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by

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Abstract

We present an analytical lower bound of multipartite concurrence based on the generalized Bloch representation of density matrices. It is shown that the lower bound can be used as an effective entanglement witness of genuine multipartite entanglement. Tight lower and upper bounds for multipartite tangles are also derived.

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I. INTRODUCTION

Quantum entanglement, as the remarkable nonlocal feature of quantum mechanics, is recognized as a valuable resource in the rapidly expanding field of quantum information science, with various applications [1, 2] such as quantum computation , quantum teleportation, dense coding, quantum cryptographic schemes, quantum radar, entanglement swapping and remote states preparation.

Quantum states without entanglement are called separable states, which constitute a convex subset of all the quantum states. States that are not biseparable with respect to any partitions are said to be genuinely multipartite entangled. Genuinely multipartite entanglement is a kind of important type of entanglement, which offers significant advantage in quantum information processing tasks [3]. In particular, it is the basic ingredient in measurement-based quantum computation [4], and is beneficial in various quantum communication protocols [5], including secret sharing [6] (cf. [7]). Despite its importance, characterization and detection of this kind of resource turn out to be rather hard and only a few results have been proposed [8–11].

Quantifying quantum entanglement is a basic and longer standing problem in quantum information theory. A measure of quantum entanglement can be used to detect and classify entanglement of quantum states. In this paper, we use the multipartite concurrence [12] to investigate the multipartite entanglement. Let $\mathcal{H}_i$, $i = 1, 2, \cdots, N$, be d-dimensional vector spaces. The concurrence of a N-partite pure state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ is defined by

$$C_N(|\psi\rangle\langle\psi|) = 2^{1-\frac{N}{2}} \sqrt{(2^N - 2) - \sum_{\alpha} Tr\{\rho_\alpha^2\}}, \tag{1}$$

where α labels all the different reduced density matrices. If we list all the $2^N - 2$ reduced matrices in the following way: $\{\rho_1, \rho_2, \cdots, \rho_N, \rho_{12}, \rho_{13}, \cdots, \rho_{1N}, \rho_{23}, \cdots, \rho_{12\cdots N-1}, \cdots, \rho_{23\cdots N}\}$, (1) can be reexpressed as

$$C_N(|\psi\rangle\langle\psi|) = 2^{1-\frac{N}{2}} \sqrt{(2^N - 2) - 2 \sum_{k=1}^{2^{N-1}-1} Tr\{\rho_k^2\}}. \tag{2}$$

For a mixed multipartite quantum state, $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, the corresponding concurrence of (1) is given by the convex roof:

$$C_N(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_N(|\psi_i\rangle\langle\psi_i|). \tag{3}$$

The correlation tensors of the generalized Bloch representation of a quantum state paly significant roles in quantum information theory. In [13–16], separable conditions for both bi- and multi-partite quantum states are introduced by studying the norm of the correlation tensors. In [17, 18], the authors present a multipartite entanglement measure for N-qubit and N-qudit pure states, using the norm of the correlation tensors. In [9], the authors have introduced a general framework for detecting genuine multipartite entanglement and non full separability in multipartite quantum systems of arbitrary dimensions based also on the correlation tensors. In [19], we have found that the norms of the correlation tensors are closely related to the maximal violation of a kind of multipartite (multi-setting ?) Bell inequalities.

In the following, we first reform the concurrence for multipartite pure states in terms of the norms of the correlation tensors. The correlation tensors are then used to derive a lower bound of concurrence for mixed multipartite quantum states. The lower bound also provides a fully separable condition for multipartite quantum states. We further show that genuine multipartite entanglement can be detected by the bound. We also investigate the multipartite tangle. Tight lower and upper bounds are derived.

II. LOWER BOUND OF MULTIPARTITE CONCURRENCE

We first consider the concurrence of multipartite pure states $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ in terms of the generalized Bloch representation of $|\psi\rangle\langle\psi|$. Let $\{\lambda_{\alpha_k}\}$ be the $SU(d)$ generators.

The generalized Bloch representation for any quantum states $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ is given by [15],

$$
\begin{aligned}
\rho \;=\; & \frac{1}{d^N}\left(\otimes_{j=1}^{N} I_d + \sum_{\{\mu_1\}}\sum_{\alpha_1} \mathcal{T}_{\alpha_1}^{\{\mu_1\}} \lambda_{\alpha_1}^{\{\mu_1\}} + \sum_{\{\mu_1\mu_2\}}\sum_{\alpha_1\alpha_2} \mathcal{T}_{\alpha_1\alpha_2}^{\{\mu_1\mu_2\}} \lambda_{\alpha_1}^{\{\mu_1\}} \lambda_{\alpha_2}^{\{\mu_2\}} \right. \\
& + \sum_{\{\mu_1\mu_2\mu_3\}} \sum_{\alpha_1\alpha_2\alpha_3} \mathcal{T}_{\alpha_1\alpha_2\alpha_3}^{\{\mu_1\mu_2\mu_3\}} \lambda_{\alpha_1}^{\{\mu_1\}} \lambda_{\alpha_2}^{\{\mu_2\}} \lambda_{\alpha_3}^{\{\mu_3\}} \\
& + \cdots + \sum_{\{\mu_1\mu_2\cdots\mu_M\}} \sum_{\alpha_1\alpha_2\cdots\alpha_M} \mathcal{T}_{\alpha_1\alpha_2\cdots\alpha_M}^{\{\mu_1\mu_2\cdots\mu_M\}} \lambda_{\alpha_1}^{\{\mu_1\}} \lambda_{\alpha_2}^{\{\mu_2\}} \cdots \lambda_{\alpha_M}^{\{\mu_M\}} \\
& \left. + \cdots + \sum_{\alpha_1\alpha_2\cdots\alpha_N} \mathcal{T}_{\alpha_1\alpha_2\cdots\alpha_N}^{\{1,2,\cdots,N\}} \lambda_{\alpha_1}^{\{1\}} \lambda_{\alpha_2}^{\{2\}} \cdots \lambda_{\alpha_N}^{\{N\}} \right),
\end{aligned} \tag{4}
$$

where $\{\mu_1\mu_2\cdots\mu_M\}$ is a subset of $\{1, 2, \cdots, N\}$, $\lambda_{\alpha_k}^{\{\mu_k\}} = I_d \otimes I_d \otimes \cdots \otimes \lambda_{\alpha_k} \otimes I_d \otimes \cdots \otimes I_d$ with λ_{α_k} appearing at the μ_kth position and

$$
\mathcal{T}_{\alpha_1\alpha_2\cdots\alpha_M}^{\{\mu_1\mu_2\cdots\mu_M\}} = \frac{d^M}{2^M} \mathrm{Tr}[\rho \lambda_{\alpha_1}^{\{\mu_1\}} \lambda_{\alpha_2}^{\{\mu_2\}} \cdots \lambda_{\alpha_M}^{\{\mu_M\}}], \tag{5}
$$

which can be viewed as the entries of the tensors $\mathcal{T}^{\{\mu_1\mu_2\cdots\mu_M\}}$.

After some algebraic calculations, we have

$$
\begin{aligned}
\mathrm{Tr}\rho^2 \;=\; & \frac{1}{d^{2N}} (d^N + 2d^{N-1} \sum_{k_1 \in \{1,2,\cdots N\}} ||\mathcal{T}^{k_1}||^2 + 2^2 d^{N-2} \sum_{k_1 k_2} ||\mathcal{T}^{k_1 k_2}||^2 + \cdots \\
& + 2^M d^{N-M} \sum_{k_1\cdots k_M} ||\mathcal{T}^{k_1\cdots k_M}||^2 + \cdots + 2^N ||\mathcal{T}^{1\cdots N}||^2)
\end{aligned} \tag{6}
$$

and

$$
\begin{aligned}
\mathrm{Tr}\rho_{k_1\cdots k_M}^2 \;=\; & \frac{1}{d^{2M}} (d^M + 2d^{M-1} \sum_{j \in \{1,2,\cdots M\}} ||\mathcal{T}^{k_j}||^2 + \cdots + 2^2 d^{M-2} \sum_{j,l} ||\mathcal{T}^{k_j k_l}||^2 \\
& + \cdots + 2^M ||\mathcal{T}^{k_1\cdots k_M}||^2)
\end{aligned} \tag{7}
$$

for any $1 \leq M \leq N-1$, where $\rho_{k_1\cdots k_M}$ is the M-partite reduced density matrix supporting on $\mathcal{H}_{k_1} \otimes \mathcal{H}_{k_2} \otimes \cdots \otimes \mathcal{H}_{k_M}$.

For any $1 \leq M \leq N-1$, we get

$$\begin{aligned}
&\sum_{k_1,k_2,\cdots,k_M} (\mathrm{Tr}\rho^2 - \mathrm{Tr}\rho^2_{k_1\cdots k_M}) \\
&= C_N^M(\frac{d^N}{d^{2N}} - \frac{d^M}{d^{2M}}) + (C_N^M \frac{2d^{N-1}}{d^{2N}} - C_{N-1}^{M-1}\frac{2d^{M-1}}{d^{2M}}) \sum_{k_1\in\{1,2,\cdots N\}} ||\mathcal{T}^{k_1}||^2 \\
&+(C_N^M \frac{2^2 d^{N-2}}{d^{2N}} - C_{N-2}^{M-2}\frac{2^2 d^{M-2}}{d^{2M}}) \sum_{k_1 k_2} ||\mathcal{T}^{k_1 k_2}||^2 + \cdots \\
&+(C_N^M \frac{2^l d^{N-l}}{d^{2N}} - C_{N-l}^{M-l}\frac{2^l d^{M-l}}{d^{2M}}) \sum_{k_1 k_2\cdots k_l} ||\mathcal{T}^{k_1 k_2\cdots k_l}||^2 + \cdots \\
&+(C_N^M \frac{2^M d^{N-M}}{d^{2N}} - C_{N-M}^{0}\frac{2^M}{d^{M}}) \sum_{k_1 k_2\cdots k_M} ||\mathcal{T}^{k_1 k_2\cdots k_M}||^2 \\
&+C_N^M \frac{2^{M+1} d^{N-M+1}}{d^{2N}} \sum_{k_1 k_2\cdots k_{M+1}} ||\mathcal{T}^{k_1 k_2\cdots k_{M+1}}||^2 + \cdots \\
&+C_N^M \frac{2^N}{d^{2N}} ||\mathcal{T}^{1\cdots N}||^2,
\end{aligned}$$

where $C_N^M = M!/N!(M-N)!$.

By substituting the equation above into (1), for pure states $|\psi\rangle$ we have

$$\begin{aligned}
&2^{N-2} C_N^2(|\psi\rangle\langle\psi|) \\
&= \frac{1}{d^N}[-(d+1)^N + d^N + 2^N - 1] + \frac{2}{d^{N+1}}[-(d+1)^{N-1} + 2^N - 1] \sum_{k_1\in\{1,2,\cdots N\}} ||\mathcal{T}^{k_1}||^2 \\
&+ \frac{2^2}{d^{N+2}}[-(d+1)^{N-2} + 2^N - 1] \sum_{k_1 k_2} ||\mathcal{T}^{k_1 k_2}||^2 + \cdots \\
&+ \frac{2^M}{d^{N+M}}[-(d+1)^{N-M} + 2^N - 1] \sum_{k_1 k_2\cdots k_M} ||\mathcal{T}^{k_1 k_2\cdots k_M}||^2 + \cdots \\
&+ \frac{2^{N-1}}{d^{2N-1}}[-(d+1) + 2^N - 1] \sum_{k_1 k_2\cdots k_{N-1}} ||\mathcal{T}^{k_1 k_2\cdots k_{N-1}}||^2 \\
&+ \frac{(2^N-2)2^N}{d^2 N} ||\mathcal{T}^{1\cdots N}||^2.
\end{aligned} \tag{8}$$

Since $\mathrm{Tr}\rho^2 = 1$ for any pure state $\rho = |\psi\rangle\langle\psi|$, from (6) we have

$$\begin{aligned}
\sum_{k_1\in\{1,2,\cdots N\}} ||\mathcal{T}^{k_1}||^2 \;=\; & \frac{d^{2N} - d^N}{2d^{N-1}} - \frac{2^2 d^{N-2}}{2d^{N-1}} \sum_{k_1 k_2} ||\mathcal{T}^{k_1 k_2}||^2 - \cdots \\
& - \frac{2^M d^{N-M}}{2d^{N-1}} \sum_{k_1\cdots k_M} ||\mathcal{T}^{k_1\cdots k_M}||^2 - \cdots - \frac{2^N}{2d^{N-1}} ||\mathcal{T}^{1\cdots N}||^2.
\end{aligned} \tag{9}$$

Substituting (9) into (8), we obtain an alternative relation about the the concurrence,

$$
\begin{aligned}
2^{N-2}C_N^2(|\psi\rangle\langle\psi|) \; = \; & [2^N - \frac{(d+1)^N}{d^N} - \frac{1}{d^N}(d^N-1)(d+1)^{N-1}] \\
& + \frac{2^2}{d^{N+2}}[(d+1)^{N-1} - (d+1)^{N-2}]\sum_{k_1k_2}||\mathcal{T}^{k_1k_2}||^2 + \cdots \\
& + \frac{2^M}{d^{N+M}}[(d+1)^{N-1} - (d+1)^{N-M}]\sum_{k_1\cdots k_M}||\mathcal{T}^{k_1\cdots k_M}||^2 + \cdots \\
& + \frac{2^N}{d^{2N}}[(d+1)^{N-1} - 1]||\mathcal{T}^{1\cdots N}||^2. \qquad (10)
\end{aligned}
$$

Formula (10) gives a sufficient and necessary condition for the fully separability of multipartite pure states. In particular, as the tensors $\mathcal{T}^{\{\mu_1\mu_2\cdots\mu_M\}}$ in (4) are mean values of the observables $\lambda_{\alpha_1}^{\{\mu_1\}}\lambda_{\alpha_2}^{\{\mu_2\}}\cdots\lambda_{\alpha_M}^{\{\mu_M\}}$, (10) also gives an experimental way to measure the concurrence of a pure multipartite state. From (10) we can now derive the lower bound for multipartite concurrence of any mixed states ρ.

Theorem 1: For any mixed quantum state $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, we have

$$
\begin{aligned}
C_N(\rho) \; \geq \; & -2^{1-N/2}[-2^N + \frac{(d+1)^N}{d^N} + \frac{1}{d^N}(d^N-1)(d+1)^{N-1}]^{\frac{1}{2}} \\
& + 2^{1-N/2}\{\frac{2^2}{d^{N+2}}[(d+1)^{N-1} - (d+1)^{N-2}]\sum_{k_1k_2}||\mathcal{T}^{k_1k_2}||^2 + \cdots \\
& + \frac{2^M}{d^{N+M}}[(d+1)^{N-1} - (d+1)^{N-M}]\sum_{k_1\cdots k_M}||\mathcal{T}^{k_1\cdots k_M}||^2 + \cdots \\
& + \frac{2^N}{d^{2N}}[(d+1)^{N-1} - 1]||\mathcal{T}^{1\cdots N}||^2\}^{\frac{1}{2}}. \qquad (11)
\end{aligned}
$$

Proof: For simplicity we denote $C = -2^N + \frac{(d+1)^N}{d^N} + \frac{1}{d^N}(d^N-1)(d+1)^{N-1}$, and C_α the coefficient of $||\mathcal{T}^\alpha||^2$ in (10) for $\alpha \in \{k_1k_2, k_1k_2k_3, \cdots, 1\cdots N\}$, which are nonnegative numbers depending only on N and d.

Assume that $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ is the optimal decomposition such that (3) attains the minimum. We have that

$$
\begin{aligned}
C_N(\rho) \; = \; & \sum_i p_i C_N(|\psi_i\rangle) = 2^{1-N/2}\sum_i p_i\{-C + \sum_\alpha C_\alpha||\mathcal{T}_i^\alpha||^2\}^{\frac{1}{2}} \\
\geq \; & 2^{1-N/2}[\sum_i p_i\sqrt{\sum_\alpha C_\alpha||\mathcal{T}_i^\alpha||^2} - \sqrt{C}] \\
\geq \; & 2^{1-N/2}[\sqrt{\sum_\alpha C_\alpha(\sum_i p_i||\mathcal{T}_i^\alpha||)^2} - \sqrt{C}] \\
\geq \; & 2^{1-N/2}[\sqrt{\sum_\alpha C_\alpha||\mathcal{T}^\alpha||^2} - \sqrt{C}],
\end{aligned}
$$

where we have used the inequalities $\sqrt{a-b} \geq \sqrt{a} - \sqrt{b}$ for $a \geq b \geq 0$ and $\sum_i \sqrt{\sum_j x_{ij}^2} \geq \sqrt{\sum_j (\sum_i x_{ij})^2}$ for real and nonnegative x_{ij}. ■

The lower bound (11) can be used to estimate the concurrence for multipartite quantum states with arbitrary dimension. It is also a kind of entanglement witness for fully separability. Moreover, this multipartite concurrence can be employed to detect the genuine multipartite entanglement. It has been shown that an N-partite quantum state $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ is genuine multipartite entangled if [20]

$$C_N(\rho) > \begin{cases} 2^{1-\frac{N}{2}} \sqrt{2^N - 4 + \dfrac{2}{d} - 2 \displaystyle\sum_{k=1}^{\frac{N-1}{2}} \dfrac{C_N^k}{d^k}}, & \text{for odd } N, \\ 2^{1-\frac{N}{2}} \sqrt{2^N - 4 + \dfrac{2}{d} - 2 \displaystyle\sum_{k=1}^{\frac{N}{2}-1} \dfrac{C_N^k}{d^k} - \dfrac{C_N^{\frac{N}{2}}}{d^{\frac{N}{2}}}}, & \text{for even } N. \end{cases}$$

Since the concurrence $C_N(\rho)$ is difficult to compute, our lower bound can employed to detect the genuine multipartite entanglement.

As an example, let us consider tripartite case. From (12) $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ is genuinely multipartite entangled if $C_3(\rho) > \sqrt{2 - \frac{2}{d}}$. For a three-qubit GHZ state mixed with noise, $\rho_{GHZ} = \frac{x}{8} I + (1-x)|GHZ\rangle\langle GHZ|$, where $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, we have from Theorem 1, $C_3(\rho_{GHZ}) \geq$???. Therefore, ρ_{GHZ} is genuinely multipartite entangled for $x < 0.0496$.

III. BOUNDS ON MULTIPARTITE TANGLE

We now consider the multipartite tangle that is tightly related concurrence. From the squared I-concurrence for bipartite quantum systems [21], we introduce the multipartite squared I-concurrence. For a multipartite pure quantum state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ the multipartite squared I-concurrence is defined by the square of the multi-concurrence,

$$\tau_N(|\psi\rangle\langle\psi|) = C_N^2(|\psi\rangle\langle\psi|) = 2^{2-N}[(2^N - 2) - \sum_\alpha Tr\{\rho_\alpha^2\}], \tag{12}$$

where α labels all the different reduced density matrices. For a mixed multipartite quantum state, $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, the corresponding multipartite squared I-concurrence is then given by the convex roof:

$$\tau_N(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i \tau_N(|\psi_i\rangle\langle\psi_i|). \tag{13}$$

Such defined multipartite squared I-concurrence has the following properties: (i) $\tau_N(\rho) = 0$ if and only if ρ is fully separable; (ii) $\tau_N(\rho)$ is invariant under local unitary transformation of ρ; (iii) $\tau_N(\rho) \geq C_N^2(\rho)$. By property (i) above, a multipartite state in not separable if $\tau_N(\rho) > 0$. In the following, we present valid lower and upper bounds for $\tau_N(\rho)$.

Theorem 2: For any mixed quantum state $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, we have

$$\begin{aligned}
\tau_N(\rho) \ \geq \ & -2^{2-N}[-2^N + \frac{(d+1)^N}{d^N} + \frac{1}{d^N}(d^N-1)(d+1)^{N-1}] \\
& + 2^{2-N}\{\frac{2^2}{d^{N+2}}[(d+1)^{N-1} - (d+1)^{N-2}]\sum_{k_1k_2}||\mathcal{T}^{k_1k_2}||^2 + \cdots \\
& + \frac{2^M}{d^{N+M}}[(d+1)^{N-1} - (d+1)^{N-M}]\sum_{k_1\cdots k_M}||\mathcal{T}^{k_1\cdots k_M}||^2 + \cdots \\
& + \frac{2^N}{d^{2N}}[(d+1)^{N-1} - 1]||\mathcal{T}^{1\cdots N}||^2\}; \qquad (14)
\end{aligned}$$

$$\tau_N(\rho) \ \leq \ 2^{2-N}(2^N - 2 - \sum_\alpha Tr\rho_\alpha^2). \qquad (15)$$

Proof: We still use the simplified notions C and C_α used in proving the Theorem 1. Assume that $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ is the optimal decomposition such that (13) attains the minimum. We have that

$$\begin{aligned}
\tau_N(\rho) \ &= \ \sum_i p_i C_N^2(|\psi_i\rangle) = 2^{2-N}\sum_i p_i\{-C + \sum_\alpha C_\alpha||\mathcal{T}_i^\alpha||^2\} \\
&= \ 2^{2-N}[\sum_\alpha C_\alpha(\sum_i p_i||\mathcal{T}_i^\alpha||)^2 - C] \\
&\geq \ 2^{2-N}[\sum_\alpha C_\alpha||\mathcal{T}^\alpha||^2 - C],
\end{aligned}$$

where we have used the triangle inequality for the Hilbert-Schmidt norm.

On the other hand, by the definition of $\tau_N(\rho)$, we have

$$\begin{aligned}
\tau_N(\rho) \ &\leq \ \sum_i p_i\tau_N(|\psi_i\rangle) = 2^{2-N}(2^N - 2 - \sum_{\alpha,i} p_i Tr(\rho_\alpha^i)^2) \\
&\leq \ 2^{2-N}[2^N - 2 - \sum_\alpha Tr(\sum_i p_i\rho_\alpha^i)^2] \\
&= \ 2^{2-N}(2^N - 2 - \sum_\alpha Tr\rho_\alpha^2),
\end{aligned}$$

which gives the upper bound. ■

From the proof of Theorem 2, one has that for pure states the lower and upper bounds are exact. Thus the lower and upper bounds (14) for $\tau_N(\rho)$ are tight.

Example: Consider the randomly generated three-qubit state, $\rho = \frac{1-p}{8}I_8 + p|\psi\rangle\langle\psi|$ with $0 \leq p \leq 1$, where I_8 is the 8×8 identity matrix. To check the efficiency of the bounds of $\tau_3(\rho)$ in Theorem 2, we first compute all the norms of the correlation tensors and then derive the upper and lower bounds. To compare the validity of the estimation of $\tau_3(\rho)$, we take $p = 0.97, 0.98$ and 0.995 sequentially. For weakly mixed states (with large p), the bounds provide an excellent estimation for tangle, see Fig. 1.

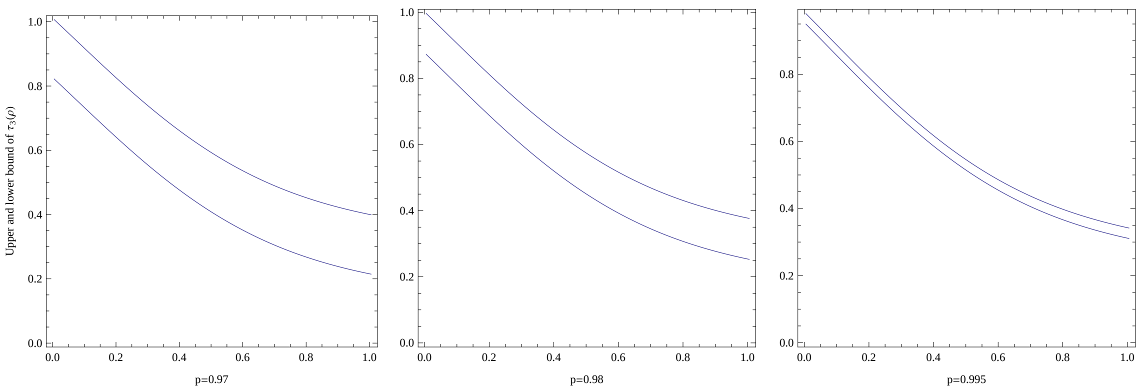

FIG. 1: Upper and lower bounds of tangle for $\rho = \frac{1-p}{8} I_8 + p|\psi\rangle\langle\psi|$, where $|\psi\rangle$ are randomly generated pure states.

Remark Our bounds are given by the norms of the correlation tensors. As the Hilbert-Schmidt norm is invariant under local unitary transformation, the bounds give experimentally feasible way in identify both separability and genuine multipartite entanglement. Further more, as has been discussed in [9, 15], partial knowledge of the correlation tensors may also allow us to detect entanglement and estimate the degree of entanglement.

IV. CONCLUSIONS AND DISCUSSIONS

It is a basic and fundamental question in quantum entanglement theory to compute the concurrence for multipartite quantum systems. Since the concurrence is defined by taking the optimization over all the ensemble decompositions of a mixed quantum states, it is formidable to derive an analytical formula. We have derived an analytical and experimentally feasible formula for multipartite concurrence of any multipartite pure quantum states by using generalized Bloch representation of density matrices. We have then obtained a lower bound of concurrence for any mixed multipartite quantum states. Genuine multipartite entanglement can be detected by using this bound. We have also investigated the multipartite tangle. Tight lower and upper bounds are obtained. The approach used in this manuscript can also be used to investigate the k-separability of multipartite quantum systems. Future research on the construction of genuine multipartite entanglement criteria in terms of the lower bound of multipartite squared I-concurrence and the k norm would be also interesting.

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