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Rectifiability of varifolds with locally bounded
first variation with respect to anisotropic surface
energies

by

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RECTIFIABILITY OF VARIFOLDS WITH LOCALLY BOUNDED FIRST VARIATION WITH RESPECT TO ANISOTROPIC SURFACE ENERGIES

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ABSTRACT. We extend Allard's celebrated rectifiability theorem to the setting of varifolds with locally bounded first variation with respect to an anisotropic integrand. In particular, we identify a sufficient and necessary condition on the integrand to obtain the rectifiability of every d-dimensional varifold with locally bounded first variation and positive d-dimensional density. In codimension one, this condition is shown to be equivalent to the strict convexity of the integrand with respect to the tangent plane.

1. INTRODUCTION

Allard's rectifiability Theorem, [1], asserts that every d-varifold in $\mathbb{R}^n$ with locally bounded (isotropic) first variation is d-rectifiable when restricted to the set of points in $\mathbb{R}^n$ with positive lower d-dimensional density. It is a natural question whether this result holds for varifolds whose first variation with respect to an anisotropic integrand is locally bounded.

More specifically, for an open set $\Omega \subset \mathbb{R}^n$ and a positive C^1 integrand

$$F : \Omega \times G(n,d) \to \mathbb{R}_{>0} := (0, +\infty),$$

where $G(n,d)$ denotes the Grassmannian of d-planes in $\mathbb{R}^n$, we define the *anisotropic energy* of a d-varifold $V \in \mathbb{V}_d(\Omega)$ as

$$\mathbf{F}(V, \Omega) := \int_{\Omega \times G(n,d)} F(x,T) \, dV(x,T). \tag{1.1}$$

We also define its *anisotropic first variation* as the order one distribution whose action on $g \in C_c^1(\Omega, \mathbb{R}^n)$ is given by

$$\begin{aligned}
\delta_F V(g) &:= \frac{d}{dt} \mathbf{F}\big(\varphi_t^{\#} V\big)\Big|_{t=0} \\
&= \int_{\Omega \times G(n,d)} \Big[\langle d_x F(x,T), g(x) \rangle + B_F(x,T) : Dg(x) \Big] dV(x,T),
\end{aligned}$$

where $\varphi_t(x) = x + tg(x)$, $\varphi_t^{\#} V$ is the image varifold of V through φ_t, $B_F(x,T) \in \mathbb{R}^n \otimes \mathbb{R}^n$ is an explicitly computable $n \times n$ matrix and $\langle A, B \rangle := \operatorname{tr} A^* B$ for $A, B \in \mathbb{R}^n \otimes \mathbb{R}^n$, see Section 2 below for the precise definitions and the relevant computations. We have then the following:

Question. Is it true that for every $V \in \mathbb{V}_d(\Omega)$ such that $\delta_F V$ is a Radon measure in Ω, the associated varifold V_* defined as[1]

$$V_* := V \llcorner \{x \in \Omega : \Theta_*^d(x, V) > 0\} \times G(n,d) \tag{1.2}$$

is d-rectifiable?

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[1] Here $\Theta_*^d(x, V)$ is the lower d-dimensional density of V at the point x, see Section 2.

We will show that this is true if (and only if in the case of autonomous integrands) F satisfies the following *atomic condition* at every point $x \in \Omega$.

Definition 1.1. For a given integrand $F \in C^1(\Omega \times G(n,d))$, $x \in \Omega$ and a Borel probability measure $\mu \in \mathcal{P}(G(n,d))$, let us define

$$A_x(\mu) := \int_{G(n,d)} B_F(x,T)d\mu(T) \in \mathbb{R}^n \otimes \mathbb{R}^n. \tag{1.3}$$

We say that F verifies the *atomic condition* (AC) at x if the following two conditions are satisfied:

(i) $\dim \operatorname{Ker} A_x(\mu) \leq n - d$ for all $\mu \in \mathcal{P}(G(n,d))$,
(ii) if $\dim \operatorname{Ker} A_x(\mu) = n - d$, then $\mu = \delta_{T_0}$ for some $T_0 \in G(n,d)$.

The following Theorem is the main result of this paper, see again Section 2 for the relevant (standard) definitions:

Theorem 1.2. *Let $F \in C^1(\Omega \times G(n,d), \mathbb{R}_{>0})$ be a positive integrand and let us define*

$$\mathscr{V}_F(\Omega) = \Big\{ V \in \mathbb{V}_d(\Omega) : \delta_F V \text{ is a Radon measure} \Big\}. \tag{1.4}$$

Then we have the following:

(i) *If F satisfies the atomic condition at every $x \in \Omega$, then for every $V \in \mathscr{V}_F(\Omega)$ the associated varifold V_* defined in (1.2) is d-rectifiable.*
(ii) *Assume that F is autonomous, i.e. that $F(x,T) \equiv F(T)$: then every V_* associated to a varifold $V \in \mathscr{V}_F(\mathbb{R}^d)$ is d-rectifiable if and only if F satisfies the atomic condition.*

For the area integrand, $F(x,T) \equiv 1$, it is easy to verify that $B_F(x,T) = T$ where we are identifying $T \in G(n,d)$ with the matrix $T \in (\mathbb{R}^n \otimes \mathbb{R}^n)_{\text{sym}}$ representing the orthogonal projection onto T, see Section 2. Since T is positive semidefinite (i.e. $T \geq 0$), it is easy to check that the (AC) condition is satisfied. In particular Theorem 1.2 provides a new independent proof of Allard's rectifiability theorem.

Since the atomic condition (AC) is essentially necessary to the validity of the rectifiability Theorem 1.2, it is relevant to relate it to the previous known notions of *ellipticity* (or *convexity*) of F with respect to the "plane" variable T. This task seems to be quite hard in the general case. For $d = (n-1)$ we can however completely characterize the integrands satisfying (AC). Referring again to Sections 2 and 5 for a more detailed discussion, we recall here that in this case the integrand F can be equivalently thought as a positive one-homogeneous even function $G : \Omega \times \mathbb{R}^n \to \mathbb{R}_{>0}$ via the identification

$$G(x, \lambda\nu) := |\lambda| F(x, \nu^{\perp}) \qquad \text{for all } \lambda \in \mathbb{R} \text{ and } \nu \in \mathbb{S}^{n-1}. \tag{1.5}$$

The atomic condition then turns out to be equivalent to the *strict convexity* of G, more precisely:

Theorem 1.3. *An integrand $F : C^1(\Omega \times G(n, n-1), \mathbb{R}_{>0})$ satisfies the atomic condition at x if and only if the function $G(x, \cdot)$ defined in (1.5) is strictly convex.*

As we said, we have not been able to obtain a simple characterization in the general situation when $2 \leq d \leq (n-2)$ (while for $d = 1$ the reader can easily verify that an analogous version of Theorem 1.3 holds true). In this respect, let us recall that the study of convexity notions for integrands defined on the d-dimensional Grassmannian is an active field of research, where several basic questions are still open, see [7, 6] and the survey [4].

Beside its theoretical interest, the above Theorem has some relevant applications in the study of existence of minimizers of geometric variational problems defined on class of rectifiable sets. It can be indeed shown that given an **F**-minimizing sequence of sets, the limit of the varifolds naturally associated to them is **F**-stationary (i.e. it satisfies $\delta_F V = 0$) and has density bounded away from zero. Hence, if F satisfies (AC), this varifold is rectifiable and it can be shown that its support minimizes **F**, see the forthcoming papers [11, 15] and also [9, 12, 8] for similar results obtained with different techniques.

We conclude this introduction with a brief sketch of the proof of Theorem 1.2. The original proof of Allard in [1] (see also [18] for a quantitative improvement under slightly more general assumptions) for varifolds with locally bounded variations with respect to the area integrand heavily relies on the monotonicity formula, which is strongly linked to the isotropy of the area integrand, [3]. A completely different strategy must hence be used to prove Theorem 1.2.

The idea is to use the notion of *tangent measure* introduced by Preiss, [20], in order to understand the local behavior of a varifold V with locally bounded first variation. Indeed at $\|V_*\|$ almost every point[2], condition (AC) is used to show that every tangent measure is translation invariant along *at least d* (fixed) directions, while the positivity of the lower d-dimensional density ensures that there exists at least one tangent measure that is invariant along *at most d* directions. The combination of these facts allows to show that the "Grassmannian part" of the varifold V_* at x is a Dirac delta δ_{T_x} on a fixed plane T_x, see Lemma 3.1. A key step is then to show that $\|V_*\| \ll \mathcal{H}^d$: this is achieved by using ideas borrowed from [2] and [14]. Once this is obtained, a simple rectifiability criterion, based on the results in [20] and stated in Lemma 2.2, allows to show that V_* is d-rectifiable.

2. Notations and preliminary results

We work on an open set $\Omega \subset \mathbb{R}^n$ and we set $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$, $B_r = B_r(0)$ and $B := B_1(0)$. Given a d-dimensional vector space T, we will denote $B_r^d(x) = B_r(x) \cap (x + T)$ and similarly for B_r^d and B^d.

For a matrix $A \in \mathbb{R}^n \otimes \mathbb{R}^n$, A^* denotes its transpose. Given $A, B \in \mathbb{R}^n \otimes \mathbb{R}^n$ we define $A : B = \operatorname{tr} A^* B = \sum_{ij} A_{ij} B_{ij}$, so that $|A|^2 = A : A$.

2.1. **Measures and rectifiable sets.** We denote by $\mathcal{M}_+(\Omega)$ (respectively $\mathcal{M}(\Omega, \mathbb{R}^m)$, $m \geq 1$) the set of positive (resp. $\mathbb{R}^m$-valued) Radon measure on Ω. Given a Radon measure μ we denote by $\operatorname{spt}\mu$ its support. For a Borel set E, $\mu \llcorner E$ is the restriction of μ to E, i.e. the measure defined by $[\mu \llcorner E](A) = \mu(E \cap A)$. For a $\mathbb{R}^m$-valued Radon measure $\mu \in \mathcal{M}(\Omega, \mathbb{R}^m)$ we denote by $|\mu| \in \mathcal{M}_+(\Omega)$ its total variation and we recall that, for all open sets U,

$$|\mu|(U) = \sup \left\{ \int \langle \varphi(x), d\mu(x) \rangle : \quad \varphi \in C_c^\infty(U, \mathbb{R}^m), \quad \|\varphi\|_\infty \leq 1 \right\}.$$

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[2]Here $\|V_*\|$ is the projection on $\mathbb{R}^n$ of the measure V_*, see Section 2.

Eventually, we denote by $\mathcal{H}^d$ the d-dimensional Hausdorff measure and for a d-dimensional vector space $T \subset \mathbb{R}^n$ we will often identify $\mathcal{H}^d \llcorner T$ with the d-dimensional Lebesgue measure $\mathcal{L}^d$ on $T \approx \mathbb{R}^d$.

A set $K \subset \mathbb{R}^n$ is said to be d-rectifiable if it can be covered, up to an $\mathcal{H}^d$-negligible set, by countably many C^1 d-dimensional submanifolds. Given a d-rectifiable set K, we denote $T_x K$ the approximate tangent space of K at x, which exists for $\mathcal{H}^d$-almost every point $x \in K$, [21, Chapter 3]. A positive Radon measure $\mu \in \mathcal{M}_+(\Omega)$ is said to be d-rectifiable if there exists a d-rectifiable set $K \subset \Omega$ such that $\mu = \theta \mathcal{H}^d \llcorner K$ for some Borel function $\theta : \mathbb{R}^n \to \mathbb{R}_{>0}$.

For $\mu \in \mathcal{M}_+(\Omega)$ we consider its lower and upper d-dimensional densities at x:

$$\Theta^d_*(x, \mu) = \liminf_{r \to 0} \frac{\mu(B_r(x))}{\omega_d r^d}, \qquad \Theta^{d*}(x, \mu) = \limsup_{r \to 0} \frac{\mu(B_r(x))}{\omega_d r^d},$$

where $\omega_d = \mathcal{H}^d(B^d)$ is the measure of the d-dimensional unit ball in $\mathbb{R}^d$. In case these two limits are equal, we denote by $\Theta^d(x, \mu)$ their common value. Note that if $\mu = \theta \mathcal{H}^d \llcorner K$ with K rectifiable then $\theta(x) = \Theta^d_*(x, \mu) = \Theta^{d*}(x, \mu)$ for μ-a.e. x, see [21, Chapter 3].

If $\eta : \mathbb{R}^n \to \mathbb{R}^n$ is a Borel map and μ is a Radon measure, we let $\eta_\# \mu = \mu \circ \eta^{-1}$ be the push-forward of μ through η. Let $\eta^{x,r} : \mathbb{R}^n \to \mathbb{R}^n$ be the dilation map, $\eta^{x,r}(y) = (y - x)/r$. For a positive Radon measure $\mu \in \mathcal{M}_+(\Omega)$, $x \in \mathrm{spt}\mu \cap \Omega$ and $r \ll 1$, we define

$$\mu_{x,r} = \frac{1}{\mu(B_r(x))} (\eta^{x,r}_\# \mu) \llcorner B. \tag{2.1}$$

The normalization in (2.1) implies, by the Banach-Alaoglu Theorem, that for every sequence $r_i \to 0$ there exists a subsequence $r_{i_j} \to 0$ and a Radon measure $\sigma \in \mathcal{M}_+(B)$, called *tangent measure* to μ at x, such that

$$\mu_{x,r_{i_j}} \xrightarrow{*} \sigma.$$

We collect all tangent measures to μ at x into $\mathrm{Tan}(x, \mu) \subset \mathcal{M}_+(B)$. The next Lemma shows that $\mathrm{Tan}(x, \mu)$ is not trivial at μ-almost every point where μ has positive lower d-dimensional density and that furthermore there is always a tangent measure which looks at most d-dimensional on a prescribed ball (a similar argument can be used to show that $\mathrm{Tan}(x, \mu)$ is always not trivial at μ almost every point without any assumption on the d-dimensional density, see [5, Corollary 2.43]).

Lemma 2.1. *Let $\mu \in \mathcal{M}_+(\Omega)$ be a Radon measure. Then for every $x \in \Omega$ such that $\Theta^d_*(x, \mu) > 0$ and for every $t \in (0, 1)$, there exists a tangent measure $\sigma_t \in \mathrm{Tan}(x, \mu)$ satisfying*

$$\sigma_t(\overline{B_t}) \geq t^d. \tag{2.2}$$

Proof. Step 1: We claim that for every $x \in \Omega$ such that $\Theta^d_*(x, \mu) > 0$, it holds

$$\limsup_{r \to 0} \frac{\mu(B_{tr}(x))}{\mu(B_r(x))} \geq t^d, \qquad \forall t \in (0, 1). \tag{2.3}$$

More precisely, we are going to show that

$$\big\{ x \in \mathrm{spt}\mu : (2.3) \text{ fails} \big\} \subset \big\{ x \in \mathrm{spt}\mu : \Theta^d_*(x, \mu) = 0 \big\},$$

which clearly implies that (2.3) holds for μ almost every $x \in \Omega$ with positive lower d-dimensional density. Let indeed $x \in \mathrm{spt}\mu$ be such that (2.3) fails, then there exist $t_0 \in (0, 1)$, $\bar{\varepsilon} > 0$, and $\bar{r} > 0$ such that

$$\mu(B_{t_0 r}(x)) \leq (1 - \bar{\varepsilon}) t_0^d \, \mu(B_r(x)) \qquad \text{for all } r \leq \bar{r}.$$

Iterating this inequality, we deduce that

$$\mu(B_{t_0^k \bar{r}}(x)) \leq (1 - \bar{\varepsilon})^k t_0^{kd} \mu(B_{\bar{r}}(x)) \qquad \text{for all } k \in \mathbb{N}$$

and consequently

$$\Theta_*^d(x, \mu) \leq \lim_{k \to \infty} \frac{\mu(B_{t_0^k \bar{r}}(x))}{\omega_d (t_0^k \bar{r})^d} = 0.$$

Step 2: Let now x be a point satisfying (2.3) and let $t \in (0, 1)$: there exists a sequence $r_j \downarrow 0$ (possibly depending on t and on x), such that

$$t^d \leq \limsup_{r \to 0} \frac{\mu(B_{tr}(x))}{\mu(B_r(x))} = \lim_{j \to \infty} \frac{\mu(B_{tr_j}(x))}{\mu(B_{r_j}(x))} = \lim_{j \to \infty} \mu_{x, r_j}(B_t)$$

where μ_{x, r_j} is defined as in (2.1). Up to extracting a (not relabelled) subsequence,

$$\mu_{x, r_j} \overset{*}{\rightharpoonup} \sigma_t \in \mathrm{Tan}(x, \mu).$$

By upper semicontinuity:

$$\sigma_t(\overline{B_t}) \geq \limsup_j \mu_{x, r_j}(\overline{B_t}) \geq t^d$$

which is (2.2). $\qquad \square$

In order to prove Theorem 1.2 we need the following rectifiability criterion which is essentially [16, Theorem 4.5], see also [19, Theorem 16.7]. For the sake of readability, we postpone its proof to Appendix B.

Lemma 2.2. *Let $\mu \in \mathcal{M}_+(\Omega)$ be a Radon measure such that the following two properties hold:*

(i) *For μ-a.e. $x \in \Omega$, $0 < \Theta_*^d(x, \mu) \leq \Theta^{d*}(x, \mu) < +\infty$.*
(ii) *For μ-a.e. $x \in \Omega$ there exists $T_x \in G(n, d)$ such that every $\sigma \in \mathrm{Tan}(x, \mu)$ is translation invariant along T_x, i.e.*

$$\int \partial_e \varphi \, d\sigma = 0 \qquad \text{for every } \varphi \in C_c^1(B) \text{ and every } e \in T_x.$$

Then μ is d-rectifiable, i.e. $\mu = \theta \mathcal{H}^d \llcorner K$ for some d-rectifiable set K and Borel function $\theta : \mathbb{R}^n \to \mathbb{R}_{>0}$. Furthermore $T_x K = T_x$ for μ-a.e. x.

2.2. **Varifolds and integrands.** We denote by $G(n, d)$ the Grassmannian of (unoriented) d-dimensional linear subspaces in $\mathbb{R}^n$ (often referred to as d-planes) and given any set $E \subset \mathbb{R}^n$ we denote by $G(E) = E \times G(n, d)$ the (trivial) Grassmannian bundle over E. We will often identify a d-dimensional plane $T \in G(n, d)$ with the matrix $T \in (\mathbb{R}^n \otimes \mathbb{R}^n)_{\mathrm{sym}}$ representing the *orthogonal projection* onto T.

A d-varifold on Ω is a positive finite Radon measure V on $G(\Omega)$ and we will denote with $\mathbb{V}_d(\Omega)$ the set of all d-varifolds on Ω.

Given a diffeomorphism $\psi \in C^1(\Omega, \mathbb{R}^n)$, we define the push-forward of $V \in \mathbb{V}_d(\Omega)$ with respect to ψ as the varifold $\psi^\# V \in \mathbb{V}_d(\psi(\Omega))$ such that

$$\int_{G(\psi(\Omega))} \Phi(x, S) d(\psi^\# V)(x, S) = \int_{G(\Omega)} \Phi(\psi(x), d_x \psi(S)) J\psi(x, S) dV(x, S),$$

for every $\Phi \in C_c^0(G(\psi(\Omega)))$. Here $d_x \psi(S)$ is the image of S under the linear map $d_x \psi(x)$ and

$$J\psi(x, S) := \sqrt{\det \left(\left(d_x \psi|_S \right)^* \circ d_x \psi|_S \right)}$$

denotes the d-Jacobian determinant of the differential $d_x \psi$ restricted to the d-plane S, see [21, Chapter 8]. Note that the push-forward of a varifold V is *not* the same as

the push-forward of the Radon measure V through a map ψ defined on $G(\Omega)$ (the latter being denoted with an expressly different notation: $\psi_\# V$).

To a varifold $V \in \mathbb{V}_d(\Omega)$, we associate the measure $\|V\| \in \mathcal{M}_+(\Omega)$ defined by

$$\|V\|(A) = V(G(A)) \qquad \text{for all } A \subset \Omega \text{ Borel.}$$

Hence $\|V\| = \pi_\# V$, where $\pi : \Omega \times G(n,d) \to \Omega$ is the projection onto the first factor and the push-forward is intended in the sense of Radon measures. By the disintegration theorem for measures, see for instance [5, Theorem 2.28], we can write

$$V(dx, dT) = \|V\|(dx) \otimes \mu_x(dT),$$

where $\mu_x \in \mathcal{P}(G(n,d))$ is a (measurable) family of parametrized non-negative measures on the Grassmannian such that $\mu_x(G(n,d)) = 1$.

A d-dimensional varifold $V \in \mathbb{V}_d(\Omega)$ is said d-rectifiable if there exist a d-rectifiable set K and a Borel function $\theta : \mathbb{R}^n \to \mathbb{R}_{>0}$ such that $V = \theta \mathcal{H}^d \llcorner (K \cap \Omega) \otimes \delta_{T_x K}$.

We will use the notation

$$\Theta_*^d(x, V) = \Theta_*^d(x, \|V\|) \qquad \text{and} \qquad \Theta^{d*}(x, V) = \Theta^{d*}(x, \|V\|)$$

for the upper and lower d-dimensional densities of $\|V\|$. In case $\Theta_*^d(x, V) = \Theta^{d*}(x, V)$, we denote their common value $\Theta^d(x, V)$.

As already specified in the introduction, we will associate to any d-varifold V, its "at most d-dimensional" part V_* defined as

$$V_* := V \llcorner \{x \in \Omega : \Theta_*^d(x, V) > 0\} \times G(n,d).$$

Note that

$$\|V_*\| = \|V\| \llcorner \{x \in \Omega : \Theta_*^d(x, V) > 0\}$$

and thus, by the Lebesgue-Besicovitch differentiation Theorem [5, Theorem 2.22], for $\|V_*\|$ almost every point (or equivalently for $\|V\|$ almost every x with $\Theta_*^d(x, V) > 0$)

$$\lim_{r \to 0} \frac{\|V_*\|(B_r(x))}{\|V\|(B_r(x))} = 1. \tag{2.4}$$

In particular,

$$\Theta_*^d(x, V_*) > 0 \qquad \text{for } \|V_*\|\text{-a.e. } x. \tag{2.5}$$

Let $\eta^{x,r}(y) = (y - x)/r$, as in (2.1) we define

$$V_{x,r} := \frac{r^d}{\|V\|(B_r(x))} \big((\eta^{x,r})^\# V \big) \llcorner G(B),$$

where the additional factor r^d is due to the presence of the d-Jacobian determinant of the differential $d\eta^{x,r}$ in the definition of push-forward of varifolds. Note that, with the notation of (2.1):

$$\|V_{x,r}\| = \|V\|_{x,r}.$$

The following Lemma is based on a simple Lebesgue point argument, combined with the separability of $C_c^0(G(\Omega))$, see for instance [10, Proposition 9].

Lemma 2.3. *For $\|V\|$-almost every point $x \in \Omega$ and every sequence $r_j \to 0$ there is a subsequence r_{j_i} such that*

$$V_{x, r_{j_i}}(dy, dT) = \|V\|_{x, r_{j_i}}(dy) \otimes \mu_{x + y r_{j_i}}(dT) \overset{*}{\to} \sigma(dy) \otimes \mu_x(dT) =: V^\infty(dy, dT)$$

with $\sigma \in \mathrm{Tan}(x, \|V\|)$.

We call any varifold $V^\infty \in \mathbb{V}_d(B)$ arising as in Lemma 2.3 a *tangent varifold* to V at x and we collect them into $\mathrm{Tan}(x, V)$. The key point of the Lemma above is that the "Grassmannian" part V_y^∞ of a tangent varifold V^∞ equals μ_x for every $y \in \Omega$: it therefore neither depends on the space variable y, nor on the chosen blow-up sequence (r_j). Informally, we could write:

$$\text{“}\mathrm{Tan}(x, V) = \mathrm{Tan}(x, \|V\|) \otimes \mu_x(dT)\text{”}.$$

We furthermore note that, as a consequence of (2.4),

$$\mathrm{Tan}(x, \|V_*\|) = \mathrm{Tan}(x, \|V\|) \quad \text{and} \quad \mathrm{Tan}(x, V_*) = \mathrm{Tan}(x, V) \quad \|V_*\|\text{-a.e.} \quad (2.6)$$

The anisotropic integrand that we consider is a C^1 function

$$F : G(\Omega) \longrightarrow \mathbb{R}_{>0}.$$

Since our results are local in nature, up to restricting to a compactly contained open subset of Ω, we can assume the existence of two positive constants λ, Λ such that

$$0 < \lambda \le F(x, T) \le \Lambda < \infty \qquad \text{for all } (x, T) \in G(\Omega).$$

Given $x \in \Omega$, we will also consider the "frozen" integrand

$$F_x : G(n, d) \to (0, +\infty), \qquad F_x(T) := F(x, T). \qquad (2.7)$$

As in (1.1), we define the *anisotropic energy* of $V \in \mathbb{V}_d(\Omega)$ as

$$\mathbf{F}(V, \Omega) := \int_{G(\Omega)} F(x, T) \, dV(x, T).$$

For a vector field $g \in C_c^1(\Omega, \mathbb{R}^n)$, we consider the family of functions $\varphi_t(x) = x + t g(x)$, and we note that they are diffeomorphisms of Ω into itself for t small enough. The *anisotropic first variation* is defined as

$$\delta_F V(g) := \frac{d}{dt} \mathbf{F}\big(\varphi_t^\# V, \Omega\big)\Big|_{t=0}.$$

It can be easily shown, see Appendix A, that

$$\delta_F V(g) = \int_{G(\Omega)} \Big[\langle d_x F(x, T), g(x) \rangle + B_F(x, T) : Dg(x) \Big] dV(x, T), \qquad (2.8)$$

where the matrix $B_F(x, T) \in \mathbb{R}^n \otimes \mathbb{R}^n$ is uniquely defined by

$$\begin{aligned} B_F(x, T) : L &:= F(x, T)(T : L) + \big\langle d_T F(x, T), T^\perp \circ L \circ T + (T^\perp \circ L \circ T)^* \big\rangle \\ &=: F(x, T)(T : L) + C_F(x, T) : L \qquad \text{for all } L \in \mathbb{R}^n \otimes \mathbb{R}^n. \end{aligned} \qquad (2.9)$$

Note that, via the identification of a d-plane T with the orthogonal projection onto it, $G(n, d)$ can be thought as a subset of $\mathbb{R}^n \otimes \mathbb{R}^n$ and this gives the natural identification:

$$\mathrm{Tan}_T G(n, d) = \big\{ S \in \mathbb{R}^n \otimes \mathbb{R}^n : S^* = S, \quad T \circ S \circ T = 0, \quad T^\perp \circ S \circ T^\perp = 0 \big\},$$

see Appendix A for more details. We are going to use the following properties of $B_F(x, T)$ and $C_F(x, T)$, which immediately follow from (2.9):

$$|B_F(x, T) - B_F(x, S)| \le \|F\|_{C^1} |S - T| \qquad \text{for all } T, S \in G(n, d), \qquad (2.10)$$

$$C_F(x, T) : v \otimes w = 0 \quad \text{for all } v, w \in T. \qquad (2.11)$$

We also note that trivially

$$|\delta_F V(g)| \le \|F\|_{C^1(\mathrm{spt} g)} \|g\|_{C^1} \|V\|(\mathrm{spt}(g)), \qquad (2.12)$$

and that, if F_x is the frozen integrand (2.7), then (2.8) reduces to

$$\delta_{F_x} V(g) = \int_{G(\Omega)} B_F(x, T) : Dg(y) \, dV(y, T).$$

We say that a varifold V has *locally bounded anisotropic first variation* if $\delta_F V$ is a Radon measure on Ω, i.e. if

$$|\delta_F V(g)| \leq C(K)\|g\|_\infty \quad \text{for all } g \in C_c^1(\Omega, \mathbb{R}^n) \text{ with } \mathrm{spt}\, g \subset K \subset\subset \Omega.$$

Furthermore, we will say that V is **F**-*stationary* if $\delta_F V = 0$.

We conclude this section with the following simple result which shows that every tangent varifold to a varifold having locally bounded anisotropic first variation is $\mathbf{F}_x$-stationary.

Lemma 2.4. *Let $V \in \mathbb{V}_d(\Omega)$ be a d-dimensional varifold with locally bounded anisotropic first variation. Then, for $\|V\|$-almost every point, every $W \in \mathrm{Tan}(x, V)$ is $\mathbf{F}_x$-stationary, i.e. $\delta_{F_x} W = 0$. Moreover, if $W(dy, dT) = \sigma(dy) \otimes \mu_x(dT)$ for some $\sigma \in \mathrm{Tan}(x, \|V\|)$ (which by Lemma 2.3 happens $\|V\|$-a.e. x), then*

$$\partial_e \sigma = 0 \qquad \text{for all } e \in T_x := \mathrm{Im} A_x(\mu_x) \tag{2.13}$$

in the sense of distributions, where $A_x(\mu_x)$ is defined in (1.3).

Proof. Let x be a point such that the conclusion of Lemma 2.3 holds true and such that

$$\limsup_{r \to 0} \frac{|\delta_F V|(B_r(x))}{\|V\|(B_r(x))} = C_x < +\infty. \tag{2.14}$$

Note that, by Lemma 2.3 and Lebesgue-Besicovitch differentiation Theorem, [21, Theorem 4.7], this is the case for $\|V\|$-almost every point. We are going to prove the Lemma at every such a point.

Let r_i be a sequence such that $V_{x,r_i}(dy, dT) \overset{*}{\rightharpoonup} W(dy, dT) = \sigma(dy) \otimes \mu_x(dT)$, $\sigma \in \mathrm{Tan}(x, \|V\|)$. For $g \in C_c^1(B, \mathbb{R}^n)$, we define $g_i := g \circ \eta^{x,r_i} \in C_c^1(B_{r_i}(x), \mathbb{R}^n)$ and we compute

$$\delta_{F_x} V_{x,r_i}(g) = \int_{G(B)} B_F(x, T) : Dg(y) \, dV_{x,r_i}(y, T)$$

$$= \frac{r_i^d}{\|V\|(B_{r_i}(x))} \int_{G(B_{r_i}(x))} B_F(x, T) : Dg\left(\eta^{x,r_i}(z)\right) J\eta^{x,r_i}(z, T) \, dV(z, T)$$

$$= \frac{r_i}{\|V\|(B_{r_i}(x))} \int_{G(B_{r_i}(x))} B_F(x, T) : Dg_i(z) \, dV(z, T)$$

$$= r_i \frac{\delta_{F_x} V(g_i)}{\|V\|(B_{r_i}(x))} = r_i \frac{\delta_F V(g_i) + \delta_{(F_x-F)} V(g_i)}{\|V\|(B_{r_i}(x))}.$$

Combining (2.12), (2.14) and since $r_i \|Dg_i\|_{C^0} = \|Dg\|_{C^0}$, we get

$$|\delta_{F_x} V_{x,r_i}(g)| \leq r_i \frac{|\delta_F V|(B_{r_i}(x))\|g\|_\infty}{\|V\|(B_{r_i}(x))}$$

$$+ r_i \frac{\|F - F_x\|_{C^1(B_{r_i}(x))}\|g_i\|_{C^1}\|V\|(B_{r_i}(x))}{\|V\|(B_{r_i}(x))}$$

$$\leq r_i C_x \|g\|_\infty + o_{r_i}(1)\|g\|_{C^1} \to 0,$$

which implies $\delta_{F_x} W = 0$. Hence, recalling the Definition (1.3) of $A_x(\mu)$, for every $g \in C_c^1(B, \mathbb{R}^n)$:

$$0 = \delta_{F_x} W(g) = \int_B A_x(\mu_x) : Dg(y) \, d\sigma(y).$$

Therefore $[A_x(\mu_x)]^* D\sigma = 0$ in the sense of distributions, which is equivalent to (2.13), since $\mathrm{Ker} A_x(\mu_x)^* = (\mathrm{Im} A_x(\mu_x))^\perp$. $\qquad\square$

3. Intermediate lemmata

To prove the sufficiency part of Theorem 1.2, there are two key steps:

(i) Show that the "Grassmannian" part of the varifold V_* is concentrated on a single plane;

(ii) Show that $\|V_*\| \ll \mathcal{H}^d$.

In this section we prove these steps, in Lemma 3.1 and Lemma 3.4 respectively.

Lemma 3.1. *Let F be an integrand satisfying condition (AC) at every x in Ω and let $V \in \mathscr{V}_F(\Omega)$, see (1.4). Then, for $\|V_*\|$-a.e. $x \in \mathbb{R}^n$, $\mu_x = \delta_{T_0}$ for some $T_0 \in G(n,d)$.*

Proof. Let $t \leq t(d) \ll 1$ to be fixed later. By Lemmata 2.3, 2.4 and 2.1 and by (2.6), for $\|V_*\|$-a.e. x there exist a sequence $r_i \to 0$ and a tangent measure σ such that

$$\|V_*\|_{x,r_i} \overset{*}{\rightharpoonup} \sigma, \qquad (V_*)_{x,r_i} \overset{*}{\rightharpoonup} \sigma \otimes \mu_x, \qquad \sigma(\overline{B_t}) \geq t^d$$

and

$$\partial_e \sigma = 0 \quad \text{for all } e \in T_x = \text{Im} A_x(\mu_x).$$

Let us now show that if $t(d)$ is sufficiently small, then $\mu_x = \delta_{T_0}$. Assume by contradiction that μ_x is not a Dirac delta: from the (AC) condition of F, this implies that $\dim(\text{Ker}(A_x(\mu_x))) < n - d$ and consequently that $\dim(T_x) > d$. This means that σ is invariant by translation along at least $d + 1$ directions and therefore there exists $Z \in G(d+1, n)$, a probability measure $\gamma \in \mathcal{P}(Z^\perp)$ defined in the linear space $Z^\perp$ and supported in $B^{n-d-1}_{1/\sqrt{2}}$, and a constant $c \in \mathbb{R}$, such that we can decompose σ in the cylinder $B^{d+1}_{1/\sqrt{2}} \times B^{n-d-1}_{1/\sqrt{2}} \subset Z \times Z^\perp$ as

$$\sigma \llcorner B^{d+1}_{1/\sqrt{2}} \times B^{n-d-1}_{1/\sqrt{2}} = c\mathcal{H}^{d+1} \llcorner (Z \cap B^{d+1}_{1/\sqrt{2}}) \otimes \gamma,$$

where $c \leq 2^{(d+1)/2} \omega_{d+1}^{-1}$ since $\sigma(B_1) \leq 1$. Taking $t(d) < \frac{1}{2\sqrt{2}}$, the ball $\overline{B_t}$ is contained in the cylinder $B^{d+1}_{1/\sqrt{2}} \times B^{n-d-1}_{1/\sqrt{2}}$ and hence

$$t^d \leq \sigma(\overline{B_t}) \leq \sigma(B^{d+1}_t \times B^{n-d-1}_{1/\sqrt{2}}) \leq C(d)t^{d+1},$$

which is a contradiction if $t(d) \ll 1$. $\qquad \square$

The next Lemma is inspired by the "Strong Constancy Lemma" of Allard [2, Theorem 4], see also [14].

Lemma 3.2. *Let $F_j : G(B) \to \mathbb{R}_{>0}$ be a sequence of C^1 integrands and let $V_j \in \mathbb{V}_d(G(B))$ be a sequence of d-varifolds equi-compactly supported in B (i.e. such that $\text{spt}\|V_j\| \subset K \subset\subset B$) with $\|V_j\|(B) \leq 1$. If there exist $N > 0$ and $S \in G(n,d)$ such that*

(1) $|\delta_{F_j} V_j|(B) + \|F_j\|_{C^1(G(B))} \leq N$,

(2) $\delta_j := \int_{G(B)} |T - S| dV_j(z, T) \to 0$ *as $j \to \infty$,*

then, up to subsequences, there exists $\gamma \in L^1(B^d, \mathcal{H}^d \llcorner B^d)$ such that for every $0 < t < 1$

$$\left| (\Pi_S)_\# \big(F_j(z, S)\|V_j\| \big) - \gamma \mathcal{H}^d \llcorner B^d \right| (B^d_t) \longrightarrow 0, \tag{3.1}$$

where $\Pi_S : \mathbb{R}^n \to S$ denotes the orthogonal projection onto S (which in this Lemma we do not identify with S).

Proof. To simplify the notation let us simply set $\Pi = \Pi_S$; we will also denote with a prime the variables in the d-plane S so that $x' = \Pi(x)$. Let $u_j = \Pi_\#\big(F_j(z, S)\|V_j\|\big) \in \mathcal{M}_+(B^d)$: then

$$\langle u_j, \varphi \rangle = \int_{G(B)} \varphi(\Pi(z)) F_j(z, S) dV_j(z, T) \qquad \text{for all } \varphi \in C_c^0(B^d).$$

Let $e \in S$ and, for $\varphi \in C_c^1(B^d)$, let us denote by D' the gradient of φ with respect to the variables in S, so that $\Pi^*(D'\varphi)(\Pi(z)) = D(\varphi(\Pi(z)))$. We then have in the sense of distributions

$$
\begin{aligned}
-\langle \partial'_e u_j, \varphi \rangle = \langle u_j, \partial'_e \varphi \rangle &= \int_{G(B)} \langle D'\varphi(\Pi(z)), e \rangle F_j(z, S) \, dV_j(z, T) \\
&= \int_{G(B)} \langle D'\varphi(\Pi(z)), e \rangle (F_j(z, S) - F_j(z, T)) \, dV_j(z, T) \\
&\quad + \int_{G(B)} F_j(z, T)(S - T) : e \otimes \Pi^*(D'\varphi)(\Pi(z)) \, dV_j(z, T) \\
&\quad + \int_{G(B)} \big(C_{F_j}(z, S) - C_{F_j}(z, T)\big) : e \otimes \Pi^*(D'\varphi)(\Pi(z)) \, dV_j(z, T) \quad (3.2) \\
&\quad - \int_{G(B)} \langle d_z F_j(z, T), e\, \varphi(\Pi(z)) \rangle \, dV_j(z, T) \\
&\quad + \int_{G(B)} \langle d_z F_j(z, T), e\, \varphi(\Pi(z)) \rangle \, dV_j(z, T) \\
&\quad + \int_{G(B)} \big(F_j(z, T)T + C_{F_j}(z, T)\big) : e \otimes D(\varphi(\Pi(z))) \, dV_j(z, T),
\end{aligned}
$$

where we have used that

$$\mathrm{Id} : e \otimes \Pi^*(D'\varphi)(\Pi(z)) = S : e \otimes \Pi^*(D'\varphi)(\Pi(z)) = \langle D'\varphi(\Pi(z)), e \rangle$$

and $C_{F_j}(z, S) : e \otimes \Pi^*(D'\varphi)(\Pi(z)) = 0$, since $D'\varphi$ and e belong to S, see (2.11). Let us define the distributions

$$
\begin{aligned}
\langle X_j^e, \psi \rangle := \int_{G(B)} \Big(&(F_j(z, S) - F_j(z, T))\mathrm{Id} + F_j(z, T)(S - T) \\
&+ (C_{F_j}(z, S) - C_{F_j}(z, T)) \Big) : e \otimes \Pi^*\psi(\Pi(z)) \, dV_j(z, T)
\end{aligned}
$$

and

$$
\begin{aligned}
\langle f_j^e, \varphi \rangle &:= \int_{G(B)} \langle d_z F_j(z, T), e\, \varphi(\Pi(z)) \rangle \, dV_j(z, T), \\
\langle g_j^e, \varphi \rangle &:= -\int_{G(B)} \Big(\langle d_z F_j(z, T), e\, \varphi(\Pi(z)) \rangle \\
&\qquad + \big(F_j(z, T)T + C_{F_j}(z, T)\big) : e \otimes \Pi^* D'\varphi(\Pi(z)) \Big) \, dV_j(z, T) \\
&= -\delta_{F_j} V_j(e\, \varphi \circ \Pi).
\end{aligned}
$$

By their very definition, X_j^e are vector valued Radon measures in $\mathcal{M}(B_1^d, \mathbb{R}^d)$ and, by the uniform bound on the C^1 norm of the F_j, (2.10) and assumption (2):

$$\sup_{|e|=1} |X_j^e|(B_1^d) \le CN\delta_j \to 0. \tag{3.3}$$

Moreover, by the mass bound $\|V_j\|(B) \leq 1$ and assumption (1), f_j^e and g_j^e are also Radon measures satisfying

$$\sup_j \sup_{|e|=1} |f_j^e|(B_1^d) + |g_j^e|(B_1^d) < +\infty. \tag{3.4}$$

Letting e vary in an orthonormal base $\{e_1, \ldots, e_d\}$ of S, we can re-write (3.2) as

$$D'u_j = \operatorname{div}'X_j + f_j + g_j, \tag{3.5}$$

where $X_j = (X_j^{e_1}, \ldots, X_j^{e_d}) \in \mathbb{R}^d \otimes \mathbb{R}^d$, $f_j = (f_j^{e_1}, \ldots, f_j^{e_d})$ and $g_j = (g_j^{e_1}, \ldots, g_j^{e_d})$.

Let us now choose an arbitrary sequence $\varepsilon_j \downarrow 0$ and a family of smooth approximation of the identity $\psi_{\varepsilon_j}(x') = \varepsilon_j^{-d}\psi(x'/\varepsilon_j)$, with $\psi \in C_c^\infty(B_1)$, $\psi \geq 0$. To prove (3.1) it is enough to show that $\{v_j := u_j \star \psi_{\varepsilon_j}\}$ is precompact in $L_{\mathrm{loc}}^1(B_1^d)$. Note that by convolving (3.5) we get that v_j solves

$$Dv_j = \operatorname{div}Y_j + h_j, \tag{3.6}$$

where, to simplify the notation, we have set $D = D'$, $\operatorname{div} = \operatorname{div}'$ and

$$Y_j := X_j \star \psi_{\varepsilon_j} \in C_c^\infty(B_1^d, \mathbb{R}^n \otimes \mathbb{R}^n), \qquad h_j = (f_j + g_j) \star \psi_{\varepsilon_j} \in C_c^\infty(B_1^d, \mathbb{R}^n)$$

are smooth functions compactly supported in B_1^d. Note that, by (3.3), (3.4) and the positivity of u_j

$$v_j \geq 0, \qquad \int |Y_j| \to 0 \qquad \text{and} \qquad \sup_j \int |h_j| < +\infty.$$

We can solve the system (3.6) by taking another divergence and inverting the Laplacian using the potential theoretic solution (note that all the functions involved are compactly supported):

$$v_j = \Delta^{-1}\operatorname{div}(\operatorname{div}Y_j) + \Delta^{-1}\operatorname{div}h_j. \tag{3.7}$$

Recall that

$$\Delta^{-1}w = E \star w, \tag{3.8}$$

with $E(x) = -c_d|x|^{2-d}$ if $d \geq 3$ and $E(x) = c_2 \log|x|$ if $d = 2$, for some positive constants c_d, depending just on the dimension. Hence, denoting by P.V. the principal value,

$$\Delta^{-1}\operatorname{div}(\operatorname{div}Y_j)(x) = K \star Y_j(x) := \text{P.V.}\, c_d \int_{\mathbb{R}^d} \frac{(x-y) \otimes (x-y)}{|x-y|^{d+2}} : Y_j(y)dy,$$

and

$$\Delta^{-1}\operatorname{div}h_j(x) = G \star h_j(x) := c_d \int_{\mathbb{R}^d} \Big\langle \frac{x-y}{|x-y|^d}, h_j(y) \Big\rangle dy.$$

By the Frechet-Kolomogorov compactness theorem, the operator $h \mapsto G \star h : L_c^1(B_1^d) \to L_{\mathrm{loc}}^1(\mathbb{R}^d)$ is compact (where $L_c^1(B_1^d)$ are the L^1 functions with compact support in B_1^d). Indeed, for $M \geq 1$, we have

$$\int_{B_M^d} |G \star h(x+v) - G \star h(x)|dx \leq C|v| \log\left(\frac{eM}{|v|}\right) \int_{B_1^d} |h|dx, \qquad \forall v \in B_1^d. \tag{3.9}$$

In particular, $\{b_j := G \star h_j\}$ is precompact in $L_{\mathrm{loc}}^1(\mathbb{R}^d)$. The first term is more subtle: the kernel K defines a Calderon-Zygmund operator $Y \mapsto K \star Y$ on Schwarz functions that can be extended to a bounded operator from L^1 to $L^{1,\infty}$, [17, Chapter 4]. In particular we can bound the quasi-norm of $a_j := K \star Y_j$ as

$$[a_j]_{L^{1,\infty}(\mathbb{R}^d)} := \sup_{\lambda > 0} \lambda|\{|a_j| > \lambda\}| \leq C \int_{B_1^d} |Y_j| \to 0. \tag{3.10}$$

Moreover, $K \star Y_j \overset{*}{\rightharpoonup} 0$ in the sense of distributions, since $\langle K \star Y_j, \varphi \rangle = \langle Y_j, K \star \varphi \rangle \to 0$ for $\varphi \in C_c^1(\mathbb{R}^d)$. We can therefore write

$$0 \le v_j = a_j + b_j,$$

with $a_j \to 0$ in $L^{1,\infty}$ by (3.10), $a_j \overset{*}{\rightharpoonup} 0$ in the sense of distributions and $\{b_j\}$ precompact in L^1_{loc} by (3.9). Lemma 3.3 below implies that v_j is strongly precompact in L^1_{loc}, which is the desired conclusion. $\qquad \square$

Lemma 3.3. *Let $\{v_j\}, \{a_j\}, \{b_j\} \subset L^1(\mathbb{R}^d)$ such that*

   (i) $0 \le v_j = a_j + b_j$,
   (ii) $\{b_j\}$ *strongly precompact in* L^1_{loc},
   (iii) $a_j \to 0$ *in* $L^{1,\infty}$ *and* $a_j \overset{*}{\rightharpoonup} 0$ *in the sense of distributions.*

Then $\{v_j\}$ is strongly precompact in L^1_{loc}.

Proof. It is enough to show that $\chi |a_j| \to 0$ in L^1 for $\chi \in C_c^\infty(\mathbb{R}^n)$, $\chi \ge 0$. The first condition implies that $a_j^- \le |b_j|$, hence the sequence $\{\chi a_j^-\}$ is equi-integrable and thus, by (iii) and Vitali convergence Theorem, it converges to zero in L^1_{loc}, hence

$$\int \chi |a_j| = \int \chi a_j + 2 \int \chi a_j^- \to 0,$$

where the first integral goes to zero by (iii). $\qquad \square$

The following Lemma is a key step in the proof of Theorem 1.2:

Lemma 3.4. *Let F be an integrand satisfying condition (AC) at every x in Ω and let $V \in \mathscr{V}_F(\Omega)$, see (1.4). Then $\|V_*\| \ll \mathcal{H}^d$.*

Proof. Since by (2.5), $\Theta_*^d(\cdot, V) > 0$ $\|V_*\|$-a.e., classical differentiation theorems for measures imply that

$$\mathcal{H}^d \llcorner \{\Theta_*^d(\cdot, V) > \lambda\} \le \frac{1}{\lambda} \|V_*\| \qquad \forall \lambda > 0,$$

see [19, Theorem 6.9]. Hence $\mathcal{H}^d \llcorner \{\Theta_*^d(\cdot, V) > 0\}$ is a σ-finite measure and by the Radon-Nikodym Theorem

$$\|V_*\| = f \mathcal{H}^d \llcorner \{\Theta_*^d(\cdot, V) > 0\} + \|V_*\|^s \tag{3.11}$$

for some psitive Borel function f and $\|V_*\|^s$ is concentrated on a set $E \subset \{\Theta_*^d(\cdot, V) > 0\}$ such that $\mathcal{H}^d(E) = 0$: in particular $\mathcal{H}^d(\Pi(E)) = 0$ whenever Π is an orthogonal projection onto a d-dimensional subspace of $\mathbb{R}^n$. Hence $\|V_*\|^s$ and $f \mathcal{H}^d \llcorner \{\Theta_*^d(\cdot, V) > 0\}$ are mutually singular Radon measure (the fact that they are Radon measures follows trivially from (3.11)).

We are going to show that $\|V_*\|^s = 0$, which clearly concludes the proof. To this aim, let us assume by contradiction that $\|V_*\|^s > 0$ and let us choose a point $\bar{x} \in \Omega$ and a sequence of radii $r_j \to 0$ such that:

   (i)

$$\lim_{j \to \infty} \frac{\|V_*\|^s(B_{r_j}(\bar{x}))}{\|V_*\|(B_{r_j}(\bar{x}))} = \lim_{j \to \infty} \frac{\|V_*\|(B_{r_j}(\bar{x}))}{\|V\|(B_{r_j}(\bar{x}))} = 1. \tag{3.12}$$

   (ii) There exists $\sigma \in \mathrm{Tan}(\bar{x}, \|V\|) = \mathrm{Tan}(\bar{x}, \|V_*\|) = \mathrm{Tan}(\bar{x}, \|V_*\|^s)$, with $\sigma \llcorner B_{1/2} \ne 0$.
   (iii)

$$\limsup_{j \to \infty} \frac{|\delta_F V|(B_{r_j}(\bar{x}))}{\|V\|(B_{r_j}(\bar{x}))} \le C_{\bar{x}} < +\infty. \tag{3.13}$$

(iv)

$$V_j := V_{\bar{x}, r_j} \overset{*}{\rightharpoonup} \sigma \otimes \delta_S, \tag{3.14}$$

where $S \in G(n, d)$ and $\partial_e \sigma = 0$ for every $e \in S$.

Here the first, second and third conditions hold $\|V_*\|^s$-a.e. by simple measure theoretic arguments and by (2.4) and (2.6), and the fourth one holds $\|V_*\|^s$-a.e. as well by combining Lemma 2.4, Lemma 3.1 and (2.6).

Fix a smooth cutoff function χ with $0 \leq \chi \leq 1$, $\mathrm{spt}(\chi) \subset B_1$ and $\chi = 1$ in $B_{1/2}$ and define $W_j := \chi V_j$ so that

$$\|W_j\| = \chi f \mathcal{H}^d \llcorner \{\Theta_*^d(\cdot, V) > 0\} + \|W_j\|^s$$

where $\|W_j\|^s = \chi \|V_*\|^s$. In particular

$$(\Pi_S)_\# \|W_j\|^s \qquad \text{is concentrated on} \qquad E_j := \Pi_S \left(\frac{E - \bar{x}}{r_j}\right) \cap B_1^d, \tag{3.15}$$

and thus

$$\mathcal{H}^d(E_j) = 0. \tag{3.16}$$

Note furthermore that

$$\sup_j |\delta_{F_j} W_j|(\mathbb{R}^d) < +\infty, \tag{3.17}$$

where $F_j(z, T) = F(\bar{x} + r_j z, T)$. Indeed for $\varphi \in C_c^\infty(B_1, \mathbb{R}^n)$

$$|\delta_{F_j} W_j(\varphi)| = |\delta_{F_j}(\chi V_j)(\varphi)|$$

$$= \left| \int r_j \langle d_x F(\bar{x} + r_j z, T), \chi(z)\varphi(z) \rangle dV_j(z, T) \right.$$

$$+ \int B_F(\bar{x} + r_j z, T) : D\varphi(z) \chi(z) dV_j(z, T) \Big|$$

$$= \left| \int r_j \langle d_x F(\bar{x} + r_j z, T), \chi\varphi \rangle dV_j(z, T) \right.$$

$$+ \int B_F(\bar{x} + r_j z, T) : D(\chi\varphi)(z) dV_j(z, T)$$

$$- \int B_F(\bar{x} + r_j z, T) : D\chi(z) \otimes \varphi(z) dV_j(z, T) \Big|$$

$$\leq |\delta_{F_j} V_j(\chi\varphi)| + \|F\|_{C^1} \|V_j\|(B_1) \|D\chi\|_\infty \|\varphi\|_\infty$$

$$\leq r_j \frac{|\delta_F V|(B_{r_j}(\bar{x}))}{\|V\|(B_{r_j}(\bar{x}))} \|\varphi\|_\infty + \|F\|_{C^1} \|V_j\|(B_1) \|D\chi\|_\infty \|\varphi\|_\infty,$$

so that (3.17) follows from (3.13) and the fact that $\|V_j\|(B_1) \leq 1$. Finally, by (3.14),

$$\lim_j \int_{G(B_1)} |T - S| dW_j(z, T) = \lim_j \int_{G(B_1)} |T - S| \chi(z) dV_j(z, T)$$

$$= \int_{G(B_1)} |T - S| \chi(z) d\delta_S(T) d\sigma(z) = 0.$$

Hence the sequence $\{W_j\}$ satisfies the assumptions of Lemma 3.2 and thus we deduce the existence of $\gamma \in L^1(\mathcal{H}^d \llcorner B_1^d)$ such that, along a (not relabelled) subsequence, for every $0 < t < 1$

$$\left| (\Pi_S)_\#(F(\bar{x} + r_j(\cdot), S) \|W_j\|) - \gamma \mathcal{H}^d \llcorner B^d \right|(B_t^d) \longrightarrow 0. \tag{3.18}$$

By (3.12) we can substitute $\|W_j\|^s$ for $\|W_j\|$ in (3.18) to get

$$\left| (\Pi_S)_\#(F(\bar{x} + r_j(\cdot), S) \|W_j\|^s) - \gamma \mathcal{H}^d \llcorner B_1^d \right|(B_t^d) \longrightarrow 0.$$

By point (ii) above, $F(\bar{x} + r_j(\cdot), S)\|W_j\|^s \overset{*}{\rightharpoonup} F(\bar{x}, S)\chi\sigma$ with $\sigma \llcorner B_{1/2} \neq 0$. Recalling that $F(\bar{x}, S) > 0$ we then have

$$0 < \left|(\Pi_S)_{\#}(F(\bar{x}, S)\chi\sigma)\right|(B_{1/2}^d)$$

$$\leq \liminf_{j \to \infty} \left|(\Pi_S)_{\#}(F(\bar{x} + r_j(\cdot), S)\|W_j\|^s)\right|(B_{1/2}^d)$$

$$= \liminf_{j \to \infty} \left|(\Pi_S)_{\#}(F(\bar{x} + r_j(\cdot), S)\|W_j\|^s)\right|(E_j \cap B_{1/2}^d)$$

$$\leq \limsup_{j \to \infty} \left|(\Pi_S)_{\#}(F(\bar{x} + r_j(\cdot), S)\|W_j\|^s) - \gamma\mathcal{H}^d \llcorner B^d\right|(E_j \cap B_{1/2}^d) = 0,$$

since $(\Pi_S)_{\#}\|W_j\|^s$ is concentrated on E_j and $\mathcal{H}^d(E_j) = 0$, see (3.15) and (3.16). This contradiction concludes the proof. $\qquad\square$

4. Proof of Theorem 1.2

Proof of Theorem 1.2. Step 1: Sufficiency. Let F be a C^1 integrand satisfying the (AC) condition at every $x \in \Omega$ and let $V \in \mathscr{V}_F(\Omega)$, we want to apply Lemma 2.2 to $\|V_*\|$. Note that, according to Lemma 3.4 and (2.5),

$$\mathcal{H}^d \llcorner \{x \in \Omega : \Theta_*^d(x, \|V_*\|) > 0\} \ll \|V_*\| \ll \mathcal{H}^d \llcorner \{x \in \Omega : \Theta_*^d(x, \|V_*\|) > 0\}.$$

Since, by [19, Theorem 6.9], $\mathcal{H}^d\big(\{x \in \Omega : \Theta^{d*}(x, \|V_*\|) = +\infty\}\big) = 0$, we deduce that

$$0 < \Theta_*^d(x, \|V_*\|) \leq \Theta^{d*}(x, \|V_*\|) < +\infty \qquad \text{for } \|V_*\|\text{-a.e. } x \in \Omega,$$

hence assumption (i) of Lemma 2.2 is satisfied. By Lemma 3.1, $V_* = \|V_*\| \otimes \delta_{T_x}$ for some $T_x \in G(n, d)$, and, combining this with Lemma 2.4 and (2.6), for $\|V_*\|$-almost every $x \in \Omega$ every $\sigma \in \text{Tan}(x, \|V_*\|)$ is invariant along the directions of T_x, so that also assumption (ii) of Lemma 2.2 is satisfied. Hence

$$\|V_*\| = \theta\mathcal{H}^d \llcorner (K \cap \Omega),$$

for some rectifiable set K and Borel function θ. Moreover, again by Lemma 2.2, $T_xK = T_x$ for $\|V_*\|$-almost every x. This proves that V_* is d-rectifiable.

Step 2: Necessity. Let us now assume that $F(x, T) \equiv F(T)$ does not depend on the point, but just on the tangent plane and let us suppose that F does not verify the atomic condition (AC). We will show the existence of a varifold $V \in \mathscr{V}_F(\mathbb{R}^n)$, with positive lower d-dimensional density (namely $V = V_*$), which is not d-rectifiable. Indeed the negation of (AC) means that there exists a probability measure μ on $G(n, d)$, such that one of the following cases happens:

1) $\dim\big(\text{Ker}(A(\mu))\big) = \dim\big(\text{Ker}(A^*(\mu))\big) > n - d$
2) $\dim\big(\text{Ker}(A(\mu))\big) = \dim\big(\text{Ker}(A^*(\mu))\big) = n - d$ and $\mu \neq \delta_{T_0}$,

where $A(\mu) := \int_G B_F(T) \, d\mu(T)$ and $B_F(T) \in \mathbb{R}^n \otimes \mathbb{R}^n$ is constant in x. Let $W := \text{Im}A(\mu)$, $k = \dim W \leq d$ and let us define the varifold

$$V(dx, dT) := \mathcal{H}^k \llcorner W(dx) \otimes \mu(dT) \in \mathbb{V}_{\text{loc}}(\mathbb{R}^n).$$

Clearly V is not d-rectifiable since either $k < d$ or $\mu \neq \delta_W$. We start by noticing that $V = V_*$, indeed for $x \in W$

$$\Theta^d(x, V) = \lim_{r \to 0} \frac{\mathcal{H}^k(B_r(x) \cap W)}{\omega_d r^d} = \begin{cases} 1 & \text{if } k = d \\ +\infty & \text{if } k < d. \end{cases} \tag{4.1}$$

Let us now prove that $V \in \mathscr{V}_F(\mathbb{R}^d)$. For every $g \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$, we have

$$\delta_F V(g) = \int_W A(\mu) : Dg \, d\mathcal{H}^k = -\langle g, A^*(\mu)D(\mathcal{H}^k \llcorner W)\rangle = 0$$

since $D(\mathcal{H}^k \llcorner W) \in W^\perp = \operatorname{Im}A(\mu)^\perp = \operatorname{Ker}A^*(\mu)$. Hence V is $\mathbf{F}$-stationary and in particular $V \in \mathscr{V}_F(\mathbb{R}^d)$ which, together with (4.1) concludes the proof. $\qquad\square$

5. Proof of Theorem 1.3

In this section we prove Theorem 1.3. As explained in the introduction, it is convenient to identify the Grassmannian $G(n, n-1)$ with the projective space $\mathbb{RP}^{n-1} = \mathbb{S}^{n-1}/\pm$ via the map

$$\mathbb{S}^{n-1} \ni \pm\nu \mapsto \nu^\perp.$$

Hence an $(n-1)$-varifold V can be thought as a positive Radon measure $V \in \mathcal{M}_+(\Omega \times \mathbb{S}^{n-1})$ even in the $\mathbb{S}^{n-1}$ variable, i.e. such that

$$V(A \times S) = V(A \times (-S)) \qquad \text{for all } A \subset \Omega, \, S \subset \mathbb{S}^{n-1}.$$

In the same way, we identify the integrand $F : \Omega \times G(n, n-1) \to \mathbb{R}_{>0}$ with a positively one homogeneous even function $G : \Omega \times \mathbb{R}^n \to \mathbb{R}_{>0}$ via the equality

$$G(x, \lambda\nu) := |\lambda| F(x, \nu^\perp) \qquad \text{for all } \lambda \in \mathbb{R} \text{ and } \nu \in \mathbb{S}^{n-1}. \tag{5.1}$$

Note that $G \in C^1(\Omega \times (\mathbb{R}^n \setminus \{0\}))$ and that by one-homogeneity:

$$\langle d_e G(x, e), e \rangle = G(x, e) \qquad \text{for all } e \in \mathbb{R}^n \setminus \{0\}. \tag{5.2}$$

With these identifications, it is a simple calculation to check that:

$$\delta_F V(g) = \int_{\Omega \times \mathbb{S}^{n-1}} \langle d_x G(x, \nu), g(x) \rangle \, dV(x, \nu)$$
$$+ \int_{\Omega \times \mathbb{S}^{n-1}} \Big(G(x, \nu)\operatorname{Id} - \nu \otimes d_\nu G(x, \nu) \Big) : Dg(x) \, dV(x, \nu),$$

see for instance [2, Section 3] or [13, Lemma A.4]. In particular, under the correspondence (5.1)

$$B_F(x, T) = G(x, \nu)\operatorname{Id} - \nu \otimes d_\nu G(x, \nu) =: B_G(x, \nu), \qquad T = \nu^\perp.$$

Note that $B_G(x, \nu) = B_G(x, -\nu)$ since $G(x, \nu)$ is even. Hence the atomic condition at x can be re-phrased as:

(i) $\dim \operatorname{Ker}A_x(\mu) \leq 1$ for all *even* probability measures $\mu \in \mathcal{P}_{\text{even}}(\mathbb{S}^{n-1})$,
(ii) if $\dim \operatorname{Ker}A_x(\mu) = 1$ then $\mu = (\delta_{\nu_0} + \delta_{-\nu_0})/2$ for some $\nu_0 \in \mathbb{S}^{n-1}$,

where

$$A_x(\mu) = \int_{\mathbb{S}^{n-1}} B_G(x, \nu) d\mu(\nu).$$

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Since the (AC) condition deals only with the behavior of the frozen integrand $G_x(\nu) = G(x, \nu)$, for the whole proof x is fixed and for the sake of readability we drop the dependence on x.

Step 1: Sufficiency. Let us assume that $G : \mathbb{R}^n \to \mathbb{R}$ is even, one-homogeneous and strictly convex. We will show that the requirements (i) and (ii) in the (AC) condition are satisfied. First note that, by one-homogeneity, the strict convexity of G is equivalent to:

$$G(\nu) > \langle d_\nu G(\bar\nu), \nu \rangle \qquad \text{for all } \bar\nu, \nu \in \mathbb{S}^{n-1} \text{ and } \nu \neq \pm\bar\nu. \tag{5.3}$$

Plugging $-\nu$ in (5.3) and exploiting the fact that G is even we obtain

$$G(\nu) > |\langle d_\nu G(\bar\nu), \nu \rangle| \qquad \text{for all } \bar\nu, \nu \in \mathbb{S}^{n-1} \text{ and } \nu \neq \pm\bar\nu. \tag{5.4}$$

Let now $\mu \in \mathcal{P}_{\text{even}}(\mathbb{S}^{n-1}))$ be an even probability measure,

$$A(\mu) = \int_{\mathbb{S}^{n-1}} \Big(G(\nu)\text{Id} - \nu \otimes \mathrm{d}_\nu G(\nu) \Big) d\mu(\nu)$$

and assume there exists $\bar{\nu} \in \text{Ker}A(\mu) \cap \mathbb{S}^{n-1}$. We then have

$$0 = \langle d_\nu G(\bar{\nu}), A(\mu)\bar{\nu} \rangle$$
$$= \int_{\mathbb{S}^{n-1}} \Big\{ (G(\bar{\nu})G(\nu) - \langle d_\nu G(\bar{\nu}), \nu \rangle \langle d_\nu G(\nu), \bar{\nu} \rangle \Big\} d\mu(\nu)$$
$$\geq \int_{\mathbb{S}^{n-1}} \Big\{ G(\bar{\nu})G(\nu) - \big| \langle d_\nu G(\bar{\nu}), \nu \rangle \big| \big| \langle d_\nu G(\nu), \bar{\nu} \rangle \big| \Big\} d\mu(\nu)$$

where we have used (5.2). Inequality (5.4) implies however that the integrand in the last line of the above equation is strictly positive, unless $\nu = \pm\bar{\nu}$ for all $\nu \in \text{spt}\mu$, which immediately implies that the (AC) condition is satisfied.

Step 2: Necessity. Let us assume that G (or equivalently F) satisfies the (AC) condition, let $\nu, \bar{\nu} \in \mathbb{S}^{n-1}$, $\nu \neq \pm\bar{\nu}$ and define

$$\mu = \frac{1}{4} \big(\delta_\nu + \delta_{-\nu} + \delta_{\bar{\nu}} + \delta_{-\bar{\nu}} \big).$$

Then the matrix

$$A(\mu) = \frac{1}{2} B_G(\nu) + \frac{1}{2} B_G(\bar{\nu})$$

has full rank. In particular the vectors $A(\mu)\nu, A(\mu)\bar{\nu}$ are linearly independent. On the other hand

$$2A(\mu)\nu = B_G(\bar{\nu})\nu = G(\bar{\nu})\nu - \langle d_\nu G(\bar{\nu}), \nu \rangle \bar{\nu}$$
$$2A(\mu)\bar{\nu} = B_G(\nu)\bar{\nu} = G(\nu)\bar{\nu} - \langle d_\nu G(\nu), \bar{\nu} \rangle \nu$$

and thus, these two vectors are linearly independent if and only if

$$G(\nu)G(\bar{\nu}) - \langle d_\nu G(\bar{\nu}), \nu \rangle \langle d_\nu G(\nu), \bar{\nu} \rangle \neq 0.$$

Since G is positive and even, the above equation implies that

$$G(\nu)G(\bar{\nu}) - \langle d_\nu G(\bar{\nu}), \nu \rangle \langle d_\nu G(\nu), \bar{\nu} \rangle > 0 \qquad \text{for all } \nu \neq \pm\bar{\nu}. \tag{5.5}$$

We now show that (5.5) implies (5.3) and thus the strict convexity of G (actually Step 1 of the proof shows that they are equivalent). Let $\bar{\nu}$ be fixed and let us define the linear projection

$$P_{\bar{\nu}}\nu = \frac{\langle d_\nu G(\bar{\nu}), \nu \rangle}{G(\bar{\nu})} \bar{\nu}.$$

We note that by (5.2) $P_{\bar{\nu}}$ is actually a projection, i.e. $P_{\bar{\nu}} \circ P_{\bar{\nu}} = P_{\bar{\nu}}$. Hence, setting $\nu_t = t\nu + (1-t)P_{\bar{\nu}}\nu$ for $t \in [0,1]$, we have $P_{\bar{\nu}}\nu_t = P_{\bar{\nu}}\nu$. Thus

$$\nu_t - P_{\bar{\nu}}\nu_t = t(\nu - P_{\bar{\nu}}\nu). \tag{5.6}$$

Hence, if we define $g(t) = G(\nu_t)$, we have, for $t \in (0,1)$,

$$tg'(t) = t\langle d_\nu G(\nu_t), \nu - P_{\bar{\nu}}\nu \rangle = \langle d_\nu G(\nu_t), \nu_t - P_{\bar{\nu}}\nu_t \rangle > 0,$$

where in the second equality we have used equation (5.6) and the last inequality follows from (5.5) with $\nu = \nu_t$, and $t > 0$. Hence, exploiting also the one-homogeneity of G,

$$G(\nu) = g(1) > g(0) = G(P_{\bar{\nu}}\nu) = \frac{\langle d_\nu G(\bar{\nu}), \nu \rangle}{G(\bar{\nu})} G(\bar{\nu}) = \langle d_\nu G(\bar{\nu}), \nu \rangle$$

which proves (5.3) and concludes the proof. $\qquad\qquad\square$

Appendix A. First variation with respect to anisotropic integrands

In this section we compute the **F**-first variation of a varifold V. To this end we recall that, by identifying a d-plane T with the orthogonal projection onto T, we can embed $G(n,d)$ into $\mathbb{R}^n \otimes \mathbb{R}^n$. Indeed we have

$$G(n,d) \approx \Big\{ T \in \mathbb{R}^n \otimes \mathbb{R}^n : T \circ T = T, \quad T^* = T, \quad \operatorname{tr} T = d \Big\}. \qquad (A.1)$$

With this identification, let $T(t) \in G(n,d)$ be a smooth curve such that $T(0) = T$. Differentiating the above equalities we get

$$T'(0) = T'(0) \circ T + T \circ T'(0) \qquad (T'(0))^* = T'(0) \qquad \operatorname{tr} T'(0) = 0. \qquad (A.2)$$

In particular from the first equality above we obtain

$$T \circ T'(0) \circ T = 0, \qquad T^\perp \circ T'(0) \circ T^\perp = 0.$$

Hence

$$\operatorname{Tan}_T G(n,d) \subset \big\{ S \in \mathbb{R}^n \otimes \mathbb{R}^n : S^* = S, \quad T \circ S \circ T = 0, \quad T^\perp \circ S \circ T^\perp = 0 \big\}.$$

Since $\dim \operatorname{Tan}_T G(n,d) = \dim G(n,d) = d(n-d)$ the above inclusion is actually an equality. To compute the anisotropic first variation of a varifold we need the following simple Lemma:

Lemma A.1. *Let $T \in G(n,d)$ and $L \in \mathbb{R}^n \otimes \mathbb{R}^n$, and let us define $T(t) \in G(n,d)$ as the orthogonal projection onto $(\operatorname{Id} + tL)(T)$ (recall the identification (A.1)). Then*

$$T'(0) = T^\perp \circ L \circ T + (T^\perp \circ L \circ T)^* \in \operatorname{Tan}_T G(n,d).$$

Proof. One easily checks that $T(t)$ is a smooth function of T for t small. Since

$$T(t) \circ (\operatorname{Id} + tL) \circ T = (\operatorname{Id} + tL) \circ T,$$

differentiating we get

$$T'(0) \circ T = (\operatorname{Id} - T) \circ L \circ T = T^\perp \circ L \circ T. \qquad (A.3)$$

Using that $(T'(0))^* = T'(0)$, $T^* = T$, the first equation in (A.2) and (A.3), one obtains

$$\begin{aligned} T'(0) &= T'(0) \circ T + T \circ T'(0) \\ &= T'(0) \circ T + (T'(0) \circ T)^* = T^\perp \circ L \circ T + (T^\perp \circ L \circ T)^*, \end{aligned}$$

and this concludes the proof. $\qquad \square$

We are now ready to compute the first variation of an anisotropic energy:

Lemma A.2. *Let $F \in C^1(\Omega \times G(n,d))$ and $V \in \mathbb{V}_d(\Omega)$, then for $g \in C^1_c(\Omega, \mathbb{R}^n)$ we have*

$$\delta_F V(g) = \int_{G(\Omega)} \Big[\langle d_x F(x,T), g(x) \rangle + B_F(x,T) : Dg(x) \Big] dV(x,T), \qquad (A.4)$$

where the matrix $B_F(x,T) \in \mathbb{R}^n \otimes \mathbb{R}^n$ is uniquely defined by

$$B_F(x,T) : L := F(x,T)(T:L) + \big\langle d_T F(x,T), T^\perp \circ L \circ T + (T^\perp \circ L \circ T)^* \big\rangle \qquad (A.5)$$

for all $L \in \mathbb{R}^n \otimes \mathbb{R}^n$.

Proof. For $g \in C_c^1(\Omega, \mathbb{R}^n)$ let $\varphi_t(x) = x + tg(x)$ which is a diffeomorphism of Ω into itself for $t \ll 1$. We have

$$
\begin{aligned}
\delta_F V(g) &= \frac{d}{dt} \mathbf{F}(\varphi_t^\# V)\Big|_{t=0} \\
&= \frac{d}{dt} \int_{G(\Omega)} F(\varphi_t(x), d\varphi_t(T)) J\varphi_t(x, T) dV(x, T)\Big|_{t=0} \\
&= \int_{G(\Omega)} \frac{d}{dt} F(\varphi_t(x), T) dV(x, T)\Big|_{t=0} + \int_{G(\Omega)} \frac{d}{dt} F(x, d\varphi_t(T)) dV(x, T)\Big|_{t=0} \\
&\quad + \int_{G(\Omega)} F(x, T) \frac{d}{dt} J\varphi_t(x, T)\Big|_{t=0} dV(x, T).
\end{aligned}
$$

Equation (A.4) then follows by the definition of $B_F(x, T)$, (A.5), and the equalities

$$
\frac{d}{dt} F(\varphi_t(x), T)\Big|_{t=0} = \langle d_x F(x, T), g(x) \rangle \tag{A.6}
$$

$$
\frac{d}{dt} J\varphi_t(x, T)\Big|_{t=0} = T : Dg(x). \tag{A.7}
$$

$$
\frac{d}{dt} F(x, d\varphi_t(T))\Big|_{t=0} = \langle d_T F(x, T), T^\perp \circ Dg(x) \circ T + (T^\perp \circ Dg(x) \circ T)^* \rangle \tag{A.8}
$$

Here (A.6) is trivial, (A.7) is a classical computation, see for instance [21, Section 2.5], and (A.8) follows from Lemma A.1. □

Appendix B. Proof of Lemma 2.2

In this Section we prove Lemma 2.2. Let us start recalling the following rectifiability criterion due to Preiss, see [20, Theorem 5.3].

Theorem B.1. *Let μ be a measure on $\mathbb{R}^n$ and assume that at μ-a.e. x the following two conditions are satisfied:*

(I) *If we set $\alpha = \alpha_d = 1 - 2^{-d-6}$ and*

$$
E_r(x) := \left\{ z \in B_r(x) : \ \exists s \in (0, r) \ satisfying \ \frac{\mu(B_s(z))}{\omega_d s^d} \leq \alpha \frac{\mu(B_r(x))}{\omega_d r^d} \right\},
$$

then

$$
\liminf_{r \to 0} \frac{\mu(E_r(x))}{\mu(B_r(x))} = 0;
$$

(II) *If we set $\beta = \beta_d = 2^{-d-9} d^{-4}$ and*

$$
F_r(x) := \sup_{T \in G(n,d)} \left\{ \inf_{z \in (x+T) \cap B_r(x)} \frac{\mu(B_{\beta r}(z))}{\mu(B_r(x))} \right\},
$$

then

$$
\liminf_{r \to 0} F_r(x) > 0.
$$

Then μ is a d-rectifiable measure.

Proof of Lemma 2.2. By replacing μ with $\mu \llcorner \Omega'$, where $\Omega' \subset\subset \Omega$, we can assume that μ is defined on the whole $\mathbb{R}^n$. We are going to prove that μ verifies conditions (I) and (II) in Theorem B.1.

Let us start by verifying condition (I). Given $\varepsilon, m > 0$, let

$$
E(\varepsilon, m) := \left\{ z \in \mathbb{R}^n : \ \frac{\mu(B_r(z))}{\omega_d r^d} > m \ \text{for all } r \in (0, \varepsilon) \right\},
$$

and, for $\alpha = \alpha_d$ as in Theorem B.1 and $\gamma \in (1, 1/\alpha)$, set

$$\widehat{E}(\varepsilon, m) := E(\varepsilon, \alpha\gamma m) \setminus \bigcup_{k=1}^{\infty} E\left(\frac{\varepsilon}{k}, m\right).$$

If x is such that $0 < \Theta^d_*(x, \mu) < +\infty$, then $x \in \widehat{E}(\bar\varepsilon, \bar m)$ for some positive $\bar\varepsilon$ and $\bar m$ such that $\alpha\gamma\bar m < \Theta^d_*(x, \mu) < \bar m$, hence

$$\{0 < \Theta^d_*(x, \mu) < +\infty\} \subset \bigcup_{m > 0} \bigcup_{\varepsilon > 0} \widehat{E}(\varepsilon, m).$$

Let now $x \in \widehat{E}(\varepsilon, m)$ be a density point for $\widehat{E}(\varepsilon, m)$:

$$\lim_{r \to 0} \frac{\mu(B_r(x) \setminus \widehat{E}(\varepsilon, m))}{\mu(B_r(x))} = 0. \tag{B.1}$$

Note that $x \in \widehat{E}(\varepsilon, m)$ implies that $\alpha\gamma m \le \Theta^d_*(x, \mu) \le m < \gamma m$. Hence, if $(r_k)_k$ is a sequence verifying $r_k \to 0$, $r_k < \varepsilon$ and such that $\Theta^d_*(x, \mu) = \lim_k \mu(B_{r_k}(x))/\omega_d r_k^d$, then, for k large enough,

$$E_{r_k}(x) \subset B_{r_k}(x) \setminus E(\varepsilon, \alpha\gamma m) \subset B_{r_k}(x) \setminus \widehat{E}(\varepsilon, m),$$

which, together with (B.1), proves that μ verifies condition (I).

We now verify condition (II). Let x be a point such that all the tangent measures at x are translation invariant in the directions of T_x and such that $0 < \Theta^d_*(x, \mu) \le \Theta^{d*}(x, \mu) < +\infty$. Note that the latter condition implies that for every $\sigma \in \text{Tan}(x, \mu)$

$$\frac{\Theta^d_*(x, \mu)}{\Theta^{d*}(x, \mu)} t^d \le \sigma(B_t) \le \frac{\Theta^{d*}(x, \mu)}{\Theta^d_*(x, \mu)} t^d \qquad \text{for all } t \in (0, 1).$$

In particular, $0 \in \text{spt}\,\sigma$ for all $\sigma \in \text{Tan}(x, \mu)$. Let us choose a sequence $r_i \to 0$ and $z_{r_i} \in (x + T_x) \cap B_{r_i}(x)$, such that

$$\liminf_{r \to 0} \left\{ \inf_{z \in (x+T_x) \cap B_r(x)} \frac{\mu(B_{\beta r}(z))}{\mu(B_r(x))} \right\} = \lim_{i \to \infty} \frac{\mu(B_{\beta r_i}(z_{r_i}))}{\mu(B_{r_i}(x))}$$

$$\ge \lim_{i \to \infty} \mu_{x, r_i}\left(B_\beta\left(\frac{z_{r_i} - x}{r_i}\right)\right),$$

where μ_{x, r_i} is defined in (2.1) and $\beta = \beta_d$ is as in Theorem B.1. Up to subsequences we have that

$$\lim_{i \to \infty} \mu_{x, r_i} \xrightarrow{*} \sigma \in \text{Tan}(x, \mu) \qquad \text{and} \qquad \lim_{i \to \infty} \frac{z_{r_i} - x}{r_i} = z \in \bar{B} \cap T_x.$$

Hence

$$\liminf_{r \to 0} \left\{ \inf_{z_r \in (x+T_x) \cap B_r(x)} \frac{\mu(B_{\beta r}(z_r))}{\mu(B_r(x))} \right\} \ge \sigma(B_\beta(z)).$$

Let $z' \in B_{\beta/2}(z) \cap T_x$ such that $B_{\beta/2}(z') \subset B_\beta(z) \cap B$. Since σ is translation invariant in the directions of T_x

$$\sigma(B_\beta(z)) \ge \sigma(B_{\frac{\beta}{2}}(z')) = \sigma(B_{\frac{\beta}{2}}(0)) > 0,$$

where in the last inequality we have used that $0 \in \text{spt}\,\sigma$. Thus

$$\liminf_{r \to 0} F_r(x) \ge \liminf_{r \to 0} \left\{ \inf_{z \in (x+T_x) \cap B_r(x)} \frac{\mu(B_{\beta r}(z))}{\mu(B_r(x))} \right\} > 0,$$

implying that also condition (II) in Theorem B.1 is satisfied. Hence μ is d-rectifiable. In particular for μ-a.e. x, $\text{Tan}(x, \mu) = \{\omega_d^{-1} \mathcal{H}^d \llcorner (T_x K \cap B)\}$. Since, by assumption,

μ is invariant along the directions of T_x, this implies that $T_x = T_x K$ and concludes the proof. □

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