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by

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The problems of genuine multipartite entanglement detection and classification are challenging. We show that a multipartite quantum state is genuine multipartite entangled if the multipartite concurrence is larger than certain quantities given by the number and the dimension of the subsystems. This result also provides a classification of various genuine multipartite entanglement. Then, we present a lower bound of the multipartite concurrence in terms of bipartite concurrences. While various operational approaches are available for providing lower bounds of bipartite concurrences, our results give an effective operational way to detect and classify the genuine multipartite entanglement. As applications, the genuine multipartite entanglement of tripartite systems is analyzed in detail.

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Quantum entanglement, as the remarkable nonlocal feature of quantum mechanics, is recognized as a valuable resource in the rapidly expanding field of quantum information science, with various applications [1, 2] in such as quantum computation, quantum teleportation, dense coding, quantum cryptographic schemes, quantum radar, entanglement swapping and remote states preparation. A bipartite quantum state without entanglement is called separable. A multipartite quantum state that is not separable with respect to any bi-partitions is said to be genuinely multipartite entangled [3–5]. Genuine multipartite entanglement is an important type of entanglement, which offers significant advantages in quantum tasks comparing with bipartite entanglement [6]. In particular, it is the basic ingredient in measurement-based quantum computation [7], and is beneficial in various quantum communication protocols [8], including secret sharing [9] (cf. [10]), among multi-parties. Despite its importance, characterization and detection of this kind of resource turns out to be quite difficult. Recently some methods such as linear and nonlinear entanglement witnesses [4, 5, 11–18], generalized concurrence for multipartite genuine entanglement [19–22], and Bell-like inequalities [23] have been proposed. Nevertheless, the problem remains far from being satisfactorily solved.

Quantifying entanglement is also a basic and long standing problem in quantum information theory [24–27]. Estimation of any quantum entanglement measures can be used to judge the separability of a given state. From the norms of the correlation tensors in the generalized Bloch representation of a quantum state, separable conditions for both bi- and multi-partite quantum states are presented in [28–31]; a multipartite entanglement measure for N-qubit and N-qudit pure states is given in [32, 33]; a general framework for detecting genuine multipartite entanglement and non full separability in multipartite quantum systems of arbitrary dimensions has been introduced in [5]. In [34] it has been shown that the norms of the correlation tensors has a close relationship to the maximal violation of a kind of multi Bell inequalities.

In this Letter, we investigate the genuine multipartite entanglement in terms of the norms of the correlation tensors and multipartite concurrence. We show that if the multipartite concurrence is larger than a constant given by the number and dimension of the subsystems, the state must be genuine multipartite entangled. To implement the criteria, we investigate the relationship between the bipartite concurrence and the multipartite concurrence. An effective lower bound of multipartite concurrence is derived to detect genuine multipartite entanglement.

Let $\mathcal{H}_i$, $i=1,2,...,N$, denote d-dimensional Hilbert spaces. The concurrence of an N-partite quantum pure state $|\psi\rangle\in\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_N$ is defined by [25, 26],

$$C_N(|\psi\rangle\langle\psi|)=2^{1-\frac{N}{2}}\sqrt{(2^N-2)-\sum_\alpha Tr\{\rho_\alpha^2\}},\qquad(1)$$

where α labels all the different reduced density matrices of $|\psi\rangle\langle\psi|$. Any N-partite pure state that can be written as $|\psi\rangle=|\phi_A\rangle\otimes|\phi_{\bar{A}}\rangle$ is called bi-separable, where A denotes a certain subset of $\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_N$ and $\bar{A}$ stands for the complement of A. States that are not bi-separable with respect to any bipartition are said to be genuine multipartite entangled.

For an N-partite mixed quantum state, $\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i|\in\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_N$, the corresponding concurrence is given by the convex roof:

$$C_N(\rho)=\min_{\{p_i,|\psi_i\rangle\}}\sum_i p_i C_N(|\psi_i\rangle\langle\psi_i|), \qquad (2)$$

where the minimization runs over all ensembles of pure state decompositions of ρ. A genuine multipartite entangled mixed state is defined to be one that cannot be written as a convex combination of biseparable pure states.

Theorem An N-partite quantum state $\rho\in\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_N$ is genuine multipartite entangled if

$$C_N(\rho)>\begin{cases} 2^{1-\frac{N}{2}}\sqrt{2^N-4+\frac{2}{d}-2\sum_{k=1}^{\frac{N-1}{2}}\frac{\binom{N}{k}}{d^k}}, & \text{for odd } N,\\ 2^{1-\frac{N}{2}}\sqrt{2^N-4+\frac{2}{d}-2\sum_{k=1}^{\frac{N}{2}-1}\frac{\binom{N}{k}}{d^k}-\frac{\binom{N}{\frac{N}{2}}}{d^{\frac{N}{2}}}}, & \text{for even } N,\end{cases} \qquad (3)$$

where $\binom{N}{k}=N!/(k!(N-k)!)$.

Proof A general multipartite state $\rho\in\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_N$ can be written as [30],

$$\begin{aligned}\rho \;=\; &\frac{1}{d^N}(\otimes_{j=1}^N I_d+\sum_{\{\mu_1\}}\sum_{\alpha_1}\mathcal{T}_{\alpha_1}^{\{\mu_1\}}\lambda_{\alpha_1}^{\{\mu_1\}}+\sum_{\{\mu_1\mu_2\}}\sum_{\alpha_1\alpha_2}\mathcal{T}_{\alpha_1\alpha_2}^{\{\mu_1\mu_2\}}\lambda_{\alpha_1}^{\{\mu_1\}}\lambda_{\alpha_2}^{\{\mu_2\}}+\sum_{\{\mu_1\mu_2\mu_3\}}\sum_{\alpha_1\alpha_2\alpha_3}\mathcal{T}_{\alpha_1\alpha_2\alpha_3}^{\{\mu_1\mu_2\mu_3\}}\lambda_{\alpha_1}^{\{\mu_1\}}\lambda_{\alpha_2}^{\{\mu_2\}}\lambda_{\alpha_3}^{\{\mu_3\}}\\ &+\cdots+\sum_{\{\mu_1\mu_2\cdots\mu_M\}}\sum_{\alpha_1\alpha_2\cdots\alpha_M}\mathcal{T}_{\alpha_1\alpha_2\cdots\alpha_M}^{\{\mu_1\mu_2\cdots\mu_M\}}\lambda_{\alpha_1}^{\{\mu_1\}}\lambda_{\alpha_2}^{\{\mu_2\}}\cdots\lambda_{\alpha_M}^{\{\mu_M\}}+\cdots+\sum_{\alpha_1\alpha_2\cdots\alpha_N}\mathcal{T}_{\alpha_1\alpha_2\cdots\alpha_N}^{\{1,2,\cdots,N\}}\lambda_{\alpha_1}^{\{1\}}\lambda_{\alpha_2}^{\{2\}}\cdots\lambda_{\alpha_N}^{\{N\}}),\end{aligned}$$

where λ_{α_k} are the $SU(d)$ generators, $\{\mu_1\mu_2\cdots\mu_M\}$ is a subset of $\{1,2,\cdots,N\}$, $\lambda_{\alpha_k}^{\{\mu_k\}}=I_d\otimes I_d\otimes\cdots\otimes\lambda_{\alpha_k}\otimes I_d\otimes\cdots\otimes I_d$ with λ_{α_k} appearing at the μ_kth position, I_d is the $d\times d$ identity matrix and

$$\mathcal{T}_{\alpha_1\alpha_2\cdots\alpha_M}^{\{\mu_1\mu_2\cdots\mu_M\}}=\frac{d^M}{2^M}\mathrm{Tr}[\rho\lambda_{\alpha_1}^{\{\mu_1\}}\lambda_{\alpha_2}^{\{\mu_2\}}\cdots\lambda_{\alpha_M}^{\{\mu_M\}}],$$

which can be viewed as the entries of tensors $\mathcal{T}^{\{\mu_1\mu_2\cdots\mu_M\}}$.

We start with an N-partite pure quantum state $|\psi\rangle$. Let $||\bullet||$ denote the Euclidean norm for a tensor. After tedious but straightforward computation, one obtains that for odd N,

$$\begin{aligned}\sum_{\alpha=1}^{2^N-2}Tr\rho_\alpha^2 \;=\; 2[\;&\binom{N}{1}\frac{1}{d}+\binom{N-1}{0}\frac{2}{d^2}\sum_{k_1\in\{1,2,\cdots N\}}||\mathcal{T}^{k_1}||^2\\ +\;&\binom{N}{2}\frac{1}{d^2}+\binom{N-1}{1}\frac{2}{d^3}\sum||\mathcal{T}^{k_1}||^2+\binom{N-2}{0}\frac{2^2}{d^{2*2}}\sum_{k_1k_2}||\mathcal{T}^{k_1k_2}||^2\\ +\;&\cdots+\\ +\;&\binom{N}{\frac{N-1}{2}}\frac{1}{d^{\frac{N-1}{2}}}+\binom{N-1}{\frac{N-1}{2}-1}\frac{2}{d^{\frac{N-1}{2}}+1}\sum||\mathcal{T}^{k_1}||^2+\binom{N-2}{\frac{N-1}{2}-2}\frac{2^2}{d^{\frac{N-1}{2}+2}}\sum_{k_1k_2}||\mathcal{T}^{k_1k_2}||^2\\ +\;&\cdots+\binom{N-\frac{N-1}{2}}{0}\frac{2^{\frac{N-1}{2}}}{d^{2*\frac{N-1}{2}}}\sum_{k_1\cdots k_{\frac{N-1}{2}}}||\mathcal{T}^{k_1\cdots k_{\frac{N-1}{2}}}||^2];\end{aligned}$$

while for even N,

$$\begin{aligned}
\sum_{\alpha=1}^{2^N-2} Tr\rho_\alpha^2 = & \; 2[\; \binom{N}{1}\frac{1}{d} + \binom{N-1}{0}\frac{2}{d^2}\sum_{k_1\in\{1,2,\cdots N\}}||\mathcal{T}^{k_1}||^2 \\
& + \binom{N}{2}\frac{1}{d^2} + \binom{N-1}{1}\frac{2}{d^3}\sum||\mathcal{T}^{k_1}||^2 + \binom{N-2}{0}\frac{2^2}{d^{2*2}}\sum_{k_1k_2}||\mathcal{T}^{k_1k_2}||^2 \\
& + \cdots\cdots \\
& + \binom{N}{\frac{N-2}{2}}\frac{1}{d^{\frac{N-2}{2}}} + \binom{N-1}{\frac{N-2}{2}-1}\frac{2}{d^{\frac{N-2}{2}}+1}\sum||\mathcal{T}^{k_1}||^2 + \binom{N-2}{\frac{N-2}{2}-2}\frac{2^2}{d^{\frac{N-2}{2}+2}}\sum_{k_1k_2}||\mathcal{T}^{k_1k_2}||^2 \\
& + \cdots + \binom{N-\frac{N-2}{2}}{0}\frac{2^{\frac{N-2}{2}}}{d^{2*\frac{N-2}{2}}}\sum_{k_1\cdots k_{\frac{N-2}{2}}}||\mathcal{T}^{k_1\cdots k_{\frac{N-2}{2}}}||^2] \\
& + \binom{N}{\frac{N}{2}}\frac{1}{d^{\frac{N}{2}}} + \binom{N-1}{\frac{N}{2}-1}\frac{2}{d^{\frac{N}{2}}+1}\sum||\mathcal{T}^{k_1}||^2 + \binom{N-2}{\frac{N-2}{2}-2}\frac{2^2}{d^{\frac{N-2}{2}+2}}\sum_{k_1k_2}||\mathcal{T}^{k_1k_2}||^2 \\
& + \cdots + \binom{N-\frac{N}{2}}{0}\frac{2^{\frac{N}{2}}}{d^{2*\frac{N}{2}}}\sum_{k_1\cdots k_{\frac{N}{2}}}||\mathcal{T}^{k_1\cdots k_{\frac{N}{2}}}||^2.
\end{aligned}$$

Assume that $|\psi\rangle \in \mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_N$ be $R_M|\bar{R}_M$ separable, where $R_M=\{\mathcal{H}_{i_1}\otimes\mathcal{H}_{i_2}\otimes\cdots\otimes\mathcal{H}_{i_M}\}$ and $\bar{R}_M$ be the complement of R_M. Without loss of generality, we assume that $1\le M\le[\frac{N}{2}]$. As $|\psi\rangle$ is a pure state, we have $Tr(\rho_{R_M}^2)=1$, namely,

$$\frac{2}{d^{M+1}}\sum_{j\in\{1,2,\cdots M\}}||\mathcal{T}^{i_j}||^2 + \frac{2^2}{d^{M+2}}\sum_{j,l}||\mathcal{T}^{i_ji_l}||^2 + \cdots + \frac{2^M}{d^{2M}}||\mathcal{T}^{i_1\cdots i_M}||^2 = 1-\frac{1}{d^M}.$$

Thus one has that for odd N,

$$\sum_{\alpha=1}^{2^N-2} Tr\rho_\alpha^2 \ge 2[\binom{N}{1}\frac{1}{d} + \binom{N}{2}\frac{1}{d^2} + \cdots + \binom{N}{\frac{N-1}{2}}\frac{1}{d^{\frac{N-1}{2}}}] + 2(1-\frac{1}{d^M}),$$

while for even N,

$$\sum_{\alpha=1}^{2^N-2} Tr\rho_\alpha^2 \ge 2[\binom{N}{1}\frac{1}{d} + \binom{N}{2}\frac{1}{d^2} + \cdots + \binom{N}{\frac{N-2}{2}}\frac{1}{d^{\frac{N-2}{2}}}] + \binom{N}{\frac{N}{2}}\frac{1}{d^{\frac{N}{2}}} + 2(1-\frac{1}{d^M}).$$

Therefore, we have

$$\begin{aligned}
C_N(|\psi\rangle\langle\psi|) = & \; 2^{1-\frac{N}{2}}\sqrt{(2^N-2)-\sum_\alpha Tr\{\rho_\alpha^2\}} \\
\le & \begin{cases} 2^{1-\frac{N}{2}}\sqrt{2^N-4+\frac{2}{d^M}-2\sum_{k=1}^{\frac{N-1}{2}}\binom{N}{k}\frac{1}{d^k}}, & \text{for odd } N; \\ 2^{1-\frac{N}{2}}\sqrt{2^N-4+\frac{2}{d^M}-2\sum_{k=1}^{\frac{N}{2}-1}\binom{N}{k}\frac{1}{d^k}-\binom{N}{\frac{N}{2}}\frac{1}{d^{\frac{N}{2}}}}, & \text{for even } N. \end{cases}
\end{aligned}$$

Let $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ be a mixed state decomposable in terms of an ensemble of biseparable states $|\psi_i\rangle$. By the convexity of the concurrence, $C_N(\rho) \le \sum_i p_i C_N(|\psi_i\rangle\langle\psi_i|)$. Using the previous inequality, we get

$$C_N(\rho) \le \begin{cases} \sum_i p_i 2^{1-\frac{N}{2}} \sqrt{2^N - 4 + \frac{2}{d^{M_i}} - 2\sum_{k=1}^{\frac{N-1}{2}} \binom{N}{k} \frac{1}{d^k}}, & \text{for odd } N; \\ \sum_i p_i 2^{1-\frac{N}{2}} \sqrt{2^N - 4 + \frac{2}{d^{M_i}} - 2\sum_{k=1}^{\frac{N}{2}-1} \binom{N}{k} \frac{1}{d^k} - \binom{N}{\frac{N}{2}} \frac{1}{d^{\frac{N}{2}}}}, & \text{for even } N; \end{cases}$$

$$\le \begin{cases} 2^{1-\frac{N}{2}} \sqrt{2^N - 4 + \frac{2}{d} - 2\sum_{k=1}^{\frac{N-1}{2}} \binom{N}{k} \frac{1}{d^k}}, & \text{for odd } N; \\ 2^{1-\frac{N}{2}} \sqrt{2^N - 4 + \frac{2}{d} - 2\sum_{k=1}^{\frac{N}{2}-1} \binom{N}{k} \frac{1}{d^k} - \binom{N}{\frac{N}{2}} \frac{1}{d^{\frac{N}{2}}}}, & \text{for even } N, \end{cases}$$

where in the last inequality we have used the fact that $\frac{2}{d} \ge \frac{2}{d^M}$ for any $M \ge 1$. ■

Remark 1: The lower bound for the multipartite concurrence presented in the above theorem together with the fact that $C_N(\rho) = 0$ for fully separable states supply a kind of classification for multipartite entanglement which only depends the dimensions and the number of subsystems.

The lower bound (3) of the multipartite concurrence $C_N(\rho)$ of a state ρ presents a sufficient condition for a state to be genuine multipartite entangled. Besides, if we take $N = 3$ and $d = 2$ and consider any bi-separable pure state $|\psi_{123}\rangle = |\psi_{12}\rangle \otimes |\psi_3\rangle$ with $|\psi_{12}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, then the concurrence $C_3(|\psi_{123}\rangle)$ is 1, which is just the maximal value of the bound (3) for any bi-separable states. Thus this bound is tight in this case.

The theorem gives an effective way to detect genuine multipartite entanglement by estimating the multipartite concurrence of a state. Generally, it is difficult to calculate analytically the multipartite concurrence of a given state. Nevertheless, there have been many results on the lower bounds of the multipartite concurrence for mixed states [35–38]. From our theorem these bounds give rise to criteria of the genuine multipartite entanglement. To employ our theorem for detailed applications, we first present a new lower bound of multipartite concurrence in the following.

For a pure N-partite quantum state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, the bipartite concurrence with respect to the bipartite decomposition $\alpha|\bar{\alpha}$ is defined by

$$C_2^\alpha(|\psi\rangle\langle\psi|) = \sqrt{2(1 - Tr\{\rho_\alpha^2\})}, \quad (4)$$

where $\rho_\alpha = Tr_{\bar{\alpha}}\{|\psi\rangle\langle\psi|\}$ is the reduced density matrix of $\rho = |\psi\rangle\langle\psi|$ by tracing over the subsystem $\bar{\alpha}$. For a mixed multipartite quantum state, $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$, the corresponding bipartite concurrence is given by

$$C_2^\alpha(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_2^\alpha(|\psi_i\rangle\langle\psi_i|), \quad (5)$$

where the minimization runs over all ensembles of pure state decompositions of ρ. We have the following results:

Proposition For any mixed multipartite quantum state $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, the multipartite concurrence (2) is bounded by

$$C_N(\rho) \ge 2^{\frac{1-N}{2}} \sqrt{\sum_{\alpha=1}^{2^N-2} (C_2^\alpha(\rho))^2}. \quad (6)$$

Proof We start the proof with a pure state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$. According to the definition of the multipartite concurrence, one obtains that

$$\begin{aligned} C_N(|\psi\rangle\langle\psi|) &= 2^{1-\frac{N}{2}} \sqrt{(2^N - 2) - \sum_{\alpha=1}^{2^N-2} Tr\{\rho_\alpha^2\}} \\ &= 2^{1-\frac{N}{2}} \sqrt{\sum_{\alpha=1}^{2^N-2} x_\alpha^2}, \end{aligned}$$

where we have set $x_\alpha = \sqrt{1 - Tr\{\rho_\alpha^2\}}$.

For any mixed state $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, assume that $\{p_i, |\psi_i\rangle\}$ is the optimal ensemble of pure state decomposition such that $C_N(\rho) = \sum_i p_i C_N(|\psi_i\rangle\langle\psi_i|)$. Using the Minkowski inequality, one derives that

$$
\begin{aligned}
C_N(\rho) &= \sum_i p_i C_N(|\psi_i\rangle\langle\psi_i|) = 2^{1-\frac{N}{2}} \sum_i p_i \sqrt{\sum_{\alpha=1}^{2^N-2} x_{i\alpha}^2} \\
&\geq 2^{1-\frac{N}{2}} \sqrt{\sum_{\alpha=1}^{2^N-2} (\sum_i p_i x_{i\alpha})^2} \\
&\geq 2^{\frac{1-N}{2}} \sqrt{\sum_{\alpha=1}^{2^N-2} (C_2^{\alpha}(\rho))^2},
\end{aligned}
$$

which proves the proposition. ■

Remark 2: (6) is a kind of monogamy inequality [39] for multipartite entanglement in terms of the difference between total entanglement and the bipartite entanglement. Let $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ for instance. (6) is then represented by $(C_3(\rho))^2 \geq (C_2^A(\rho))^2 + (C_2^B(\rho))^2 + (C_2^C(\rho))^2$. Combining the monogamy inequality derived in [39] one has $(C_3(\rho))^2 \geq (C_2(\rho_{AB}))^2 + (C_2(\rho_{AC}))^2 + (C_2(\rho_{BC}))^2$, where ρ_{AB}, ρ_{AC} and ρ_{BC} are the reduced matrices of ρ.

In (6) the lower bound of multipartite concurrence $C_N(\rho)$ is given by the concurrences of bipartite partitions. These bipartite concurrences can be estimated in many operational approaches [27, 37, 38].

By using our theorem and proposition we now investigate the genuine multipartite entanglement by detailed examples. Let us consider the state in three qutrits systems,

$$
\rho_{GGHZ} = \frac{x}{27} I_{27} + (1-x)|GGHZ\rangle\langle GGHZ|,
$$

where $|GGHZ\rangle$ is a generalized GHZ state, $|GGHZ\rangle = (|000\rangle + |111\rangle + |222\rangle)/\sqrt{3}$. By using the lower bounds for bipartite states [27], it is direct to obtain the lower bound (6) of the concurrence $C_3(\rho)$. The genuine multipartite entanglement is then detected for $0 < x < 0.16515$, which is better than the result from the theorem 1 in [5]: $0 < x < 0.10557$ (or that in [17] as they are of the same power for detecting genuine multipartite entanglement of ρ_{GGHZ}).

As another example we consider $\rho_{GHZ} = \frac{x}{8} I_8 + (1 - x)|GHZ\rangle\langle GHZ|$, where $|GHZ\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$. The lower bound in (3) is given by $\sqrt{2 - \frac{2}{d}} = 1$. By using the lower bound for bipartite concurrence in [37], we have $C_3(\rho) \geq \sqrt{\frac{3}{2}\frac{1-3x}{1+3x}}$. From our theorem the genuine multipartite entanglement is detected for $x < 0.033$. If we employ the lower bound of concurrence in [35], genuine multipartite entanglement is detected by our theorem for $x < 0.1468$, which is also better than the range $x < 0.13$ obtained by using the theorem 1 in [5]. One may always enhance the power of detecting genuine multipartite entanglement by employing better lower bounds of multipartite concurrence. Here for the state ρ_{GHZ}, its lower bound of concurrence from [36] is given by $C_3(\rho_{GHZ}) \geq -\frac{1}{2} + \frac{3-3x}{4} + \frac{2-2x+x^2}{4\sqrt{2}}$. Therefore for $0 \leq x \leq 0.190211$, the lower bound from our proposition is better than that from [36] (see Fig. 1).

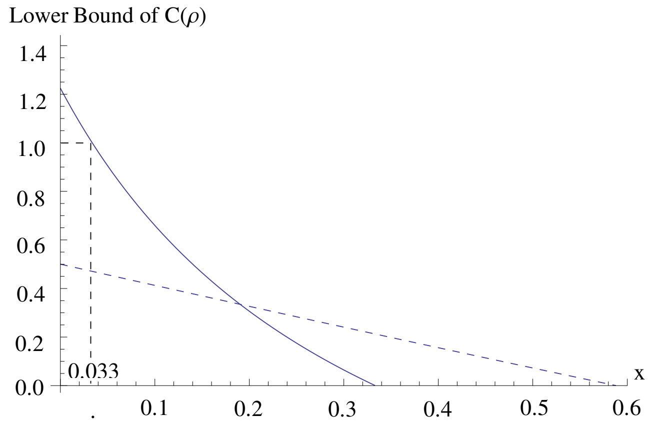

FIG. 1: Lower bound of concurrence from proposition (solid line) and that from [36] (dashed line). The solid line shows that for $x < 0.033$, $C_3(\rho_{GHZ}) > 1$ and the state is genuine multipartite entangled. For $0 \leq x \leq 0.190211$, the lower bound from our proposition supplies a better estimation of concurrence than that from [36].

Let us further consider the Dur-Cirac-Tarrach state [3],

$$
\rho_{DCT} = \sum_{\sigma=\pm} \lambda_0^{\sigma} |\psi_0^{\sigma}\rangle\langle\psi_0^{\sigma}| + \sum_{k=1}^{3} \lambda_k (|\psi_k^+\rangle\langle\psi_k^+| + |\psi_k^-\rangle\langle\psi_k^-|),
$$

where $|\psi_0^{\pm}\rangle = \frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle), |\psi_j^{\pm}\rangle = \frac{1}{\sqrt{2}}(|j\rangle_{AB}|0\rangle_C \pm |(3-j)\rangle_{AB}|0\rangle_C)$, $|j\rangle_{AB} = |j_1\rangle_A|j_2\rangle_B$ with $j = j_1 j_2$ in binary notation. Take $\lambda_0^+ = \frac{1}{6}$, $\lambda_0^- = \frac{1}{2}$, $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{18}$. From Ref. [36] the lower bound of concurrence is given by $C(\rho_{DCT}) \geq 0.3143$, where the difference of a constant factor $\sqrt{2}$ in defining the concurrence for pure states has already been taken into account. From our proposition and using the lower bound for bipartite concurrence in [37], we obtain $C(\rho_{DCT}) \geq 0.3499$. Therefore, the lower bound presented in the proposition is better than the lower bound in Refs. [36] in detecting the full separability of the three-qubit mixed state ρ_{DCT}.

In summary, for tripartite quantum systems, a state $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ is genuine multipartite entangled if $C_3(\rho) > \sqrt{2 - \frac{2}{d}}$. We have the relationship between the property of entanglement and the value of concurrence, see Fig. 2. Here we show the detailed processes of detecting genuine entanglement for arbitrary quantum states $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ by using the theorem and the proposition. One can detect the genuine multipartite entanglement for quantum states in any N partite systems with arbitrary dimensions.

Step 1: Treat $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ in terms of bipartite cuts: $\mathcal{H}_1|\mathcal{H}_2\mathcal{H}_3$, $\mathcal{H}_2|\mathcal{H}_1\mathcal{H}_3$ and $\mathcal{H}_3|\mathcal{H}_1\mathcal{H}_2$. Compute the lower bound of concurrence for "bipartite" quantum state ρ. For the example ρ_{GGHZ}, we have selected the lower bound for concurrence given in [27]. Indeed, any valid lower bound for bipartite concurrence (such as that in [37, 40–42]) is adoptable to detect genuine multipartite entanglement.

Step 2: By the proposition, one can compute the lower bound of concurrence $C_3(\rho)$ (denoted as $LC_3(\rho)$) by summing all the squared lower bounds of "bipartite" concurrence and then taking a square root.

Step 3: Compare $LC_3(\rho)$ derived in the above step (or that has been derived directly from the lower bound of $C_3(\rho)$ such as that in [25, 26, 43]) with the lower bound in the theorem for $N = 3$, i.e. $\sqrt{2-\frac{2}{d}}$. If $LC_3(\rho) > \sqrt{2-\frac{2}{d}}$, genuine multipartite entanglement is detected.

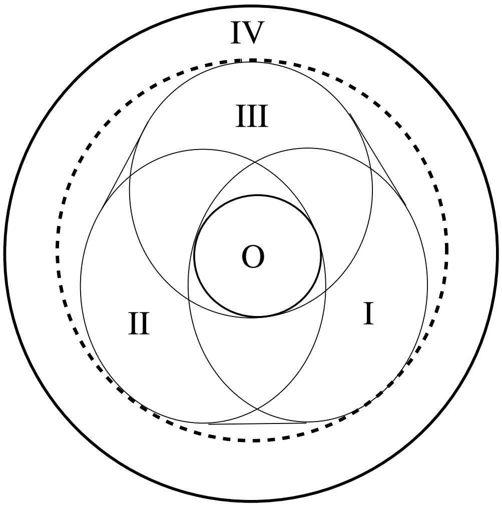

FIG. 2: Quantum states located in the area between the largest circle and the dashed circle are genuine multipartite entangled with concurrence $C(\rho) > \sqrt{2-\frac{2}{d}}$, part IV. Bipartite separable states, $0 < C(\rho) \leq \sqrt{2-\frac{2}{d}}$, in parts I, II, III and their linear superposition. And $C(\rho) = 0$, fully separable states in part 0. The rest undetected part is genuine multipartite entangled with $0 < C(\rho) \leq \sqrt{2-\frac{2}{d}}$.

To detect the genuine multipartite entanglement and measure the multipartite entanglement are basic and fundamental problems in quantum information science. We have investigated the relations between genuine multipartite entanglement and the multipartite concurrence. It has been shown that if the multipartite concurrence is larger than a constant depending only on the dimensions and the number of the subsystems, the state must be genuine multipartite entangled. We have also derived an analytical and effective lower bound of multipartite concurrence, which contributes not only to the detection of genuine multipartite entanglement, but also to the estimation of multipartite entanglement. In [44], the quantum k-separability for multipartite quantum systems have been studied. Our method can be also applied to this issue. Besides, the detection of genuine multipartite entanglement for continuous variable systems [45] may be similarly investigated by bounding the multipartite concurrence.

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