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the Bloch representation of density matrices

by

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Abstract

The correlation matrices or tensors in the Bloch representation of density matrices are encoded with entanglement properties. In this paper, based on the Bloch representation of density matrices, we give some new separability criteria for bipartite and multipartite quantum states. Theoretical analysis and some examples show that the proposed criteria can be more efficient than the previous related criteria.

1 Introduction

Quantum entanglement is a fascinating phenomenon in quantum physics. It can be seen as a physical resource like energy with applications from quantum teleportation to quantum cryptography [1]. In the last years, much work has been devoted to understanding entanglement, but there are still many problems unsolved. One of them is to determine whether a given quantum state is entangled or separable. This problem is extremely difficult to solve, and has been proved as a nondeterministic polynomial-time hard problem [2]. Nevertheless, a variety of operational criteria for separability of quantum states have been proposed in the last decades. Among them are the positive partial transpose (PPT) criterion or Peres-Horodecki criterion [3], realignment criteria [4, 5], covariance matrix criteria [6, 7] and so on; see, e.g., [8] for a comprehensive survey.

The Bloch representation [9, 10] of density matrices stands as an important role in quantum information. The correlation matrices or tensors in the Bloch representation are encoded with entanglement properties [11], which can be exploited to study quantum entanglement. In [12], by making use of correlation matrices, Vicente obtained the correlation matrix criterion for bipartite quantum states, which can be more efficient than the PPT criterion [3] and the computable cross norm or realignment (CCNR) criterion [4] in many different situations. After that, this criterion was used to give the analytical lower bounds for the entanglement measures: concurrence and tangle [13],

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which are good supplement to the lower bounds based on PPT and CCNR criteria. By the matricizations of tensors, the correlation matrix criterion was generalized to detect non-full-separability of multipartite states [14]. Later, this multipartite criterion was extended and improved to be a much more general case [15]. Meanwhile, by the standard tensor norm and the norms of matricizations of tensors, some genuine entanglement conditions were derived. In [11], some simple geometrical methods based on correlation tensors were presented to detect various multipartite entanglement. By bounding tensor norms for partially separable states and states of limited dimension, Klöckl and Huber [16] studied the detection of multipartite entanglement in an experimentally feasible way. In many cases, only few definite measurements are needed. Recently, Li et al. [17] presented some separability criteria under the combination of correlation matrices and the Bloch vectors of reduced density matrices, which can be stronger than the correlation matrix criterion [12] by examples.

This paper is further devoted to an investigation of entanglement detection in terms of Bloch representations of density matrices. On one hand, by adding some parameters, a more general separability criterion for bipartite states is presented, which can outperform the corresponding criteria given in [12, 17]. On the other hand, the presented bipartite separability criterion is extended to the multipartite case. An example shows that the new multipartite separability criterion can be better than the corresponding criteria obtained in [14, 15, 17].

The remainder of the paper is organized as follows. In Section 2, we achieve the new separability criteria for bipartite states. Theoretical analysis and some examples are exploited to illustrate the efficiency of the presented criteria. In Section 3, the new separability criterion obtained in Section 2 is extended to the multipartite case. Meanwhile, an example is used to verify the performance of the proposed criterion. In Section 4, some concluding remarks are given.

2 Separability criteria for bipartite states

Let $\lambda_i^{(d)}, i = 1, 2, \dots, d^2 - 1$ be the traceless Hermitian generators of $SU(d)$ satisfying the orthogonality relation $\text{Tr}(\lambda_i^{(d)} \lambda_j^{(d)}) = 2\delta_{ij}$. Then any state ρ in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ can be represented as [10]

$$\rho = \frac{1}{d_1 d_2} \left(I_{d_1} \otimes I_{d_2} + \sum_{i=1}^{d_1^2-1} r_i \lambda_i^{(d_1)} \otimes I_{d_2} + \sum_{j=1}^{d_2^2-1} s_j I_{d_1} \otimes \lambda_j^{(d_2)} + \sum_{i=1}^{d_1^2-1} \sum_{j=1}^{d_2^2-1} t_{ij} \lambda_i^{(d_1)} \otimes \lambda_j^{(d_2)} \right), \quad (1)$$

where

$$r_i = \frac{d_1}{2} \text{Tr}(\rho \lambda_i^{(d_1)} \otimes I_{d_2}), s_j = \frac{d_2}{2} \text{Tr}(\rho I_{d_1} \otimes \lambda_j^{(d_2)}), t_{ij} = \frac{d_1 d_2}{4} \text{Tr}(\lambda_i^{(d_1)} \otimes \lambda_j^{(d_2)}). \quad (2)$$

Denote by $\|\cdot\|_{\text{tr}}$, $\|\cdot\|_2$ and $E_{p \times q}$ the trace norm (the sum of singular values), the spectral norm (the maximum singular value) and the $p \times q$ matrix with all entries being 1, respectively. By defining $r = (r_1, \dots, r_{d_1^2-1})^T$, $s = (s_1, \dots, s_{d_2^2-1})^T$ and $T = (t_{ij})$,

we construct the following matrix

$$\mathcal{S}_{\alpha,\beta}^m(\rho) = \begin{pmatrix} \alpha\beta E_{m \times m} & \beta\omega_m(s)^T \\ \alpha\omega_m(r) & T \end{pmatrix}, \quad (3)$$

where α and β are nonnegative real numbers, m is a given natural number, and, for any column vector x ,

$$\omega_m(x) = \underbrace{\begin{pmatrix} x & \cdots & x \end{pmatrix}}_{m \text{ columns}}. \quad (4)$$

Using $\mathcal{S}_{\alpha,\beta}^m(\rho)$, we can get the following separability criterion for bipartite states.

Theorem 2.1. *If the state ρ in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ is separable, then*

$$\|\mathcal{S}_{\alpha,\beta}^m(\rho)\|_{\text{tr}} \leq \frac{1}{2} \sqrt{(2m\beta^2 + d_1^2 - d_1)(2m\alpha^2 + d_2^2 - d_2)}. \quad (5)$$

Proof. Since ρ is separable, from [12, (17)], it follows that there exist vectors $u_i \in \mathbb{R}^{d_1-1}$ and $v_j \in \mathbb{R}^{d_2-1}$ such that

$$T = \sum_i p_i u_i u_i^T, r = \sum_i p_i u_i, s = \sum_i p_i v_i, \quad (6)$$

where

$$\|u_i\|_2 = \sqrt{\frac{d_1(d_1-1)}{2}}, \|v_j\|_2 = \sqrt{\frac{d_2(d_2-1)}{2}}, p_i \geq 0, \sum_i p_i = 1. \quad (7)$$

Thus, the matrix $\mathcal{S}_{\alpha,\beta}^m(\rho)$ can be written as

$$\begin{aligned} \mathcal{S}_{\alpha,\beta}^m(\rho) &= \sum_i p_i \begin{pmatrix} \alpha\beta E_{m \times m} & \beta\omega_m(v_i)^T \\ \alpha\omega_m(u_i) & u_i v_i^T \end{pmatrix} \\ &= \sum_i p_i \begin{pmatrix} \beta E_{m \times 1} \\ u_i \end{pmatrix} \begin{pmatrix} \alpha E_{1 \times m} & v_i^T \end{pmatrix} := \sum_i p_i \bar{u}_i \bar{v}_i^T, \end{aligned} \quad (8)$$

and then

$$\begin{aligned} \|\mathcal{S}_{\alpha,\beta}^m(\rho)\|_{\text{tr}} &\leq \sum_i p_i \|\bar{u}_i \bar{v}_i^T\|_{\text{tr}} = \sum_i p_i \|\bar{u}_i\|_2 \|\bar{v}_i\|_2 \\ &= \frac{1}{2} \sqrt{(2m\beta^2 + d_1^2 - d_1)(2m\alpha^2 + d_2^2 - d_2)}, \end{aligned} \quad (9)$$

where we have used the equality, for any vectors $|a\rangle$ and $|b\rangle$,

$$||a\rangle\langle b|||_{\text{tr}} = ||a\rangle\|_2 ||b\rangle\|_2. \quad (10)$$

□

When α and β are chosen to be 0, Theorem 2.1 reduces to the correlation matrix criterion in [12]: if ρ in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ is separable, then

$$\|T\|_{\text{tr}} \leq \frac{1}{2} \sqrt{(d_1^2 - d_1)(d_2^2 - d_2)}. \quad (11)$$

If we choose $\alpha = \beta = m = 1$, then Theorem 2.1 becomes the separability criterion given in [17, Corollary 2]: any separable state ρ in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ must satisfy

$$\left\| \begin{pmatrix} 1 & s^T \\ r & T \end{pmatrix} \right\|_{\text{tr}} \leq \frac{1}{2} \sqrt{(2 + d_1^2 - d_1)(2 + d_2^2 - d_2)}. \quad (12)$$

For simplicity, we call these criteria in (11) and (12) the V-B and L-B criteria, respectively.

For the case $d_1 = d_2$, the following result can help us find separability criteria from Theorem 2.1, that are stronger than the V-B and L-B criteria.

Proposition 2.1. *If $d_1 = d_2$ and $\alpha = \beta$, then Theorem 2.1 becomes stronger when m gets larger.*

Proof. From [12, Lemma 1], it is easy to get, for any state ρ in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$,

$$\|\mathcal{S}_{\alpha,\alpha}^{m+1}(\rho)\|_{\text{tr}} \geq \alpha^2 + \|\mathcal{S}_{\alpha,\alpha}^m(\rho)\|_{\text{tr}}. \quad (13)$$

Hence, if the inequality

$$\|\mathcal{S}_{\alpha,\alpha}^{m+1}(\rho)\|_{\text{tr}} \leq \frac{1}{2}(2(m+1)\alpha^2 + d_1^2 - d_1) \quad (14)$$

holds, then, by (13),

$$\|\mathcal{S}_{\alpha,\alpha}^m(\rho)\|_{\text{tr}} \leq \frac{1}{2}(2(m+1)\alpha^2 + d_1^2 - d_1) - \alpha^2 = \frac{1}{2}(2m\alpha^2 + d_1^2 - d_1). \quad (15)$$

The proof is completed. \square

For the case $d_1 = d_2$, it follows from Proposition 2.1 that Theorem 2.1 with $\alpha = \beta$ is more efficient when m gets larger. In particular, the L-B criterion is better than the V-B criterion, and Theorem 2.1 with $\alpha = \beta = 1$ and $m \geq 2$ is stronger than the L-B criterion. Thus, Theorem 2.1 is the best one among them.

However, for the case $d_1 \neq d_2$, the following example shows that Theorem 2.1 with $\alpha = \beta$ may be weaker than the V-B criterion.

Example 2.1. The following 2×4 bound entangled state is due to [18]:

$$\rho = \frac{1}{7b+1} \begin{pmatrix} b & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1+b) & 0 & 0 & \frac{1}{2}\sqrt{1-b^2} \\ b & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & \frac{1}{2}\sqrt{1-b^2} & 0 & 0 & \frac{1}{2}(1+b) \end{pmatrix}, \quad (16)$$

where $0 < b < 1$. To verify the efficiency of the presented criteria, we consider the state

$$\rho_x = x|\xi\rangle\langle\xi| + (1-x)\rho, \quad (17)$$

where $|\xi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The V-B criterion, L-B criterion and Theorem 2.1 with $\alpha = \beta = 1, m = 2$ can detect the entanglement in ρ_x for $0.2293 \leq x \leq 1$, $0.2841 \leq x \leq 1$ and $0.3181 \leq x \leq 1$, respectively. Thus, the V-B criterion is most efficient among them.

Nonetheless, by special selection of α and β , Theorem 2.1 with $m \geq 1$ can be always stronger than the V-B criterion from the following proposition.

Proposition 2.2. *If α and β are selected to satisfy*

$$\alpha\sqrt{d_1(d_1-1)} = \beta\sqrt{d_2(d_2-1)}, \quad (18)$$

then Theorem 2.1 becomes stronger when m gets larger.

Proof. For any state ρ , from [12, Lemma 1], we get

$$\|\mathcal{S}_{\alpha,\beta}^{m+1}(\rho)\|_{\text{tr}} \geq \alpha\beta + \|\mathcal{S}_{\alpha,\beta}^m(\rho)\|_{\text{tr}}. \quad (19)$$

If the inequality

$$\|\mathcal{S}_{\alpha,\beta}^{m+1}(\rho)\|_{\text{tr}} \leq \frac{1}{2}\sqrt{(2(m+1)\beta^2 + d_1^2 - d_1)(2(m+1)\alpha^2 + d_2^2 - d_2)} \quad (20)$$

holds, then, from (19),

$$\begin{aligned} \|\mathcal{S}_{\alpha,\beta}^m(\rho)\|_{\text{tr}} &\leq \frac{1}{2}\sqrt{(2(m+1)\beta^2 + d_1^2 - d_1)(2(m+1)\alpha^2 + d_2^2 - d_2)} - \alpha\beta \\ &= \left\| \begin{pmatrix} \beta E_{(m+1) \times 1} \\ \sqrt{\frac{d_1(d_1-1)}{2}} \end{pmatrix} \begin{pmatrix} \alpha E_{1 \times (m+1)} & \sqrt{\frac{d_2(d_2-1)}{2}} \end{pmatrix} \right\|_{\text{tr}} - \alpha\beta \\ &= \left\| \begin{pmatrix} \alpha\beta E_{(m+1) \times (m+1)} & \beta\sqrt{\frac{d_2(d_2-1)}{2}} E_{(m+1) \times 1} \\ \alpha\sqrt{\frac{d_1(d_1-1)}{2}} E_{1 \times (m+1)} & \sqrt{\frac{d_1 d_2 (d_1-1)(d_2-1)}{4}} \end{pmatrix} \right\|_{\text{tr}} - \alpha\beta \\ &= (m+1)\alpha\beta + \sqrt{\frac{d_1 d_2 (d_1-1)(d_2-1)}{4}} - \alpha\beta \\ &= \left\| \begin{pmatrix} \alpha\beta E_{m \times m} & \beta\sqrt{\frac{d_2(d_2-1)}{2}} E_{(m+1) \times 1} \\ \alpha\sqrt{\frac{d_1(d_1-1)}{2}} E_{1 \times (m+1)} & \sqrt{\frac{d_1 d_2 (d_1-1)(d_2-1)}{4}} \end{pmatrix} \right\|_{\text{tr}} \\ &= \frac{1}{2}\sqrt{(2m\beta^2 + d_1^2 - d_1)(2m\alpha^2 + d_2^2 - d_2)}, \end{aligned} \quad (21)$$

where the equality (10) has been used in the first and fifth equalities, and, in the third and fourth equalities, we have employed the fact that the trace norm of a Hermitian positive semidefinite matrix is equal to its trace. \square

From Proposition 2.2, Theorem 2.1 with the condition (18) is stronger than the V-B criterion. Let us go back to Example 2.1. For simplicity, if we choose

$$\alpha = \sqrt{\frac{2}{d_1(d_1 - 1)}}, \beta = \sqrt{\frac{2}{d_2(d_2 - 1)}}, m = 1, \quad (22)$$

then Theorem 2.1 can detect the entanglement in ρ_x for $0.2235 \leq x \leq 1$. Thus, Theorem 2.1 is better than the V-B and L-B criteria.

3 Separability criteria for multipartite states

Let \mathcal{S} be a $f_1 \times \cdots \times f_N$ tensor, A and \bar{A} be two nonempty subsets of $\{1, \dots, N\}$ satisfying $A \cup \bar{A} = \{1, \dots, N\}$. Then we denote by $\mathcal{S}^{A|\bar{A}}$ the A, \bar{A} matricization of \mathcal{S} ; see [15] for a detail. This matricization is a generalization of mode- n matricization in the multilinear algebra [19].

For any state ρ in $\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_N}$, we import a natural number m and nonnegative real parameters $\alpha_1, \dots, \alpha_N$, and define

$$\delta_{k_i}^{(d_i)} = \begin{cases} \frac{2\alpha_i}{d_i} I_{d_i}, & 1 \leq k_i \leq m \\ \lambda_{k_i-m}^{(d_i)}, & m+1 \leq k_i \leq d_i^2 + m - 1 \end{cases}, i = 1, \dots, N. \quad (23)$$

The tensor used in this section is $\mathcal{W}_{\alpha_1, \dots, \alpha_N}^{(m)}$ with elements

$$w_{k_1 \dots k_N} = \frac{d_1 \cdots d_N}{2^N} \text{Tr}(\rho \delta_{k_1}^{(d_1)} \otimes \cdots \otimes \delta_{k_N}^{(d_N)}), 1 \leq k_i \leq d_i^2 + m - 1. \quad (24)$$

Clearly, if $m = 0$, the tensor $\mathcal{W}_{\alpha_1, \dots, \alpha_N}^{(m)}$ reduces to the correlation tensor in [14]. When $m = \alpha_1 = \cdots = \alpha_N = 1$, the tensor $\mathcal{W}_{\alpha_1, \dots, \alpha_N}^{(m)}$ becomes the tensor with a constant multiple in [17].

An n partite state ρ in $\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$ is separable (or fully separable) [20] if and only if it can be written in the form

$$\rho = \sum_i p_i \rho_i^1 \otimes \cdots \otimes \rho_i^n, \quad (25)$$

where the probabilities $p_i > 0$, $\sum_i p_i = 1$, and $\rho_i^1, \dots, \rho_i^n$ are pure states of subsystems.

In the following, we give the full separability criterion based on $\mathcal{W}_{\alpha_1, \dots, \alpha_N}^{(m)}$.

Theorem 3.1. *If the state ρ in $\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_N}$ is fully separable, then, for any subset A of $\{1, \dots, N\}$, we have*

$$\left\| \left(\mathcal{W}_{\alpha_1, \dots, \alpha_N}^{(m)}(\rho) \right)^{A|\bar{A}} \right\|_{\text{tr}} \leq \prod_{k=1}^N \sqrt{\frac{1}{2}(2m\alpha_k^2 + d_k^2 - d_k)}. \quad (26)$$

Proof. Without loss of generality, we assume that

$$A = \{q_1, \dots, q_M\}, q_1 < \cdots < q_M, \quad (27)$$

$$\bar{A} = \{q_{M+1}, \dots, q_N\}, q_{M+1} < \cdots < q_N. \quad (28)$$

Since ρ is fully separable, then from [14] there exist vectors $u_i^{(k)} \in \mathbb{R}^{d_k^2-1}$ such that

$$\mathcal{W}_{\alpha_1, \dots, \alpha_N}^{(m)}(\rho) = \sum_i p_i \begin{pmatrix} \alpha_1 E_{m \times 1} \\ u_i^{(1)} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} \alpha_N E_{m \times 1} \\ u_i^{(N)} \end{pmatrix}, \quad (29)$$

where

$$\|u_i^{(k)}\|_2 = \sqrt{\frac{d_k(d_k - 1)}{2}}. \quad (30)$$

Thus,

$$\begin{aligned} \left\| \left(\mathcal{W}_{\alpha_1, \dots, \alpha_N}^{(m)}(\rho) \right)^{A|\bar{A}} \right\|_{\text{tr}} &= \left\| \sum_i p_i \bigotimes_{l=1}^M \begin{pmatrix} \alpha_{q_l} E_{m \times 1} \\ u_i^{(q_l)} \end{pmatrix} \bigotimes_{p=M+1}^N \begin{pmatrix} \alpha_{q_p} E_{m \times 1} \\ u_i^{(q_p)} \end{pmatrix} \right\|_{\text{tr}}^T \\ &\leq \sum_i p_i \left\| \bigotimes_{l=1}^M \begin{pmatrix} \alpha_{q_l} E_{m \times 1} \\ u_i^{(q_l)} \end{pmatrix} \bigotimes_{p=M+1}^N \begin{pmatrix} \alpha_{q_p} E_{m \times 1} \\ u_i^{(q_p)} \end{pmatrix} \right\|_{\text{tr}}^T \\ &= \sum_i p_i \left\| \bigotimes_{l=1}^M \begin{pmatrix} \alpha_{q_l} E_{m \times 1} \\ u_i^{(q_l)} \end{pmatrix} \right\|_2 \left\| \bigotimes_{p=M+1}^N \begin{pmatrix} \alpha_{q_p} E_{m \times 1} \\ u_i^{(q_p)} \end{pmatrix} \right\|_2 \\ &= \prod_{k=1}^N \sqrt{\frac{1}{2}(2m\alpha_k^2 + d_k^2 - d_k)}, \end{aligned} \quad (31)$$

where we have used the equality (10). \square

For the case $\alpha_1 = \cdots = \alpha_N = 0$, it is easy to get that Theorem 3.1 reduces to the criterion given in [15, Theorem 4], which has an important improvement on the corresponding criterion given in [14]. If $\alpha_1 = \cdots = \alpha_N = 1$ and $m = 1$, then Theorem 3.1 becomes [17, Corollary 3]. For simplicity, these criteria in [15], [14] and [17] are called the V-M, H-M and L-M criteria, respectively.

We now give a tripartite example to verify the efficiency of Theorem 3.1. In the tripartite case, the V-M criterion is equivalent to the H-M criterion obviously. By our calculations, we find that $m = 1$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0.1$ can often lead to relatively strong results.

Example 3.1. Consider a perturbation of the tripartite GHZ state [7]:

$$|\phi'_{GHZ}\rangle = \frac{1}{\gamma}(|000\rangle + \epsilon|110\rangle + |111\rangle), \quad (32)$$

where ϵ is a given real parameter, and γ denotes the normalization. We consider the mixture of this state with the maximally mixed state:

$$\rho_{GHZ'}^x = \frac{1-x}{8}I_8 + x|\phi'_{GHZ}\rangle\langle\phi'_{GHZ}|. \quad (33)$$

Table 1 displays the detection results with different values of ϵ . Clearly, Theorem 3.1 is more efficient than the V-M, H-M and L-M criteria.

ϵ	V-M (H-M) criteria	L-M criteria	Theorem 3.1
0	$0.3536 \leq x \leq 1$	$0.4118 \leq x \leq 1$	$0.3307 \leq x \leq 1$
10^{-5}	$0.3536 \leq x \leq 1$	$0.4118 \leq x \leq 1$	$0.3307 \leq x \leq 1$
10^{-1}	$0.3424 \leq x \leq 1$	$0.4118 \leq x \leq 1$	$0.3281 \leq x \leq 1$
1	$0.3274 \leq x \leq 1$	$0.4256 \leq x \leq 1$	$0.3243 \leq x \leq 1$

Table 1: *Entanglement conditions of ρ_{GHZ}^x with different values of ϵ from the V-M (H-M) criterion, the L-M criterion and Theorem 3.1 with $\alpha_1 = \alpha_2 = \alpha_3 = 0.1$ and $m = 1$.*

4 Conclusion

Correlation matrices or tensors in the Bloch representation of quantum states contain entanglement properties. In this paper, based on the Bloch representation of quantum states, we give some new separability criteria including the V-B, L-B, V-M, H-M and L-M criteria as special cases. For bipartite cases, by choosing some special involved parameters, the presented criteria are stronger than the V-B and L-B criteria. For multipartite cases, a state as an example shows that the presented criterion can be more efficient than the V-M, H-M and L-M criteria.

Appendix.

In [17], the authors also presented some other separability criteria [17, Theorems 1-2] for bipartite and multipartite states by using Bloch vectors and correlation matrices or tensors. We now show that these criteria at most as good as the corresponding V-B, L-B, V-M and L-M criteria, respectively.

For example, if we set $\tilde{T} = (\tilde{t}_{kl}) = \mathcal{S}_{1,1}^1(\rho)$, it was shown by [17, Theorem 1] that any separable state ρ in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ satisfies

$$\left| \sum_{kl} m_{kl} \tilde{t}_{kl} \right| \leq \frac{1}{2} \sqrt{(2 + d_1^2 - d_1)(2 + d_2^2 - d_2)} \|M\|_2, \quad (34)$$

where $M = (m_{ij})$ is any real $d_1^2 \otimes d_2^2$ matrix. From (34) and [21, Page 21], we get

$$\max_{M \neq 0} \frac{\left| \sum_{kl} m_{kl} \tilde{t}_{kl} \right|}{\|M\|_2} = \max_{M \neq 0} \frac{|\text{Tr}(M^T \tilde{T})|}{\|M\|_2} = \|\tilde{T}\|_{\text{tr}} \leq \frac{1}{2} \sqrt{(2 + d_1^2 - d_1)(2 + d_2^2 - d_2)}, \quad (35)$$

which implies that the L-B criterion is at least as good as the criterion (34). Other cases can be proved similarly.

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