\mathsf{T}^*_{(k',y)}}$. Then

$$V^H_{(h,x)} \cap \left(\mathsf{T}^*_{(h,x)} \times Y_x^{G_x}\right) \subset \overline{(V_{(h,x)})_K} \cap \big(\overline{\mathsf{T}^*_{(k',y)}} \times \overline{(Y_x)_{G_y}}\big) \subset \overline{\mathcal{U}\big(G(k,y)\big)}.$$

Considering the G-saturations of both sides of this inclusion, it follows that an open neighborhood of (h,x) in $\mathcal{U}\big(G(k,x)\big)$ is contained in $\mathcal{U}\big(G(h,x)\big) \cap \overline{\mathcal{U}\big(G(k,y)\big)}$. This completes the proof. □

6.4. **Whitney Condition B.** Here, we complete the proof of Theorem 6.4 by demonstrating that the the stratifications of $\Lambda_0\mathsf{G}$ and $|\Lambda\mathsf{G}|$ are Whitney B-regular. The proof follows [Pfl01, Thm. 4.3.7] and [FPS15, Prop. 4.23]. Roughly, the proof involves giving a parameterization of a neighborhood of a point in $\Lambda\mathsf{G}$ and its tangent space sufficient to describe the secants of points in neighboring strata. Note that in our argument we use that the pieces satisfy the condition of frontier, which was shown above.

Proposition 6.18. *Let G be a proper Lie groupoid. The orbit Cartan type stratifications of the loop space $\Lambda\mathsf{G}$ and the inertia space $|\Lambda\mathsf{G}|$ both satisfy Whitney's condition B.*

Proof. Because the claim is local, we may assume that the groupoid G is given by the product of $O \times O \rightrightarrows O$ and $G \ltimes Y$ where G is the isotropy group G_x of some point $x \in \mathsf{G}_0$, O is an open neighborhood of x in its orbit, and Y is a slice through x. Let $(h,x) \in \Lambda\mathsf{G}$, $H = \mathsf{Z}_G(h)$, and $V_{(h,x)}$ a slice at (h,x) for the G-action on $G \times Y$ of the form $\exp(B_{(h,x)})$, where $B_{(h,x)}$ is a ball around the origin in the normal space $N_{(h,x)}$. Now let us denote by $\mathcal{Z}$ the decomposition of $\Lambda_0\mathsf{G}$ obtained by taking the saturations of the sets defined

through Eq. (6.6), which amounts to taking their products with O. Let R be the piece of $\mathcal{Z}$ containing (h,x), i.e. the set of points of the form $((o,o),(l,z))$ where $o \in O$ and $(l,z) \in V^H_{(h,x)} \cap \big(\mathsf{T}^*_{(h,x)} \times Y^G\big)$. We show that for any stratum $S \in \mathcal{Z}$ with $(h,x) \in \overline{S}$, Whitney's condition B is satisfied at (h,x) for the pair of strata (R,S). To describe the stratum S in some more detail, consider an orbit $G(k,y)$ for $(k,y) \in S$. As in the proof of Proposition 6.17, we may choose the representative (k,y) of the orbit $G(k,y)$ such that $(k,y) \in V_{(h,x)}$, $h \in \mathsf{T}_{(k,y)} \leq \mathsf{T}_{(h,x)}$, and $h \in \overline{\mathsf{T}^*_{(k',y)}}$ for some k'. In particular, we then have $K \leq H$ for the isotropy group $K := \mathsf{Z}_{G_y}(k)$ of (k,y) and $G_y \leq G$. As shown above, S coincides with the connected component of $\mathcal{U}\big(G(k,y)\big)$ containing (k,y).

Suppose now that $((u_i,u_i),(h_i,x_i))_{i\in\mathbb{N}}$ is a sequence in R and $((o_i,o_i),(k_i,y_i))_{i\in\mathbb{N}}$ a sequence in S, and that both sequences converge to $((x,x),(h,x))$. Assume in addition that in a smooth chart around $((x,x),(h,x))$ the secant lines

$$\ell_i = \overline{((u_i,u_i),(h_i,x_i)),((o_i,o_i),(k_i,y_i))}$$

converge to a straight line ℓ, and the tangent spaces $T_{((o_i,o_i),(k_i,y_i))}S$ converge to a subspace τ. Then we must show that $\ell \subset \tau$.

Note that the hypotheses imply that $((x,x),(h,x)) \in \mathcal{U}\big(G(h,x)\big) \cap \overline{\mathcal{U}\big(G(k,y)\big)}$. By the proof of Proposition 6.17 and the choices of (k,y) and $\mathsf{T}_{(k,y)} \subset K$ we obtain the relation

$$V^H_{(h,x)} \cap \Big(\mathsf{T}^*_{(h,x)} \times Y^G\Big) \subset \overline{(V_{(h,x)})_K} \cap \big(\overline{\mathsf{T}^*_{(k',y)}} \times \overline{(Y)_{G_y}}\big). \tag{6.10}$$

Denote by $\mathfrak{g}_x$ the Lie algebra of G, by $\mathfrak{h}$ the Lie algebra of H, and let $\mathfrak{m}$ denote the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}_x$ with respect to the initially chosen bi-invariant metric on G. Then there is a neighborhood $U \subset \mathsf{G}_0 \cong O \times_H V_{(h,x)}$ of (h,x) such that

$$\Psi\colon\ U \longrightarrow O \times \mathfrak{m} \times N_{(h,x)},\ [o, \exp_{|\mathfrak{m}} \xi, \exp_{(h,x)}(v)] \longmapsto (o,\xi,v)$$

is a smooth chart at $((x,x),(h,x))$, where $\exp_{|\mathfrak{m}}$ denotes the restriction of the exponential map of the Lie group G to $\mathfrak{m}$, and $\exp_{(h,x)}$ the exponential function restricted to the open ball $B_{(h,x)} \subset N_{(h,x)}$. After possibly shrinking U there is an open neighborhood Q of H in G such that

$$\Psi\left(O \times Q\left(V^H_{(h,x)} \cap (\mathsf{T}^*_{(h,x)} \times Y^G)\right)\right) \subset O \times \mathfrak{m} \times \left(N^H_{(h,x)} \cap T_{(h,x)}(\mathsf{T}^*_{(h,x)} \times Y^G)\right).$$

We may assume that the sequences $((u_i,u_i),(h_i,x_i))_{i\in\mathbb{N}}$ and $((o_i,o_i),(k_i,y_i))_{i\in\mathbb{N}}$ are contained in U. Since $((o_i,o_i),(k_i,y_i)) \in \mathcal{U}\big(G(k,y)\big)$, one knows that

$$\Psi((o_i,o_i),(k_i,y_i)) \in O \times \mathfrak{m} \times H\left((V_{(h,x)})_K \cap \left(\mathsf{T}^{**}_{(k,y)} \times Y^G\right)\right).$$

Recall that $\mathsf{T}^{**}_{(k,y)}$ consists of a finite collection of pairwise disjoint $\simeq$ classes in $\mathsf{T}_{(k,y)}$. Moreover, by Lemma 6.10, each such $\simeq$ class is disjoint from the closures of the other classes. By passing to a subsequence, we may assume without loss of generality that each k_i is in one fixed class, i.e.

$$((o_i,o_i),(k_i,y_i)) \in O \times G\left((V_{(h,x)})_K \cap \left(\mathsf{T}^*_{(k',y)} \times Y^G\right)\right)$$

for all i and some fixed $k' \in \mathsf{T}_{(k,y)}$.

Note that $\lim o_i = \lim u_i = x$. Moreover, each piece of $\mathcal{Z}$ is a product of a piece in $GV_{(h,x)} \subset G \times Y$ with the diagonal in $O \times O$, so we may project onto $G \times Y$ and ignore the O-factor. Choose $\tilde{l}_i \in G$ such that $(\tilde{k}_i,\tilde{y}_i) := \tilde{l}_i(k_i,y_i) \in (V_{(h,x)})_K$ for all $i \in \mathbb{N}$. Put $(\tilde{h}_i,\tilde{x}_i) := l_i(h_i,x_i)$. After possibly passing to a subsequence, $(\tilde{l}_i)_{i\in\mathbb{N}}$ converges to some

$\tilde{l} \in H$, the secant lines $\tilde{\ell}_i = \overline{(\tilde{h}_i, \tilde{x}_i), (\tilde{k}_i, \tilde{y}_i)}$ converge to a straight line $\tilde{\ell}$, and the tangent spaces $T_{(\tilde{k}_i,\tilde{y}_i)}S$ converge to a subspace $\tilde{\tau}$. By definition, and since $\tilde{l}_i T_{(k_i,y_i)}S = T_{(\tilde{k}_i,\tilde{y}_i)}S$ for all i, one obtains $\tilde{\ell} = \tilde{l}\ell$, and $\tilde{\tau} = \tilde{l}\tau$. Hence, the first claim is shown, if $\tilde{\ell} \subset \tilde{\tau}$. Without loss of generality we may therefore assume that for all $i \in \mathbb{N}$

$$(k_i, y_i) \in (V_{(h,x)})_K \cap \big(\mathsf{T}^*_{(k',y)} \times Y^G\big), \tag{6.11}$$

and then show $\ell \subset \tau$ for the sequences $(k_i, y_i)_{i\in\mathbb{N}}$ and $(h_i, x_i)_{i\in\mathbb{N}}$.

Eq. (6.11) now means in particular that

$$\Psi(k_i, y_i) \in \{0\} \times \Big((N_{(h,x)})_K \cap \exp^{-1}_{(h,x)} \big(\mathsf{T}^*_{(k',y)} \times Y^G\big)\Big).$$

Since $\overline{\mathsf{T}^*_{(k',y)}}$ is an open and closed subset of a closed subgroup of G and also contains h, the set

$$V := N_{(h,x)} \cap T_{(h,x)}\Big(\big(\overline{\mathsf{T}^*_{(k',y)}}\big) \times Y^G\Big)$$

is a subspace of $N_{(h,x)}$. Let W be the orthogonal complement of the invariant space V^H in V with respect to the H-invariant scalar product induced from $V_{(h,x)}$. Then the image under the chart Ψ of every element of $G\big(V^H_{(h,x)} \cap (\mathsf{T}^*_{(h,x)} \times Y^G)\big) \cap U$ and every (k_i, y_i) is contained in

$$\mathfrak{m} \times (W_K \cup \{0\}) \times V^H.$$

With respect to this decomposition, (h, x) has coordinates $(0, 0, 0)$, each element of $G\left(V^H_{(h,x)} \cap (\mathsf{T}^*_{(h,x)} \times Y^G)\right)$ has coordinates contained in $\mathfrak{m} \times 0 \times V^H$, and each sequence element (k_i, y_i) has coordinates contained in $\{0\} \times W_K \times V^H$. In particular, let

$$\Psi(k_i, y_i) = (0, w_i, v_i)$$

for every i. Since W_K is invariant under multiplication by non-vanishing scalars, we have

$$(\xi, w, v) := \lim_{i\to\infty} \frac{\Psi(k_i, y_i) - \Psi(h_i, x_i)}{\|\Psi(k_i, y_i) - \Psi(h_i, x_i)\|} \in \mathfrak{m} \times \overline{W_K} \times V^H \ .$$

By compactness of the unit sphere in W, the sequence $\frac{w_i}{\|w_i\|}$ converges to some $\hat{w} \in SW$ after possibly passing to a subsequence. Then $w = \|w\|\hat{w}$. Since W_K is invariant by non-vanishing scalars, we have

$$\mathfrak{m} \times \operatorname{span} \hat{w} \times V^H \subset \tau,$$

and

$$\ell = \operatorname{span} (\xi, \hat{w}, v) \subset \tau,$$

proving the first claim.

Now let us show that the orbit Cartan type stratification of $|\Lambda\mathsf{G}|$ satisfies Whitney's condition B as well. To this end let us first choose a Hilbert basis of H-invariant polynomials $p_1, \ldots, p_\kappa \colon \big(N^H_{(h,x)}\big)^\perp \to \mathbb{R}$ of the orthogonal complement of the invariant space $N^H_{(h,x)}$ in $N_{(h,x)}$. Next let $p_{\kappa+1}, \ldots, p_N \colon N^H_{(h,x)} \to \mathbb{R}$ with $N = \kappa + \dim N^H_{(h,x)}$ be a linear coordinate system of the invariant space. We can even choose these p_i in such a way that $p_{\kappa+1}, \ldots, p_{\kappa+\dim V^H}$ is a linear coordinate system of V^H. By construction, $p_1, \ldots, p_N$ then is a Hilbert basis of the normal space $N_{(h,x)}$. Denote by $p \colon N_{(h,x)} \to \mathbb{R}^N$ the corresponding Hilbert map. Recall that p induces a chart of $|\Lambda\mathsf{G}|$ over $G\backslash U$ by

$$\widehat{\Psi} \colon G\backslash U \to \mathbb{R}^N,\ G\exp_{(h,x)}(v) \mapsto p(v).$$

Note that by H-invariance of p and since for every orbit in U there is a representative in $V_{(h,x)}$, the chart $\widehat{\Psi}$ is well-defined indeed. A decomposition of $\widehat{U} := \widehat{\Psi}(G\backslash U)$ inducing the orbit Cartan type stratification on $G\backslash U$ is given by

$$\widehat{\mathcal{Z}} := \big\{\widehat{\Psi}(G\backslash(S \cap (G \times Y)) \mid S \in \mathcal{Z}\big\}.$$

Let $\widehat{S} \in \widehat{\mathcal{Z}}$ denote the stratum containing the orbit $G(h,x)$, and $\widehat{S} \in \mathcal{Z}$ a stratum $\neq \widehat{R}$ such that $G(h,x)$ lies in the closure of $\widehat{S}$. Now consider sequences of orbits $\big(G(h_i,x_i)\big)_{i\in\mathbb{N}}$ in $\widehat{R}$ and $\big(G(k_i,y_i)\big)_{i\in\mathbb{N}}$ in $\widehat{S}$ such that both sequences converge to $G(h,x)$. Moreover, assume that the sequence of secants $\overline{\widehat{\Psi}(G(h_i,x_i)),\widehat{\Psi}(G(k_i,y_i))}$ converges to a line $\widehat{\ell}$, and that the sequence of tangent spaces $T_{\widehat{\Psi}(G(k_i,y_i))}\widehat{S}$ converges to some subspace $\widehat{\tau} \subset \mathbb{R}^N$. Using notation from before, we can choose representatives (h_i,x_i) and (k_i,y_i) having coordinates in $\mathfrak{m} \times (W_K \cup \{0\}) \times V^H \subset N_{(h,x)}$ such that

$$\begin{aligned} \Psi(h_i,x_i) &= (0,0,v_i') \in \{0\} \times \{0\} \times V^H \text{ and} \\ \Psi(k_i,y_i) &= (0,w_i,v_i) \in \{0\} \times W_K \times V^H. \end{aligned} \tag{6.12}$$

Next observe that by the Tarski–Seidenberg Theorem, the stratum $\widehat{S}$ is semialgebraic as the image of the semialgebraic set $(W_K \times V^H) \cap B_{(h,x)}$ under the Hilbert map p. By the same argument, $p(W_K)$ is semialgebraic, too, and an analytic manifold, since $p(W_K) \cong \mathsf{N}_H(K)\backslash W_K \cong H\backslash W_{(K)}$. Moreover, the equality

$$\widehat{S} = (p(W_K) \times V^H) \cap p(B_{(h,x)})$$

holds true, where we have canonically identified V^H with its image under the Hilbert map p. By Eq. (6.12), this entails that

$$\widehat{\tau} = \lim_{i\to\infty} T_{\widehat{\Psi}(G(k_i,y_i))}\widehat{S} = \lim_{i\to\infty} T_{p(w_i)}p(W_K) \times V^H. \tag{6.13}$$

Since $p(W_K)$ is semialgebraic and an analytic manifold, [Łoj65, Prop. 3, p. 103] by Łojasiewicz entails that $p(W_K)$ satisfies Whitney's condition B over the origin. This means after possibly passing to subsequences, that $\ell_{W_K} \subset \tau_{W_K}$, where ℓ_{W_K} is the limit line of the secants $\overline{p(w_i),0}$, and τ_{W_K} the limit of the tangent spaces $T_{p(w_i)}p(W_K)$ for $i \to \infty$. By Eqs. (6.12) and (6.13) this entails that

$$\widehat{\ell} \subset \ell_{W_K} \times V^H \subset \tau_{W_K} \times V^H = \widehat{\tau}.$$

This finishes the proof. □

Note that $\Lambda_0\mathsf{G}$ is clearly topologically locally trivial based on its description in slices. Therefore, by Proposition 3.1, the inertia groupoid $\Lambda\mathsf{G}$ is a differentiable stratified groupoid. Now, recall that the loop space $\Lambda_0\mathsf{G}$ is a differentiable subspace of the smooth manifold G_1, and the space of arrows $\Lambda\mathsf{G}_1 = \mathsf{G}_1 {}_s\!\times_t \Lambda_0\mathsf{G}$ is a differentiable subspace of the smooth manifold $\mathsf{G}_1 {}_s\!\times_t \mathsf{G}_1$. Similarly, if $x, y \in \mathsf{G}_0$ are in the same orbit, then the slices Y_x and Y_y for G can be chosen such that $\mathsf{G}_{|Y_x}$ and $\mathsf{G}_{|Y_y}$ are isomorphic by [PPT14, Lemma 5.1]. This defines a diffeomorphism between the arrow spaces of $\mathsf{G}_{|Y_x}$ and $\mathsf{G}_{|Y_y}$, both smooth manifolds, whose restriction defines an isomorphism between $\Lambda\mathsf{G}_{|Y_x}$ and $\Lambda\mathsf{G}_{|Y_y}$. It follows that the inertia groupoid satisfies conditions (LT3) to (LT4) in Definitions 2.6 and 2.28, hence is locally translation. Moreover, by Proposition 6.14, the inertia groupoid satisfies the local contractibility hypothesis of Definition 5.7.

It is straightforward to verify that a weak equivalence $f\colon \mathsf{G} \to \mathsf{H}$ of proper Lie groupoids induces a weak equivalence $\Lambda\mathsf{G} \to \Lambda\mathsf{H}$ given by the restriction of f_1 to the loop spaces. In particular, because the stratification of $\Lambda_0\mathsf{G}$ is defined in terms of slices for G, and the representation of the isotropy group on a slice is Morita invariant, the stratification of $\Lambda_0\mathsf{G}$ is obviously the pullback of the stratification of $\Lambda_0\mathsf{H}$ via f_1. Moreover, as the stratification of the inertia space $|\Lambda\mathsf{G}|$ can be defined locally in terms of the actions of isotropy groups on slices, it is as well Morita invariant. This means that the isomorphism between $|\Lambda\mathsf{G}|$ and $|\Lambda\mathsf{H}|$ from Proposition 2.23 is an isomorphism of differentiable stratified spaces. We summarize these observations in the following.

Theorem 6.19. *Let G be a proper Lie groupoid. Then the inertia groupoid $\Lambda\mathsf{G}$ is a proper differentiable stratified groupoid fulfilling Whitney's condition B. Moreover the inertia groupoid $\Lambda\mathsf{G}$ is locally translation and satisfying the local contractibility hypotheses. Finally, the inertia space $|\Lambda\mathsf{G}|$ inherits from $\Lambda_0\mathsf{G}$ via the canoncial projection $\pi : \Lambda_0\mathsf{G} \to |\Lambda\mathsf{G}|$ a stratification also fulfilling Whitney's condition B.*

Appendix A. Differentiable stratified spaces

In this appendix we describe the category of differentiable stratified spaces used throughout this paper. Our notion of differentiable spaces is that of [NGSdS03] to which we refer the reader for more details. For the definition of stratified spaces we follow Mather [Mat73] and [Pfl01, Chap. 1], except that we relax the assumption that the spaces under consideration are Hausdorff and only require that they are locally Hausdorff. Hence, a stratified space with smooth structure as defined in [Pfl01, Chap. 1] or a differentiable stratified space as defined in [FPS15] corresponds to a Hausdorff differentiable stratified space as defined here. Note that in addition to [Pfl01, FPS15] various other concepts of structure sheaves respectively structure algebras of smooth functions over stratified spaces have been introduced in the literature. See for example the work by Kreck [Kre10] on stratifolds, by Lusala–Śniatycki [LŚ11, Sec. 4] on stratified subcartesian spaces, by Watts [Wat12, Wat15] on differential spaces, and finally by Somberg–Vân Lê–Vanzura [SLV15, LSV13] on smooth structures on locally conic stratified spaces.

A.1. **Differentiable spaces.**

Definition A.1 ([NGSdS03, Chap. 3])**.** Let $(X, \mathcal{O})$ be a locally $\mathbb{R}$-ringed space which we always assume to be commutative. One says that $(X, \mathcal{O})$ is an *affine differentiable space*, if there is a closed ideal $\mathfrak{a} \subset \mathcal{C}^\infty(\mathbb{R}^n)$ such that $(X, \mathcal{O})$ is isomorphic as a ringed space to the real spectrum of $\mathcal{C}^\infty(\mathbb{R}^n)/\mathfrak{a}$ equipped with its structure sheaf, which associates to each open set its localization over that set. Here, we consider the unique topology with respect to which $\mathcal{C}^\infty(\mathbb{R}^n)$ is a Frechét algebra. A locally $\mathbb{R}$-ringed space $(X, \mathcal{O})$ is a *differentiable space* if, for each $x \in X$, there is an open neighborhood U of X such that the restriction $(U, \mathcal{O}|_U)$ is an affine differentiable space. A differentiable space is *reduced* if for each open subset U of X the map $\mathcal{O}(U) \to \mathcal{C}(U)$ defined by the evaluation map is injective.

A *morphism of differentiable spaces* $(f, \varphi)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map $f\colon X \to Y$ and a morphism $\varphi\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of $\mathbb{R}$-algebras such that for each $x \in X$ the induced morphism on the stalks $\varphi_x\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is local, i.e. maps the the maximal ideal $\mathfrak{m}_y \subset \mathcal{O}_{Y,f(x)}$ to the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$.

Note that if $(X, \mathcal{C}_X^\infty)$ and $(Y, \mathcal{C}_Y^\infty)$ are reduced, a morphism of differentiable spaces $(f, \varphi)\colon (X, \mathcal{C}_X^\infty) \to (Y, \mathcal{C}_Y^\infty)$ is fully determined by the map $f\colon X \to Y$. The sheaf morphism φ is given in this case over each open $V \subset Y$ by the pullback map $f^* : \mathcal{C}_Y^\infty(V) \to \mathcal{C}_X^\infty(f^{-1}(V))$, $g \mapsto g \circ f_{|V}$. We therefore sometimes call a morphism between reduced differentiable spaces a *smooth map*, and just denote it by the underlying map f.

By [NGSdS03, Thm. 3.23], a differentiable space $(X, \mathcal{O})$ is reduced if and only if each $x \in X$ is contained in an open neighborhood V isomorphic as a differentiable space to a locally closed subset of the affine space $\mathbb{R}^n$ with structure sheaf given by restrictions of smooth functions from $\mathbb{R}^n$. We refer to such a V as an *affine neighborhood of* x, and call an embedding $\iota : V \hookrightarrow \mathbb{R}^n$ such that $(\iota, \iota^*) : (V, \mathcal{O}_{|V}) \to (\iota(V), \mathcal{C}^\infty_{|\iota(V)})$ is an isomorphism of locally ringed spaces a *singular chart* (of *rank* n) for X.

We often denote the structure sheaf of a reduced differentiable space X by $\mathcal{C}_X^\infty$ or shortly by $\mathcal{C}^\infty$, if no confusion can arise. By a smooth submanifold of a differentiable space X, we mean a differentiable subspace whose differentiable structure is that of a smooth manifold in the usual sense.

A.2. **Stratified spaces.** Let X be a paracompact separable locally Hausdorff topological space. A *decomposition* $\mathcal{Z}$ of X is a locally finite partition of X into locally closed subspaces such that each $S \in \mathcal{Z}$ is a countable union of smooth (not necessarily Hausdorff) manifolds such that the condition of frontier is satisfied:

(CF) If $R \cap \overline{S} \neq \emptyset$ for $R, S \in \mathcal{Z}$, then $R \subset \overline{S}$.

If $R \subset \overline{S}$, one writes $R \leq S$ and says that R is incident to S. The incidence relation is an order relation on $\mathcal{Z}$. The elements of a decomposition $\mathcal{Z}$ are called its *pieces*.

In the following we provide a generalization of the definition of a stratification by Mather [Mat73] to the case of a locally Hausdorff space; cf. also [Pfl01, Sec. 1.2].

Definition A.2. Let X be a locally Hausdorff topological space. A *stratification* of X is an assignment to each $x \in X$ of a germ $\mathcal{S}_x$ of subsets of X at x such that for each $x \in X$ there is a Hausdorff neighborhood U of x in X and a decomposition $\mathcal{Z}$ of U with the property that for each $y \in U$ the germ $\mathcal{S}_y$ is equal to the germ at y of the piece of $\mathcal{Z}$ containing y. The set X along with the stratification $\mathcal{S}$ is called a *stratified space*. If, moreover, X is a differentiable space, and for each $x \in X$, the germ $\mathcal{S}_x$ is that of a smooth submanifold of X, then we say X is a *differentiable stratified space*.

A continuous function $f\colon (X, \mathcal{S}) \to (Y, \mathcal{R})$ is a *morphism of stratified spaces* if, for each $x \in X$ with $f(x) = y$, there are open Hausdorff neighborhoods U of x and V of y with $U \subset f^{-1}(V)$ and decompositions of U and V inducing their respective stratifications such that for every $z \in U$ contained in the piece S of U, there is an open neighborhood O of z in U such that $f_{|S \cap O}$ maps into the piece of V containing $f(z)$.

If $(X, \mathcal{O}_X, \mathcal{S})$ and $(Y, \mathcal{O}_Y, \mathcal{R})$ are differentiable stratified spaces, a function $f\colon X \to Y$ is a *differentiable stratified morphism* if it is simultaneously a morphism of differentiable spaces and a morphism of stratified spaces.

Note that the definition of a differentiable stratified space coincides, for a Hausdorff space X, with that of a stratified space with $\mathcal{C}^\infty$ structure defined in [Pfl01, Section 1.3]; see also [FPS15, Section 2].

If $(X, \mathcal{Z})$ is a decomposed space, then the decomposition induces a stratification by assigning to $x \in X$ the germ at x of the piece containing x; two decompositions of X

are *equivalent* if they induce the same stratification. The *depth* of $x \in X$ with respect to the decomposition $\mathcal{Z}$ is the maximum k such that $x \in S_0 < S_1 < \cdots < S_k$ for $S_i \in \mathcal{Z}$.

Now, let $(X, \mathcal{S})$ be a stratified space and let $x \in X$. The proof of [Mat73, Lemma 2.1] (see also [Pfl01, Lem. 1.5.2]) is local, hence can be executed on a Hausdorff neighborhood of x. It therefore extends to our case and demonstrates that the depth of x coincides for any decomposition of a Hausdorff neighborhood of x inducing $\mathcal{S}$. Hence, we may define the *depth of x with respect to $\mathcal{S}$* to be the depth with respect to any such decomposition. In the same way, the proofs of [Mat73, Lem. 2.2] and [Pfl01, Prop. 1.2.7]) extend to the situation of locally Hausdorff stratified space. So X admits a decomposition $\mathcal{Z}$ that induces $\mathcal{S}$ and is maximal in the sense that for every Hausdorff open subset U of X, the restriction of $\mathcal{Z}$ to U is coarser than any decomposition of U that induces $\mathcal{S}$. We will often refer to $\mathcal{Z}$ simply as the *maximal decomposition of X*. Its pieces are called the *strata* of X. Note that if X is a (Hausdorff) differentiable stratified space, the strata of $\mathcal{Z}$ are obviously (Hausdorff) smooth manifolds.

We recall the following from [Pfl01, Section 1.4.1].

Definition A.3. A stratified space $(X, \mathcal{S})$ is *topologically locally trivial* if for every $x \in X$ in the stratum S of X, there is a neighborhood U, a stratified space $(F, \mathcal{S}^F)$, a point $o \in F$, and an isomorphism of stratified spaces $h\colon U \to (S \cap U) \times F$ such that $h^{-1}(y, o) = y$ for all $y \in S \cap U$, and such that $\mathcal{S}^F_o$ is the germ of the set $\{o\}$.

A.3. **Fibered products.** By [NGSdS03, Theorem 7.6], the fibred product of differentiable spaces has a unique differentiable space structure with respect to which the projection maps are morphism of differentiable spaces. We now demonstrate that the same holds true for differentiable stratified spaces. Let X, Y, and Z be differentiable stratified spaces with respective stratifications $\mathcal{S}^X$, $\mathcal{S}^Y$, and $\mathcal{S}^Z$. Suppose $f\colon X \to Z$ and $g\colon Y \to Z$ are differentiable stratified mappings. If f is in addition a stratified submersion, then we define a stratification of the fibred product $X\,{}_f\!\times_g Y$ as follows. Let $(x, y) \in X\,{}_f\!\times_g Y$, let $P \subset X$ be a subset whose germ $[P]_x = \mathcal{S}^X_x$, and let $R \subset Y$ such that $[R]_y = \mathcal{S}^Y_y$. Then we assign to $(x, y) \in X\,{}_f\!\times_g Y$ the germ $\mathcal{S}_{(x,y)} := [P\,{}_f\!\times_g R]_{(x,y)}$. We refer to $\mathcal{S}$ as the *induced stratification* of $X\,{}_f\!\times_g Y$ by the stratifications $\mathcal{S}^X$ and $\mathcal{S}^Y$.

Lemma A.4. *Suppose X, Y, and Z are differentiable stratified spaces and $f\colon X \to Z$ and $g\colon Y \to Z$ are differentiable stratified mappings. If f is in addition a stratified submersion, then the induced stratification $\mathcal{S}$ is a stratification of $X\,{}_f\!\times_g Y$.*

Of course, $X\,{}_f\!\times_g Y$ and $Y\,{}_g\!\times_f X$ are isomorphic, so the same holds true if we assume that g is a stratified submersion.

Proof. For simplicity, we work with the maximal decompositions $\mathcal{Z}^X$, $\mathcal{Z}^Y$, and $\mathcal{Z}^Z$ of X, Y, and Z, respectively, which is clearly sufficient as the definition of the fibred product is local. Then the fact that $\mathcal{Z}^X$ and $\mathcal{Z}^Y$ are partitions of X and Y immediately implies that $\mathcal{Z} := \{P\,{}_f\!\times_g R \mid P \in \mathcal{Z}^X, R \in \mathcal{Z}^Y\}$ is a partition of $X\,{}_f\!\times_g Y$. That each $P \in \mathcal{Z}^X$ and $R \in \mathcal{Z}^Y$ is locally closed implies that $P \times R$ is locally closed in $X \times Y$ and hence $P\,{}_f\!\times_g R = (P \times R) \cap (X\,{}_f\!\times_g Y)$ is locally closed in $X\,{}_f\!\times_g Y$. Given $(x, y) \in X\,{}_f\!\times_g Y$, let U_x and U_y be open neighborhoods of x in X and y in Y, respectively, that each intersect finitely many elements $\mathcal{Z}^X$ and $\mathcal{Z}^Y$, and then $(U_x \times U_y) \cap (X\,{}_f\!\times_g Y)$ is an open neighborhood of (x, y) in $X\,{}_f\!\times_g Y$ that evidently meets finitely many elements of $\mathcal{Z}$. Therefore, $\mathcal{Z}$ is a locally finite partition of $X\,{}_f\!\times_g Y$ into locally closed sets.

Now, let $P \in \mathcal{Z}^X$ and $R \in \mathcal{Z}^Y$, and choose connected components P_0 of P and R_0 of R. Then as f and g are stratified mappings, there is a piece $S \in \mathcal{Z}^Z$ with $f(P_0), g(R_0) \subset S$; see [Pfl01, 1.2.10]. Moreover, as f is a stratified submersion, $f_{|P_0}$ is by definition a submersion. Then by [Lan02, Prop. 2.5 & 2.6] and the fact that $f_{|P_0}$ is a submersion implies that $f_{|P_0}$ is transversal to $g_{|R_0}$, we have that $P_0 \,{}_f\!\times_g R_0$ is a smooth submanifold of $P_0 \times R_0$ and hence of the differentiable space $X \,{}_f\!\times_g Y$. Hence each connected component of $P \,{}_f\!\times_g R$ is a smooth manifold.

Finally, suppose $(P \,{}_f\!\times_g R) \cap \overline{(P' \,{}_f\!\times_g R')} \neq \emptyset$ for $P, P' \in \mathcal{Z}^X$ and $R, R' \in \mathcal{Z}^Y$. Choose $(x, y) \in (P \,{}_f\!\times_g R) \cap \overline{(P' \,{}_f\!\times_g R')}$, and then for any open neighborhoods U_x and U_y of x and y in X and Y, respectively, $U_x \times Y_y$ intersects $P' \,{}_f\!\times_g R'$. It follows that $P \cap \overline{P'}, R \cap \overline{R'} \neq \emptyset$ so that $P \subset \overline{P'}$ and $R \subset \overline{R'}$, hence $P \,{}_f\!\times_g R \subset \overline{(P' \,{}_f\!\times_g R')}$. That is, $\mathcal{Z}$ satisfies the condition of frontier and hence is a decomposition of $X \,{}_f\!\times_g Y$. □

A.4. **Tangent space.** Assume that $(X, \mathcal{C}^\infty)$ is a differentiable space. Then, given a point $x \in X$, the maximal ideal $\mathfrak{m}_x \subset \mathcal{C}^\infty_x$ in the stalk at x is finitely generated, hence the quotient space $\mathfrak{m}_x/\mathfrak{m}_x^2$ is a finite dimensional real vector space. One calls this space the *Zariski cotangent space* $T^*_x X$ of $(X, \mathcal{C}^\infty)$ at x, and its dual $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ the *Zariski tangent space* $T_x X$.

Remark A.5. There is another notion of a tangent bundle for a differentiable stratified space $(X, \mathcal{C}^\infty)$, namely the *stratified tangent space* $(T^{\mathrm{st}} X, \mathcal{C}^\infty)$. If $(X, \mathcal{C}^\infty)$ fulfills Whitney's condition A, then $(T^{\mathrm{st}} X, \mathcal{C}^\infty)$ is a differentiable stratified space as well. See [Pfl01, Sec. 2.1] for more details on the stratified tangent bundle

A.5. **Differential forms.** Let $(X, \mathcal{C}^\infty)$ denote a reduced differentiable stratified space. Let U be an affine open subset of X and $\iota : U \hookrightarrow \mathbb{R}^n$ be a singular chart of X. Denote by $\mathcal{I}_\iota$ the sheaf of smooth functions vanishing on $\overline{\iota(U)}$. Then define the sheaf Ω^k_ι for $k = 0$ as $\iota^{-1}(\mathcal{C}^\infty/\mathcal{I}_\iota) \cong \mathcal{C}^\infty_{|U}$ and for $k \in \mathbb{N}^*$ as the following inverse image sheaf

$$\Omega^k_\iota := \iota^{-1}\big(\Omega^k_{\mathbb{R}^n}/(\mathcal{I}_\iota \Omega^k_{\mathbb{R}^n} + d\mathcal{I}_\iota \wedge \Omega^{k-1}_{\mathbb{R}^n})\big).$$

Observe that by construction the exterior differential factors through the Ω^k_ι, hence we obtain a differential graded algebra $\big(\Omega^\bullet_\iota, d\big)$. If $\kappa : V \hookrightarrow \mathbb{R}^m$ is another singular chart of X, there exists a unique sheaf isomorphism $\eta_{\iota,\kappa} : \Omega^\bullet_{|\kappa(U\cap V)} \to \Omega^\bullet_{|\iota(U\cap V)}$ extending the isomorphism of sheaves $\eta_{\iota,\kappa} : (\mathcal{C}^\infty/\mathcal{I}_\kappa)_{|\kappa(U\cap V)} \to (\mathcal{C}^\infty/\mathcal{I}_\iota)_{|\iota(U\cap V)}$. One concludes that the cocycle condition

$$\eta_{\kappa,\iota} = \eta_{\kappa,\lambda} \circ \eta_{\lambda,\iota} \tag{A.1}$$

is fulfilled if $\lambda : V \hookrightarrow \mathbb{R}^l$ denotes a third singular chart of X. Hence the sheaves Ω^k_ι glue to a globally defined sheaf Ω^k_X of so-called *abstract k-forms* on X in such a way that the gluing maps preserve d. So we obtain a sheaf complex $\big(\Omega^\bullet_X, d\big)$ of differential graded algebras. The complex of global sections $\big(\Omega^\bullet(X), d\big)$ will be called the *Grauert–Grothendieck complex* of X.

Remark A.6. For $X \subset \mathbb{C}^n$ a complex space, the construction of the complex $\Omega^\bullet(X)$ within the analytic category goes back to Grauert [GK64] and Grothendieck [Gro66].

Let us now describe how one can represent elements of $\Omega^k(X)$. To this end assume to be given an open covering $\mathcal{U}$ of X by coordinate domains and a family $(\kappa_U)_{U\in\mathcal{U}}$ of singular charts $\kappa_U : U \hookrightarrow \widetilde{U} \subset \mathbb{R}^{n_U}$ such that $\widetilde{U}$ is open and contains $\kappa_U(U)$ as

a relatively closed subset. An element of $\Omega^k(X)$ can then be represented as a family $([\omega_U])_{U\in\mathcal{U}}$, where $\omega_U \in \Omega^k(\widetilde{U})$ and where one has for any two $U, V \in \mathcal{U}$ over the overlap $U \cap V$

$$\eta_{\kappa_V,\kappa_U}\big([\omega_U]\big) = [\omega_V].$$

Hereby, $[\omega_U]$ denotes the equivalence class of ω_U in $\Omega^k_{\kappa_U}$.

This representation allows for the following useful construction. Assume that S is a stratum of X, and let $\iota_S : S \hookrightarrow X$ denote the canonical embedding. Given an element $\omega = ([\omega_U])_{U\in\mathcal{U}} \in \Omega^k(X)$ one observes that for any two $U, V \in \mathcal{U}$ the pulled back forms $\iota^*_{S\cap U}\kappa^*_U(\omega_U)$ and $\iota^*_{S\cap V}\kappa^*_V(\omega_V)$ coincide on the overlap $U \cap V$, hence glue together to a global form on S which we denote by $\iota^*_S\omega \in \Omega^k(S)$.

By construction, each of the sheaves Ω^k_X carries the structure of a $\mathcal{C}^\infty$-module in a natural way. This observation entails the following result

Proposition A.7. *The Grauert–Grothendieck complex $\big(\Omega^\bullet(X), d\big)$ of a differentiable stratified space $(X, \mathcal{C}^\infty)$ is a complex of fine sheaves.*

Finally in this section, we will define the pull-back morphism $f^* : \Omega^k_Y \to \Omega^k_X$ associated to a smooth map $f : X \to Y$ between reduced differentiable stratified spaces $(X, \mathcal{C}^\infty_X)$ and $(Y, \mathcal{C}^\infty_Y)$. By the preceeding proposition and the construction of the Grauert–Grothendieck complex it suffices to consider the case where $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are affine. Choose open neighorhoods $U \subset \mathbb{R}^n$ of X and $V \subset \mathbb{R}^m$ of Y such that X is closed in U and Y in V. Choose a smooth function $F : U \to V$ such that $F_{|X} = f$. For $\omega \in \Omega^k(V)$ representing an abstract k-form on Y we put

$$f^*([\omega]) := [F^*\omega] \in \Omega^k(X).$$

Since F^* maps the vanishing ideal $I_Y \subset \mathcal{C}^\infty(V)$ to the vanishing ideal $I_X \subset \mathcal{C}^\infty(U)$ and since F^* commutes with d, F^* maps the $I_Y\Omega^k(V) + dI_Y \wedge \Omega^k(V)$ to $I_X\Omega^k(U) + dI_X \wedge \Omega^k(U)$. Moreover, if $\widetilde{F} : U \to V$ is another smooth function such that $\widetilde{F}_{|X} = f$, then $F^*g - \widetilde{F}^*g \in I_X$ and $F^*dg - \widetilde{F}^*dg \in dI_X$ for all $g \in \mathcal{C}^\infty(V)$, which entails that $F^*\omega - \widetilde{F}^*\omega \in I_X\Omega^k(U) + dI_X \wedge \Omega^k(U)$. This proves that $[F^*\omega]$ neither depends on the particular choice of the representative of $[\omega]$ nor on the particular smooth F extending f to an open neighborhood of X. Hence $f^* : \Omega^k(Y) \to \Omega^k(X)$ is well-defined. Obviously, d commutes with f^*, since it commutes with F^*.

References

[AC07] A. Adem and F.R. Cohen. Commuting elements and spaces of homomorphisms. *Math. Ann.*, 338(3):587–626, 2007.

[AG12] A. Adem and J. M. Gómez. Equivariant K-theory of compact Lie group actions with maximal rank isotropy. *J. Topol.*, 5(2):431–457, 2012.

[ALR07] A. Adem, J. Leida, and Y. Ruan. *Orbifolds and stringy topology*, volume 171 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2007.

[AS09] I. Androulidakis and G. Skandalis. The holonomy groupoid of a singular foliation. *J. Reine Angew. Math.*, 626:1–37, 2009.

[BGNX12] K. Behrend, G. Ginot, B. Noohi, and P. Xu. String topology for stacks. *Astérisque*, (343):xiv+169, 2012.

[BM93] H. Boualem and P. Molino. Modèles locaux saturés de feuilletages riemanniens singuliers. *C. R. Acad. Sci. Paris Sér. I Math.*, 316(9):913–916, 1993.

[Bou98] N. Bourbaki. *General topology. Chapters 1–4*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation.

[Bry87] J.-L. Brylinski. Cyclic homology and equivariant theories. *Ann. Inst. Fourier (Grenoble)*, 37(4):15–28, 1987.

[BtD95] T. Bröcker and T. tom Dieck. *Representations of compact Lie groups*, volume 98 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. Translated from the German manuscript, Corrected reprint of the 1985 translation.

[BX11] K. Behrend and P. Xu. Differentiable stacks and gerbes. *J. Symplectic Geom.*, 9(3):285–341, 2011.

[CPS04] M.-J. Choi, D.H. Park, and D.Y. Suh. The existence of semialgebraic slices and its applications. *J. Korean Math. Soc.*, 41(4):629–646, 2004.

[DCPV11] C. De Concini, C. Procesi, and M. Vergne. Infinitesimal index: cohomology computations. *Transform. Groups*, 16(3):717–735, 2011.

[DCPV13] C. De Concini, C. Procesi, and M. Vergne. The infinitesimal index. *J. Inst. Math. Jussieu*, 12(2):297–334, 2013.

[Deb01] Claire Debord. Holonomy groupoids of singular foliations. *J. Differential Geom.*, 58(3):467–500, 2001.

[DJZ04] W. Domitrz, S. Janeczko, and M. Zhitomirskii. Relative Poincaré lemma, contractibility, quasi-homogeneity and vector fields tangent to a singular variety. *Illinois J. Math.*, 48(3):803–835, 2004.

[DK00] J.J. Duistermaat and J.A.C. Kolk. *Lie groups*. Universitext. Springer-Verlag, Berlin, 2000.

[FOR09] R.L. Fernandes, J.-P. Ortega, and T.S. Ratiu. The momentum map in Poisson geometry. *Amer. J. Math.*, 131(5):1261–1310, 2009.

[FPS15] C. Farsi, M.J. Pflaum, and C. Seaton. Stratifications of inertia spaces of compact Lie group actions. *J. Singul.*, 13:107–140, 2015.

[FS14] T. Friedmann and R.P. Stanley. Counting conjugacy classes of elements of finite order in Lie groups. *European J. Combin.*, 36:86–96, 2014.

[GK64] H. Grauert and H. Kerner. Deformationen von Singularitäten komplexer Räume. *Math. Ann.*, 153:236–260, 1964.

[God58] R. Godement. *Topologie Algébrique et Theorie des Faisceaux*. Hermann, Paris, 1958.

[GPS12] J.M. Gómez, A. Pettet, and J. Souto. On the fundamental group of $\mathrm{Hom}(\mathbb{Z}^k, G)$. *Math. Z.*, 271(1-2):33–44, 2012.

[Gro66] A. Grothendieck. On the De Rham cohomology of algebraic varieties. *Publ. Math. IHES*, 29:351–359, 1966.

[Kre10] M. Kreck. *Differential Algebraic Topology From Stratifolds to Exotic Spheres*, volume 110 of *Graduate Studies in Mathematics*. American Math. Society, 2010.

[Lan02] S. Lang. *Introduction to differentiable manifolds*. Universitext. Springer-Verlag, New York, second edition, 2002.

[Łoj65] S. Łojasiewicz. Ensembles semi-analytiques. In *Mimeographié*. Institute des Hautes Études Scientifique, Bures-sur-Yvette, France, 1965.

[LŚ11] T. Lusala and J. Śniatycki. Stratified subcartesian spaces. *Canad. Math. Bull.*, 54(4):693–705, 2011.

[LSV13] H.V. Lê, P. Somberg, and J. Vanžura. Smooth structures on pseudomanifolds with isolated conical singularities. *Acta Math. Vietnam.*, 38(1):33–54, 2013.

[LT97] E. Lerman and S. Tolman. Hamiltonian torus actions on symplectic orbifolds and toric varieties. *Trans. Amer. Math. Soc.*, 349(10):4201–4230, 1997.

[Mac05] K.C.H. Mackenzie. *General theory of Lie groupoids and Lie algebroids*, volume 213 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005.

[Mat73] J.N. Mather. Stratifications and mappings. In *Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971)*, pages 195–232. Academic Press, New York, 1973.

[Mel91] R.B. Melrose. Pseudodifferential operators, corners and singular limits. In *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, pages 217–234. Math. Soc. Japan, Tokyo, 1991.

[Mol88] P. Molino. *Riemannian foliations*, volume 73 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1988. Translated from the French by Grant Cairns, With appendices by Cairns, Y. Carrière, É. Ghys, E. Salem and V. Sergiescu.

[MROD93] J. Margalef Roig and E. Outerelo Domínguez. Lie group actions over manifolds with corners. *Math. Japon.*, 38(3):577–582, 1993.

[NGSdS03] J.A. Navarro González and J.B. Sancho de Salas. *C^∞-differentiable spaces*, volume 1824 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2003.

[NW14] J. Noguchi and J. Winkelmann. *Nevanlinna theory in several complex variables and Diophantine approximation*, volume 350 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Tokyo, 2014.

[Par12] P.-É. Paradan. On the structure of $K_G(T_GM)$. arXiv:1209.3852, 2012.

[Par13] D.H. Park. On orbit types of semialgebraically proper actions. *Arch. Math. (Basel)*, 101(1):33–41, 2013.

[Pfl01] M.J. Pflaum. *Analytic and geometric study of stratified spaces*, volume 1768 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.

[PPT07] M. J. Pflaum, H. B. Posthuma, and X. Tang. An algebraic index theorem for orbifolds. *Adv. Math.*, 210(1):83–121, 2007.

[PPT14] M.J. Pflaum, H. Posthuma, and X. Tang. Geometry of orbit spaces of proper Lie groupoids. *J. für die Reine und Angewandte Mathematik*, 25:11351153, 2014.

[Pra85] Jean Pradines. How to define the differentiable graph of a singular foliation. *Cahiers Topologie Géom. Différentielle Catég.*, 26(4):339–380, 1985.

[PS02] D.H. Park and D.Y. Suh. Linear embeddings of semialgebraic G-spaces. *Math. Z.*, 242(4):725–742, 2002.

[PV09] P.-É. Paradan and M. Vergne. Index of transversally elliptic operators. *Astérisque*, (328):297–338 (2010), 2009.

[PY07] D. Panyushev and O. Yakimova. Symmetric pairs and associated commuting varieties. *Math. Proc. Cambridge Philos. Soc.*, 143(2):307–321, 2007.

[Ren80] J. Renault. *A groupoid approach to C^*-algebras*, volume 793 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[Ric79] R.W. Richardson. Commuting varieties of semisimple Lie algebras and algebraic groups. *Compositio Math.*, 38(3):311–327, 1979.

[RPSAW05] J. I. Royo Prieto, M. Saralegi-Aranguren, and R. Wolak. Top-dimensional group of the basic intersection cohomology for singular Riemannian foliations. *Bull. Pol. Acad. Sci. Math.*, 53(4):429–440, 2005.

[Sch80] G.W. Schwarz. Lifting smooth homotopies of orbit spaces. *Inst. Hautes Études Sci. Publ. Math.*, (51):37–135, 1980.

[Sch01] T. Schmah. Torus actions on symplectic orbi-spaces. *Proc. Amer. Math. Soc.*, 129(4):1169–1177, 2001.

[Shi97] M. Shiota. *Geometry of Subanalytic and Semialgebraic Sets*, volume 150 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1997.

[Sja05] R. Sjamaar. A de Rham theorem for symplectic quotients. *Pacific J. Math.*, 220(1):153–166, 2005.

[SL91] R. Sjamaar and E. Lerman. Stratified symplectic spaces and reduction. *Ann. of Math. (2)*, 134(2):375–422, 1991.

[SLV15] P. Somberg, H.V. Lê, and J. Vanžura. Poisson smooth structures on stratified symplectic spaces. In Pierre Cartier, A.D.R. Choudary, and Michel Waldschmidt, editors, *Mathematics in the 21st Century*, volume 98 of *Springer Proceedings in Mathematics & Statistics*, pages 181–204. Springer Basel, 2015.

[Ste74] P. Stefan. Accessible sets, orbits, and foliations with singularities. *Proc. London Math. Soc. (3)*, 29:699–713, 1974.

[Sus73] H. J. Sussmann. Orbits of families of vector fields and integrability of distributions. *Trans. Amer. Math. Soc.*, 180:171–188, 1973.

[TGS08] E. Torres Giese and D. Sjerve. Fundamental groups of commuting elements in Lie groups. *Bull. Lond. Math. Soc.*, 40(1):65–76, 2008.

[Tu04] J.-L. Tu. Non-Hausdorff groupoids, proper actions and K-theory. *Doc. Math.*, 9:565–597 (electronic), 2004.

[Ver96] Michele Vergne. Equivariant index formulas for orbifolds. *Duke Math. J.*, 82(3):637–652, 1996.

[Wat12] J. Watts. *Diffeologies, Differential Spaces, and Symplectic Geometry*. ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)–University of Toronto (Canada).

[Wat15] J. Watts. The differential structure of an orbifold. arXiv:1503.01740, 2015.

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