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THE GRAUERT–GROTHENDIECK COMPLEX ON DIFFERENTIABLE SPACES AND A SHEAF COMPLEX OF BRYLINSKI

MARKUS J. PFLAUM, HESSEL POSTHUMA, AND XIANG TANG

Dedicated to Henri Laufer on the occasion of his 70th birthday.

Abstract. We use the Grauert–Grothendieck complex on differentiable spaces to study basic relative forms on the inertia space of a compact Lie group action on a manifold. We prove that the sheaf complex of basic relative forms on the inertia space is a fine resolution of Bryliski's sheaf of functions on the inertia space.

1. Introduction

In his paper [Bry87b], Jean-Luc Brylinski studies the cyclic homology theory of the smooth crossed-product algebra $\mathscr{A} := \mathscr{C}^\infty_{\mathrm{c}}(G \times M)$ associated to a manifold M which carries a smooth action of a Lie group G. The crossed-product algebra carries the convolution product $*$ defined by

$$(1.1) \qquad \big(u * v\big)(g,x) := \int_M u(gh, h^{-1}x) \cdot v(h^{-1}, x)\, d\mu(h) \quad \text{for } u, v \in \mathscr{A},\ (g,x) \in G \times M.$$

The symbol μ hereby denotes a fixed left-invariant Haar measure on G. Brylinski asserts in his article that the Hochschild homology $HH_k(\mathscr{A})$ of the (topological) algebra $\mathscr{A}$ coincides naturally with the space $\Omega^k_{\mathrm{b.r.}}(M/G)$ of so-called *basic relative k-forms*. The sheaf complex of basic relative forms will be constructed in Section 3. Despite it being a sheaf complex over the orbit space M/G, basic relative forms are defined as forms on the so-called *loop space*

$$(1.2) \qquad \Lambda_0(G \ltimes M) := \bigcup_{g \in G} \{g\} \times M^g \subset G \times M$$

which essentially consists of a disjoint union of the fixed point manifolds M^g, $g \in G$ of the smooth action $G \times M \to M$ together with an appropriate topology on it. In Section 3 we will also see that the loop space carries even the structure of a differentiable stratified space. Brylinski has also claimed in [Bry87b] that the sheaf complex of basic relative forms is a resolution of a certain sheaf $\mathscr{B}$ on the orbit space M/G. The section spaces of that sheaf over $O \subset M/G$ open are given by

$$(1.3) \quad \mathscr{B}(O) := \big\{ f \in \mathscr{C}^\infty_{\Lambda_0}(s^{-1}_{|\Lambda_0}\pi^{-1}(O)) \mid f \text{ is } G\text{-invariant and } f(g,-) \text{ locally constant for all } g \in G \big\}.$$

Hereby, the map $s_{|\Lambda_0} : \Lambda_0(G \ltimes M) \to M$ is given by $s_{|\Lambda_0}(g,p) = p$, $\pi : M \to G/M$ is the orbit projection, the sheaf $\mathscr{C}^\infty_{\Lambda_0}$ is defined as the sheaf of continuous functions on the loop space which are locally restrictions of smooth functions on $G \times M$, the G-action on $\Lambda_0(G \ltimes M)$ is given by the diagonal action with conjugation in the first coordinate, and the function $f(g,-)$ for $f \in \mathscr{C}^\infty(\Lambda_0(G \ltimes M))$ and $g \in G$ is the map $M^g \to \mathbb{R}$, $p \to f(g,p)$. Since the sheaf $\mathscr{B}$ has been defined first in [Bry87b], we call it *Brylinskis's sheaf.*

For the proof that $HH_k(\mathscr{A})$ is isomorphic to $\Omega^k_{\mathrm{b.r.}}(M/G)$, Brylinski refers to the unpublished paper [Bry87a], a proof of the second claim is missing.

The purpose of these notes is to shed some light onto Brylinski's sheaf $\mathscr{B}$ and the sheaf complex of basic relative forms. We interpret the latter as a certain subcomplex of the Grauert-Grothendieck complex of differential forms on the loop space of the G-manifold M. Then, in Theorem 3.1,we show that it is what it is claimed to be: an acyclic complex of fine sheaves whose cohomology in degree zero coincides with Brylinski's sheaf $\mathscr{B}$.

Let us mention that the cyclic homology theory of crossed product algebras has been studied also by Block–Getzler [BG94] and Nistor [Nis93]. Moreover, Farsi–Pflaum–Seaton have described in [FPS15b, FPS15a] the stratification theory of the loop space and proved a de Rham Theorem for it.

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2. Differential forms

In this section, we will define the Grauert–Grothendieck complex of differential forms on a differentiable space $(X, \mathscr{O})$ which we allow to be non-reduced. To this end consider first an affine open subset of X and let $\iota : U \hookrightarrow \widetilde{U} \subset \mathbb{R}^n$ be a singular chart. Denote by $\mathscr{I}_\iota$ the ideal sheaf in $\mathscr{C}^\infty_{\widetilde{U}}$ such that $\mathscr{O}_{|U} \cong \iota^{-1}(\mathscr{O}_\iota)$, where $\mathscr{O}_\iota$ is the restriction of the sheaf $\mathscr{C}^\infty_{\widetilde{U}}/\mathscr{I}_\iota$ to $\iota(U)$. We then define Ω^0_ι as the sheaf $\mathscr{O}_\iota$ and, for $k \in \mathbb{N}^*$, the sheaf Ω^k_ι as the inverse image sheaf

$$\Omega^k_\iota := \iota^{-1}\big(\Omega^k_{\widetilde{U}}/\mathscr{I}_\iota\Omega^k_{\widetilde{U}} + d\mathscr{I}_\iota \wedge \Omega^{k-1}_{\widetilde{U}}\big) = \iota^*\big(\Omega^k_{\widetilde{U}}/\mathscr{I}_\iota\Omega^k_{\widetilde{U}} + d\mathscr{I}_\iota \wedge \Omega^{k-1}_{\widetilde{U}}\big)\,. \tag{2.1}$$

The latter equality holds true because $\mathscr{O}_{|U} \cong \iota^{-1}(\Omega^0_\iota)$, and because the sheaf

$$\Omega^k_{\widetilde{U}}/\mathscr{I}_\iota\Omega^k_{\widetilde{U}} + d\mathscr{I}_\iota \wedge \Omega^{k-1}_{\widetilde{U}}$$

is an $\mathscr{O}_\iota$-module. By construction, the exterior differential d descends to sheaf morphisms $d : \Omega^k_\iota \to \Omega^{k-1}_\iota$ so that we obtain a complex of sheaves $\big(\Omega^\bullet_\iota, d\big)$ over U. In the following we will show that the sheaf complexes $\big(\Omega^\bullet_\iota, d\big)$ glue to a sheaf complex $\big(\Omega^\bullet, d\big)$ on X, when ι runs through a singular atlas of $(X, \mathscr{O})$. To construct the gluing maps we will proceed by proving a sequence of lematas.

Lemma 2.1. *Let $\iota : U \hookrightarrow \widetilde{U} \subset \mathbb{R}^n$ and $\kappa : U \hookrightarrow \widehat{U} \subset \mathbb{R}^m$ be two singular charts of $(X, \mathscr{O})$ for which there exists a smooth embedding $H : \widehat{U} \hookrightarrow \widetilde{U}$ such that $H(\widehat{U})$ is closed in $\widetilde{U}$ and such that the pullback $H^* : \mathscr{C}^\infty_{\widetilde{U}} \to \mathscr{C}^\infty_{\widehat{U}}$ induces an isomorphism of locally ringed spaces $\big(H_{|\kappa(U)}, H^*\big) : (\kappa(U), \mathscr{O}_\kappa) \to (\iota(U), \mathscr{O}_\iota)$. Then there exists a unique isomorphism of sheaf complexes $\eta_{\kappa,\iota} : \Omega^\bullet_\iota \to \Omega^\bullet_\kappa$ such that $\eta_{\kappa,\iota}(f) = H^*f$ for all $f \in \mathscr{O}_\iota(V)$ with $V \subset U$ open.*

Proof. By construction we have the following canonical identifications

$$\Omega^k_\iota(U) \cong \Omega^k(\widetilde{U})/\big(J_\iota\Omega^k(\widetilde{U}) + dJ_\iota \wedge \Omega^{k-1}(\widetilde{U})\big), \quad \text{where } J_\iota := \mathscr{I}_\iota(\widetilde{U}) \tag{2.2}$$

and

$$\Omega^k_\kappa(U) \cong \Omega^k(\widehat{U})/\big(J_\kappa\Omega^k(\widehat{U}) + dJ_\kappa \wedge \Omega^{k-1}(\widehat{U})\big), \quad \text{where } J_\kappa := \mathscr{J}_\kappa(\widehat{U}). \tag{2.3}$$

Denote by $I \subset \mathcal{C}^\infty(\widetilde{U})$ the vanishing ideal of $H(\widehat{U})$. The pull-back morphism $H^* : \Omega^k(\widetilde{U}) \to \Omega^k(\widehat{U})$ then is surjective with kernel $I\Omega^k(\widetilde{U}) + dI \wedge \Omega^{k-1}(\widetilde{U})$, since $H(\widehat{U})$ is a closed submanifold of $\widetilde{U}$. Since $H^*J_\iota \subset J_\kappa$ and since d commutes with H^*, pull-back by H induces a surjective map denoted by the same symbol

$$H^* : \Omega^k(\widetilde{U})/\big(J_\iota\Omega^k(\widetilde{U}) + dJ_\iota \wedge \Omega^{k-1}(\widetilde{U})\big) \longrightarrow \Omega^k(\widehat{U})/\big(J_\kappa\Omega^k(\widehat{U}) + dJ_\kappa \wedge \Omega^{k-1}(\widehat{U})\big).$$

This map is injective since $I\Omega^k(\widetilde{U}) + dI \wedge \Omega^{k-1}(\widetilde{U})$ is contained in $J_\iota\Omega^k(\widetilde{U}) + dJ_\iota \wedge \Omega^{k-1}(\widetilde{U})$ and since $H^*J_\iota = J_\kappa$. After choosing for each open set $V \subset U$ an open $\widehat{V} \subset \widehat{U}$ such that $V = \kappa^{-1}(\widehat{V})$, pull-back via $H_{|\widehat{V}}$ induces in the same way isomorphisms $H^*_V : \Omega^k_\iota(V) \to \Omega^k_\kappa(V)$ for each $k \in \mathbb{N}$ and $V \subset U$ open. The family of isomorphisms H^*_V then defines the desired sheaf isomorphism $\eta_{\kappa,\iota}$. Since each H^*_V commutes with the differentials, $\eta_{\kappa,\iota}$ is a morphism of sheaf complexes, indeed. Moreover, since $\Omega^\bullet_\iota$ is generated as a sheaf of differential graded algebras by Ω^0_ι, $\eta_{\kappa,\iota}$ is uniquely determined by its action on Ω^0_ι. This finishes the proof. □

Lemma 2.2. *Under the assumption of the preceding lemma let $G : \check{U} \hookrightarrow \widetilde{U}$ be a second smooth embedding defined on an open neighborhood $\check{U} \subset \mathbb{R}^n$ of $\kappa(U)$ such that $\big(G_{|\kappa(U)}, G^*\big) : (\kappa(U), \mathscr{O}_\kappa) \to (\iota(U), \mathscr{O}_\iota)$ is an isomorphism of locally ringed spaces . Then $H^*_V : \Omega^k_\iota(V) \to \Omega^k_\kappa(V)$ and $G^*_V : \Omega^k_\iota(V) \to \Omega^k_\kappa(V)$ coincide for all open $V \subset U$ which means that the sheaf isomorphism $\eta_{\kappa,\iota}$ does not depend on the particular embedding inducing an isomorphism between $(\kappa(U), \mathscr{O}_\kappa)$ and $(\iota(U), \mathscr{O}_\iota)$.*

Proof. After possibly shrinking $\widetilde{U}$ we can assume that $G(\check{U})$ is closed in $\widetilde{U}$ as well. For every smooth function $f \in \mathcal{C}^\infty(\widetilde{U})$ we then have

$$(H^*f)_{|\widetilde{U}\cap\check{U}} - (G^*f)_{|\widetilde{U}\cap\check{U}} \in \mathscr{J}_\kappa(\widetilde{U}\cap\check{U}), \text{ and } (H^*df)_{|\widetilde{U}\cap\check{U}} - (G^*df)_{|\widetilde{U}\cap\check{U}} \in d\mathscr{J}_\kappa(\widetilde{U}\cap\check{U}).$$

That implies that the actions of H^* and G^* on $\Omega^k_\iota(U)$ coincide. Likewise $H^*_V = G^*_V$ for all open $V \subset U$, hence $\eta_{\kappa,\iota}$ is independent of the particularly chosen embedding H. □

Next let $\iota : U \hookrightarrow \mathbb{R}^n$ and $\kappa : V \hookrightarrow \mathbb{R}^m$ be two singular charts of X defined on open $U, V \subset X$. We will construct a sheaf morphism $\eta_{\kappa,\iota} : \Omega^k_{\iota|U\cap V} \to \Omega^k_{\kappa|U\cap V}$. Let $x \in U \cap V$. By the embedding theorem A.3 there exists a singular chart $\lambda : W_x \hookrightarrow \mathbb{R}^{\operatorname{rk} x}$ defined over an open neighborhood $W_x \subset U \cap V$ of x. Moreover, after possibly shrinking W_x, there exist embeddings $H : \widetilde{W}_x \hookrightarrow \mathbb{R}^n$ and $G : \widetilde{W}_x \hookrightarrow \mathbb{R}^m$ of an open neighborhood $\widetilde{W}_x$ of $\lambda(x)$ such that $\iota_{|W_x} = H \circ \lambda$ and $\kappa_{|W_x} = G \circ \lambda$, and such that H^* induces an isomorphism from $(\iota(U), \mathscr{O}_\iota)$ to $(\lambda(U), \mathscr{O}_\lambda)$ and G^* one from $(\kappa(U), \mathscr{O}_\kappa)$ to $(\lambda(U), \mathscr{O}_\lambda)$. By Lemma 2.1 we obtain isomorphisms of sheaf complexes $\eta_{\lambda,\iota} : \Omega^\bullet_{\iota|W_x} \to \Omega^\bullet_\lambda$ and $\eta_{\lambda,\kappa} : \Omega^\bullet_{\kappa|W_x} \to \Omega^\bullet_\lambda$. Put $\eta^{W_x}_{\kappa,\iota} := (\eta_{\lambda,\kappa})^{-1} \circ \eta_{\lambda,\iota}$. Then $\eta^{W_x}_{\kappa,\iota}$ is a sheaf isomorphism from $\Omega^\bullet_{\iota|W_x}$ to $\Omega^\bullet_{\kappa|W_x}$ which by Lemma 2.2 does not depend on the particular choice of λ and the embeddings H and G. Moreover, if y is another point of $U \cap V$, an argument using Lemma 2.2 and the embedding theorem A.3 again shows that the sheaf isomorphisms $\eta^{W_x}_{\kappa,\iota}$ and $\eta^{W_y}_{\kappa,\iota}$ coincide on the overlap $W_x \cap W_y$. This proves the next lemma.

Lemma 2.3. *Given two singular charts $\iota : U \hookrightarrow \mathbb{R}^n$ and $\kappa : V \hookrightarrow \mathbb{R}^m$ of $(X, \mathscr{O})$ there exists a unique sheaf morphism $\eta_{\kappa,\iota} : \Omega^\bullet_{\kappa|U\cap V} \to \Omega^\bullet_{\iota|U\cap V}$ such that*

$$\eta_{\kappa,\iota|W_x} = (\eta_{\lambda,\kappa})^{-1} \circ \eta_{\lambda,\iota}$$

for each $x \in U \cap V$ and each singular chart $\lambda : W_x \hookrightarrow \mathbb{R}^{\operatorname{rk} x}$ defined on a sufficiently small open neighborhood $W_x \subset U \cap V$ of x.

Application of Lemma 2.2 and the embedding theorem A.3 a last time entails the final lemma.

Lemma 2.4. *Assume that $\iota : U \hookrightarrow \mathbb{R}^n$, $\kappa : V \hookrightarrow \mathbb{R}^m$, and $\lambda : W \hookrightarrow \mathbb{R}^l$ are three singular charts of X. Then the following cocycle condition holds true over the intersection $U \cap V \cap W$:*

$$\eta_{\kappa,\iota} = \eta_{\kappa,\lambda} \circ \eta_{\lambda,\iota}. \tag{2.4}$$

The cocycle condition holding true for any triple of singular charts entails that the sheaf complexes Ω^k_ι glue together to a global sheaf Ω^k_X on X. This sheaf is the *sheaf of abstract k-forms on X*. We sometimes denote it briefly by Ω^k. By construction, each of the $\eta_{\kappa,\iota}$ commutes with the exterior differential d, hence the operators $d : \Omega^k_\kappa \to \Omega^{k+1}_\kappa$ glue together to a sheaf morphism $d : \Omega^k_X \to \Omega^{k+1}_X$. So finally we obtain a sheaf complex $\big(\Omega^\bullet_X, d\big)$ of commutative differential graded algebras. The complex of global sections $\big(\Omega^\bullet(X), d\big)$ will be called the *Grauert–Grothendieck complex* of X.

Remark 2.5. For $X \subset \mathbb{C}^n$ a complex space or algebraic variety, the complex $\Omega^\bullet(X)$ of abstract forms on X has been first constructed by Grauert [GK64] and Grothendieck [Gro66] using sheaves of holomorphic respectively regular functions. In [Spa71, §4.], Spallek has given a construction of the sheaf of k-forms of class $\mathscr{C}^\infty$ on an affine differentiable space $(X, \mathscr{O})$ which essentially corresponds to the one given here in Eq. (2.1). It is claimed in [Spa71] that given a (singular) atlas for $(X, \mathscr{O})$, the sheaves of k-forms over affine domains induce a global sheaf of k-forms on X. Several details of the corresponding construction, in particular a verification of the cocycle condition, are missing.

Remark 2.6. Recall that a morphism of differentiable spaces $(f,\varphi):(X,\mathscr{O}_X)\to(Y,\mathscr{O}_Y)$ consists of a continuous map $f:X\to Y$ and a morphism of sheaves $\varphi:\mathscr{O}_Y\to f_*\mathscr{O}_X$. Using the construction of the sheaves of abstract forms on X and Y one extends the morphism φ in a unique way to a morphism of sheaves of commutative differential algebras $\varphi:\Omega^\bullet_Y\to f_*\Omega^\bullet_X$. One concludes that forming the sheaf complex of abstract forms is a functor defined on the category of differentiable spaces.

3. Basic relative forms

Let us first recall the notion of *relative forms* associated to a smooth map $p:M\to N$ between manifolds M and N. By relative forms one understands sections of the sheaf complex $\Omega^\bullet_{M\xrightarrow{p}N}$ defined as the quotient sheaf

$$\Omega^\bullet_M/d(p^{-1}\mathscr{C}^\infty_N)\wedge\Omega^\bullet_M$$

together with the differential induced by the differential on $\Omega^\bullet_M$. If $p:M\to N$ is a surjective submersion, the space of global sections $\Omega^k_{M\xrightarrow{p}N}(M)$ can be identified with the space of smooth families $(\omega_y)_{y\in N}$ of forms $\omega_y\in\Omega^k\big(p^{-1}(y)\big)$. The differential acts fiberwise on $(\omega_y)_{y\in N}$ which means that

$$d\big((\omega_y)_{y\in N}\big)=(d\omega_y)_{y\in N}\ .$$

If the underlying map p is projection onto the first factor of a product $N\times M$, one can identify $\Omega^k_{M\xrightarrow{p}N}$ with the sheaf of smooth sections $\Gamma^\infty(-,\wedge^k s^*T^*M)$, where $s:N\times M\to M$ is projection onto the second factor. More precisely, one has in this case a sequence of sheaf morphisms, whose composition is an isomorphism:

$$\Gamma^\infty(-,\wedge^k s^*T^*M)\hookrightarrow\Omega^k_{N\times M}\longrightarrow\Omega^k_{N\times M\xrightarrow{p}N}=\Omega^k_{N\times M}/d(p^{-1}\mathscr{C}^\infty_N)\wedge\Omega^{k-1}_{N\times M}\ . \tag{3.1}$$

Note that even though $\Gamma^\infty(-,\wedge^k s^*T^*M)$ is a subsheaf of Ω^k_M for each k, one does not obtain that way a subsheaf complex of $\Omega^\bullet_M$ since the exterior derivative on $\Omega^\bullet_M$ does in general not map $\Gamma^\infty(-,\wedge^k s^*T^*M)$ to $\Gamma^\infty(-,\wedge^{k-1}s^*T^*M)$. The correct differential on $\Gamma^\infty(-,\wedge^k s^*T^*M)$ acts fiberwise as explained above.

After these preliminary remarks we now assume that G is a compact Lie group acting on a smooth manifold M. By $G\ltimes M$ we denote the corresponding action groupoid. More precisely, this is the Lie groupoid with arrow space $G\times M$, object space M, source map $s:G\times M\to M$, $(g,p)\mapsto p$, target map $t:G\times M\to M$, $(g,p)\mapsto gp$ and multiplication $m:(G\ltimes M)^{(2)}\to G\times M$, $((h,gp),(g,p))\to(hg,p)$ defined on the fibered product $(G\ltimes M)^{(2)}:=(G\times M)\,{}_s\!\times_t(G\times M)$. The unit map of the action groupoid is $u:M\to G\times M$, $p\mapsto(e,p)$ with e denoting the identity element of G, and the inversion map is $G\times M\to G\times M$, $(g,p)\mapsto(g^{-1},gp)$. We will denote the orbit space of the action groupoid $G\ltimes M$ by M/G, and the orbit map by $\pi:M\to M/G$. For more details on Lie groupoids see [MM03].

The space $\Lambda_0(G\ltimes M)$ defined in Eq. (1.2) corresponds to the loop space of the action groupoid $G\ltimes M$ that means to the space of all $(g,p)\in G\times M$ such that $s(g,p)=t(g,p)$. Since $\Lambda_0(G\ltimes M)$ is a closed subspace of the manifold $G\times M$, the loop space inherits from the ambient manifold the structure of a differentiable space. We denote the canonical embedding by $\iota:\Lambda_0(G\ltimes M)\hookrightarrow G\times M$. Since by its definition the loop space is locally semialgebraic, it carries a minimal Whitney B stratification, so $\Lambda_0(G\ltimes M)$ becomes a differentiable stratified space. For simplicity, we denote the loop space shortly by Λ_0. Moreover, we denote the structure sheaf of smooth functions on Λ_0 by $\mathscr{C}^\infty_{\Lambda_0}$, and the sheaf of smooth functions on $G\times M$ vanishing on Λ_0 by $\mathscr{J}$. Now consider the Grauert-Grothendieck complex $\Omega^\bullet_{\Lambda_0}$ of differential forms on the loop space. Following Brylinski, we define the sheaf $\Omega^k_{\Lambda_0\to G}$ of *relative forms on the loop space* as the quotient sheaf

$$\Omega^k_{\Lambda_0\to G}:=\iota^{-1}\Big(\Omega^k_{G\times M\to G}\big/\big(\mathscr{J}\,\Omega^k_{G\times M\to G}+d\mathscr{J}\wedge\Omega^{k-1}_{G\times M\to G}\big)\Big)\ . \tag{3.2}$$

The graded sheaf $\Omega^\bullet_{\Lambda_0\to G}$ inherits from $\Omega^\bullet_{G\times M\to G}$ a differential turning it into a sheaf complex. Let us give another representation of the sheaf or relative forms on Λ_0. To this end observe that the pull-back bundle s^*T^*M induces a monomorphism of bundles $\wedge^k s^*T^*M \hookrightarrow \wedge^k T^*(G\times M)$, hence the following morphism of sheaves:

$$\Gamma^\infty_{\Lambda_0}(-,\wedge^k s^*T^*M) \longrightarrow \Omega^k_{\Lambda_0} = \iota^{-1}\Big(\Omega^k_{G\times M}/\big(\mathscr{J}\Omega^k_{G\times M} + d\mathscr{J}\wedge\Omega^{k-1}_{G\times M}\big)\Big). \tag{3.3}$$

Here, $\Gamma^\infty_{\Lambda_0}(-,\wedge^k s^*T^*M)$ stands for the sheaf of smooth sections of the vector bundle $\wedge^k s^*T^*M \to G\times M$ over the subspace Λ_0. In other words, the section space $\Gamma^\infty_{\Lambda_0}(U,E)$ for $U\subset\Lambda_0$ open can be identified with the quotient space $\Gamma^\infty(\widetilde{U},E)/\mathscr{J}(\widetilde{U})\Gamma^\infty(\widetilde{U},E)$, where $\widetilde{U}\subset G\times M$ open is chosen so that $\widetilde{U}\cap\Lambda_0 = U$. Since the composition of sheaf morphisms in (3.1) is an isomorphism, the sheaf morphism (3.3) induces a canoncial identification

$$\Omega^k_{\Lambda_0\to G}(U)\cong\Gamma^\infty(\widetilde{U},\wedge^k s^*T^*M)/\big(\mathscr{J}(\widetilde{U})\ \Gamma^\infty(\widetilde{U},\wedge^k s^*T^*M) + d\mathscr{J}(\widetilde{U})\wedge\Gamma^\infty(\widetilde{U},\wedge^{k-1}s^*T^*M)\big),$$

where $\widetilde{U}$ is chosen as before. For a section $\omega\in\Gamma^\infty(\widetilde{U},\wedge^k s^*T^*M)$ we denote its image in $\Omega^k_{\Lambda_0\to G}(U)$ by $[\omega]$. Since Λ_0 is the union of the fibers $\{g\}\times M^g$, $g\in G$, the relative form $[\omega]$ can be identified with the smooth family $(\omega_g)_{g\in G}$ of restrictions $\omega_g := \omega_{|M^g}$, and any smooth family $(\omega_g)_{g\in G}$ of forms $\omega_g\in\Omega^k(M^g)$ gives rise to a unique relative form on the loop space. Under this identification the differential of $[\omega]=(\omega_g)_{g\in G}$ is given by the smooth family $(d\omega_g)_{g\in G}$.

Next recall that the G-action on M gives rise for each $p\in M$ to the normal space $N_pM := T_pM/T_p\mathcal{O}_p$, where $\mathcal{O}_p$ denotes the G-orbit through p. The family N^* associating to each $p\in M$ the conormal space $N^*_pM\subset T^*M$ is a smooth generalized vector subbundle of the cotangent bundle T^*M in the sense of Drager–Lee–Park–Richardson [DLPR12]. Note that under the isomorphism between the tangent and cotangent bundle induced by a riemannian metric on M the generalized vector bundle N^* becomes a generalized distribution in the sense of Stefan–Suessmann, cf. [Ste74, Sus73]. The restriction of N^* to an orbit or a stratum of fixed orbit type is a vector bundle; see [PPT14]. After these preparatory remarks it is clear that $\wedge^k N^*$, $k\in\mathbb{N}^*$, is a smooth generalized vector subbundle of $\wedge^k T^*M$. Likewise, if one puts $N_pM^g := T_pM^g/(T_p\mathcal{O}_p\cap T_pM^g)$ for $g\in G$, the alternating power $\wedge^k(NM^g)^*$ is a smooth generalized vector subbundle of $\wedge^k T^*M^g$ for every $k\in\mathbb{N}^*$. The space $\Omega^k_{\text{h.r.}}(U)$ of *horizontal relative k-forms* over $U\subset\Lambda_0$ open is now defined as

$$\begin{aligned}\Omega^k_{\text{h.r.}}(U) &:= \big\{[\omega]=(\omega_g)_{g\in G}\in\Omega^k_{\Lambda_0\to G}(U)\mid \omega_{(g,p)}\in\wedge^k N^*_p \text{ for all } (g,p)\in\Lambda_0\big\} = \\ &= \big\{[\omega]=(\omega_g)_{g\in G}\in\Omega^k_{\Lambda_0\to G}(U)\mid \omega_g\in\Gamma^\infty(U\cap M^g,\wedge^k(NM^g)^*\big\}\ .\end{aligned}$$

Obviously, we thus obtain a subsheaf $\Omega^k_{\text{h.r.}}\subset\Omega^k_{\Lambda_0\to G}$ having these spaces as its section spaces. Now observe that the G-action on the cotangent bundle T^*M coming from the G-action on M induces a G-action on relative forms, hence we can speak of *invariant relative k-forms*. These are exactly those $[\omega]\in\Omega^k_{\text{h.r.}}(U)$ which satisfy

$$\omega_{(hgh^{-1},hx)}(hv_1,\ldots,hv_k) = \omega_{(g,x)}(v_1,\ldots,v_k) \tag{3.4}$$

for all $(g,x)\in U$ and $h\in G$ such that $(hgh^{-1},hx)\in U$ and $v_1,\ldots,v_k\in N_x$. Note that by definition invariance of $[\omega]$ does not depend on the particular choice of a representative. If one writes $[\omega]$ as a smooth family $(\omega_g)_{g\in G}$ of forms ω_g on $U\cap M^g$ and if U is G-invariant, invariance of $[\omega]$ can be equivalently expressed by

$$h^*\omega_{hgh^{-1}} = \omega_g \quad \text{for all } g,h\in G. \tag{3.5}$$

This relation implies in particular that the differential on $\Omega^\bullet_{\Lambda_0\to G}$ maps an invariant family $(\omega_g)_{g\in G}$ over a G-invariant open U to the invariant family $(d\omega_g)_{g\in G}$. Since the G-action on T^*M leaves the conormal bundle N^* invariant, we can even speak of *invariant horizontal relative k-forms*. Now we are ready to put for $O\subset M/G$ open

$$\Omega^k_{\text{b.r.}}(O) := \big\{[\omega]\in\Omega^k_{\text{h.r.}}(s^{-1}_{|\Lambda_0}\pi^{-1}(O))\mid [\omega] \text{ is invariant}\big\}. \tag{3.6}$$

These spaces are the section spaces of a sheaf $\Omega^k_{\text{b.r.}}$. Observe that the sheaf $\Omega^k_{\text{b.r.}}$ is defined over the orbit space M/G, not the loop space. Following Brylinski again, we call sections of $\Omega^k_{\text{b.r.}}$ *basic relative k-forms*. In case we need to clarify the action groupoid underlying a sheaf of basic relative

k-forms we will denote that sheaf more clearly by $\Omega^k_{G\ltimes M\text{-b.r.}}$. The differential d maps $\Omega^k_{\text{b.r.}}$ to $\Omega^{k+1}_{\text{b.r.}}$. This follows from Cartan's magic formula since it entails for $[\omega] = (\omega_g)_{g\in G} \in \Omega^k_{\text{b.r.}}(O)$, $g \in G$, and every element ξ of the Lie algebra $\mathfrak{g}_g$ of the centralizer $G_g := \mathsf{Z}_G(g)$ of g the equality

$$i_{\xi_{M^g}} d\omega_g = \mathscr{L}_{\xi_{M^g}}\omega_g - di_{\xi_{M^g}}\omega_g = 0,$$

where ξ_{M^g} denotes the fundamental vector field of ξ on M^g. So we finally obtain a complex of sheaves $(\Omega^\bullet_{\text{b.r.}}, d)$ over the orbit space M/G. Since each of the $\Omega^k_{\text{b.r.}}$ is in a natural way a $\mathscr{C}^\infty_{M/G}$-module, $(\Omega^\bullet_{\text{b.r.}}, d)$ is even a complex of fine sheaves. To formulate our main result let us remind the reader that we denote by $\mathscr{B}$ Brylinski's sheaf over the orbit space M/G and that this sheaf has section spaces given by (1.3).

Example 3.1. As an example let us consider the S^1-action on $\mathbb{R}^2$ by rotation. We parametrize S^1 by $e^{2\pi i\theta}$, where $\theta \in \mathbb{R}$. In coordinates, the action is expressed as

$$\mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2,\ \big(\theta,(x,y)\big) \mapsto \big(\cos(\theta)x - \sin(\theta)y, \sin(\theta)x + \cos(\theta)y\big)\ .$$

The loop space is given by

$$\Lambda_0(\mathrm{S}^1 \ltimes \mathbb{R}^2) = \big\{(e^{2\pi i\theta},(x,y)) \in \mathrm{S}^1\times\mathbb{R}^2 \mid \theta = 0 \text{ or } x = y = 0\big\}.$$

The minimal stratification of $\Lambda_0(\mathrm{S}^1 \ltimes \mathbb{R}^2)$ is given by the decomposition into the strata $S_0 = \{1\} \times (\mathbb{R}^2\backslash\{(0,0)\}$, $S_1 = (\mathrm{S}^1\backslash\{1\} \times \{(0,0)\}$, and $S_2 = \{(1,(0,0))\}$. Using the given parametrization of S^1 it is clear that in a neighborhood of the subset $\{1\}\times\mathbb{R}^2 \subset \Lambda_0(\mathrm{S}^1 \ltimes \mathbb{R}^2)$ the loop space looks like a neighborhood of $\{0\}\times\mathbb{R}^2$ in the space

$$\Lambda_0' := \big(\mathbb{R}\times\{(0,0)\}\big) \cup \big(\{0\}\times\mathbb{R}^2\big)\ .$$

The loop space is smooth around each point of the stratum S_1, hence to describe the sheaf of smooth functions on $\Lambda_0(\mathrm{S}^1 \ltimes \mathbb{R}^2)$ we need to only understand how smooth funtions on Λ_0' look around a neighborhhod of the origin. To this end let $I \subset \mathscr{C}^\infty(\mathbb{R}^3)$ denote the ideal of smooth functions vanishing on Λ_0'. If $f \in I$, then $f = \theta g$ for some $g \in \mathscr{C}^\infty(\mathbb{R}^3)$ and with $\theta : \mathbb{R}^3 \to \mathbb{R}$ denoting here the projection onto the first coordinate. Since f vanishes on the θ-axis, g does so, too, hence $g = xf_1 + yf_2$ for some $f_1, f_2 \in \mathscr{C}^\infty(\mathbb{R}^3)$. We obtain the representation $f = \theta x f_1 + \theta y f_2$. Therefore, the differential graded ideal $I\Omega^\bullet(\mathbb{R}^3) + dI \wedge \Omega^\bullet(\mathbb{R}^3) \subset \Omega^\bullet(\mathbb{R}^3)$ consists of all sums of forms of the form

$$\theta x\omega_1 + \theta y\omega_2 + xd\theta\wedge\omega_3 + yd\theta\wedge\omega_4 + \theta dx\wedge\omega_3 + \theta dy\wedge\omega_4,$$

where $\omega_1, \omega_2 \in \Omega^k(\mathbb{R}^3)$ and $\omega_3, \omega_4 \in \Omega^{k-1}(\mathbb{R}^3)$. Now observe that in the space of relative forms $\Omega^k_{\mathbb{R}^3 \xrightarrow{\theta} \mathbb{R}}(\mathbb{R}^3)$ the form $d\theta$ vanishes. Moreover, the pullback of a relative form of degree $k \geqslant 1$ to the stratum $\mathbb{R}^*\times\{(0,0\}$ vanishes as well. One concludes that, for $k \geqslant 1$ the space

$$\Omega^k_{\Lambda_0' \xrightarrow{\theta} \mathbb{R}}(\Lambda_0') := \Omega^k_{\mathbb{R}^3 \xrightarrow{\theta} \mathbb{R}}(\mathbb{R}^3)/I\Omega^k_{\mathbb{R}^3 \xrightarrow{\theta} \mathbb{R}}(\mathbb{R}^3) + dI \wedge \Omega^{k-1}_{\mathbb{R}^3 \xrightarrow{\theta} \mathbb{R}}(\mathbb{R}^3)$$

of relative k-forms on Λ_0' can be identified with the space of smooth families $(\omega_\theta)_{\theta\in\mathbb{R}}$, where

$$\omega_\theta \in \begin{cases} \Omega^k(\mathbb{R}^2), & \text{if } \theta = 0,\\ \{0\}, & \text{else.}\end{cases}$$

So one concludes that $\Omega^k_{\Lambda_0(\mathrm{S}^1\ltimes\mathbb{R}^2)\to\mathrm{S}^1}\big(\Lambda_0(\mathrm{S}^1 \ltimes \mathbb{R}^2)\big) \cong \Omega^k(\mathbb{R}^2)$ for $k \geqslant 1$ and that, under this ismorphism, $\Omega^k_{\text{b.r.}}\big(\mathbb{R}^2/\mathrm{S}^1\big) \cong \Omega^k_{\text{bas}}\big(\mathbb{R}^2/\mathrm{S}^1\big)$, where the latter denotes the space of basic k-forms on $\mathbb{R}^2$. In case $k = 0$, one has $\Omega^0_{\text{b.r.}}\big(\mathbb{R}^2/\mathrm{S}^1\big) \cong \mathscr{C}^\infty(\mathbb{R}^2/\mathrm{S}^1)$. Finally in this example, Brylinski's sheaf can be identified with the sheaf (over the orbit space) of G-invariant locally constant functions on $\mathbb{R}^2$.

Theorem 3.1. *Let G be a compact Lie group acting on a manifold M. The sheaf complex $(\Omega^\bullet_{\text{b.r.}}, d)$ of basic relative forms together with the natural monomorphism of sheaves*

$$d_{-1} : \mathscr{B} \hookrightarrow \Omega^0_{\text{b.r.}} = \pi_*\big(\mathscr{C}^\infty_{\Lambda_0}\big)^G$$

then forms a fine resolution of Brylinski's sheaf $\mathscr{B}$.

Proof. Let $O \subset M/G$ be open, and $f \in \mathscr{B}(O)$. By definition through Eq. (1.3) f then is a smooth G-invariant function on $s^{-1}_{|\Lambda_0}\pi^{-1}(O)$, and $f(g,-) : M^g \to \mathbb{R}$ is locally constant for all $g \in G$. Hence $df(g,-) \in \Omega^1(M^g)$ vanishes for every g. This entails that $\mathscr{B} \hookrightarrow \Omega^\bullet_{\text{b.r.}}$ is a cochain complex of sheaves over the orbit space M/G.

It remains to show that that cochain complex of sheaves is exact, meaning that for each orbit $\mathscr{O} \in M/G$ the complex of stalks $\mathscr{B}_{\mathscr{O}} \hookrightarrow \Omega^\bullet_{\text{b.r.},\mathscr{O}}$ is exact. To this end we proceed in several steps. In the first step we consider the case, where M is a finite dimensional vector space V carrying a linear G-action, and where $\mathscr{O} = \{0\}$, the orbit through the origin. Choose a G-invariant scalar product on V. Assume that $B \subset V$ is an open ball around the origin, and consider the homothety $h : [0,1] \times V \to V$, $(t,v) \to tv$. The homothety h leaves each of the subsets $B^g \subset B$ invariant, and commutes with the G-action. Hence

$$K : \Omega^k_{\text{b.r.}}(B) \to \Omega^{k-1}_{\text{b.r.}}(B),\ \omega = (\omega_g)_{g\in G} \mapsto K\omega := \begin{cases} \Lambda_0(G \ltimes M) \ni (g,p) \mapsto \omega_g(0), & \text{for } k = 0, \\ \left(\int_0^1 h_t^*(\xi_t \lrcorner \omega_g)\right)_{g\in G}, & \text{for } k \geqslant 1, \end{cases}$$

is a well-defined operator, where $\Omega^{-1}_{\text{b.r.}}$ is Brylinski's sheaf $\mathscr{B}$, h_t equals $h(t,-) : B \to B$, $v \mapsto tv$, and $\xi_t : B \to TB$ is the vector field given by $\xi_t := \partial_t h_t$. Cartan's magic formula implies that

$$(\omega_g)_{g\in G} = dK(\omega_g)_{g\in G} + Kd(\omega_g)_{g\in G} \quad \text{for all } (\omega_g)_{g\in G} \in \Omega^k_{\text{b.r.}}(B) \text{ and } k \in \mathbb{N},$$

since h_1^* acts by identity on $\left(\omega_g\right)_{g\in G}$, $\left(h_0^*\omega_g\right)_{g\in G} = 0$ for $k \geqslant 1$, and $h_0^*\omega = \omega(-,0)$ for $k = 0$. Hence $\mathscr{B}_{\mathscr{O}} \hookrightarrow \Omega^\bullet_{\text{b.r.},\mathscr{O}}$ is exact when $\mathscr{O} = \{0\}$.

In the second step we come back to the general case of a G-action on an arbitrary manifold M. Choose a G-invariant riemannian metric η on M. Let $p \in M$ be a point, $\mathscr{O}$ the orbit through p, and $N_p := T_pM/T_p\mathscr{O}$ the normal space to $\mathscr{O}$ at p. Via the riemannian metric we can identify N_p with the orthogonal complement of $T_p\mathscr{O}$ in T_pM. The isotropy group G_p acts in a natural way on N_p, cf. [Pfl01, Sec. 4.2.5]. Choose an open ball B around the origin of N_p with radius smaller than the injectivity radius at p. The slice theorem [Pfl01, Thm. 4.2.6] entails that the G-action on M induces an action of the isotropy group G_p on the slice $S_p := \exp(B)$ and that the exponential map intertwines the G_p-actions on N_p and S_p. Moreover, the exponential map procvides an equivariant diffeomorphism between $G \times_{G_p} B$ and the G-saturation $U := G \cdot S_p$ of the slice. It therefore suffices to verify the claim for the case where M has the form $G \times_H B$ with $H \subset G$ being a compact subgroup and B an open H-invariant ball around the origin of a finite dimensional H-representation space V, and where $\mathscr{O}$ is the orbit through the point $[e,0] \in G \times_H B$. From now on we will consider only this setting. Note that here and in the following we will denote by $[g,v]$ the equivalence class of a point $(g,v) \in G \times B$ in $G \times_H B$.

In the third step we provide a description of the tangent bundle $T(G \times_H V)$. To this end choose a bi-invariant riemannian metric on G. Let $\mathscr{Q}$ be the foliation of G given by the orbits of the canonical right action of H on G. For each $g \in G$ let E_g be the orthogonal complement of the tangent space $T_g\mathscr{O}$ to the leaf of $\mathscr{Q}$ through g. One thus obtains a vector bundle $E \to G$ which is invariant under the left action of G on TG and invariant under the right action of H. The latter follows from the fact that the right action of H on G maps leaves of $\mathscr{Q}$ to leaves, since $g\exp(t\xi)h = gh\exp\left(t\,\mathrm{Ad}_{h^{-1}}(\xi)\right)$ for all $g \in G$, $h \in H$, $\xi \in \mathfrak{h}$, and $t \in \mathbb{R}$. One concludes that E can be identified with the trivial bundle $G \times \mathfrak{m}$, where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. Now call two elements $(\Xi,(v,X)), (\mathrm{Z},(w,Y)) \in E \times TV = E \times V \times V$ *equivalent*, if there is an $h \in H$ with $\Xi = \mathrm{Z}h$ and $h(v,X) = (w,Y)$. We denote by $[\Xi,(v,X)]$ the equivalence class of $(\Xi,(v,X)) \in E \times TV$. One checks immediately that the quotient space of $E \times TV$ by this equivalence relation can be canonically identified with the tangent bundle $T(G \times_H V)$. Under this identification, an element of the tangent space $T_{[g,v]}(G \times_H V) \cong \mathfrak{m} \times V$ over the footpoint $[g,v] \in G \times_H V$ has a unique representation of the form $[(g,\xi),(v,X)]$ with $\xi \in \mathfrak{m}$ and $X \in V$. For later purposes let us remark that if $[g,v] = [g',v']$, i.e. if $(g',v') = (gh^{-1},hv)$ for some $h \in H$, then $[(g,\xi),(v,X)] = [(g',\xi'),(v',X')]$ with $\xi' = \xi h^{-1}$ and $v' = hv$. In the following step we will denote the equivalence class $[(g,\xi),(v,X)]$ shortly by $[\xi,X]_{(g,v)}$.

The fourth step consists in verifying that the embedding map $\iota : B \hookrightarrow M := G \times_H B$, $v \mapsto [e, v]$ induces an isomorphism between the sheaves $\Omega^k_{G\ltimes M\text{-b.r.}}$ and $\Omega^k_{H\ltimes B\text{-b.r.}}$. Observe that both sheaves live on the same topological space, since the orbit spaces M/G and B/H are canonically isomorphic since ι is a Morita equivalence. The isomorphism is given by pullback via ι. More precisely, for $O \subset M/G$ and a basic relative form $\omega = (\omega_g)_{g\in G} \in \Omega^k_{G\ltimes M\text{-b.r.}}(O)$ let $\iota^*\omega$ be the family $\big(\iota^*_{O,h}\omega_h\big)_{h\in H}$, where $\iota_{O,h}$ denotes the restriction of the embedding ι to $\big(\pi^{-1}_{H\ltimes B}(O)\big)^h$ with $\pi_{H\ltimes B} : B \to B/H$ being the orbit map of the groupoid $H \ltimes B$. Obviously, $\iota^*_{O,h}\omega_h$ is a $\mathsf{Z}_H(h)$-invariant horizontal form on $\big(\pi^{-1}_{H\ltimes B}(O)\big)^h$, and the family $\big(\iota^*_{O,h}\omega_h\big)_{h\in H}$ is H-invariant. Hence $\iota^*\omega \in \Omega^k_{H\ltimes B\text{-b.r.}}(O)$, so we obtain a morphism of sheaf complexes $\iota^* : \Omega^\bullet_{G\ltimes M\text{-b.r.}} \to \Omega^\bullet_{H\ltimes B\text{-b.r.}}$. Let us show that it is an isomorphism. To this end note first that for every $g \in G$ the invariant space M^g is given by $M^g = \{[f, v] \in G \times_H B \mid g \in fH_vf^{-1}\}$. Now let $(\varrho_h)_{h\in H} \in \Omega^k_{H\ltimes B\text{-b.r.}}(O)$, and define $\omega \in \Omega^k_{G\ltimes M\text{-b.r.}}(O)$ by

$$\omega_{g,[f,v]}\big([\xi_1, X_1]_{(f,v)}, \dots, [\xi_k, X_k]_{(f,v)}\big) := \varrho_{f^{-1}gf,v}(f^{-1}gfX_1, \dots, f^{-1}gfX_k),$$

where $g \in G$, $[f, v] \in M^g \cap s^{-1}_{\Lambda_0(G\ltimes M)}\pi^{-1}_{|G\ltimes M}(O)$, and $[\xi_1, X_1]_{(f,v)}, \dots, [\xi_k, X_k]_{(f,v)} \in T_{(f,v)}M$. Since $\varrho = (\varrho_h)_{h\in H}$ is H-invariant, one obtains for every $h \in H$ the equality

$$\begin{aligned}\omega_{g,[fh^{-1},hv]}&\big([\xi_1h^{-1}, hX_1]_{(fh^{-1},hv)}, \dots, [\xi_kh^{-1}, hX_k]_{(fh^{-1},hv)}\big) = \\ &= \varrho_{hf^{-1}gfh^{-1},hv}(hf^{-1}gfX_1, \dots, hf^{-1}gfX_k) = \\ &= \varrho_{f^{-1}gf,v}(f^{-1}gfX_1, \dots, f^{-1}gfX_k).\end{aligned}$$

This shows that ω_g is independant of the choices made, and an element of $\Omega^k(M^g \cap s^{-1}_{\Lambda_0(G\ltimes M)}\pi^{-1}_{|G\ltimes M}(O))$. Moreover, ω_g is a horizontal form by construction. The family $\omega = (\omega_g)_{g\in G}$ is also G-invariant. To verify this let $h \in G$, and observe that by definition

$$\omega_{hgh^{-1},[hf,v]}\big([h\xi_1, X_1]_{(hf,v)}, \dots, [h\xi_k, X_k]_{(hf,v)}\big) = \varrho_{f^{-1}gf,v}(f^{-1}gfX_1, \dots, f^{-1}gfX_k),$$

where $g \in G$, $[f, v]$, and $[\xi_1, X_1]_{(f,v)}, \dots, [\xi_k, X_k]_{(f,v)}$ are as above. This proves G-invariance of the family ω, hence $\omega \in \Omega^\bullet_{G\ltimes M\text{-b.r.}}(O)$ indeed. By construction it is clear that $\iota^*\omega = \varrho$. By G-invariance, ω is uniquely determined by $\iota^*\omega$. So ι is a sheaf isomorphism as claimed.

In the fifth and final step we show that $\mathscr{B}_{\mathscr{O}} \hookrightarrow \Omega^\bullet_{G\ltimes M\text{-b.r.},\mathscr{O}}$ is an exact sheaf complex in the case where $M = G \times_H B$ and $\mathscr{O}$ is the orbit through the point $[e, 0]$. By the second step, it suffices to consider this case. By the fourth step, the embedding $\iota : B \hookrightarrow M$ induces an isomorphis of sheaf complexes $\iota^* : \Omega^\bullet_{G\ltimes M\text{-b.r.}} \to \Omega^\bullet_{H\ltimes B\text{-b.r.}}$. By the first step, the cochain complex $\mathscr{B}_{\{0\}} \hookrightarrow \Omega^\bullet_{H\ltimes B\text{-b.r.},\{0\}}$ is exact, hence $\mathscr{B}_{\mathscr{O}} \hookrightarrow \Omega^\bullet_{G\ltimes M\text{-b.r.},\mathscr{O}}$ is so, too, since the orbit through $\iota(0)$ coincides with $\mathscr{O}$. The proof is finished. □

Appendix A. Differentiable stratified spaces

For the convenience of the reader we briefly recall here the notion of a differentiable space, mainly following [NGSdS03], and then describe what it means that a stratification is compatible with the differentiable structure.

Definition A.1. An algebra over $\mathbb{R}$ of the form $A = \mathscr{C}^\infty(\mathbb{R}^n)/J$, where $J \subset \mathscr{C}^\infty(\mathbb{R}^n)$ is a closed ideal, is called a *differentiable algebra*. By $\operatorname{spec} A$ is the maximal spectrum of a differentiable algebra A, and by $\mathscr{O}_A$ its structure sheaf, which is the sheafification of the presheaf $U \mapsto A_U$, $U \subset \operatorname{spec} A$ open, where A_U is the localization of A over the subset of elements which do not vanish over U; cf. [NGSdS03, Sec. 3.1].

A differentiable algebra carries in a natural way the structure of a Fréchet algebra. Moreover, given a differentiable algebra A, the pair $(\operatorname{spec} A, \mathscr{O}_A)$ is a commutative locally ringed space.

Definition A.2. A commutative locally $\mathbb{R}$-ringed space $(X, \mathscr{O})$ is called an *affine differentiable space*, if it is isomorphic as a commutative locally ringed space to $(\operatorname{spec} A, \mathscr{O}_A)$ for some differentiable algebra A. The ringed space $(X, \mathscr{O})$ is called a *differentiable space*, if for every $x \in X$ there exists an open neighborhood $U \subset X$ such that $(U, \mathscr{O}_{|U})$ is an affine differentiable space. By a *morphism of differentiable spaces* we understand a morphism of locally $\mathbb{R}$-ringed spaces $(f, \varphi) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ between differentiable spaces $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$.

Locally, a differentiable space $(X, \mathscr{O})$ can be embedded into euclidean space. We call an embedding $\iota : U \hookrightarrow \mathbb{R}^n$ over an open $U \subset X$ together with a morphism of differentiable algebras $\iota^* : \mathscr{C}^\infty(\iota(U)) \to \mathscr{O}(U)$ such that $\big(\iota^*(x_1)(x), \ldots, \iota^*(x_n)(x)\big) = \iota(x)$ for all $x \in U$ a *singular chart* for $(X, \mathscr{O})$ if these data induce an isomorphism of locally $\mathbb{R}$-ringed spaces

$$(\iota, \iota^*) : (U, \mathscr{O}_{|U}) \to \big(\iota(U), \mathscr{C}^\infty_{|\iota(U)}/\mathscr{J}_\iota\big),$$

where $\mathscr{J}_\iota$ is the kernel of the morphism of sheaves $\iota^* : \mathscr{C}^\infty_{|\iota(U)} \to \mathscr{O}_{|U}$. By a *singular atlas* of $(X, \mathscr{O})$ we understand a family $\mathfrak{A}$ of singular charts such that the family of domains $\{\mathrm{dom}(\iota)\}_{(\iota,\iota^*)\in\mathfrak{A}}$ is an open cover of X. If $(X, \mathscr{O})$ is reduced, the embedding $\iota : U \hookrightarrow \mathbb{R}^n$ completely determines the singular chart (ι, ι^*), because ι^* then is just pullback by the embedding. By abuse of language we denote a singular chart even in the non-reduced case just by the embedding $\iota : U \hookrightarrow \mathbb{R}^n$.

The following result shows that around a point of a differentible space the minimal embedding dimension is given by the dimension of the Zariski tangent space.

Theorem A.3 ([Pfl01, Prop. 1.3.10 & Corollaries])**.** *Let $(X, \mathscr{O})$ be differentiable space, and $x \in X$ a point. Then there exists an open affine neighborhood W of x together with a singular chart $\lambda : W \hookrightarrow \mathbb{R}^{\mathrm{rk}\, x}$, where $\mathrm{rk}\, x$ is the dimension of the Zariski tangent space $T_x X$ at x. Moreover, if $\iota : U \hookrightarrow \mathbb{R}^n$ is another singular chart defined on an open affine neighborhood of x, then $\mathrm{rk}\, x \leqslant n$, and there exists an open affine neighborhood $V \subset W \cap U$ of x, an open neighborhood $\widetilde{V} \subset \mathbb{R}^{\mathrm{rk}\, x}$ of $\lambda(x)$ and a smooth embedding $H : \widetilde{V} \hookrightarrow \mathbb{R}^n$ such that $\iota(V)$ is closed in $\widetilde{V}$ and such that $H \circ \iota_{|V} = \kappa_{|V}$.*

Definition A.4. A stratification $\mathscr{S}$ of the topological space X underlying a differentiable space $(X, \mathscr{O})$ is said to be *compatible with* $(X, \mathscr{O})$ or just *with* $\mathscr{O}$ if for each stratum $S \in \mathscr{S}$ and singular chart $\iota : U \hookrightarrow \mathbb{R}^n$ the image $\iota(S \cap U)$ of the stratum under ι is a submanifold of $\mathbb{R}^n$. We call a differentiable space $(X, \mathscr{O})$ together with a compatible stratification $\mathscr{S}$ of X a *differentiable stratified space.*

Example A.1. Typical examples of differentiable stratified spaces are real or complex algebraic varieties and orbit spaces of compact Lie group actions on manifolds. In [PPT14] it has been shown that the orbit space of a proper Lie groupoid carries the structure of a differentiable stratified space in a natural way.

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