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by

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Abstract

We present a convergence result for solutions of the vector-valued Allen-Cahn Equation. In the spirit of the work of Luckhaus and Sturzenhecker we establish convergence towards a distributional formulation of multi-phase mean-curvature flow using sets of finite perimeter. Like their result, ours relies on the assumption that the time-integrated energies of the approximations converge to those of the limit.

Keywords: Mean curvature flow, Allen-Cahn Equation

Mathematical Subject Classification: 35A15, 35K57, 35K93

Introduction

Motion by mean curvature is an important geometric evolution equation and arises in various problems in geometry, physics and the sciences. Its multi-phase version for example is a popular model for the evolution of grain boundaries in polycrystals undergoing heat treatment, already motivated in [33]. The Allen-Cahn Equation (1) is a well-established phase-field approximation for (multi-phase) mean-curvature flow [1], replacing sharp interfaces by diffused transition layers.

The derivation of motion by mean curvature as the singular limit of the Allen-Cahn Equation has a long history and is well-understood in the two-phase case: The first formal asymptotic expansions were constructed by Rubinstein, Sternberg and Keller [36]. Convergence for a smooth evolution was proved independently by De Mottoni and Schatzman [14] and Chen [11]. Bronsard and Kohn [9] used the gradient flow structure of (1) to prove compactness, and convergence to motion by mean curvature in the radially symmetric case. For the long-time behavior past singularities the following two well-established notions of weak solutions have proven to be useful for understanding the singular limit of (1): viscosity solutions [18, 12] and Brakke's varifold-solutions [7].

Viscosity solutions on the one hand are are based on the level-set formulation [32] and the well-known geometric comparison principle of two-phase mean-curvature flow. Evans, Soner and Souganidis [17] rigorously proved the convergence towards the viscosity solution – at least if the level-set of the viscosity solution does not develop an interior but remains "thin". Barles, Soner and Souganidis [6] showed in particular that this holds true for mean-convex or star shaped intitial conditions.

Brakke's varifold-solutions [7] on the other hand are based on the gradient flow structure of mean-curvature flow. Ilmanen proved convergence towards Brakke's formulation [21] in the two-phase case by translating Huisken's celebrated monotonicity formula [20] to the phase-field framework of (1).

While the question of convergence of (1) seems to be almost settled in the two-phase case, little is known in the multi-phase case. Even the work of Ilmanen [21] seems not to apply since he makes use of comparison techniques at a crucial point. Bronsard and Reitich [10] carried out a formal asymptotic expansion at a triple junction and proved short-time existence; but rigorous long-time convergence results past singularities seem not yet available.

Our proof is of variational nature in the sense that it is based on the gradient flow structure of the Allen-Cahn Equation and mean-curvature flow and we use some techniques known from the analytical

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study of the static analogue of (1), which goes back to Modica and Mortola [30]. Modica [29] and Sternberg [37] provided the convergence of the Ginzburg-Landau Energy (see (2) for a definition in the multi-phase case) towards a multiple of the perimeter functional in the sense of Γ-convergence. Kohn and Sternberg [22] were able to construct *local* minimizers of the Ginzburg-Landau Energy (2) based on the above Γ-convergence. Furthermore, it turns out that the the convergence of the Ginzburg-Landau Energy towards the perimeter functional is even stronger: Luckhaus and Modica [25] proved that also the first variations of the energies converge towards the mean curvature – the first variation of the perimeter functional – by the clever use of a classical argument of Reshetnyak [34]. A year later, Baldo [5] extended the Γ-convergence of the energies (2) to the multi-phase case.

From a conceptional viewpoint, our proof is closely related to a number of other convergence proofs for implicit time-discretizations in the spirit of De Giorgi's *minimizing movements* [13]. Luckhaus and Sturzenhecker [26] established the convergence of the time-discretization proposed by Almgren, Taylor and Wang [2] and Luckhaus and Sturzenhecker [26] towards a distributional solution of mean-curvature flow, see (11) and (12) for a multi-phase version of this formulation. Recently, Otto and the first author [23] proved convergence of the thresholding scheme of Merriman, Bence and Osher [27, 28] in the multi-phase case based on the minimizing movements interpretation of Esedoğlu and Otto [16]. Over the last decades, this variational viewpoint has proven to be flexible enough to study a tremendous amount of problems such as the Stefan Problem [26] and its anisotropic variant [19], the Mullins-Sekerka Flow [35] and its multi-phase variant [8], volume-preserving mean-curvature flow [31, 24], the evolution of martensitic phase transitions [15], and many more.

Our main result, Theorem 1.2, establishes the convergence of solutions of (1) for a general class of potentials and any space dimension. Like the results of Luckhaus and Sturzenhecker [26] and the first author and Felix Otto [23], also ours is only a *conditional* convergence result in the sense that we have to make the assumption that the time-integrated energy of the approximations converges to the time-integrated energy of the limit, see (9). Although, this is a very natural assumption, it is not guaranteed by the a priori estimates coming from the energy-dissipation equality (24).

The main idea of our proof is to multiply the Allen-Cahn Equation

$$\partial_t u_\varepsilon = \Delta u_\varepsilon - \frac{1}{\varepsilon^2}\partial_u W(u_\varepsilon) \tag{1}$$

with $\varepsilon\,(\xi\cdot\nabla)\,u_\varepsilon$, integrate in space and time and pass to the limit $\varepsilon\downarrow 0$. To this end we extend the above mentioned argument of Luckhaus and Modica [25] to the multi-phase case and obtain the curvature-term $\int_\Sigma H\,\xi\cdot\nu$ from the right-hand side. The more delicate part, and the core of this paper, is how to pass to the limit in the velocity-term $\int_\Sigma V\,\xi\cdot\nu$. The difficulty is that one has to pass to the limit in a *product* of weakly converging terms, the normal and the velocity. We overcome this difficulty by "freezing" the normal and introducing an appropriate approximation (46) of the tilt-excess. After doing so it turns out that the new nonlinearity with the frozen normal can be written as a derivative of a compact quantity. The technique of freezing the normal was used before in [23], where the authors introduce an approximation of the energy-excess.

To work with the tilt-excess instead of the energy-excess along the sequence seems very natural to us in this particular problem and might be interesting in other cases too. The only extra difficulty is that one has to argue why one can pass to the limit in the nonlinear quantity (46). However, our problem seems to be much simpler than the one in [23] because we do not have to work on multiple time scales.

The structure of the paper is as follows. In Section 1 we introduce the notation and state our main result, Theorem 1.2. In Section 3 we prove compactness of the solutions together with bounds on the normal velocities. In Section 2 we cite some preliminary results that will turn out to be useful. We display a general chain rule of Ambrosio and Dal Maso [3] which we use to identify the nonlinearities in the multi-phase case as derivatives. Furthermore, we repeat the application of De Giorgi's structure result from [23] to handle the excess. In Section 4 we pass to the limit in the equation. Since this is the most difficult part, we give a short overview over the idea of the proof first. We then present our extension of [25] in Proposition 4.1 to handle the curvature-term and prove the convergence of the velocity-term in Proposition 4.5, which is the main novelty and the core of the paper. We conclude with the proof of the main result, Theorem 1.2.

1 Main result

The Allen-Cahn Equation (1) describes a system of fast reaction and slow diffusion and is the (by the factor $\frac{1}{\varepsilon}$ accelerated) L^2-gradient flow of the Ginzburg-Landau energy

$$E_\varepsilon(u_\varepsilon) = \int \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon)\, dx. \tag{2}$$

For convenience we will work with periodic boundary conditions for u, i.e. on the flat torus $[0,\Lambda)^d$ for some $\Lambda > 0$ and write $\int dx$ short for $\int_{[0,\Lambda)^d} dx$.

Here the (unknown) order parameter $u_\varepsilon \colon \mathbb{R}^d \to \mathbb{R}^N$ is vector-valued and $W \colon \mathbb{R}^N \to [0,\infty)$ is a smooth multi-well potential with finitely many zeros at $u = \alpha_1, \ldots, \alpha_P \in \mathbb{R}^N$. We will furthermore impose a polynomial growth condition on W at ∞: There exist constants $0 < c < C < \infty$, $R < \infty$ and an exponent $p \geq 2$ such that

$$c|u|^p \leq W(u) \leq C|u|^p \quad \text{for } |u| \geq R. \tag{3}$$

For convenience we further assume the following coercivity of W: There exists a number $M < \infty$ such that W is smaller inside the box $[-M,M]^N$ than outside

$$\max_{[-M,M]^N} W \leq \min_{\mathbb{R}^N \setminus [-M,M]^N} W. \tag{4}$$

By now it is a classical result due to Baldo [5] that these energies Γ-converge w.r.t. the L^1-topology to an *optimal partition energy* given by

$$E(\chi) = \frac{1}{2} \sum_{1\leq i,j\leq P} \sigma_{ij} \int \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)| \right), \tag{5}$$

for a partition $\chi_1, \ldots, \chi_P \colon [0,\Lambda)^d \to \{0,1\}$ satisfying the compatibility condition $\sum_{1\leq i\leq P} \chi_i = 1$ a.e. Note that for $\chi_i = \mathbf{1}_{\Omega_i}$ we can also rewrite the limiting energy in terms of the interfaces $\Sigma_{ij} := \partial_* \Omega_i \cap \partial_* \Omega_j$ between the phases, where ∂_* denotes the reduced boundary:

$$E(\chi) = \frac{1}{2} \sum_{1\leq i,j\leq P} \sigma_{ij} \mathcal{H}^{d-1}(\Sigma_{ij}).$$

The link between u_ε and χ is given by

$$u_\varepsilon \to u := \sum_{1\leq j\leq P} \chi_i \alpha_i. \tag{6}$$

The constants σ_{ij} are the geodesic distances with respect to the metric $2W(u)\langle\cdot,\cdot\rangle$, i.e.

$$\sigma_{ij} = d(\alpha_i, \alpha_j), \tag{7}$$

where the geodesic distance is defined as

$$d(u,v) := \inf \left\{ \int_0^1 \sqrt{2W(\gamma)} |\dot\gamma| ds : \gamma \colon [0,1] \to \mathbb{R}^n \text{ a } C^1 \text{ curve with } \gamma(0) = u, \gamma(1) = v \right\}. \tag{8}$$

The surface tensions satisfy the triangle inequality

$$\sigma_{ij} \leq \sigma_{ik} + \sigma_{kj} \quad \text{for all } i,j,k$$

and clearly

$$\sigma_{ii} = 0, \quad \sigma_{ij} > 0 \quad \text{for } i \neq j, \quad \text{and} \quad \sigma_{ij} = \sigma_{ji}.$$

We will want to localize both the Ginzburg-Landau Energy and the optimal partition energy. Given $\eta \in C([0,\Lambda)^d)$ let

$$E_\varepsilon(\eta, u_\varepsilon) := \int \eta \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) dx,$$

$$E(\eta,\chi) := E(\eta,u) := \frac{1}{2} \sum_{1\leq i,j\leq P} \sigma_{ij} \int \eta \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)| \right).$$

For our result we will impose

$$\int_0^T E_\varepsilon(u_\varepsilon)\,dt \to \int_0^T E(\chi)\,dt \tag{9}$$

ruling out a certain loss of surface area in the limit $\varepsilon \downarrow 0$. Under this assumption we will establish convergence towards the following distributional formulation of mean-curvature flow, see [26, 23].

Definition 1.1 (Motion by mean curvature)**.** Fix some finite time horizon $T < \infty$, a $P \times P$-matrix of surface tensions σ as above and initial data $\chi^0 \colon [0,\Lambda)^d \to \{0,1\}^P$ with $E_0 := E(\chi^0) < \infty$ and $\sum_{1\le i\le P} \chi_i^0 = 1$. We say that

$$\chi \colon (0,T) \times [0,\Lambda)^d \to \{0,1\}^P$$

with $\operatorname{ess\,sup}_t E(\chi) < \infty$ and $\sum_{1\le i\le P} \chi_i = 1$ *moves by mean curvature* if there exist densities V_i with

$$\int_0^T \int V_i^2 \, |\nabla \chi_i| \, dt < \infty \tag{10}$$

satisfying the following properties:

1. For all $\xi \in C_0^\infty((0,T)\times[0,\Lambda)^d, \mathbb{R}^d)$

$$\sum_{1\le i,j\le P} \sigma_{ij} \int_0^T \int \left(\nabla\cdot\xi - \nu_i \cdot \nabla\xi\, \nu_i - V_i\, \xi\cdot\nu_i \right) \frac{1}{2} \left(|\nabla\chi_i| + |\nabla\chi_j| - |\nabla(\chi_i+\chi_j)| \right) dt = 0, \tag{11}$$

   where ν_i is the inner normal of χ_i, i.e. it is the density of $\nabla\chi_i$ with respect to $|\nabla\chi_i|$.

2. The functions V_i are the normal velocities of the interfaces in the sense that for all $\zeta \in C^\infty([0,T]\times[0,\Lambda)^d)$ with $\zeta(T) = 0$ we have

$$\int_0^T \int \partial_t\zeta\, \chi_i \, dx\, dt + \int \zeta(0)\chi_i^0 \, dx = -\int_0^T \int \zeta\, V_i\, |\nabla\chi_i| \, dt. \tag{12}$$

Our main result is the following theorem.

Theorem 1.2. *Let W satisfy* (3) *and* (4) *and let $T < \infty$ be an arbitrary finite time horizon. Given a sequence of initial data $u_\varepsilon^0 \colon [0,\Lambda)^d \to [-M,M]^N$ approximating a partition χ^0, in the sense that*

$$u_\varepsilon^0 \to \sum_{1\le i\le P} \chi_i \alpha_i \quad \text{a.e. and} \quad E_0 := E(\chi^0) = \lim_{\varepsilon\downarrow 0} E_\varepsilon(u_\varepsilon^0) < \infty,$$

there exists a subsequence $\varepsilon \downarrow 0$ such that the solutions u_ε of (1) *with initial datum u_ε^0 converge to a time-dependent partition χ. If the convergence assumption* (9) *holds, then χ moves by mean curvature according to Definition 1.1.*

Remark 1.3. For any partition $\chi^0 \in BV\left([0,\Lambda)^d;\{0,1\}^P\right)$ it is possible to choose u_ε^0 with $u_\varepsilon^0 \to \sum_{1\le i\le P}\chi_i\alpha_i$ in L^1 and $E_\varepsilon(u_\varepsilon^0) \to E_0(\chi)$ by the Γ-convergence result [5]. The bound $u_\varepsilon^0 \in [-M,M]^N$ can be achieved by componentwise truncation, see also (30), which only decreases the energy by the coercivity assumption (4).

Throughout the paper we will make use of the following notations: ∂_t denotes the time-derivative, ∇ the spatial gradient of a function defined on real space $\mathbb{R}^d \ni x$, $\partial_u W(u)$ denotes the gradient of W at a point $u \in \mathbb{R}^N$ in state space. For the functions ϕ_i we will abuse the notation ∂_u in order to use the generalized chain rule below, see Lemma 2.2. We will write $A \lesssim B$ if there exists a generic constant $C < \infty$ depending only on d, N and W through the surface tensions (7) such that $A \le C\,B$.

2 Preliminary considerations

In the two-phase case the surface tension is given by the value $\phi(\alpha_2)$ of the primitive $\phi(u) := \int_{\alpha_1}^{u} \sqrt{2W}$ and most limiting arguments are done using the composition $\phi \circ u_\varepsilon$ rather than u_ε itself. For instance, if one wants to pass to the limit in expressions like $\int \eta\sqrt{2W(u_\varepsilon)}\nabla u_\varepsilon$ it is tremendously helpful to notice that the non-linearity in u_ε has the form $\nabla(\phi \circ u_\varepsilon)$.

In the multi-phase case, unfortunately, it is not possible to define the potential in such a clean way, because there are too many choices of paths between two states. Instead, we are forced to work with the geodesic distances

$$\phi_i(u) := d(u, \alpha_i), \text{ where } d \text{ was defined in (8).} \tag{13}$$

However, these "primitives" are only Lipschitz-continuous in general, because there could be multiple geodesics between u and α_i. Thus we cannot naively use the classical chain rule anymore.

Fortunately, there is the following result of Ambrosio and Dal Maso [3] on the validity of a chain rule for Lipschitz functions:

Theorem 2.1 (Ambrosio, Dal Maso [3]; Corollary 3.2)**.** *Let $p \in [1, \infty]$, $u \in W^{1,p}(\Omega; \mathbb{R}^n)$, and let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a Lipschitz continuous function such that $f(0) = 0$. Then $v := f \circ u \in W^{1,p}(\Omega; \mathbb{R}^k)$. Furthermore, for almost every $x \in \Omega$ the restricition of the function f to the affine space*

$$T_x^u := \left\{ y \in \mathbb{R}^n : y = u(x) + (z \cdot D)u \text{ for some } z \in \mathbb{R}^d \right\}$$

is differentiable at $u(x)$ and

$$Dv = D(f|_{T_x^u})(u)Du \quad a.e. \quad in\ \Omega.$$

The important thing to note is that, if we want to use the chain rule, the proper notion of differential of f depends on the function u we precompose with. Another minor issue is that, if f is real valued, $D(f|_{T_x^u})(u)$ is a linear form on $\operatorname{span}(\{\partial_1 u, \ldots, \partial_d u\})$ rather than on $\mathbb{R}^n$. The following lemma serves to fix and justify our somewhat abusive notation ignoring these problems.

Lemma 2.2. *For any $\epsilon > 0$ let $\Pi^\varepsilon(t,x)$ be the orthogonal projection in $\mathbb{R}^n$ onto the subspace $\dot{T}_{t,x}^\varepsilon := \operatorname{span}(\{\partial_1 u_\varepsilon, \ldots, \partial_d u_\varepsilon, \partial_t u_\varepsilon\})$. Let $T_{t,x}^\varepsilon := u_\varepsilon(t,x) + \dot{T}_{t,x}^\varepsilon$. For $1 \le i \le P$ we define the map $\partial_u \phi_i(u_\varepsilon) : [0,T] \times [0,\Lambda)^d \to \operatorname{Lin}(\mathbb{R}^n; \mathbb{R})$ to be*

$$\partial_u \phi_i(u_\varepsilon)(t,x)v := D\big(\phi_i|_{T_{t,x}^\varepsilon}\big)(u_\varepsilon(t,x))\Pi^\varepsilon(t,x)v$$

for any $v \in \mathbb{R}^n$ (up to a set of $\mathcal{L}^{d+1}$-measure zero). Then the chain rule is valid with the pair $\partial_u \phi_i(u_\varepsilon)$ and $(\partial_t, \nabla)u_\varepsilon$: For almost every $(t,x) \in [0,T] \times [0,\Lambda)^d$ we have

$$\nabla(\phi_i \circ u_\varepsilon) = \partial_u \phi_i(u_\varepsilon)\nabla u_\varepsilon \quad and \quad \partial_t(\phi_i \circ u_\varepsilon) = \partial_u \phi_i(u_\varepsilon)\partial_t u_\varepsilon. \tag{14}$$

Furthermore, we can control the modulus of $\partial_u \phi_i(u_\varepsilon)$ almost everywhere in time and space:

$$|\partial_u \phi_i(u_\varepsilon)| \le \sqrt{2W(u_\varepsilon)}. \tag{15}$$

Proof. That the chain rule holds follows immediately from Theorem 2.1 and the obvious fact that $\Pi^\varepsilon(t,x)\nabla u_\varepsilon(t,x) = \nabla u_\varepsilon(t,x)$. Let (t,x) be a point such that $\phi_i|_{T_{t,x}^\varepsilon}$ is differentiable in $u := u_\varepsilon(t,x)$, let $v \in \dot{T}_{t,x}^\varepsilon$ and $h > 0$. Using the triangle inequality of d and comparing the length of geodesics to straight lines we get

$$|\phi_i(u + hv) - \phi_i(u)| \le d(u + hv, u) \le \int_0^1 \sqrt{2W(u + thv)}h|v|\,dt.$$

Continuity of W implies that we can pass to the limit $h \to 0$ to get

$$\left| D\phi_i|_{T_{t,x}^\varepsilon}(u)v \right| \le \sqrt{2W(u)}|v|,$$

which for all vectors of the form $v = \Pi^\varepsilon(t,x)\tilde{v}$ for some $\tilde{v} \in \mathbb{R}^n$ gives

$$\left| \partial_u \phi_i|_{T_{t,x}^\varepsilon}(u)\Pi^\varepsilon(t,x) \right| \le \sqrt{2W(u)}.$$

□

It was already observed by Baldo, see Proposition 2.2 in [5], that the optimal partition energy (5) can be written as a (measure-theoretic) supremum using the "primitives" ϕ_i defined in (13). We will use this fact in the following form: Given $\varepsilon > 0$ there exists a scale $r > 0$ such that

$$\sum_{B\in\mathscr{B}_r} \left\{ E(\eta_B, u) - \max_{1\le i\le P} \int \eta_B \left|\nabla\left(\phi_i \circ u\right)\right| \right\} \le \varepsilon, \tag{16}$$

where η_B is a cutoff for B in the ball $2B$ with the same center but with the double radius and the covering $\mathscr{B}_r$ is given by

$$\mathscr{B}_r := \{B_r(i) \colon i \in \mathscr{L}_r\} \tag{17}$$

of $[0,\Lambda)^d$, where $\mathscr{L}_r = [0,\Lambda)^d \cap \frac{r}{\sqrt{d}}\mathbb{Z}^d$ is a regular grid of midpoints on $[0,\Lambda)^d$. Let us note that each summand in (16) is non-negative:

$$0 \le E(u, \eta_B) - \max_{1\le i\le P} \int \eta_B \left|\nabla\left(\phi_i \circ u\right)\right|.$$

This is the same covering as in Definition 5.1 in [23]. A nice feature is that by construction, for each $n \ge 1$ and each $r > 0$, the covering

$$\{B_{nr}(i) \colon i \in \mathscr{L}_r\} \quad \text{is locally finite,} \tag{18}$$

in the sense that for each point in $[0,\Lambda)^d$, the number of balls containing this point is bounded by a constant $c(d,n)$ which is independent of r.

The general idea of our proof is to use some regularity of the limit in order to estimate the error terms. To this end we use the following fact, which is a direct consequence of [23, Lemma 5.2 and Lemma 5.3]. It ensures that in BV-partitions on small scales generically only one interface between two phases is present and that it looks flat in the sense that the variation of the normal is small.

Lemma 2.3. *For every $\varepsilon > 0$ and $\chi : [0,\Lambda)^d \to \{0,1\}^P$ with $\sum_{1\le i\le P} \chi_i = 1$, there exists an $r_0 > 0$ such that for all $r \le r_0$ and there exist unit vectors $\nu_B \in \mathbb{S}^{d-1}$ for all $B \in \mathscr{B}_r$ such that*

$$\sum_{B\in\mathscr{B}_r} \min_{i\ne j} \left\{ \int \eta_B \left|\nu_i - \nu_B\right|^2 \left|\nabla\chi_i\right| + \int \eta_B \left|\nu_j + \nu_B\right|^2 \left|\nabla\chi_j\right| + \sum_{k\notin\{i,j\}} \int \eta_B \left|\nabla\chi_k\right| \right\} \lesssim \varepsilon E(\chi). \tag{19}$$

3 Compactness

3.1 Results

Proposition 3.1. *Given initial data $u_\varepsilon^0 \in [-M,M]^N$ with $u_\varepsilon^0 \to \sum_i \chi_i^0 \alpha_i$ and*

$$E_\varepsilon(u_\varepsilon^0) \to E(\chi^0) < \infty, \tag{20}$$

for any sequence there exists a subsequence $\varepsilon \downarrow 0$ such that the solutions u_ε of (1) *converge:*

$$u_\varepsilon \to u \quad \text{a.e. in } (0,T)\times[0,\Lambda)^d, \tag{21}$$

where the function is given by $u = \sum_i \chi_i \alpha_i$ with a partition $\chi \in BV((0,T)\times[0,\Lambda)^d;\{0,1\}^P)$. Furthermore we have

$$\operatorname*{ess\,sup}_t E(\chi) \le E_0 \tag{22}$$

and the compositions (13) *are uniformly bounded in $BV((0,T)\times[0,\Lambda)^d)$ and converge:*

$$\phi_i \circ u_\varepsilon \to \phi_i \circ u \quad \text{in } L^1((0,T)\times[0,\Lambda)^d). \tag{23}$$

The time compactness stated in this Proposition stems from the following basic estimate coming from the gradient flow structure of the Allen-Cahn Equation and implies BV-bounds on the compositions (13).

Lemma 3.2. *Let u_ε be a solution to* (1)*. Then*

$$E_\varepsilon(u_\varepsilon(T)) + \int_0^T \int \varepsilon \left|\partial_t u_\varepsilon\right|^2 dx\, dt = E_\varepsilon(u_\varepsilon(0)). \tag{24}$$

Another easy consequence of Lemma 3.2 is the following Hölder-continuity of the volumes of the phases Ω_i, saying that $|\Omega_i(t)\Delta\Omega_i(s)| \lesssim E_0\sqrt{t-s}$ for any $s < t$.

Lemma 3.3. *For any i, ϕ_i as in* (13) *and any $0 \le s < t \le T$ we have*

$$\int |\phi_i(u_\varepsilon)(t) - \phi_i(u_\varepsilon)(s)|\, dx \lesssim E_\varepsilon(u_\varepsilon^0)\sqrt{t-s}. \tag{25}$$

For the limit we have in particular that for a.e. $0 \le s < t \le T$

$$\int |\chi(t) - \chi(s)|\, dx \lesssim E_0\sqrt{t-s}. \tag{26}$$

While the compactness statement, Proposition 3.1, did not rely on the convergence assumption (9) we will need to assume it in the following, starting with the existence of the normal velocities.

Proposition 3.4. *In the situation of Proposition 3.1, given the convergence assumption* (9)*, for every $1 \le i \le P$ the measure $\partial_t\chi_i$ is absolutely continuous w.r.t. $|\nabla\chi_i|\, dt$ and the density V_i is square-integrable:*

$$\int_0^T \int V_i^2 \left|\nabla\chi_i\right| dt \lesssim E_0. \tag{27}$$

Furthermore, equation (12) *holds.*

The following lemma shows that – up to a further subsequence – the convergence assumption can be refined to pointwise a.e. in time and can be localized by a smooth test function in space. We furthermore argue that the convergence assumption assures equipartition of energy as $\varepsilon \downarrow 0$.

Lemma 3.5. *Given $u_\varepsilon \to u$ and the convergence assumption* (9)*, by passing to a further subsequence if necessary, we have*

$$\lim_{\varepsilon\downarrow 0} E_\varepsilon(u_\varepsilon) = E(u) \quad \text{for a.e. } \; 0 \le t \le T \tag{28}$$

and for any smooth test function $\zeta \in C^\infty([0,\Lambda)^d)$ we have

$$E(\zeta,u) = \lim_{\varepsilon\downarrow 0} E_\varepsilon(\zeta,u) = \lim_{\varepsilon\downarrow 0} \int \zeta\varepsilon\left|\nabla u_\varepsilon\right|^2 dx = \lim_{\varepsilon\downarrow 0} \int \zeta\frac{2}{\varepsilon}W(u_\varepsilon)\, dx = \lim_{\varepsilon\downarrow 0} \int \zeta\sqrt{2W(u_\varepsilon)}\left|\nabla u_\varepsilon\right| dx \tag{29}$$

for a.e. $0 \le t \le T$.

Remark 3.6. The proof of Lemma 3.5 does not use the equation. In particular, (29) holds for any time-independent $u^\varepsilon \colon [0,\Lambda)^d \to \mathbb{R}^N$ with $u^\varepsilon \to u$ and $E_\varepsilon(u^\varepsilon) \to E(u) < \infty$.

3.2 Proofs

Proof of Proposition 3.1. Step 1: Existence of the solutions u_ε.
In order to prove long-time existence for the Allen-Cahn Equation (1) one has to use its structure in some way. At the example

$$\partial_t u - \Delta u = u^2$$

one can see that in reaction-diffusion equations one can have blow-up in finite time. For the convenience of the reader we give the easy long-time existence proof for (1) with De Giorgi's minimizing movements. Since $\varepsilon > 0$ is fixed, we ignore this parameter here and denote the initial conditions by u^0 and the Ginzburg-Landau Energy by E. For a fixed time-step size $h > 0$ and $n = 1, \ldots$ we set

$$u^n = \arg\min_u E(u) + \frac{1}{2h}\int \left|u - u^{n-1}\right|^2 dx.$$

The existence of minimizers u^n follows from the direct method since both E and the metric term $\frac{1}{2h}\int \left|u-u^{n-1}\right|^2 dx$ are lower-semi continuous w.r.t. weak convergence in H^1. We interpolate these functions in a piecewise constant way: $u^h(t) := u^n$ for $t\in[nh,(n+1)h)$. By comparing u^n to its predecessor u^{n-1} we obtain the energy-dissipation inequality

$$E(u^h(T)) + \int_0^{T-h}\int \left|\partial_t^h u^h\right|^2 dx\,dt \le E(u^0),$$

where $\partial_t^h u(t) = \frac{u(t+h)-u(t)}{h}$ denotes the discrete time-derivative of a function u. Thus, we obtain compactness: There exists a sequence $h\downarrow 0$ and a limit $u\colon (0,T)\times[0,\Lambda)^d\to\mathbb{R}^N$ such that

$$u^h \to u \quad a.e.$$

and furthermore

$$\operatorname*{ess\,sup}_t \int |\nabla u|^2\,dx, \qquad \int_0^T\int |\partial_t u|^2\,dx\,dt < \infty.$$

We claim that the sequence u^h (and therefore of u) is uniformly bounded. Since by assumption the initial conditions u^0 is bounded it is enough to prove that if u^{n-1} has values inside the box $[-M,M]^N$, then so does u^n.

Assuming the contrary, we construct $\tilde u$ by truncating u^n componentwise:

$$\tilde u_i = \operatorname{sign}(u_i^n)\,(M\wedge|u_i^n|)\,. \tag{30}$$

By construction and since W satisfies (4) we have $E(\tilde u)\le E(u^n)$ and furthermore since by assumption $|u_i^{n-1}|\le M$ we have

$$\int |\tilde u - u^{n-1}|^2 dx < \int |\tilde u^n - u^{n-1}|^2 dx,$$

a contradiction to the minimality of u^n. Hence the sequence u^h is uniformly bounded.

We want to pass to the limit $h\downarrow 0$ in the the Euler-Lagrange equation

$$\int_0^T\int \partial_t^{-h}\xi\cdot u^h + \Delta\xi\cdot u^h\,dx\,dt = \int_0^T\int \xi\cdot\partial_u W(u^h)\,dx\,dt$$

for all test vector fields $\xi\in C_0^\infty((0,T)\times[0,\Lambda)^d,\mathbb{R}^N)$. By the pointwise convergence

$$\partial_u W(u^h)\to\partial_u W(u) \quad a.e.$$

and the uniform bound $|u_i^h|\le M$ we can pass to the limit in the nonlinear term by Lebesgue's dominated convergence.

A standard boot-strapping argument shows that u is smooth.

Step 2: BV-compactness of $\phi_i(u_\varepsilon)$.
We first prove boundedness of $\phi_i(u_\varepsilon)$ in L^1. By the triangle inequality for d we have $\phi_i(u_\varepsilon)\le d(\alpha_i,0)+d(0,u_\varepsilon)$, so that it is sufficient to consider $d(0,u_\varepsilon)$. By the growth condition (3) on W we see

$$d(0,v)\le\int_0^1\sqrt{2W(sv)}|v|ds \lesssim |v|+|v|^{\frac{p}{2}+1}\lesssim 1+|v|^p$$

for all $v\in\mathbb{R}^n$. Hence we only need to prove L^p-boundedness of u_ε. We note that (24) implies $E_\varepsilon(u_\varepsilon(t))\le E_\varepsilon(u_\varepsilon^0)\to E_0<\infty$. The coercivity assumption (3) then gives the desired bound:

$$\int_0^T\int |u_\varepsilon|^p dxdt \overset{(3)}{\lesssim} \int_0^T\int 1+W(u_\varepsilon)dxdt\lesssim 1. \tag{31}$$

We now turn to the bounds on the derivatives. Like more than a handful of people before us, we will make use of the following simple application of Young's inequality which yields a sharp inequality for the approximate energies E_ε:

$$\int_0^T E_\varepsilon(u)\,dt = \int_0^T\int \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\,dx\,dt \geq \int_0^T\int \sqrt{2W(u)}\,|\nabla u|\,dxdt \geq \int_0^T\int |\nabla(\phi_i\circ u)|\,dxdt \tag{32}$$

for all $1 \leq i \leq P$, where we used estimate (15) of Lemma 2.2. The same reasoning works if we substitute the spatial derivative ∇u by the time derivative $\partial_t u$:

$$\int_0^T\int \frac{\varepsilon}{2}|\partial_t u|^2 + \frac{1}{\varepsilon}W(u)\,dx \geq \int_0^T\int \sqrt{2W(u)}\,|\partial_t u|\,dxdt \geq \int_0^T\int |\partial_t\phi_i(u)|\,dxdt. \tag{33}$$

In order to prove compactness of the solutions u_ε we now plug (32) and (33) into the energy-dissipation equality (24) and obtain

$$\int_0^T\int |\partial_t\phi_i(u_\varepsilon)|\,dx\,dt + \int_0^T\int |\nabla\phi_i(u_\varepsilon)|\,dx\,dt \lesssim (1+T)\,E_\varepsilon(u_\varepsilon^0). \tag{34}$$

By the Rellich compactness theorem, we thus find a subsequence $\varepsilon \downarrow 0$ and a function $v\colon (0,T)\times[0,\Lambda)^d \to \mathbb{R}$ such that

$$\phi_i(u_\varepsilon) \to v \quad \text{in } L^1([0,T]\times[0,\Lambda)^d). \tag{35}$$

Step 3: The limit v takes the form $\sum_i \chi_i\alpha_i$.
This statement is part of the classical Γ-limit result [5]. However, we take this opportunity to provide a clarification of the previously known argument.

After passing to another subsequence we can assume that the sequence u_ε generates a Young measure $p_{t,x}$. We note that

$$\int_0^T\int W(u_\varepsilon)\,dx\,dt \to 0$$

implies that u_ε tends to the zeros of W in measure: For any $\delta > 0$ we have

$$|\{(x,t)\colon\ \operatorname{dist}(u_\varepsilon,\{\alpha_1,\ldots,\alpha_P\}) \geq \delta\}| \to 0 \quad \text{as } \varepsilon \to 0.$$

Hence also the Young measure concentrates and we get

$$p_{t,x} = \sum_{1\leq j\leq P} p_{t,x}(\alpha_j)\delta_{\alpha_j}.$$

From this estimate we also get that no mass escapes to infinity, i.e. $\sum_{1\leq j\leq P} p_{t,x}(\alpha_j) = 1$.

By (35) for all $f \in C_c(\mathbb{R})$ also $f\circ\phi_i(u_\varepsilon)$ is strongly compact in L^1. Therefore Young measure theory gives us the following (a.e.) identities:

$$v = \lim \phi_i(u_\varepsilon) = \sum_{1\leq j\leq P} \phi_i(\alpha_j)p_{t,x}(\alpha_j),$$

$$f\left(\sum_{1\leq j\leq P} \phi_i(\alpha_j)p_{t,x}(\alpha_j)\right) = \lim f(\phi_i(u_\varepsilon)) = \sum_{1\leq j\leq P} f(\phi_i(\alpha_j))p_{t,x}(\alpha_j).$$

If we take f to be uniformly convex on the interval $[0, \max_{1\leq j\leq P}\phi_i(\alpha_j)]$ we see from the equality statement in Jensen's inequality that

$$\phi_i(\alpha_j) \equiv \sum_{1\leq k\leq P} \phi_i(\alpha_k)p_{t,x}(\alpha_k) \quad p_{t,x} \text{ almost surely.}$$

For $(t,x) \in [0,T] \times [0,\Lambda)^d$ (up to a set of measure zero) let α_j be such that $p_{t,x}(\alpha_j) > 0$. We then get that

$$0 = \phi_j(\alpha_j) = \sum_{1\leq k\leq P} \phi_j(\alpha_k) p_{t,x}(\alpha_k) = \sum_{k\neq j} \phi_j(\alpha_k) p_{t,x}(\alpha_k).$$

Since for $k \neq j$ we have $\phi_j(\alpha_k) > 0$ we get $p_{t,x}(\alpha_k) = 0$. Thus we get $p_{t,x}(\alpha_j) = 1$ and setting $\chi_i(t,x) := p_{t,x}(\alpha_i)$ proves the claim.

Step 4: $\chi_i \in BV$.

A similar claim is proven to be true in Prop. 2.2 in [5]. For the convenience of the reader we reproduce the proof.

Applying the Fleming-Rishel coarea formula in space and time we see for each $1 \leq i \leq P$ that

$$\begin{aligned}||(\partial_t, \nabla)\phi_i \circ u_\varepsilon||_{TV} =& \int_{-\infty}^{\infty} \mathcal{H}^d \left(\partial_* \{(t,x) : \phi_i \circ u_\varepsilon(t,x) \leq s\}\right) ds \\ \geq& \int_0^{d_i} \mathcal{H}^d \left(\partial_* \{(t,x) : \phi_i \circ u_\varepsilon(t,x) \leq s\}\right) ds \\ =& d_i ||(\partial_t, \nabla)\chi_i||_{TV},\end{aligned}$$

where we define $d_i := \min_{1\leq j\leq P, i\neq j} d(\alpha_i, \alpha_j)$. Thus $\chi_i \in BV([0,T] \times [0,\Lambda)^d)$.

For the statement $||E(\chi)||_{L^\infty([0,T])} \leq E_0$ we refer the reader to the proof of the $\Gamma - \liminf$ inequality in [5] and the energy-dissipation equality (24). □

Proof of Lemma 3.2. Differentiating the approximate energies w.r.t. time and integrating by parts show

$$\begin{aligned}\frac{d}{dt} E_\varepsilon(u_\varepsilon) =& \int \varepsilon \nabla u_\varepsilon : \nabla \partial_t u_\varepsilon + \frac{1}{\varepsilon} \partial_u W(u_\varepsilon) \cdot \partial_t u_\varepsilon \, dx \\ =& \int \varepsilon \left(-\Delta u_\varepsilon + \frac{1}{\varepsilon^2} \partial_u W(u_\varepsilon) \right) \cdot \partial_t u_\varepsilon \, dx \\ \overset{(1)}{=}& - \int \varepsilon |\partial_t u_\varepsilon|^2 \, dx.\end{aligned}$$

Integrating from 0 to T we obtain (24). □

Proof of Lemma 3.3. W.l.o.g. we assume that $s = 0$ and write $t = T$. Estimating the integral

$$\int |\phi_i(u_\varepsilon)(T) - \phi_i(u_\varepsilon)(0)| \, dx \leq \int_0^T \int |\partial_t (\phi_i \circ u_\varepsilon)| \, dx \, dt,$$

using the chain rule (14), the Lipschitz estimate (15) and the Cauchy-Schwarz inequality, we obtain

$$\int_0^T \int |\partial_t \phi_i(u_\varepsilon)| \, dx \, dt \leq \left(\int_0^T \int \varepsilon \, |\partial_t u_\varepsilon|^2 \, dx \, dt \right)^{\frac{1}{2}} \left(\int_0^T \int \frac{2}{\varepsilon} W(u_\varepsilon) \, dx \, dt \right)^{\frac{1}{2}}. \tag{36}$$

By the energy-dissipation equality (24) the first right-hand side integral is bounded by $E_\varepsilon(u_\varepsilon^0) \to E_0$ and the second right-hand side integral similarly by TE_0. Thus (25) holds.

We now turn to the proof of (26). Passing to the limit in (25) we have by (21) for a.e. $s, t \in (0,T)$

$$\int |\phi_i(u)(t) - \phi_i(u)(s)| \, dx \lesssim E_0 \sqrt{|t-s|}.$$

Since $\chi_i = 1$ if and only if $\phi_i(u) = 0$ and $|\phi_i(u)| \geq \min_{j\neq i} \sigma_{ij} > 0$ otherwise, we also have (26). □

Proof of Proposition 3.4. The strategy is the following:

1. We prove the estimate for $\phi_i \circ u$ on the left-hand side and for $E(\bullet, u)dt$ on the right-hand side.

2. We replace $\phi_i \circ u$ with u using a suitable localization of Step 4 of the proof of Proposition 3.1, i.e. of the Fleming-Rishel coarea formula.

3. We prove that $\partial_t \chi_i$ is singular to the "wrong" parts of $E(\bullet, u)$ in order to replace the right-hand side with $|\nabla \chi_i| dt$ and (12).

Step 1: For all $1 \le i \le P$ we have $\partial_t(\phi_i \circ u) \ll E(\bullet, u)dt$ and the corresponding density is square-integrable w.r.t. $E(\bullet, u)dt$.
We localize with a smooth test function $\zeta \in C_0^\infty((0,T) \times [0,\Lambda)^d; \mathbb{R}^{1+d})$ and integrate by parts

$$\int_0^T \int \phi_i(u_\varepsilon) \partial_t \zeta \, dx \, dt \overset{(14)}{=} - \int_0^T \int \zeta \, \partial_u \phi_i(u_\varepsilon) \cdot \partial_t u_\varepsilon \, dx \, dt.$$

As in (36) we use the Lipschitz estimate (15) and the Cauchy-Schwarz inequality to obtain

$$\int_0^T \int \phi_i(u_\varepsilon) \partial_t \zeta \, dx \, dt \le \left(\int_0^T \int \varepsilon \left| \partial_t u_\varepsilon \right|^2 dx \, dt \right)^{\frac{1}{2}} \left(\int_0^T \int \zeta^2 \frac{2}{\varepsilon} W(u_\varepsilon) \, dx \, dt \right)^{\frac{1}{2}}. \tag{37}$$

By the convergence (23) of the composition and the equipartition of energy (41) we can pass to the limit in this inequality and obtain

$$\int_0^T \int \phi_i(u) \partial_t \zeta \, dx \, dt \le \left(\liminf_{\varepsilon \downarrow 0} \int_0^T \int \varepsilon \left| \partial_t u_\varepsilon \right|^2 dx \, dt \right)^{\frac{1}{2}} \left(\int_0^T E(\zeta^2, u) dt \right)^{\frac{1}{2}}. \tag{38}$$

By equation (24) the first factor on the right-hand side is controlled by $\sqrt{E_0}$. From this we see that indeed $|\partial_t(\phi_i \circ u)| \ll E(\bullet, u)dt$ and by taking the supremum over the test functions ζ we see that the density is square-integrable.

Step 2: We have $d_i |\partial_t \chi_i| \le |\partial_t(\phi_i \circ u)|$ where $d_i := \min_{1 \le j \le P, i \ne j} d(\alpha_i, \alpha_j)$.
Basically, we want to use the argument of Step 4 in the proof of Proposition 3.1 for the partial derivative $\partial_t \chi_i$. This can be done by combining the slicing theorem, cf. Theorem 3.103 in [4], and with the previous argument at almost each point $x \in [0,\Lambda)^d$, which leads to

$$d_i |\partial_t \chi_i|(U) \le |\partial_t(\phi_i \circ u)|(U)$$

for all open sets $U \subset [0,T] \times [0,\Lambda)^d$. This implies that for all $\xi \in C_c([0,T] \times [0,\Lambda)^d; [0,\infty)$ we have the inequality

$$d_i |\partial_t \chi_i|(\xi) \le |\partial_t(\phi_i \circ u)|(\xi):$$

Indeed, we can approximate ξ by constants on small sets whose boundaries are negligible w.r.t. the measures on both sides. We thus get that for all $1 \le i \le P$ we have $\partial_t \chi_i \ll E(\bullet, u)dt$ and the corresponding density V_i satisfies $V_i \in L^2(E(\bullet, u)dt)$.

Step 3: We have that $|(\partial_t, \nabla)\chi_i|$ and $\frac{1}{2}(|\nabla \chi_j|_d + |\nabla \chi_k|_d - |\nabla(\chi_j + \chi_k)|_d) \, dt$ are singular for all pairwise different $1 \le i, j, k \le P$.
For a characteristic function $\chi : [0,T] \times [0,\Lambda)^d \to \mathbb{R}$ we write $|\nabla \chi|_{d+1}$ for the total variation in time and space of the partial spacial derivatives and $|\nabla \chi|_d$ for the total variation the spacial derivatives in space defined almost everywhere in time.

According to Theorem 4.17 in [4] one can decompose $\operatorname{supp} |(\partial_t, \nabla)\chi_i|$ into the pairwise disjoint sets $\tilde{\Sigma}_{i,l} := \partial_* \tilde{\Omega}_i \cap \partial_* \tilde{\Omega}_l$, $1 \le l \le P$, which are the intersections of the reduced boundaries in time and space. The excpetional sets are $\mathcal{H}^d$-negligible and hence can be ignored in all the derivatives $\operatorname{supp} |(\partial_t, \nabla)\chi_m|$, $1 \le m \le P$. Thus we only have to prove that

$$\frac{1}{2}(|\nabla \chi_j|_d + |\nabla \chi_k|_d - |\nabla(\chi_j + \chi_k)|_d) \, dt \left(\tilde{\Sigma}_{il} \right) = 0$$

for all $1 \le l \le P$.

Since $j, k \ne i$ and the interfaces are pairwise disjoint we have that

$$|(\partial_t, \nabla)\chi_j| \left(\tilde{\Sigma}_{il} \right) = 0 \text{ or } |(\partial_t, \nabla)\chi_k| \left(\tilde{\Sigma}_{il} \right) = 0.$$

In the first case we have, since restricition commutes with the total variation,

$$\frac{1}{2}\left(|\nabla\chi_j|_{d+1}+|\nabla\chi_k|_{d+1}-|\nabla(\chi_j+\chi_k)|_{d+1}\right)\left(\tilde{\Sigma}_{il}\right)=\frac{1}{2}\left(|\nabla\chi_k|_{d+1}\left(\tilde{\Sigma}_{il}\right)-|\nabla\chi_k|_{d+1}\left(\tilde{\Sigma}_{il}\right)\right)=0.$$

The analogous argument gives the same result in the second case. Finally, a straightforward generalization of Theorem 3.103 in [4] to higher dimensional slicings implies

$$|\nabla\chi_l|_{1+d}=|\nabla\chi_l|_d\,dt,$$

which proves the claim.

Step 4: Conclusion of the L^2-estimate.
Since $|\partial_t\chi_i|\leq|(\partial_t,\nabla)\chi_i|$ as measures we get from Step 2 and Step 3 that $|\partial_t\chi_i|\ll|\nabla\chi_i|_d\,dt$. Step 3 also allows to replace $E(\bullet,u)\,dt$ by $|\nabla\chi_i|_d\,dt$ in the L^2-estimate.

Step 5: Initial data.
Furthermore we have (12) for $\zeta(0)=0$ by construction of the normal velocities. To generalize to arbitrary test functions ζ and include the initial conditions, we pass to the compositions $\phi_i\circ u$ instead.

Let us first show that now an easy consequence of (25) is

$$\int_0^T\int\partial_t\zeta\,\phi_i\circ u\,dx\,dt+\int\zeta(0)\,\phi_i\circ u^0\,dx=-\int_0^T\int\zeta\,\partial_t\left(\phi_i\circ u\right)dx\,dt. \tag{39}$$

Indeed, whenever $\zeta(0)=0$, as before, by construction of the normal velocities we trivially have

$$\int_0^T\int\partial_t\zeta\,\phi_i\circ u\,dx\,dt=-\int_0^T\int\zeta\,\partial_t\left(\phi_i\circ u\right)dx\,dt=-\sum_{1\leq j\leq P}\sigma_{ij}\int_0^T\int\zeta\,V_j\,|\nabla\chi_j|\,dt.$$

To include the initial conditions, for an arbitrary $\zeta\in C^\infty([0,T]\times[0,\Lambda)^d)$, we substitute ζ in the above formula by $\eta_\delta\zeta$ with the cutoff $\eta_\delta(t)=\frac{t}{\delta}\wedge 1$ and calculate

$$\int_0^T\int\partial_t\eta_\delta\,\zeta\,\phi_i(u)\,dx\,dt=\lim_{\varepsilon\downarrow 0}\frac{1}{\delta}\int_0^\delta\int\zeta\,\phi_i(u_\varepsilon)\,dx\,dt\overset{(25)}{=}\lim_{\varepsilon\downarrow 0}\left\{\int\zeta(0)\,\phi_i(u^{0,\varepsilon})\,dx+O(\sqrt{\delta})\right\},$$

where the constant in the $O(\sqrt{\delta})$-term is uniform in ε. Passing to the limit $\delta\to 0$ we indeed obtain (39).

Now, given (39) we prove (12). Using (26) we have with the same approximation argument (12) as before on the sequence, now for the limit

$$\int_0^T\int\partial_t\zeta\,\chi_j\,dx\,dt+\int\zeta(0)\chi_j(0)\,dx=-\int_0^T\int\zeta\,V_j\,|\nabla\chi_j|\,dt\quad\text{for all } j$$

with the trace $\chi_j(0)$ instead of the initial conditions χ_j^0. Multiplying by σ_{ij} and summing over j we obtain (39) with $\phi_i(u(0))$ instead of $\phi_i(u^0)$ and thus

$$\phi_i(u^0)=\phi_i(u(0))\quad\text{a.e. and for all } i.$$

Since for $u=\sum_j\chi_j\alpha_j$ we have $\chi_i=1$ if and only if $\phi_i(u)=0$ it follows that $\chi_i^0=\chi_i(0)$ a.e. and in particular (12) holds. □

Proof of Lemma 3.5. The proof is already contained in [23] and [25]. For the convenience of the reader we reproduce the arguments here.

Step 1: Localization in time.
We first show that the integrated assumption of the convergence of the energies (9) and the Γ-convergence of E_ε to E already imply the pointwise convergence (28) – at least up to a further subsequence. We will prove

$$\lim_{\varepsilon\downarrow 0}\int_0^T|E_\varepsilon(u_\varepsilon)-E(\chi)|\,dt=0, \tag{40}$$

which after passage to a subsequence clearly implies (28).

To convince ourselves of (40) we rewrite the integral as

$$\int_0^T |E_\varepsilon(u_\varepsilon) - E(\chi)|\, dt = \int_0^T (E_\varepsilon(u_\varepsilon) - E(\chi))\, dt + 2\int_0^T (E_\varepsilon(u_\varepsilon) - E(\chi))_-\, dt.$$

The first right-hand side integral vanishes as $\varepsilon \downarrow 0$ by (9). By the lower semi-coninuity part of the Γ-convergence of E_ε to E, see [5], and by the convergence (21) of u_ε to u the integrand of the second right-hand side term tends to zero pointwise a.e. in $(0,T)$. By Lebesgue's dominated convergence also the integral vanishes in the limit and we proved (40).

Step 2: Localization in space.
We claim that the convergence (28) of the energies implies

$$\lim_{\varepsilon\downarrow 0} E_\varepsilon(\zeta, u_\varepsilon) = E(\zeta, u) \quad \text{for a.e. } 0 \le t \le T \text{ and all } \zeta \in C^\infty([0,\Lambda)^d). \tag{41}$$

Indeed, if we assume that w.l.o.g. by linearity $0 \le \zeta \le 1$, using the $\liminf$-inequality of the Γ-convergence on the domains $\{\zeta > s\}$ and the layer cake representation $\zeta = \int_0^1 \mathbf{1}_{\{\zeta > s\}}\, ds$ we obtain the inequality

$$E(\zeta, u) \le \liminf_{\varepsilon\downarrow 0} E_\varepsilon(\zeta, u_\varepsilon).$$

But the same argument works for $0 \le 1-\zeta \le 1$ instead of ζ and by the convergence (28) we have

$$E(u) - E(\zeta,u) = E(1-\zeta, u) \le \liminf_{\varepsilon\downarrow 0} E_\varepsilon(1-\zeta, u_\varepsilon) \overset{(28)}{=} E(u) - \limsup_{\varepsilon\downarrow 0} E_\varepsilon(\zeta, u_\varepsilon),$$

which is the inverse inequality and thus (41) follows.

Step 3: Equipartition of energy.
Now let us turn to (29). First we claim that (29) reduces to

$$\int \sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon|\, dx \to E(u). \tag{42}$$

Indeed, setting $a_\varepsilon^2 := \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2$ and $b_\varepsilon^2 := \frac{1}{\varepsilon} W(u_\varepsilon)$, using $a_\varepsilon^2 - b_\varepsilon^2 = (a_\varepsilon + b_\varepsilon)(a_\varepsilon - b_\varepsilon)$ and Cauchy-Schwarz

$$\int \zeta \left|a_\varepsilon^2 - b_\varepsilon^2\right| dx \le \|\zeta\|_\infty \left(\int (a_\varepsilon + b_\varepsilon)^2\, dx \right)^{\frac12} \left(\int (a_\varepsilon - b_\varepsilon)^2\, dx \right)^{\frac12}.$$

Since $(a_\varepsilon + b_\varepsilon)^2 \lesssim a_\varepsilon^2 + b_\varepsilon^2$ the first right-hand side integral stays bounded in the limit $\varepsilon \downarrow 0$ and it is enough to prove that the second right-hand side integral vanishes as $\varepsilon \downarrow 0$. Expanding the square and using the definition of a_ε and b_ε we see that the limit of the second right-hand side integral is equal to

$$E_\varepsilon(u_\varepsilon) - \int \sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon|\, dx$$

and indeed the proof of (29) reduces to (42).

We conclude by proving (42). By lower semi-continuity and Young's inequality for any cutoff $0 \le \eta \le 1$ we get

$$\begin{aligned}\int \eta\, |\nabla(\phi_i \circ u)| \le \liminf_{\varepsilon\downarrow 0} \int \eta\, |\nabla(\phi_i \circ u_\varepsilon)|\, dx &\le \liminf_{\varepsilon\downarrow 0} \int \eta \sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon|\, dx \\ &\le \liminf_{\varepsilon\downarrow 0} E_\varepsilon(\eta, u_\varepsilon) \overset{(41)}{=} E(\eta, \chi).\end{aligned}$$

Using (16) we conclude.

□

4 Convergence

In Section 3 we proved that the solutions u_ε of the Allen-Cahn Equation (1) are precompact. In this section we show how to pass to the limit in the Allen-Cahn Equation (1) and prove that the limit moves by mean curvature. Since this section is the core of the paper, we give a short idea of the proof and then pass to the rigorous derivation in the subsequent parts, first for the curvature term, and afterwards for the velocity term.

4.1 Idea of the proof

To illustrate the idea of our proof we give a short overview in the simpler two-phase case. In this setting the convergence of the curvature-term

$$\lim_{\varepsilon\downarrow 0}\int_0^T\int\left(\varepsilon\Delta u_\varepsilon-\frac{1}{\varepsilon}W'(u_\varepsilon)\right)\xi\cdot\nabla u_\varepsilon\,dx\,dt=\sigma\int\nabla\xi\colon\,(Id-\nu\otimes\nu)\,|\nabla\chi|\,dt \tag{43}$$

is by (28) literally contained in [25] and the only difficulty is to prove

$$\lim_{\varepsilon\downarrow 0}\int_0^T\int\partial_t u_\varepsilon\,\xi\cdot\varepsilon\nabla u_\varepsilon\,dx\,dt=\sigma\int_0^T\int V\,\xi\cdot\nu\,|\nabla\chi|\,dt. \tag{44}$$

Since $\partial_t u_\varepsilon \rightharpoonup V\,|\nabla\chi|\,dt$ and $\varepsilon\nabla u_\varepsilon\approx\nu$ only in a weak sense, we cannot directly pass to the limit in the product. The general idea to work around this problem is to follow the strategy of [23]: Thinking of the test vector field ξ as a localization, we "freeze" the normal along the sequence to be the fixed direction $\nu^*\in\mathbb{S}^{d-1}$ and estimate the error w.r.t. an approximation of the *tilt-excess*

$$\mathcal{E}:=\sigma\int_0^T\int|\nu-\nu^*|^2|\nabla\chi|\,dt, \tag{45}$$

measuring the (local) flatness of the reduced boundary $\partial_*\Omega$ of the limit $\Omega=\{\chi=1\}$. The main difference to the work [23] is that we measure the error w.r.t. the tilt-excess $\mathcal{E}$ instead of the energy-excess

$$\int|\nabla\chi|-\int|\nabla\chi^*|,\quad\text{where }\chi^*\text{ is a half-space in direction }\nu^*.$$

After a localization, De Giorgi's structure theorem guarantees the smallness in both cases, see Section 5 in [23]. Our approximation of the tilt-excess along the sequence is

$$\mathcal{E}_\varepsilon:=\int_0^T\int|\nu_\varepsilon-\nu^*|^2\,\varepsilon\,|\nabla u_\varepsilon|^2\,dx\,dt. \tag{46}$$

While in the case of [23] the convergence assumption trivially implies the convergence of the (approximate) energy-excess, here we have to argue why we can pass to the limit in the nonlinear term $\mathcal{E}_\varepsilon$ and connect it to $\mathcal{E}$.

Using the trivial equality $|\nu-\nu^*|^2=2(1-\nu\cdot\nu^*)$ and the convergence assumption (9) this question reduces to

$$\int_0^T\int\varepsilon\,|\nabla u_\varepsilon|\,\nabla u_\varepsilon dx\,dt\to\sigma\int_0^T\int\nabla\left(\phi\circ u\right)dt, \tag{47}$$

where $\phi(u)=\int_0^u\sqrt{2W(s)}ds$. But it is easy to check that by the equipartition of energy (29) we can replace $\varepsilon\,|\nabla u_\varepsilon|$ by $\sqrt{2W(u_\varepsilon)}$ up to an error that vanishes as $\varepsilon\downarrow 0$. Identifying the nonlinearity

$$\sqrt{2W(u_\varepsilon)}\nabla u_\varepsilon=\nabla\left(\phi\circ u_\varepsilon\right)$$

as the derivative of the compact quantities $\phi\circ u_\varepsilon\to\phi\circ u$ yields the convergence of the excess.

Now let us explain how we will use the approximate tilt-excess to supress oscillations of the *direction* of the term $\varepsilon \nabla u_\varepsilon$ on the left-hand side of (44) so that we can pass to the limit in the product. We replace the normal $\nu_\varepsilon = \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$ by a constant direction $\nu^* \in \mathbb{S}^{d-1}$ and control the difference

$$\int_0^T \int \partial_t u_\varepsilon \, \xi \cdot \varepsilon \nabla u_\varepsilon dx \, dt - \int_0^T \int \partial_t u_\varepsilon \, \xi \cdot (\varepsilon \left| \nabla u_\varepsilon \right| \nu^*) \, dx \, dt \tag{48}$$

by the following combination of the excess and the initial energy

$$\|\xi\|_\infty \left(\frac{1}{\alpha} \mathcal{E}_\varepsilon + \alpha E_0 \right)$$

for any (small) parameter $\alpha > 0$ – an immediate consequence of Young's inequality and the energy-dissipation estimate (24). We now argue as in the proof of (47). Using the equipartition (29) we can further replace $\varepsilon \left| \nabla u_\varepsilon \right|$ in the second integral in (48) by $\sqrt{2W(u_\varepsilon)}$ so that

$$\int_0^T \int \partial_t u_\varepsilon \, \xi \cdot (\varepsilon \left| \nabla u_\varepsilon \right| \nu^*) \, dx \, dt = \int_0^T \int \partial_t u_\varepsilon \, \sqrt{2W(u_\varepsilon)} \, \xi \cdot \nu^* dx \, dt + o(1). \tag{49}$$

Identifying the nonlinear term as the derivative $\partial_t \left(\phi \circ u_\varepsilon \right)$ we can pass to the limit $\varepsilon \downarrow 0$ and obtain

$$\int_0^T \int \partial_t \left(\phi \circ u_\varepsilon \right) \xi \cdot \nu^* dx \, dt \to \sigma \int_0^T \int V \, \xi \cdot \nu^* \left| \nabla \chi \right| dt.$$

As before along the sequence, now at the level of the limit, by Young's inequality we can "un-freeze" the normal, i.e. replace ν^* by ν at the expense of

$$\|\xi\|_\infty \left(\frac{1}{\alpha} \mathcal{E} + \alpha \int_0^T \int V^2 \left| \nabla \chi \right| dt \right).$$

Thus we arrive at the right-hand side of (44) – up to an error that we can handle: we localize on a scale $r > 0$ so that $\mathcal{E} \to 0$ as $r \downarrow 0$, while the second error term stays bounded by the L^2-estimate (27). We then recover the motion law (11) by sending $\alpha \downarrow 0$.

4.2 Convergence of the curvature-term

In the two-phase case, the convergence (43) of the curvature-term is contained in the work of Luckhaus and Modica [25]. In our setting, the convergence does not follow immediately from their work. We give an extension of this result by quantifying the Reshetnyak-argument.

Proposition 4.1. *Given a sequence $u_\varepsilon \to u = \sum_i \chi_i \alpha_i$ such that the energies converge in the sense of*

$$E_\varepsilon(u_\varepsilon) \to E(u). \tag{50}$$

Then also the first variations converge:

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} \int \left(\varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \partial_u W(u_\varepsilon) \right) \cdot \left(\xi \cdot \nabla \right) u_\varepsilon \, dx \\ &= \frac{1}{2} \sum_{1 \leq i,j \leq P} \sigma_{ij} \int \nabla \xi \colon \left(Id - \nu_i \otimes \nu_i \right) \frac{1}{2} \left(\left| \nabla \chi_i \right| + \left| \nabla \chi_j \right| - \left| \nabla (\chi_i + \chi_j) \right| \right). \end{aligned} \tag{51}$$

Proof. Following the lines of [25] we can rewrite the left-hand side of (51) by integrating the first term by parts and using the chain rule for the second term. With Einstein's summation convention and omitting the index ε we have

$$\int (\varepsilon \partial_i \partial_i u_k - \frac{1}{\varepsilon} \partial_k W) \, \xi_j \, \partial_j u_k \, dx = \int \left\{ - \varepsilon \, \partial_i u_k \, \partial_i \xi_j \, \partial_j u_k - \varepsilon \, \partial_i u_k \, \xi_j \, \partial_i \partial_j u_k - \frac{1}{\varepsilon} \partial_j (W(u)) \, \xi_j \right\} dx. \tag{52}$$

We can now rewrite the second term on the right-hand side and integrate by parts to see

$$-\int \varepsilon\, \partial_i u_k\, \xi_j\, \partial_i \partial_j u_k\, dx = -\int \varepsilon\, \xi_j\, \partial_j \Big\{ \frac{1}{2} \left(\partial_i u_k\right)^2 \Big\}\, dx = \int (\nabla \cdot \xi)\, \frac{\varepsilon}{2} \left|\nabla u\right|^2 dx.$$

Plugging this into (52) the left-hand side of (51) is thus equal to

$$\int \nabla\xi\colon \left(Id - N^\varepsilon \otimes N^\varepsilon\right) \varepsilon \left|\nabla u_\varepsilon\right|^2 dx + \int (\nabla\cdot\xi) \Big(\frac{1}{\varepsilon} W(u_\varepsilon) - \frac{\varepsilon}{2}\left|\nabla u_\varepsilon\right|^2\Big) dx,$$

where $N^\varepsilon := \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} \in \mathbb{R}^{P\times d}$ and $(N^\varepsilon \otimes N^\varepsilon)_{ij} := \sum_k N^\varepsilon_{ki} N^\varepsilon_{kj} \in \mathbb{R}^{d\times d}$, a slightly non-standard definition of this symbol. By the equipartition of energy (29), see also Remark 3.6, the second integral is negligible as $\varepsilon \to 0$ and up to another error that vanishes as $\varepsilon \to 0$ we can replace the first term by

$$\int \nabla\xi\colon \left(Id - N^\varepsilon \otimes N^\varepsilon\right) \sqrt{2W(u_\varepsilon)}\, \left|\nabla u_\varepsilon\right| dx.$$

Again by the equipartition of energy (29) it is enough to prove the convergence of the nonlinear term

$$\int A\colon N^\varepsilon \otimes N^\varepsilon \sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon|\, dx \to \sum_{i,j} \sigma_{ij} \int A\colon \nu_i \otimes \nu_i \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla(\chi_i + \chi_j)|\right) \tag{53}$$

for any smooth matrix field $A\colon [0,\Lambda)^d \to \mathbb{R}^{d\times d}$. By linearity we may assume w.l.o.g. $|A| \le 1$.

We prove (53) using the following two claims:

Claim 1: We choose a majority phase by introducing the function $\phi = \phi_i$ for some arbitrary $1 \le i \le P$ on the right-hand side of (53). The corresponding estimate is

$$\limsup_{\varepsilon\to 0} \left| \int A\colon N^\varepsilon \otimes N^\varepsilon \sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon|\, dx - \int A\colon \nu_\varepsilon \otimes \nu_\varepsilon \left|\nabla(\phi\circ u_\varepsilon)\right| dx \right| \lesssim E(u,\eta) - \int \eta \left|\nabla \phi(u)\right|, \tag{54}$$

where $\nu_\varepsilon := \frac{\nabla \phi(u_\varepsilon)}{|\nabla\phi(u_\varepsilon)|} \in \mathbb{R}^d$.

Claim 2: We adapt the Reshetnyak argument in [25] to our setting by turning the qualitative statements there into a quantitative statement. Under the assumption (50) we claim

$$\limsup_{\varepsilon\downarrow 0} \left| \int A\colon \nu_\varepsilon \otimes \nu_\varepsilon \left|\nabla(\phi\circ u_\varepsilon)\right| dx - \int A\colon \nu\otimes\nu \left|\nabla(\phi\circ u)\right| \right| \lesssim E(u,\eta) - \int \eta \left|\nabla(\phi\circ u)\right|. \tag{55}$$

In both cases we express the errors in terms of the “mild excess”

$$E(u,\eta) - \int \eta \left|\nabla\phi(u)\right|, \tag{56}$$

which measures the local difference of the multi-phase setting to the two-phase setting on the support of the matrix field A approximated with a cut-off η.

Decomposing an arbitrary matrix field A by a partition of unity and using the localization estimate (16) we obtain (53) and thus proved the proposition.

Proof of Claim 1: Introducing a majority phase.
First we replace the matrix $N^\varepsilon = \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$ by $\pi_{u_\varepsilon} N^\varepsilon$, where $\pi_u = \frac{\partial_u\phi}{|\partial_u \phi|} \otimes \frac{\partial_u\phi}{|\partial_u\phi|}$. Note that then the additional sum in the definition of the symbol $\pi_{u_\varepsilon} N^\varepsilon \otimes \pi_{u_\varepsilon} N^\varepsilon$ collapses:

$$\left(\pi_u N \otimes \pi_u N\right)_{ij} = \sum_{k=1}^N \left(\pi_u N\right)_{ki} \left(\pi_u N\right)_{kj} = \left(\sum_{k=1}^N \frac{(\partial_u \phi)_k^2}{|\partial_u\phi|^2}\right) \frac{(\partial_u\phi\nabla u)_i}{|\partial_u\phi||\nabla u|} \frac{(\partial_u\phi\nabla u)_j}{|\partial_u\phi||\nabla u|}$$

and using the chain rule of Ambrosio and Dal Maso Lemma 2.2 we see

$$A\colon \left(\pi_{u_\varepsilon} N^\varepsilon\right) \otimes \left(\pi_{u_\varepsilon} N^\varepsilon\right) = A\colon \nu_\varepsilon \otimes \nu_\varepsilon.$$

Two errors arise in (54). The first error when replacing N^ε by $\pi_{u_\varepsilon} N^\varepsilon$ and the second when replacing $\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|$ by $|\nabla(\phi\circ u)|$.

The first error when introducing the projection π_u is bounded by

$$\int \eta \, |(Id - \pi_{u_\varepsilon})\, N^\varepsilon|^2 \sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon| \, dx. \tag{57}$$

Since multiplication by π_u is an orthogonal projection in matrix-space and $\left|\frac{\partial_u \phi}{|\partial_u \phi|} N^\varepsilon\right| = |\pi_u N^\varepsilon| \leq 1$ we have

$$|(Id - \pi_{u_\varepsilon})\, N^\varepsilon|^2 = |N^\varepsilon|^2 - |\pi_{u_\varepsilon} N^\varepsilon|^2 \leq 1 - |\pi_{u_\varepsilon} N^\varepsilon| = 1 - \left|\frac{\partial_u \phi}{|\partial_u \phi|} N^\varepsilon\right|.$$

Multiplying this inequality with $\sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon|$ and using the Lipschitz estimate for $\phi \circ u$ (15) we see that

$$\begin{aligned} |(Id - \pi_{u_\varepsilon})\, N^\varepsilon|^2 \sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon| &\leq \left(1 - \left|\frac{\partial_u \phi}{|\partial_u \phi|} N^\varepsilon\right|\right) \sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon| \\ &\leq \sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon| - |\partial_u \phi(u_\varepsilon) \nabla u_\varepsilon| \, . \end{aligned}$$

Plugging this into (57) and using the Ambrosio-Dal Maso chain rule (2.2) again, we see that the error is controlled by

$$E_\varepsilon(u_\varepsilon, \eta) - \int \eta \, |\nabla(\phi \circ u_\varepsilon)| \, dx.$$

By the convergence of the energies (50) and lower semi-continuity of the total variation we can pass to the limit $\varepsilon \to 0$ in this expression and obtain the upper bound

$$E(u, \eta) - \int \eta \, |\nabla(\phi \circ u)| \, .$$

Finally, we turn to the second error, when substituting $\sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon|$ by $|\nabla(\phi \circ u_\varepsilon)|$ in (54). Since $|\nabla(\phi \circ u_\varepsilon)| \leq |\partial_u \phi| |\nabla u_\varepsilon| \leq \sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon|$, by Young's inequality this second error is estimated by

$$\int \eta \left| \sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon| - |\nabla(\phi \circ u_\varepsilon)| \right| dx = \int \eta \left(\sqrt{2W(u_\varepsilon)}\, |\nabla u_\varepsilon| - |\nabla(\phi \circ u_\varepsilon)| \right) dx,$$

which by the equipartition (29) and Remark 3.6 again passes to the limit as before and thus proves (54).

Proof of Claim 2: A quantitative Reshetnyak-argument for $\phi \circ u$.
We could pass to the limit in the nonlinear expression $\int A \colon \nu \otimes \nu \, |\nabla (\phi \circ u)|$ by the classical Reshetnyak argument if we knew that the mass $\int |\nabla (\phi \circ u)|$ converged. In our case we unfortunately do not know if the total variation for each $\phi_i \circ u$ converges, but we can make the error small by localizing.

Our argument for (55) can be regarded as a quantitative analogue of the classical Reshetnyak-argument [34], see also [25].

By Banach-Alaoglu and a disintegration result for measures we can find a measure μ on $[0, \Lambda)^d$ and a family of probability measures $\{p_x\}_{x \in [0,\Lambda)^d}$ on $\mathbb{S}^{d-1}$ such that

$$\int \zeta(x, \nu_\varepsilon) \, |\nabla(\phi \circ u_\varepsilon)| \, dx \to \iint \zeta(x, \tilde\nu) \, dp_x(\tilde\nu) \, d\mu(x) \tag{58}$$

for all $\zeta \in C([0,\Lambda)^d \times \mathbb{S}^{d-1})$ – at least after passage to a subsequence. But since we will identify the limit we may pass to subsequences. In particular we have

$$\int A \colon \nu_\varepsilon \otimes \nu_\varepsilon \, |\nabla(\phi \circ u_\varepsilon)| \, dx \to \int A(x) \colon \int \tilde\nu \otimes \tilde\nu \, dp_x(\tilde\nu) \, d\mu(x). \tag{59}$$

Our aim is to prove that – up to the "mild excess" (56) – the right-hand side of (59) is equal to

$$\int A \colon \nu \otimes \nu \, |\nabla(\phi \circ u)| \, .$$

On the one hand, by the lower semi-continuity of the total variation and (58) with $\zeta(x,\nu)=\eta(x)\geq 0$

$$\int \eta|\nabla(\phi\circ u)| \leq \liminf_{\varepsilon\downarrow 0}\int \eta|\nabla(\phi\circ u_\varepsilon)|\,dx = \iint \eta\,d\mu, \tag{60}$$

i. e. $|\nabla(\phi\circ u)|$ is dominated by μ.

On the other hand, by the assumption (50) the measure μ is dominated by the energy. Indeed, for any $\eta\geq 0$ we have by Young's inequality

$$\int \eta\,d\mu = \lim_{\varepsilon\downarrow 0}\int \eta\,|\nabla(\phi\circ u_\varepsilon)|\,dx \leq \liminf_{\varepsilon\downarrow 0} E_\varepsilon(u_\varepsilon,\eta) = E(\chi,\eta). \tag{61}$$

Using $|\tilde\nu\otimes\tilde\nu-\nu\otimes\nu|\leq 2\,|\tilde\nu-\nu|$ and the relation (60) between the measures $|\nabla\phi(u)|$ and μ we see

$$\begin{aligned}\left|\iint A\colon \int \tilde\nu\otimes\tilde\nu\,dp_x(\tilde\nu)\,d\mu - \int A\colon \nu\otimes\nu\,|\nabla(\phi\circ u)|\right| &\lesssim \int \eta\,(d\mu-|\nabla(\phi\circ u)|)\\ &\quad+\int\eta\int|\nu-\tilde\nu|\,dp_x(\tilde\nu)\,|\nabla(\phi\circ u)|\,.\end{aligned}$$

By (61) the first right-hand side term is estimated by the "mild excess" (56).

We are left with proving

$$\int\eta\int|\nu-\tilde\nu|\,dp_x(\tilde\nu)\,|\nabla(\phi\circ u)| \lesssim E(\chi,\eta)-\int\eta\,|\nabla(\phi\circ u)|\,. \tag{62}$$

But distributional convergence of $\nabla(\phi\circ u_\varepsilon)$ towards $\nabla(\phi\circ u)$ and (58) with $\zeta(x,\tilde\nu)=\xi(x)\cdot\tilde\nu$ yield an equality for the linear term

$$\begin{aligned}\int\xi\cdot\nu\,|\nabla(\phi\circ u)| = \int\xi\cdot\nabla(\phi\circ u) &= \lim_{\varepsilon\downarrow 0}\int\xi\cdot\nabla(\phi\circ u_\varepsilon)\,dx\\ &= \lim_{\varepsilon\downarrow 0}\int\xi\cdot\nu_\varepsilon\,|\nabla(\phi\circ u_\varepsilon)|\,dx \overset{(58)}{=} \iint\xi\cdot\int\tilde\nu\,dp_x(\tilde\nu)\,d\mu\end{aligned} \tag{63}$$

for any smooth test vector field $\xi\colon[0,\Lambda)^d\to\mathbb{R}^d$. This draws a connection between the normal ν and the expectation $\int\tilde\nu\,dp_x(\tilde\nu)$ of the measures p_x.

Therefore for any such ξ with $|\xi|\leq\eta$ we get

$$\begin{aligned}\int\xi\cdot\int(\nu-\tilde\nu)\,dp_x(\tilde\nu)\,|\nabla(\phi\circ u)| &\overset{(63)}{=} \int\xi\cdot\int\tilde\nu dp_x(\tilde\nu)\,(d\mu-|\nabla(\phi\circ u)|)\\ &\overset{(60)}{\leq} \|\xi\|_\infty\left(\int\eta\,d\mu-\int\eta\,|\nabla(\phi\circ u)|\right)\end{aligned}$$

and after taking the supremum over all such ξ we discover

$$\int\eta\int|\nu-\tilde\nu|\,dp_x(\tilde\nu)\,|\nabla(\phi\circ u)| \leq \int\eta\,d\mu-\int\eta\,|\nabla(\phi\circ u)|\,. \tag{64}$$

Finally, notice that another application (61) proves the claim (62). □

Remark 4.2. The quantitative Reshetnyak argument (57) holds also for any other Lipschitz continuous function $f(x,\tilde\nu)$ on $\mathbb{S}^{d-1}$ instead of $A(x)\colon\tilde\nu\otimes\tilde\nu$.

4.3 Convergence of the velocity-term

As in the proof of convergence in the two-phase case our main tool will be a suitable tilt-excess. However, because ∇u_ε now describes the direction of change both in physical space and in state space, some care needs to be taken in defining such an excess. It is apparent that the limiting equation only sees the direction of change in physical space explicitly. In contrast, the change of direction in state space only enters implicitly through the surface tensions, which are the lengths of geodesics connecting the wells. It is therefore natural to define an approximate tilt-excess which only fixes the change of direction in physical space.

Definition 4.3. Let $\nu^* \in \mathbb{S}^{d-1}$ and $\eta \in C^\infty\left([0,T]\times[0,\Lambda)^d;[0,1]\right)$. For $\varepsilon > 0$ and a function $u_\varepsilon \in W^{1,2}([0,T]\times[0,\Lambda)^d;\mathbb{R}^n)$ the localized tilt-excess of the i-th phase, $1 \le i \le N$, is given by

$$\mathcal{E}^i_\varepsilon(\nu^*;\eta,u_\varepsilon) := \int_0^T \int \eta \frac{1}{\varepsilon} \left|\varepsilon\nabla u_\varepsilon + \partial_u\phi_i(u_\varepsilon)\otimes\nu^*\right|^2 dxdt. \tag{65}$$

In the limit $\varepsilon = 0$ and for a partition $\chi_i = \mathbf{1}_{\Omega_i} \in BV\left([0,T]\times[0,\Lambda)^d;\{0,1\}\right)$ with $\sum_i \chi_i = 1$ we define the tilt-excess for $1 \le i,j \le P$, $i \ne j$, to be

$$\mathcal{E}^{ij}(\nu^*;\eta,u) := \int_0^T \int \eta\left|\nu_i - \nu^*\right|^2 |\nabla\chi_i|\,dt + \int_0^T \int \eta\left|\nu_j + \nu^*\right|^2 |\nabla\chi_j|\,dt + \sum_{k\notin\{i,j\}} \int_0^T \int \eta\,|\nabla\chi_k|\,dt, \tag{66}$$

where $u = \sum_{1\le i\le N}\alpha_i\chi_i$ and ν_i, as throughout the paper, is the inner normal of Ω_i.

Note that the limiting excess measures two things: Firstly, the last term measures whether mostly the interface between the i-th and the j-th phase is present. Secondly, the first two terms measure how close the interface is to being flat.

A subtle point in the definition is that χ_i falls while moving out of the corresponding phase, while ϕ_i grows. Hence their differentials have opposite directions. We choose ν^* to be the approximate inner normal of χ_i, which leads to the positive sign in $\mathcal{E}^i_\varepsilon$ and the second term in $\mathcal{E}^{ij}$ and the negative one in the first term in $\mathcal{E}^{ij}$. For a similar reason the limiting excesses are not symmetric in i and j. Instead we have $\mathcal{E}^{ij}(\nu^*;\eta,u) = \mathcal{E}^{ji}(-\nu^*;\eta,u)$.

We first make sure that we can use $\mathcal{E}^{ij}(\nu^*;\eta,\chi)$ to asymptotically bound $\mathcal{E}^i_\varepsilon(\nu^*;\eta,u_\varepsilon)$.

Lemma 4.4. *For every $1\le i,j\le P$, $i\ne j$, $\nu^*\in\mathbb{S}^{d-1}$ and $\eta\in C^\infty([0,T]\times[0,\Lambda)^d;[0,1])$ we have that*

$$\limsup_{\varepsilon\to0}\mathcal{E}^i_\varepsilon(\nu^*;\eta,u_\varepsilon) \lesssim \mathcal{E}^{ij}(\nu^*;\eta,\chi). \tag{67}$$

Using this estimate, as in the two-phase case before, we prove (44) up to an error controlled by the tilt-excess (66).

Proposition 4.5. *Given the convergence assumption (9), there exists a finite Radon measure μ on $[0,T]\times[0,\Lambda)^d$, such that for any $1\le i,j\le P$, $i\ne j$, any parameter $\alpha>0$, any direction $\nu^*\in\mathbb{S}^{d-1}$ and any test vector field $\xi\in C_0^\infty((0,T)\times\mathbb{R}^d,\mathbb{R}^d)$ we have*

$$\begin{aligned}\limsup_{\varepsilon\downarrow0}&\left|\int_0^T\int \varepsilon(\xi\cdot\nabla)u_\varepsilon\cdot\partial_t u_\varepsilon\,dx\,dt - \sigma_{ij}\int_0^T\int \xi\cdot\nu_i V_i\frac12\left(|\nabla\chi_i|+|\nabla\chi_j|-|\nabla(\chi_i+\chi_j)|\right)dt\right|\\ &\lesssim \|\xi\|_\infty\left(\frac1\alpha\mathcal{E}^{ij}(\nu^*;\eta,u)+\alpha\mu(\eta)\right).\end{aligned} \tag{68}$$

Here $\eta\in C^\infty([0,T]\times\mathbb{R}^d)$ is a smooth cut-off for the support of ξ, i. e. $\eta\ge0$ and $\eta\equiv1$ on $\operatorname{supp}\xi$.

Proof of Lemma 4.4. Expanding the square and exploiting that the Lipschitz constant of ϕ_i is bounded by $\sqrt{2W(u)}$ we see that

$$\mathcal{E}^i_\varepsilon(\nu^*;\eta,u_\varepsilon) \le \int_0^T\int \eta\left(\varepsilon|\nabla u_\varepsilon|^2 + \frac{2}{\varepsilon}W(u_\varepsilon) + 2(\nu^*\cdot\nabla)u_\varepsilon\cdot\partial_u\phi_i(u_\varepsilon)\right)dxdt.$$

By the chain rule due to Ambrosio and Dal-Maso, see Theorem 2.1 and Lemma 2.2, we can rewrite the last term as

$$(\nu^*\cdot\nabla)u_\varepsilon\cdot\partial_u\phi_i(u_\varepsilon) = \nu^*\cdot\nabla(\phi_i\circ u_\varepsilon).$$

Thus we see using the convergence assumption (9) and the convergence (23) of $\phi_i\circ u_\varepsilon$ to $\phi_i\circ u$ that

$$\begin{aligned}\limsup_{\varepsilon\to0}\mathcal{E}^i_\varepsilon(\nu^*;\eta,u_\varepsilon) &\le \limsup_{\varepsilon\to0} 2\int_0^T\int \eta\left(e_\varepsilon(u_\varepsilon)+\nu^*\cdot\nabla(\phi_i\circ u_\varepsilon)\right)dx\,dt \\ &= 2\int_0^T\left(E(\eta,u)+\nu^*\cdot\int\eta\nabla(\phi_i\circ u)\right)dt.\end{aligned} \tag{69}$$

The second term can be rewritten as

$$\nu^* \cdot \nabla(\phi_i \circ u) = \nu^* \cdot \sum_{1\leq k\leq P} \sigma_{ik} \nabla \chi_k \leq \sigma_{ij}\, \nu^* \cdot \nabla \chi_j + \sum_{k\notin\{i,j\}} \sigma_{ik} \left|\nabla \chi_k\right|,$$

while the second one can be estimated by

$$E(\eta, u) \leq \sigma_{ij} \int \eta \left|\nabla \chi_j\right| + C \sum_{k\notin\{i,j\}} \int \eta \left|\nabla \chi_k\right|$$

for some generic constant $C < \infty$, only depending on $\max_{ij} \sigma_{ij}$. Thus we can asymptotically bound the excess by

$$\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon^i(\nu^*; \eta, u_\varepsilon) \leq \sigma_{ij} \int_0^T \int \eta\, 2\left(1 + \nu_j \cdot \nu^*\right) \left|\nabla \chi_j\right| dt + C \sum_{k\notin\{i,j\}} \int_0^T \int \eta \left|\nabla \chi_k\right| dt.$$

Since $2\left(1+\nu_j \cdot \nu^*\right) = \left|\nu_j + \nu^*\right|^2$ in particular (67) holds. □

Proof of Proposition 4.5. Step 1: Replacing ∇u_ε with $\partial_u \phi_i(u_\varepsilon) \otimes \nu^*$.
Using the tilt-excess (65) and Young's inequality we see

$$\left| \int_0^T \int \left(\varepsilon(\xi\cdot\nabla)u_\varepsilon + \xi\cdot\nu^*\partial_u\phi_i(u_\varepsilon)\right)\cdot\partial_t u_\varepsilon \,dx\,dt \right| \lesssim \|\xi\|_\infty \left(\frac{1}{\alpha}\mathcal{E}_\varepsilon^i(\nu^*;\eta,u_\varepsilon) + \alpha \int_0^T\int \eta\varepsilon|\partial_t u_\varepsilon|^2 dx dt \right). \quad (70)$$

By the energy-dissipation equality (24) the sequence $\varepsilon|\partial_t u_\varepsilon|^2$ is bounded in L^1 and thus, along a subsequence, has a weak*-limit μ as Radon measures. In the limit we get, applying Lemma 4.4 along the way,

$$\limsup_{\varepsilon\downarrow 0} \left| \int_0^T \int \left(\varepsilon(\xi\cdot\nabla)u_\varepsilon + \xi\cdot\nu^*\partial_u\phi_i(u_\varepsilon)\right)\cdot\partial_t u_\varepsilon \,dx\,dt \right| \lesssim \|\xi\|_\infty \left(\frac{1}{\alpha}\mathcal{E}^{ij}(\nu^*;\eta,u) + \alpha\mu(\eta) \right).$$

Step 2: Passing to the limit in the nonlinear term.
In the second term on the left-hand side of (70) we may now use the chain rule again to see

$$-\int_0^T \int \xi\cdot\nu^*\partial_u\phi_i(u_\varepsilon)\cdot\partial_t u_\varepsilon dx dt = -\int_0^T\int \xi\cdot\nu^*\partial_t\left(\phi_i\circ u_\varepsilon\right)dx dt \to -\int_0^T\int \xi\cdot\nu^*\partial_t\left(\phi_i\circ\sum_{1\leq k\leq P}\chi_k\alpha_k\right)dt.$$

Step 3: Rewriting the limit in terms of the interface between χ_i and χ_j.
We can rewrite this limit to read

$$-\int_0^T\int \xi\cdot\nu^*\partial_t\left(\phi_i\circ\sum_{1\leq k\leq P}\chi_k\alpha_k\right)dt = -\int_0^T\int \xi\cdot\nu^*\sum_{1\leq k\leq P}\sigma_{ik}\partial_t\chi_k \overset{3.4}{=} -\int_0^T\int \xi\cdot\nu^*\sum_{1\leq k\leq P}\sigma_{ik}V_k|\nabla\chi_k|dt.$$

Thanks to the tilt-excess (66) we can now get rid of all terms except the j^{th} one: With a little help from our friends Cauchy, Schwarz and Young we arrive at

$$\left| -\int_0^T\int \xi\cdot\nu^*\sum_{1\leq k\leq P}\sigma_{ik}V_k|\nabla\chi_k|dt + \int_0^T\int \xi\cdot\nu^*\sigma_{ij}V_j\left|\nabla\chi_j\right|dt \right|$$
$$\lesssim \|\xi\|_\infty\left(\frac{1}{\alpha}\mathcal{E}^{ij}(\nu^*;\eta,u) + \alpha\int_0^T\int\eta\sum_{1\leq k\leq P}V_k^2|\nabla\chi_k|dt\right).$$

Here, due to the L^2-estimate Proposition 3.4, the right-hand side is an acceptable error term after redefining μ.

Hence we are left with a term only depending on the j^{th} phase which we can replace with (minus) the according term for the i^{th} phase: Indeed, using $\sum_k \chi_k = 1$ the error in doing so is equal to

$$\left| \int_0^T \int \xi \cdot \nu^* \sigma_{ij} \left(V_j \left| \nabla \chi_j \right| + V_i \left| \nabla \chi_i \right| \right) dt \right| = \left| \int_0^T \int \xi \cdot \nu^* \sigma_{ij} \partial_t \left(1 - \sum_{k \notin \{i,j\}} \chi_k \right) dt \right|$$
$$\lesssim \int_0^T \int |\xi| \sum_{k \notin \{i,j\}} |V_k| \left| \nabla \chi_k \right| dt,$$

which by Young's inequality is controlled by the same right-hand side as before.

Exploiting $|\nu^* - \nu_i| \, |V_i| \lesssim \frac{1}{\alpha} |\nu^* - \nu_i|^2 + \alpha |V_i|$ we now use the tilt-excess once again to "un-freeze" the approximate normal ν^* and eliminate other interfaces:

$$\left| \int_0^T \int \xi \cdot \nu^* \sigma_{ij} V_i \left| \nabla \chi_i \right| dt - \int_0^T \int \xi \cdot \nu_i \sigma_{ij} V_i \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)| \right) dt \right|$$
$$\lesssim \|\xi\|_\infty \left(\frac{1}{\alpha} \mathcal{E}^{ij}(\nu^*; \eta, u) + \alpha \int_0^T \int \eta \, V_i^2 |\nabla \chi_i| dt \right).$$

Retracing our steps we see that we arrived at the desired estimate. □

We conclude with the proof of our main result.

Proof of Theorem 1.2. We constructed the limit u of the approximations u_ε in Proposition 3.1, the normal velocity with the according L^2-bounds in Proposition 3.4. We only have to prove the motion law (11). Given a smooth test vector field $\xi \in C_0^\infty((0,T) \times [0,\Lambda)^d, \mathbb{R}^d)$, we multiply the Allen-Cahn Equation (1) by $\varepsilon \left(\xi \cdot \nabla \right) u_\varepsilon$ and integrate w.r.t. space and time:

$$\int_0^T \int \left(\varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \partial_u W(u_\varepsilon) \right) \cdot \left(\xi \cdot \nabla \right) u_\varepsilon \, dx \, dt = \int_0^T \int \varepsilon (\xi \cdot \nabla) u_\varepsilon \cdot \partial_t u_\varepsilon \, dx \, dt. \tag{71}$$

By Proposition 4.1 the convergence of the energies (28) imply the convergence of the first variations for a.e. t. By Lebesgue's dominated convergence the left-hand side of (71) converges:

$$\lim_{\varepsilon \downarrow 0} \int_0^T \int \left(\varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} \partial_u W(u_\varepsilon) \right) \cdot \left(\xi \cdot \nabla \right) u_\varepsilon \, dx \, dt$$
$$= \sum_{i,j} \sigma_{ij} \int_0^T \int \nabla \xi \colon \left(Id - \nu_i \otimes \nu_i \right) \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)| \right) dt.$$

In order to prove the convergence of the right-hand side, we proceed as in [23]. We decompose $\xi = \sum_{B \in \mathscr{B}_r} \varphi_B \xi$ with a partition of unity underlying the covering $\mathscr{B}_r$ defined in (17). Using Proposition 4.5 for $\xi_B = \varphi_B \xi$ on time intervals $0 = T_1 < \ldots < T_K = T$ and passing to the limit $K \to \infty$ we obtain the error

$$\left| \int_0^T \int \varepsilon (\xi \cdot \nabla) u_\varepsilon \cdot \partial_t u_\varepsilon \, dx \, dt - \sum_{1 \le i,j \le P} \sigma_{ij} \int_0^T \int V_i \, \xi \cdot \nu_i \frac{1}{2} \left(|\nabla \chi_i| + |\nabla \chi_j| - |\nabla (\chi_i + \chi_j)| \right) dt \right|$$
$$\lesssim \|\xi\|_\infty \left(\frac{1}{\alpha} \int_0^T \sum_{B \in \mathscr{B}_r} \min_{i,j} \min_{\nu^* \in \mathbb{S}^{d-1}} \int \eta_B \left| \nu_i - \nu^* \right|^2 |\nabla \chi_i| + \int \eta_B \left| \nu_j + \nu^* \right|^2 |\nabla \chi_j| \right.$$
$$\left. + \sum_{k \notin \{i,j\}} \int \eta_B \left| \nabla \chi_k \right| dt + \alpha \int_0^T \int \sum_{B \in \mathscr{B}_r} \eta_B \, d\mu \right),$$

where for a ball B the function η_B denotes a cutoff for B in $2B$ as in Section 2. Because of the finite overlap (18) the last term is uniformly bounded in r. Using Lemma 2.3 we see that the first term vanishes as $r \to 0$. Then taking $\alpha \to 0$ we obtain the convergence of the velocity-term and thus verified the motion law (11). □

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