Generalized Monogamy Relations of Concurrence for \(N\)-qubit Systems

by

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Quantum entanglement [1–6] is an essential feature of quantum mechanics, which distinguishes the quantum from classical world. As one of the fundamental differences between quantum entanglement and classical correlations, a key property of entanglement is that a quantum system entangled with one of other systems limits its entanglement with the remaining others. In multipartite quantum systems, there can be several inequivalent types entanglement among the subsystems and the amount of entanglement with different types might not be directly comparable to each other. The monogamy relation of entanglement is a way to characterize the different types of entanglement distribution. The monogamy relations give rise to the structures of entanglement in the multipartite setting. Monogamy is also an essential feature allowing for security in quantum key distribution [7]. Monogamy relations are not always satisfied by entanglement measures. Although the concurrence and entanglement of formation do not satisfy such monogamy inequality, it has been shown that the oth (at \( \alpha = \sqrt{2} \)) power of concurrence and oth (at \( \alpha = \sqrt{2} \)) power of entanglement of formation for \( N \)-qubit states do satisfy the monogamy relations [8].

In this paper, we study the general monogamy inequalities satisfied by the concurrence and concurrence of assistance. We show that the concurrence of multi-qubit pure states satisfies some generalized monogamy inequalities.

The concurrence for a bipartite pure state \( |\psi\rangle_{AB} \) is given by [10–12]

\[
C(|\psi\rangle_{AB}) = \sqrt{2[1-\text{Tr}(\rho_A^2)]},
\]

(1)

where \( \rho_A \) is the reduced density matrix by tracing over the subsystem \( B \), \( \rho_A = \text{Tr}_B(|\psi\rangle_{AB}\langle\psi|) \). The concurrence is extended to mixed states \( \rho = \sum_i p_i |\psi_i\rangle \langle \psi_i| \), \( 0 \leq p_i \leq 1 \), \( \sum_i p_i = 1 \), by the convex roof extension,

\[
C(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle),
\]

(2)

where the minimum is taken over all possible pure state decompositions of \( \rho_{AB} \).

For a tripartite state \( |\psi\rangle_{ABC} \), the concurrence of assistance is defined by [13]

\[
C_a(|\psi\rangle_{ABC}) \equiv C_a(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle),
\]

(3)

for all possible ensemble realizations of \( \rho_{AB} = \text{Tr}_C(|\psi\rangle_{ABC}\langle\psi|) = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i| \). When \( \rho_{AB} = |\psi\rangle_{AB}\langle\psi| \) is a pure state, then one has \( C(|\psi\rangle_{AB}) = C_a(\rho_{AB}) \).

For an \( N \)-qubit state \( |\psi\rangle_{AB_1...BN-1} \), the concurrence \( C(|\psi\rangle_{AB_1...BN-1}) \) of the state \( |\psi\rangle_{AB_1...BN-1} \), viewed as a bipartite state with partition \( A \) and \( B_1B_2...BN-1 \), satisfies the Coffman-Kundu-Wootters inequality (\( N = 3 \)) [14] and its generalization to \( N \)-qubit case. In [8, 9] it has been shown that the concurrence of a state \( \rho \) satisfies a more general monogamy inequality,

\[
C^\alpha_{AB_1B_2...BN-1} \geq C^\alpha_{AB_1} + C^\alpha_{AB_2} + ... + C^\alpha_{AB_{N-1}},
\]

where \( C^\alpha_{AB_{i+1}} = \text{Tr}_{B_{i+1}}(|\psi\rangle_{AB_{i+1}...BN-1}(\rho) \), \( C^\alpha_{AB_{i+1}} = C(|\psi\rangle_{AB_{i+1}...BN-1}) \), \( C^\alpha_{AB_{i+1}} = C(\rho_{AB_i})(i = 1, ..., N - 1) \) and \( \alpha \geq \sqrt{2} \). The dual inequality in terms of the concurrence of assistance for \( N \)-qubit states has the form [15],

\[
C^2(|\psi\rangle_{AB_1...BN-1}) \leq \sum_{i=1}^{N-1} C^2(\rho_{AB_i}).
\]

(4)

The concurrence (1) is related to the linear entropy of a state \( \rho \) [16],

\[
T(\rho) = 1 - \text{Tr}(\rho^2).
\]

For a bipartite state \( \rho_{AB} \), \( T(\rho) \) has property [17],

\[
|T(\rho_{AB}) - T(\rho_{BA})| \leq T(\rho_{AB}) \leq T(\rho_{PA}) + T(\rho_{PB}).
\]

(5)

**Theorem 1** For any \( 2 \otimes 2 \otimes ... \otimes 2 \) pure state \( |\psi\rangle = |\psi\rangle_{ABC_1C_2...CN-2} \), we have

\[
C^2(|\psi\rangle_{AB|C_1C_2...CN-2}) \geq \max \left\{ \sum_{i=1}^{N-2} [C^2(\rho_{AC_i}) - C^2(\rho_{BC_i})] \right\},
\]

(6)
where $\rho_{AB} = Tr_{C_1...C_{N-2}}(|\psi\rangle\langle\psi|)$, $\rho_{AC} = Tr_{BC_1...C_{N-1}}(|\psi\rangle\langle\psi|)$ and $\rho_{BC} = Tr_{AC_1...C_{N-1}}(|\psi\rangle\langle\psi|)$.

Theorem 2 For any $2 \otimes 2 \otimes ... \otimes 2 \otimes 2$ pure state $|\psi\rangle_{ABC_1...C_{N-2}}$, we have

$$C^2(|\psi\rangle_{AB|C_1...C_{N-2}}) \leq 2C^2(\rho_{AB}) + \sum_{i=1}^{N-2} \left( C^2(\rho_{AC_i}) + C^2(\rho_{BC_i}) \right),$$

where $\rho_{AB} = Tr_{C_1...C_{N-2}}(|\psi\rangle\langle\psi|)$, $\rho_{AC} = Tr_{BC_1...C_{N-1}}(|\psi\rangle\langle\psi|)$ and $\rho_{BC} = Tr_{AC_1...C_{N-1}}(|\psi\rangle\langle\psi|)$.

Proof] For any $2 \otimes 2 \otimes ... \otimes 2 \otimes 2$ pure state $|\psi\rangle_{ABC_1...C_{N-2}}$, one has:

$$C^2(\rho_{AB|C_1...C_{N-2}}(|\psi\rangle)) = 2(1 - Tr\rho_{AB})$$

$$\geq |2(1 - Tr(\rho_A)) - 2(1 - Tr(\rho_B))|$$

$$= |C^2_{A|BC_1...C_{N-2}} - C^2_{B|AC_1...C_{N-2}}|,$$

the first inequality is due to the left inequality in (5).

For a $2 \otimes 2 \otimes m$ quantum pure state $|\psi\rangle_{ABC}$, it has been shown that $C^2(\rho_{AB|C}) = C^2(\rho_{AB}) + \tau^2(|\psi\rangle_{ABC})$ [18], where $\tau^2(|\psi\rangle_{ABC}) = C^2_{ABC} - C^2_A - C^2_B$ is the three tangle of concurrence, $C^2_{ABC}$ is the concurrence of under bipartition $A|BC$ for pure state $|\psi\rangle_{ABC}$, $\rho_{AB|C} = Tr_{BC}(|\psi\rangle_{ABC}|\psi\rangle)$ and $AB\langle AC|C = C(\rho_{ABC})$. Namely,

$$C^2_{ABC} = C^2(\rho_{AB}) + C^2(\rho_{AC}) + C^2(\rho_{BC}).$$

Hence we have

$$C^2(\rho_{AB|C_1...C_{N-2}}(|\psi\rangle)) \geq C^2_{B|AC_1...C_{N-2}} - C^2_{A|BC_1...C_{N-2}} - C^2_{A|BC_1...C_{N-2}},$$

we can obtain another inequality in (6).

Theorem 1 shows that the entanglement contained in the pure states $|\psi\rangle_{ABC_1...C_{N-2}}$ is related to the sum of entanglement between bipartitions of the system.

The lower bound in inequalities (6) is easily calculable. As an example, let us consider the four-qubit pure state

$$|\psi\rangle_{ABCD} = \frac{1}{\sqrt{2}}(|0000\rangle + |1011\rangle).$$

We have $\rho_{ACD} = Tr_B(|\psi\rangle_{ABCD}|\psi\rangle) = \frac{1}{2}(|000\rangle\langle000| + |101\rangle\langle101|)$, $\rho_{BCD} = Tr_A(|\psi\rangle_{ABCD}|\psi\rangle) = \frac{1}{2}(|000\rangle\langle000| + |010\rangle\langle010|)$, $C(\rho_{AC}) = 0$, $C(\rho_{BD}) = 1$ and $C(\rho_{BC}) = C(\rho_{BD}) = 0$. Therefore, $C(|\psi\rangle_{ABCD}) \geq 1$, namely, the state $|\psi\rangle_{ABCD}$ saturates the inequality (6).

Theorem 1 gives a monogamy-type lower bound of $C(|\psi\rangle_{AB|C_1C_2...C_{N-2}})$. According to the subadditivity of the linear entropy, we also have:
\[ 2a^2 - b = a + b, \text{ where } a = \frac{3}{2}e_1 + \frac{2\sqrt{3}}{3}e_2, \]
\[ b = \frac{3}{2}e_1 - \frac{2\sqrt{3}}{3}e_2, \text{ and } e_1 \text{ and } e_2 \text{ are two orthogonal basic vectors in a plane.} \]

Now we consider further the generalized monogamy relations in terms of arbitrary partitions for the N-qubit generalized W-class states [19]:

\[ |W\rangle_{A_1...A_N} = a_1 |1\ldots0\rangle_{A_1...A_N} + \ldots + a_N |0\ldots1\rangle_{A_1...A_N}, \]

where \( \sum_{i=1}^{N} |a_i|^2 = 1 \). One has \( C(\rho_{A_iA_q}) = C_A(\rho_{A_iA_q}) \) \((p \neq q \in \{1, \ldots, N\})\). Then (6) and (8) give rise to

\[ \begin{align*}
& \left| \sum_{i \in I} [C(\rho_{A_iA_i}) - C(\rho_{A_jA_i})] \right| \\
& \leq C^2(\hat{\rho}_{A_iA_jA_{I\setminus\{i\}}}) \\
& \leq 2C^2(\rho_{A_iA_i}) + \sum_{i \in I} [C^2(\rho_{A_iA_i}) + C^2(\rho_{A_jA_i})],
\end{align*} \]

where \( 1 \leq i < j \leq N, \{A_iA_j\} = \{A_1...A_i-1, A_i+1...A_j-1, A_{j+1}...A_N\} \) and \( I = \{1, 2, \ldots, i-1, i+1, \ldots, j-1, j+1, \ldots, N\} \).

The inequality Eq.(10) implies that the entanglement (square of concurrence) between \( A_iA_j \) and the other qubits cannot be more than the sum of the individual entanglements between \( A_i \) and each of the \( N-1 \) remaining qubits and the the individual entanglements between \( A_j \) and each of the \( N-1 \) remaining qubits. Take 5-qubit generalized W-class states as an example, one has \( C_{ABC1C2C3}^2 \leq \sum_{\ell=(B,C_1,C_2,C_3)} C^2(\rho_{A\ell}) + \sum_{\ell=(A,C_1,C_2,C_3)} C^2(\rho_{B\ell}), \) see Fig. 1.

**FIG. 1:** Generalized monogamy for five-qubit generalized W-class states.

Now we generalize our results to the concurrence \( C_{ABC1C2...C_{N-2}} \) under partition \( ABC1 \) and \( C_{2...C_{N-2}} \) for pure state \( |\psi\rangle_{ABC1C2...C_{N-2}} \), similar to the theorem 1 and the theorem 2, we can obtain the following corollaries:

**Corollary 1:** For any N-qubit pure state \( |\psi\rangle_{ABC1C2...C_{N-2}} \), we have

\[ C^2(|\psi\rangle_{ABC1C2...C_{N-2}}) \]
\[ \geq \max \{ \sum_{i=1}^{N-2} C^2(\rho_{ACi}) - \sum_{i=1}^{N-2} C^2(\rho_{BCi}), \]
\[ \sum_{i=1}^{N-2} C^2(\rho_{BCi}) - \sum_{i=1}^{N-2} C^2(\rho_{ACi}) \} - \sum_{j \in J} C^2(\rho_{C_{ij}}), \]

where \( J = \{A, B, C_2, \ldots, C_{N-2}\} \) and \( \rho_{C_{ij}} \) is the reduced density matrix by tracing over the subsystems except for \( C_1 \) and \( j \).

[proof] For any N-qubit pure state \( |\psi\rangle_{ABC1C2...C_{N-2}} \), we have

\[ C^2(|\psi\rangle_{ABC1C2...C_{N-2}}) \]
\[ = 2(1 - Tr(r_{ABC1})) \]
\[ \geq 2(1 - Tr(r_{A_{BC1}})) - 2(1 - Tr(r_{C_{ij}})) - C^2(|\psi\rangle_{C_1|ABC2...C_{N-2}}), \]

the first inequality is due to \( Tr(\rho_{ABC1}) \geq Tr(\rho_{A}) - Tr(\rho_{BC1}) \). Combining Theorem 1 and (4), we obtain (11).

On the other hand, from the property of linear entropy, \( Tr(\rho_{ABC1}) \geq Tr(\rho_{C_{ij}}) - Tr(\rho_{AB}) \), we also can obtain the follow corollary.

**Corollary 2:** For any N-qubit pure state \( |\psi\rangle_{ABC1C2...C_{N-2}} \), we have

\[ C^2(|\psi\rangle_{ABC1C2...C_{N-2}}) \]
\[ \geq C^2(\rho_{AC1}) + C^2(\rho_{BC1}) + \sum_{i=1}^{N-2} C^2(\rho_{C_{ij}}), \]
\[ - 2C^2(\rho_{AB}) - \sum_{i=1}^{N-2} \left( C^2(\rho_{ACi}) + C^2(\rho_{BCi}) \right), \]

and

\[ C^2(|\psi\rangle_{ABC1C2...C_{N-2}}) \]
\[ \leq 2C^2(\rho_{AB}) + \sum_{i=1}^{N-2} \left( C^2(\rho_{ACi}) + C^2(\rho_{BCi}) \right) \]
\[ + \sum_{j \in J} C^2(\rho_{C_{ij}}), \]

where \( J \) and \( \rho_{C_{ij}} \) are defined as in Corollary 1.

In corollary 2, the upper bound is due to the right inequality of (4) and (5). Analogously, by use of \( T(\rho_{ABC1}) \geq T(\rho_{AC1}) - Tr(\rho_{AB}) \), \( T(\rho_{ABC1}) \geq T(\rho_{A}) - Tr(\rho_{BC1}) \) and \( T(\rho_{ABC1}) \leq T(\rho_{A}) + T(\rho_{BC1}) + T(\rho_{C_{ij}}) \), one can get more results like (11), (12) and (13).
The lower bounds in corollary 1 and corollary 2 are not equivalent. We consider the following two examples to show that corollary 1 and corollary 2 give rise to different lower bounds.

Example 1: Let us consider the pure state $|\psi\rangle_{ABCiC_2C_3C_4} = \frac{1}{\sqrt{2}}(|000000\rangle + |011000\rangle)$. We have $C(\rho_{AB}) = C(\rho_{AC}) = C(\rho_{BC}) = C(\rho_{C_2C_3C_4}) = 0$ for $i = 2, 3, 4$; $C(\rho_{BC}) = C(\rho_{C_2C_3}) = 0$ for $i = 1, 2, 3, 4$, and $C(\rho_{AC}) = C(\rho_{AC_1}) = 1$. Thus $C(|\psi\rangle) \geq 1$ from (11) and $C(|\psi\rangle) \geq 0$ from (12). Namely bound (11) is better than (12) in this case.

Example 2: For the state $|\psi\rangle_{ABCiC_2C_3C_4} = \frac{1}{\sqrt{2}}(|000000\rangle + |011000\rangle)$, it is straightforward to calculate that $C(\rho_{AB}) = C(\rho_{AC}) = C(\rho_{BC}) = C(\rho_{C_2C_3C_4}) = C(\rho_{C_1C_2}) = C(\rho_{C_1C_3}) = C(\rho_{C_1C_4}) = C(\rho_{C_1C_2C_3C_4}) = 0$, $i = 1, 2, 3, 4$ and $C(\rho_{C_2C_3C_4}) = C(\rho_{C_1C_2C_3}) = C(\rho_{C_1C_2C_4}) = 1$. Hence $C(|\psi\rangle) \geq 0$ from (11), $C(|\psi\rangle) \geq 1$ from (12) and bound (12) is better than (11) for this state.

We have presented the generalized monogamy relations of concurrence for N-qubit systems, by showing the relations among $C(|\psi\rangle_{ABC_1...C_{N-2}})$, $C_{AB}$, $C_{AC_1}$, $C_{BC_1}$, $C_{C_1C_2}$, $C_{C_1C_3}$, $C_{C_1C_4}$, $C_{C_1C_2C_3C_4}$, $2 \leq i \leq N - 2$, which give rise to the lower and upper bounds on the entanglement sharing among the partitions. Theorem 1 (Theorem 2) gives a lower (an upper) bound of $C(|\psi\rangle_{ABC|C_1...C_{N-2}})$. It has been shown that although $C(|\psi\rangle_{AB|C_1...C_{N-2}})$, $C(|\psi\rangle_{ABC|C_{N-2}})$ and $C(|\psi\rangle_{C_1|C_{N-2}})$ do not satisfy relation like $C^2(|\psi\rangle_{AB|C_1...C_{N-2}}) = C^2(|\psi\rangle_{AB|C_1...C_{N-2}})$, they do satisfy the triangle inequality: the sum of any two of them is greater or equal to the third.

Entanglement monogamy is a fundamental property of multipartite entangled states. We have presented a new kind of monogamy relations satisfied by the concurrence of N-qubit pure states under partition $AB$ and $C_1...C_{N-2}$ as well as under partition $ABC_1$ and $C_2...C_{N-2}$. These relations also give rise to a kind of trade off relations related to the lower and upper bounds of concurrences. Similar results can be obtained for concurrences under arbitrary partitions $C_{ABC_1...C_i|C_{i+1}...C_{N-2}}$ ($2 \leq i \leq N - 1$). Such restrictions on entanglement distribution may be also true for other measures of quantum correlations like negativity or quantum discord.

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