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by

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GLOBAL EXISTENCE OF THE HARMONIC MAP HEAT FLOW INTO LORENTZIAN MANIFOLDS

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Abstract. We investigate a parabolic-elliptic system for maps from a compact Riemann surface M into a Lorentzian manifold $N\times\mathbb{R}$ with a warped product metric. We prove a global existence result of the parabolic-elliptic system by assuming either some geometric conditions on the target Lorentzian manifold or small energy of the initial maps. The result implies the existence of a harmonic map in a given homotopy class with fixed boundary data.

1. introduction

Suppose (M,h) and (N,g) are compact Riemannian manifolds of dimension m and n respectively. For a map $u\in C^2(M,N)$, the energy functional of u is defined as

$$E(u)=\frac{1}{2}\int_M |\nabla u|^2 dv_h. \tag{1.1}$$

A critical point of the energy functional E in $C^2(M,N)$ is called a harmonic map. By Nash's embedding theorem [28], we can embed N isometrically into some Euclidian space $\mathbb{R}^K$ and the corresponding Euler-Lagrange equation is

$$\Delta_h u = A(u)(\nabla u,\nabla u),$$

where Δ_h is the Laplace-Beltrami operator on M with respect to h and A is the second fundamental form of $N\subset\mathbb{R}^K$. More generally, we define the tension field $\tau(u)$ as

$$\tau(u) = -\Delta_h u + A(u)(\nabla u,\nabla u).$$

Thus, u is harmonic if and only if $\tau(u)=0$.

Harmonic maps constitute one of the model problems of geometric analysis and have been widely studied in the past several decades. For example, the methods used in the study of harmonic maps can be adapted to the study of constant mean curvature surfaces, pseudo-holomorphic curves, etc. In physics, harmonic maps arise as a mathematical representation of the nonlinear sigma model. This leads to several generalizations. For example, motivated by the supersymmetric sigma model, the map can be coupled with a spinor field, see [8] and [21] and the references there in.

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From another perspective, that of general relativity, it is also natural to replace the target of the harmonic maps to Lorentzian manifolds. Recent work on minimal surfaces in anti-de-Sitter space and their applications in theoretical physics (see e.g. Alday and Maldacena[3]) shows the importance of this extension. Geometrically, the link between harmonic maps into $\mathbb{S}^4_1$ and the conformal Gauss maps of Willmore surfaces in $\mathbb{S}^3$ [5] also naturally leads to such harmonic maps. The existence of geodesics in Lorentzian manifolds was studied in [4]. Variational methods for such harmonic maps were developed in [12] and [13]. The regularity of weak solutions was studied in [34], and in [16] energy identities for harmonic map sequences were obtained.

In this paper, we shall address the existence problem for harmonic maps from Riemann surfaces to standard static Lorentzian manifolds (to be defined shortly). Let us now state our results. Let M be a compact Riemannian manifold with a smooth boundary ∂M and $N \times \mathbb{R}$ be a Lorentzian manifold equipped with a warped product metric of the following form

$$g = g_N - \beta(d\theta)^2,$$

where $(\mathbb{R}, d\theta^2)$ is the 1-dimensional Euclidean space, (N, g_N) is a n-dimensional compact Riemannian manifold embedded into $\mathbb{R}^K$ and β is a positive C^∞ function on N. Since N is compact, there exist two positive constants λ and Λ such that

$$0 < \lambda \le \beta(y) \le \Lambda < \infty, \ \forall \ y \in N.$$

In fact, a more general form of the warped product metric is

$$g = g_N - \beta(d\theta + \omega)^2, \tag{1.2}$$

where ω is a 1-form on N. A Lorentzian manifold with a metric of the form (1.2) is called a standard static manifold. For more details on such manifolds, we refer to [22, 29]. To simplify the problem, throughout this paper, we assume that $\omega = 0$; the case $\omega \neq 0$ will be discussed in future work.

For $(u, v) \in C^2(M, N \times \mathbb{R})$, we consider the following functional

$$E_g(u, v; M) = \frac{1}{2}\int_M \left\{|\nabla u|^2 - \beta(u)|\nabla v|^2\right\} dv_h, \tag{1.3}$$

which is called the Lorentzian energy of the map (u, v) on M. A critical point (u, v) in $C^2(M, N \times \mathbb{R})$ of the functional (1.3) is called a harmonic map from (M, h) into the Lorentzian manifold $(N \times \mathbb{R}, g)$. Via direct calculation, one can derive the Euler-Lagrange equations for (1.3),

$$\begin{cases} -\Delta u = A(u)(\nabla u, \nabla u) - B^\top(u)|\nabla v|^2, & in\ M \\ -div\{\beta(u)\nabla v\} = 0, & in\ M \end{cases} \tag{1.4}$$

where A is the second fundamental form of N in $\mathbb{R}^K$, $B(u) := (B^1, B^2, \cdots, B^K)$ with

$$B^j := -\frac{1}{2}\frac{\partial \beta(u)}{\partial y^j}$$

and $B^\top$ is the tangential part of B along the map u. For details, see [34].

Let us explain some notations first. For $\Omega \subset M$, we put

$$E(u,v;\Omega) = \frac{1}{2}\int_\Omega \left\{|\nabla u|^2 + |\nabla v|^2\right\} dx,$$

$$E(u;\Omega) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx,$$

and

$$E(v;\Omega) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx.$$

For brevity, we will omit Ω if the domain is clear from the context.

When the domain is a Riemann surface, the Lorentzian energy $E_g(u,v)$ is conformally invariant as in the case of a Riemannian target, and the analytical framework of Riemannian harmonic maps thus has a chance to work. As in the Riemannian case, we want to find the conditions for the existence of a harmonic map in a given homotopy class when the target manifold is Lorentzian. For the Riemannian case, Al'ber [1, 2], Eells-Sampson [11] and Hamilton [15] proved an existence result using the harmonic map heat flow when the sectional curvature of the target manifold is non-positive. These papers investigate the evolution problem associated with the Euler-lagrange equations which can be interpreted as a gradient flow for the energy functional (1.1).

In the Lorentzian case, it is thus natural to seek an analogous parabolic system. Although the flow is still a gradient flow, due to the unboundedness of the energy $E(u,v)$ caused by the Lorentzian metric, we can no longer expect the long time existence of the heat flow. In fact, because of the lack of boundedness, even the blow up analysis along the heat flow cannot be carried over from the Riemannian case. To overcome these difficulties, we consider the following parabolic-elliptic flow inspired by the construction of [9] to study the flow of Dirac harmonic maps.

Let $\phi \in C^{2+\alpha}(M,N)$, $\psi \in C^{2+\alpha}(\partial M,\mathbb{R})$ for some $0<\alpha<1$. Consider the flow

$$\begin{cases} \partial_t u = \Delta u + A(u)(\nabla u,\nabla u) - B^\top(u)|\nabla v|^2, & in\ M\times[0,T) \\ -div(\beta(u)\nabla v) = 0, & in\ M\times[0,T) \end{cases} \tag{1.5}$$

with the boundary-initial data

$$\begin{cases} u(x,t) = \phi(x), & on\ M\times\{t=0\}\cup\partial M\times\{t>0\}, \\ v(x,t) = \psi(x), & on\ \partial M\times\{t>0\}. \end{cases} \tag{1.6}$$

By standard elliptic theory, for the above (ϕ,ψ), there exists a unique solution $v \in C^{2+\alpha}(M)$ of the equation

$$\begin{cases} -div(\beta(\phi)\nabla v) = 0 & in\ M, \\ v(x) = \psi & on\ \partial M. \end{cases}$$

This v is called an extension of ψ. For simplicity, we still denote it by $\psi \in C^{2+\alpha}(M)$ and in the following, we use the extension when needed.

Now, we state our first main result, concerning the short time existence of the flow (1.5).

Theorem 1.1. *Let (M^m, h) $(m = 2)$ be a compact Riemann surface with a smooth boundary ∂M and (N, g) be another compact Riemannian manifold. Then for any*

$$\phi \in C^{2+\alpha}(M, N),\ \psi \in C^{2+\alpha}(\partial M, \mathbb{R})$$

where $0 < \alpha < 1$, the problem (1.5) and (1.6) admits a unique solution

$$u \in \cap_{0<s<T_1} C^{2+\alpha,1+\alpha/2}(M \times [0, s]),$$

and

$$v, \nabla v \in \cap_{0<s<T_1} C^{\alpha,\alpha/2}(M \times [0, s]),\ v \in L^\infty([0, T_1); C^{2+\alpha}(M)),$$

for some time $T_1 > 0$. Here, the maximum existence time T_1 is characterized by the condition that

$$\limsup_{x\in M, t\to T_1} E(u; B_r^M(x)) > \epsilon_1 \text{ for any } r > 0,$$

where ϵ_1 is the constant in Lemma 2.6 and $B_r^M(x)$ is a geodesic ball in M. Moreover, the set

$$S(u, T_1) := \{x \in M | \limsup_{t\to T_1} E(u; B_r^M(x)) > \epsilon_1 \text{ for any } r > 0\} \tag{1.7}$$

is finite and a point in it is called a singularity at the singular time T_1.

Moreover, we show that at a singular point (x, T_1), $0 < T_1 \le \infty$, after suitable space-time rescalings, a nontrivial harmonic sphere splits off.

Theorem 1.2. *Let (u, v) be the solution to (1.5) with the boundary-initial data (1.6) in Theorem 1.1. Suppose (x_0, T_1) is a singularity such that*

$$\limsup_{t\nearrow T_1} E(u(t); B_r^M(x_0)) > \epsilon_1 \quad \text{for all} \quad r > 0. \tag{1.8}$$

Then

(1) *if $x_0 \in M \setminus \partial M$, there exist sequences $t_i \nearrow T_1$, $x_i \to x_0 \in M$, $r_i \to 0$ and a nontrivial harmonic map $\widetilde{u} : \mathbb{R}^2 \to N$, such that as $i \to \infty$,*

$$u(x_i + r_i x, t_i) \to \widetilde{u}(x) \quad \text{in} \quad C^1_{loc}(\mathbb{R}^2).$$

$\widetilde{u}$ has finite energy and conformally extends to a smooth harmonic sphere.

(2) *if $x_0 \in \partial M$, we have $\frac{dist(x_i,\partial M)}{r_i} \to \infty$ and the same bubbling statement as in (1) holds.*

Next we present two main results which establish the existence of a harmonic map from M into the Lorentzian manifold $N \times \mathbb{R}$ in any given homotopy class in two cases. In the first case, we assume that the initial energy $E(\phi; M)$ is small and in the second case, we assume that the sectional curvature K_N of the Riemannian manifold N is non-positive. More precisely, we have

Theorem 1.3. *For any given* $(\phi, \psi) \in C^{2+\alpha}(M, N \times \mathbb{R})$*, there exist constants* $\overline{\epsilon}_1 \geq 0$*,* $\overline{\epsilon}_2 > 0$ *and* $\overline{\epsilon} > 0$ *which are defined by*

$$\overline{\epsilon}_1 := \inf\{E(w) | w \in W^{1,2}(M, N), w|_{\partial M} = \phi\},$$

$$\overline{\epsilon}_2 := \inf\{E(w) | w : S^2 \to N \text{ is a harmonic map and nonconstant}\},$$

and $\overline{\epsilon} := \overline{\epsilon}_1 + \overline{\epsilon}_2 > 0$ *such that if*

$$E(\phi; M) + (\Lambda - \lambda) E(\psi; M) \leq \overline{\epsilon},$$

then the parabolic-elliptic system (1.5) *and* (1.6) *admits a global solution*

$$u \in \cap_{0<s<\infty} C^{2+\alpha, 1+\alpha/2}(M \times [0, s]),$$

and

$$v, \nabla v \in \cap_{0<s<\infty} C^{\alpha, \alpha/2}(M \times [0, s]), \ v \in L^\infty([0, \infty); C^{2+\alpha}(M)).$$

Moreover, $(u(x,t), v(x,t))$ *subconverges in* C^2 *to a harmonic map* $(u_\infty, v_\infty) : M \to N \times \mathbb{R}$ *with boundary data* $u_\infty|_{\partial M} = \phi$ *and* $v_\infty|_{\partial M} = \psi$*.*

Theorem 1.3 generalizes the result for the harmonic map heat flow by Chang [6] to the Lorentzian case. Before introducing the theorem in the second case, we need the following definition.

Definition 1.1. *Let* $\rho \in C^2(N)$ *be a nonnegative function on a Riemannian manifold* (N, g) *and* $d(x, x_0)$ *be the distance between* $x \in N$ *and* $x_0 \in N$*. If* ρ *satisfies*

(1) $\nabla^2 \rho(x) > 0$ *for any* $x \in N$*;*
(2) $\rho(x) \leq C(1 + d(x, x_0))^{d_0}$ *for some positive integer* d_0 *and fixed* $x_0 \in N$*,*

we call ρ *a nonnegative strictly convex function with polynomial growth.*

When N has non-positive sectional curvature, then the squared distance function $d^2(., x_0)$ for any $x_0 \in \widetilde{N}$ is such a function on $\widetilde{N}$, where $\widetilde{N}$ is the universal covering space of (N, g), with metric $\widetilde{g} := \pi_N^* g$ being the pull-back metric on $\widetilde{N}$ and $\pi_N : \widetilde{N} \to N$ being the projection. Therefore, our subsequent results will apply to targets of non-positive sectional curvature.

We have

Theorem 1.4. *Suppose the universal covering space* $(\widetilde{N}, \widetilde{g})$ *admits a nonnegative strictly convex function* $\rho \in C^2(\widetilde{N})$ *with polynomial growth. For any given* $\phi \in C^{2+\alpha}(M, N)$*,* $\psi \in C^{2+\alpha}(\partial M, \mathbb{R})$*, the parabolic-elliptic system* (1.5) *and* (1.6) *admits a global solution*

$$u \in \cap_{0<s<\infty} C^{2+\alpha, 1+\alpha/2}(M \times [0, s]),$$

and

$$v, \nabla v \in \cap_{0<s<\infty} C^{\alpha, \alpha/2}(M \times [0, s]), \ v \in L^\infty([0, \infty); C^{2+\alpha}(M)).$$

Moreover, $(u(x,t), v(x,t))$ *subconverges in* C^2 *to a harmonic map* $(u_\infty, v_\infty) : M \to N \times \mathbb{R}$ *with boundary data* $u_\infty|_{\partial M} = \phi$ *and* $v_\infty|_{\partial M} = \psi$*.*

In the Riemannian case, such a result was first proved by Ding and Lin [10] when the universal covering of the target manifold admits a nonnegative strictly convex function with quadratic growth. The polynomial growth case was proved by Li and Zhu [24].

As already mentioned, since for a Riemannian manifold N with non-positive sectional curvature K_N, the square of the distance function on the universal covering of N is a nonnegative strictly convex function with quadratic growth, the existence theorem for harmonic maps by Al'ber [1, 2], Eells and Sampson [11], Hamilton [15] and Hildebrandt-Kaul-Widman [18] can be generalized for two-dimensional domains to the Lorentzian case as a corollary of Theorem 1.4.

Theorem 1.5. *When (N, g) is a compact Riemannian manifold with non-positive sectional curvature, the conclusions in Theorem 1.4 hold.*

The paper is organized as follows. In Section 2, we derive some a priori estimates. In Section 3, we prove a small energy regularity lemma. Also, we establish the short time existence theorem 1.1 and give a characterization of the singularities in this section. In section 4, we analyze the blow up behavior of the singularities developed by the flow and prove our Theorem 1.2. In section 5, we use the blow up analysis to get some long time existence and convergence results. Theorem 1.3 and Theorem 1.4 are proved in this section. Throughout this paper, we use C to denote a universal constant.

Notation:

$$\begin{aligned}\mathcal{V}(M_s^t; N\times\mathbb{R}) := \{(u,v) :& M\times[s,t)\to N\times\mathbb{R},\ v\in L^\infty([s,t);C^{2+\alpha}(M)),\\ & v, \nabla v\in \cap_{s<\rho<t} C^{\alpha,\alpha/2}(M\times[s,\rho]),\\ & u\in\cap_{s<\rho<t}C^{2+\alpha,1+\alpha/2}(M\times[s,\rho]).\}\end{aligned}$$

2. Some a priori estimates

First, we present a lemma which ensures that the Lorentzian energy E_g is non-increasing along the flow (1.5). This is an important property of our parabolic-elliptic flow.

Lemma 2.1. *Suppose $(u, v) \in \mathcal{V}(M_0^{T_1}; N\times\mathbb{R})$ is a solution of (1.5) and (1.6), then the Lorentzian energy $E_g(u(t), v(t))$ is non-increasing on $[0, T_1)$ and for any $0 \le s \le t < T_1$, there holds*

$$E_g(u(t), v(t)) + \int_s^t\int_M |\partial_t u|^2 dxdt \le E_g(u(s), v(s)).$$

Proof. First, we may assume $v, \nabla v, \nabla u \in C^1(M \times (0, T_1))$. By direct computations, we get

$$\begin{aligned}
\frac{\partial}{\partial t} E_g(u,v) &= \int_M \nabla u \cdot \nabla u_t + \int_M B^\top(u) \cdot u_t |\nabla v|^2 - \int_M \beta(u) \nabla v \cdot \nabla v_t \\
&= \int_{\partial M} \frac{\partial u}{\partial n} u_t - \int_M \Delta u u_t + \int_M B^\top(u) \cdot u_t |\nabla v|^2 - \int_M \beta(u) \nabla v \cdot \nabla v_t \\
(u_t|_{\partial M} = 0) \quad &= -\int_M (u_t + B^\top(u)|\nabla v|^2))^\top \cdot u_t + \int_M B^\top(u) \cdot u_t |\nabla v|^2 \\
&\quad - \int_M \beta(u) \nabla v \cdot \nabla v_t \\
&= -\int_M |u_t|^2 - \int_M \beta(u) \nabla v \cdot \nabla v_t \\
(v_t|_{\partial \Omega} = 0) \quad &= -\int_M |u_t|^2 + \int_M div(\beta(u) \nabla v) v_t \\
&= -\int_M |u_t|^2 \leq 0.
\end{aligned}$$

For the general case that $(u, v) \in \mathcal{V}(M_0^{T_1}; N \times \mathbb{R})$, this can be done just by replacing the derivative $\frac{\partial}{\partial t}$ by difference quotients in the proof. Then the conclusion of the lemma follows immediately. □

The next lemma tells us that the L^2 norms (energy) of u and v are always bounded by the initial data.

Lemma 2.2. *Suppose $(u, v) \in \mathcal{V}(M_0^{T_1}; N \times \mathbb{R})$ is a solution of (1.5) and* (1.6)*, then for any $0 \leq t < T_1$, there holds*

$$\begin{aligned}
\int_M |\nabla u|^2(\cdot, t) dx &\leq \int_M |\nabla \phi|^2 dx + (\Lambda - \lambda) \int_M |\nabla \psi|^2 dx \text{ and} \\
\int_M |\nabla v|^2(\cdot, t) dx &\leq \frac{\Lambda}{\lambda} \int_M |\nabla \psi|^2 dx.
\end{aligned}$$

Proof. Multiplying the equation of v by $v - \psi$ and integrating on M, we get

$$\begin{aligned}
0 &= \int_M div\{\beta(u) \nabla v\}(v - \psi) dx \\
&= \int_{\partial M} \beta(u) \frac{\partial v}{\partial n}(v - \psi) dx - \int_M \beta(u) \nabla v \cdot \nabla(v - \psi) dx \\
&= -\int_M \beta(u) |\nabla v|^2 dx + \int_M \beta(u) \nabla v \cdot \nabla \psi dx, \qquad (2.1)
\end{aligned}$$

where in the last equality we use the fact that $v|_{\partial \Omega} = \psi$.

By Young's inequality, we have

$$\begin{aligned}\int_M \beta(u)|\nabla v|^2dx &\leq \int_M \beta(u)|\nabla v\cdot\nabla\psi|dx\\ &\leq \frac{1}{2}\int_M \beta(u)|\nabla v|^2dx+\frac{1}{2}\int_M \beta(u)|\nabla\psi|^2dx.\end{aligned} \tag{2.2}$$

Thus we obtain

$$\int_M \beta(u)|\nabla v|^2dx \leq \int_M \beta(u)|\nabla\psi|^2dx, \tag{2.3}$$

and

$$\int_M |\nabla v|^2dx \leq \frac{\Lambda}{\lambda}\int_M |\nabla\psi|^2dx. \tag{2.4}$$

Combining (2.3) with Lemma 2.1, we have

$$\begin{aligned}\frac{1}{2}\int_M |\nabla u|^2dx &\leq E_g(u,v)+\frac{1}{2}\int_M \beta(u)|\nabla v|^2dx\\ &\leq E_g(\phi,\psi)+\frac{1}{2}\int_M \beta(u)|\nabla\psi|^2dx\\ &\leq E(\phi)+(\Lambda-\lambda)E(\psi).\end{aligned}$$

□

As a direct corollary of the above lemma, we have

Corollary 2.3. *Suppose* $(u,v)\in\mathcal{V}(M_0^{T_1};N\times\mathbb{R})$ *is a solution of (1.5) and* (1.6)*, then*

$$\int_0^{T_1}\int_M |u_t|^2dxdt \leq (1+\Lambda-\lambda)E(\phi,\psi).$$

Proof. By Lemma 2.1 and (2.3), we know that

$$\begin{aligned}\int_0^t\int_M |u_t|^2dxdt &\leq E_g(u(\cdot,0),v(\cdot,0))-E_g(u(\cdot,t),v(\cdot,t))\\ &\leq \frac{1}{2}\int_M |\nabla\phi|^2dx-\frac{\lambda}{2}\int_M |\nabla\psi|^2dx+\frac{1}{2}\int_M \beta(u)|\nabla v(\cdot,t)|^2dx\\ &\leq \frac{1}{2}\int_M |\nabla\phi|^2dx+\frac{\Lambda-\lambda}{2}\int_M |\nabla\psi|^2 \leq (1+\Lambda-\lambda)E(\phi,\psi).\end{aligned}$$

□

In Lemma 2.2, we prove that $\|\nabla v\|_{L^2(M)}$ is uniformly bounded by using an integration method. In fact, we can use the theory of second order elliptic equations of divergence form to obtain a stronger $W^{1,p}$ estimate for v along the flow. More precisely, we have

Lemma 2.4. *($W^{1,p}$ estimate for v) Suppose $(u,v) \in \mathcal{V}(M_0^{T_1}; N \times \mathbb{R})$ is a solution of (1.5) and* (1.6)*, then for any $p > 1$, $0 \le t < T_1$, we have*

$$\int_M |\nabla v|^p(\cdot, t)dx \le C \int_M |\nabla \psi|^p dx,$$

where C only depends on p, M, λ, Λ.

Proof. Set $\tilde{v} = v - \psi$, and we have $\tilde{v} = 0$ on ∂M. Since v satisfies a second order elliptic divergence equations, then

$$div(\beta(u)\nabla \tilde{v}) = -div(\beta(u)\nabla \psi).$$

Thus, by Theorem 1 in [27], we know that

$$\int_M |\nabla \tilde{v}|^p dx \le C \int_M |\beta(u)\nabla \psi|^p dx,$$

where C only depends on p, M, Λ, λ. It implies that

$$\int_M |\nabla v|^p dx \le C \int_M |\nabla \psi|^p dx.$$

□

Lemma 2.5. *Let $(u,v) \in \mathcal{V}(M_0^{T_1}; N \times \mathbb{R})$ be a solution to (1.5) and* (1.6)*. There exists a positive constant $R_0 < 1$ such that, for any $x_0 \in M$, $0 \le R \le R_0$ and $0 < s \le t < T_0$, there holds*

$$E(u(t); B_R^M(x_0)) \le E(u(s); B_{2R}^M(x_0)) + C_1 \frac{t-s}{R^2} + C_2(t-s), \tag{2.5}$$

where C_1 and C_2 depend on $\lambda, \Lambda, M, N, E(\phi), \|\psi\|_{W^{1,4}(M)}$.

Proof. Let $\eta \in C_0^\infty(B_{2R}^M(x_0))$ be a cut-off function such that $\eta(x) = \eta(|x - x_0|)$, $0 \le \eta \le 1$, $\eta|_{B_R^M(x_0)} \equiv 1$ and $|\nabla \eta| \le \frac{C}{R}$. By direct computations, we get

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_M |\nabla u|^2\eta^2 &= \int_M \langle \nabla u, \nabla u_t \rangle \eta^2 \\
&= \int_{\partial B_{2R}^M(x_0)} \frac{\partial u}{\partial n} \cdot u_t \eta^2 - \int_M \langle \Delta u, u_t \rangle \eta^2 - 2\int_M \nabla u \cdot \nabla \eta \eta u_t \\
&= \int_M \langle -u_t - B^\top(u)|\nabla v|^2, u_t \rangle \eta^2 - 2\int_M \nabla u \cdot \nabla \eta \eta u_t \\
&= -\int_M |u_t|^2\eta^2 - \int_M B^\top(u)|\nabla v|^2 \cdot u_t \eta^2 - 2\int_M \nabla u \cdot \nabla \eta \eta u_t.
\end{aligned}$$

By Lemma 2.2, Lemma 2.4 and Young's inequality, we have

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_M |\nabla u|^2\eta^2 &\le -\frac{1}{2}\int_M |u_t|^2\eta^2 + C\int_M |\nabla u|^2|\nabla \eta|^2 + C\int_M |\nabla v|^4\eta^2 \\
&\le \frac{C_1}{R^2} + C_2.
\end{aligned}$$

By integrating the above inequality from s to t, we can get (2.5).

□

Next, we derive an ϵ_1- regularity lemma.

Lemma 2.6. *Let $(\phi,\psi)\in C^{2+\alpha}(M,N\times\mathbb{R})$, $z_0=(x_0,t_0)\in M\times(0,T_1]$, denote $P^M_r(z_0):=B^M_r(x_0)\times[t_0-r^2,t_0]$. Assume that $(u,v)\in\mathcal{V}(M_0^{T_1};N\times\mathbb{R})$, then there exist two positive constants $\epsilon_1=\epsilon_1(M,N,\|\phi\|_{C^{2+\alpha}(M)},\|\psi\|_{C^{2+\alpha}(M)})>0$ and $C=C(\alpha,r,M,N,\|\phi\|_{C^{2+\alpha}(M)},\|\psi\|_{C^{2+\alpha}(M)})>0$ such that if*

$$\sup_{[t_0-4r^2,t_0]}E(u(t),B^M_{2r}(x_0))\le\epsilon_1,$$

we have

$$r\|\nabla v\|_{L^\infty(P^M_r(z_0))}+r\|\nabla u\|_{L^\infty(P^M_r(z_0))}\le C \tag{2.6}$$

and for any $0<\beta<1$,

$$\sup_{t_0-\frac{r^2}{4}\le t\le t_0}\|v(t)\|_{C^{2+\alpha}(B^M_{r/2}(x_0))}+\|u\|_{C^{\beta,\beta/2}(P^M_{r/2}(z_0))}+\|\nabla u\|_{C^{\beta,\beta/2}(P^M_{r/2}(z_0))}\le C(\beta), \tag{2.7}$$

Moreover, if

$$\sup_{x_0\in M}\sup_{[t_0-r^2,t_0]}E(u(t),B^M_r(x_0))\le\epsilon_1,$$

then

$$\sup_{t_0-\frac{r^2}{8}\le t\le t_0}\|v(t)\|_{C^{2+\alpha}(M)}+\|u\|_{C^{2+\alpha,1+\alpha/2}(M\times[t_0-\frac{r^2}{8},t_0])}\le C, \tag{2.8}$$

and

$$\|v\|_{C^{\alpha,\alpha/2}(M\times[t_0-\frac{r^2}{8},t_0])}+\|\nabla v\|_{C^{\alpha,\alpha/2}(M\times[t_0-\frac{r^2}{8},t_0])}\le C. \tag{2.9}$$

Proof. **Step 1:** We prove (2.7), (2.8) and (2.9) under the assumption that (2.6) is true.

Taking the cut-off function $\eta\in C_0^\infty(P^M_r(z_0))$ such that $0\le\eta\le1$, $\eta|_{P^M_{3r/4}(0)}\equiv1$, $|\nabla^j\eta|\le\frac{C}{r^j}, j=1,2$ and $|\partial_t\eta|\le\frac{C}{r^2}$, set $U=\eta u$, then

$$\begin{cases}\partial_tU-\Delta U=f, & in\quad P^M_r(z_0);\\ U(x,t)=0, & on\quad B^M_r(x_0)\times\{t=t_0-r^2\};\\ U(x,t)=\eta\varphi, & on\quad \partial B^M_r(x_0)\times(t_0-r^2,t_0),\end{cases}$$

where

$$f:=\eta(\partial_t-\Delta)u+u(\partial_t-\Delta)\eta-2\nabla\eta\nabla u.$$

By the standard parabolic theory, for any $1<p<\infty$, we have

$$\begin{aligned}\|U\|_{W^{2,1}_p(P^M_r(z_0))}&\le C\big(\|f\|_{L^p(P^M_r(z_0))}+\|\eta\varphi\|_{W^{2,1}_p(P^M_r(z_0))}+\|U\|_{L^p(P^M_r(z_0))}\big)\\&\le C\big(1+\|\varphi\|_{C^2(M)}\big),\end{aligned}$$

where we use the fact that $f \in L^\infty$ under the equation (1.5) and assumption (2.6). Then, by Sobolev's embedding, for any $0 < \beta = 1 - 4/p < 1$, we obtain

$$\begin{aligned}\|u\|_{C^{\beta,\beta/2}(P^M_{3r/4}(z_0))} + \|\nabla u\|_{C^{\beta,\beta/2}(P^M_{3r/4}(z_0))} &\leq \|U\|_{C^{\beta,\beta/2}(P^M_r(z_0))} + \|\nabla U\|_{C^{\beta,\beta/2}(P^M_r(z_0))} \\ &\leq C\|U\|_{W^{2,1}_p(P^M_r(z_0))} \leq C(\beta)(1 + \|\varphi\|_{C^2(M)}). \end{aligned} \tag{2.10}$$

Taking a cut-off function $\xi(x) = \xi(|x - x_0|) \in C^\infty_0(B^M_r(x_0))$ such that $0 \leq \xi \leq 1$, $\xi|_{B^M_{3r/4}} \equiv 1$ and $|\nabla^j \xi| \leq \frac{C}{r^j}$, $j = 1, 2$, set $V = \xi v$, then we have

$$\begin{cases} \Delta V = h, & \text{in } B^M_r(x_0); \\ V = \xi\psi, & \text{on } \partial B^M_r(x_0), \end{cases}$$

where $h = \Delta\xi v + 2\nabla\xi\nabla v + \xi\Delta v \in L^\infty$. By the standard elliptic estimates and Sobolev embedding, we get

$$\|v\|_{C^{1,1-2/p}(B^M_{3r/4}(X_0))} \leq C\|V\|_{W^{2,p}(B^M_r(x_0))} \leq C(1 + \|\psi\|_{C^2(M)}) \tag{2.11}$$

for any $2 < p < \infty$. Noting that (2.10) and (2.11) yields $\Delta v \in C^\alpha(B^M_{3r/4}(x_0))$, by the Schauder estimates and taking some suitable cut-off functions as before, we get

$$\|v\|_{C^{2+\alpha}(B^M_{r/2}(x_0))} \leq C(1 + \|\phi\|_{C^2(M)})(1 + \|\psi\|_{C^{2+\alpha}(M)}) \tag{2.12}$$

for any $t_0 - \frac{r^2}{4} \leq t \leq t_0$. Then (2.7) follows from (2.10) and (2.12) immediately.

To prove (2.8) and (2.9), we rewrite the equation of v as follows

$$\Delta v = \gamma(u)\nabla u\nabla v,$$

where $\gamma(u) = \frac{2B^\top(u)}{\beta(u)}$. Then for any $t_0 - \frac{r^2}{4} < t, s < t_0$, we have

$$\begin{aligned}\Delta(v(\cdot,t) - v(\cdot,s)) =& \gamma(u(\cdot,t))\nabla u(\cdot,t)\nabla(v(\cdot,t) - v(\cdot,s)) \\ &+ \left(\gamma(u(\cdot,t))\nabla u(\cdot,t) - \gamma(u(\cdot,s))\nabla u(\cdot,s)\right)\nabla v(\cdot,s) \ \mathit{in}\ M.\end{aligned}$$

Combining (2.6), (2.10) with the fact that $v(\cdot,t) - v(\cdot,s) = 0$ *on* ∂M, by the standard elliptic estimates and Sobolev embedding, we obtain

$$\|v(\cdot,t) - v(\cdot,s)\|_{C^{1+\alpha}(M)} \leq C\|\gamma(u(\cdot,t))\nabla u(\cdot,t) - \gamma(u(\cdot,s))\nabla u(\cdot,s)\|_{L^\infty(M)} \leq C|s-t|^{\alpha/2}.$$

Thus, we get $\|\nabla v\|_{C^{\alpha,\alpha/2}(M\times[t_0-\frac{r^2}{4},t_0])} + \|\nabla v\|_{C^{\alpha,\alpha/2}(M\times[t_0-\frac{r^2}{4},t_0])} \leq C$ which is (2.9) and

$$\begin{cases} \partial_t u - \Delta u \in C^{\alpha,\frac{\alpha}{2}}(M \times [t_0 - \frac{r^2}{4}, t_0]), \\ u|_{\partial M} = \phi \in C^{2+\alpha}(M). \end{cases}$$

Taking some suitable cut-off function and by the standard Schauder estimates for parabolic equations, we have $u \in C^{2+\alpha,1+\alpha/2}(M \times [t_0 - \frac{r^2}{8}, t_0])$ and

$$\begin{aligned}&\|u\|_{C^{2+\alpha,1+\alpha/2}(M\times[t_0-\frac{r^2}{8},t_0])} \\ &\leq C(\|\partial_t u - \Delta u\|_{C^{\alpha,\frac{\alpha}{2}}(M\times[t_0-\frac{r^2}{4},t_0])} + \|u\|_{C^0(M\times[t_0-\frac{r^2}{4},t_0])} + \|\phi\|_{C^{2+\alpha}(M)}) \leq C.\end{aligned}$$

Thus we get (2.8).

Step 2: Next we prove (2.6). The idea is similar as in [25, 30]. Without loss of generality, we may assume $r = \frac{1}{2}$. Choose $0 \le \rho < 1$ such that

$$(1-\rho)^2 \sup_{P^M_\rho(z_0)} |\nabla u|^2 = \max_{0\le\sigma\le 1}\{(1-\sigma)^2 \sup_{P^M_\sigma(z_0)} |\nabla u|^2\}$$

and choose $z_1 = (x_1, t_1) \in P^M_\rho(z_0)$ such that

$$|\nabla u|^2(z_1) = \sup_{P^M_\rho(z_0)} |\nabla u|^2 := e.$$

We claim that

$$(1-\rho)^2 e \le 4.$$

We proceed by contradiction. If $(1-\rho)^2 e > 4$, we set

$$\widetilde{u}(x,t) := u(x_1 + e^{-\frac{1}{2}}x, t_1 + e^{-1}t) \quad and \quad \widetilde{v}(x) := v(x_1 + e^{-\frac{1}{2}}x, t_1 + e^{-1}t).$$

Denoting

$$D_r(0) := \{x \in B_r(0) | x_1 + e^{-\frac{1}{2}}x \in B^M_1(x_0)\}$$

and

$$S_r := \{(x,t) \in B_r(0) \times [-r^2, 0] | (x_1 + e^{-\frac{1}{2}}x, t_1 + e^{-1}t) \in P^M_1(z_0)\},$$

then

$$\begin{cases} \partial_t \widetilde{u} = \Delta \widetilde{u} + A(\widetilde{u})(\nabla\widetilde{u}, \nabla\widetilde{u}) - B^\top(\widetilde{u})|\nabla\widetilde{v}|^2, & in \quad S_1; \\ -div(\beta(\widetilde{u})\nabla\widetilde{v}) = 0, & in \quad S_1, \end{cases} \tag{2.13}$$

with the boundary data

$$\begin{cases} \widetilde{u}(x,t) = \phi(x_1 + e^{-\frac{1}{2}}x), & if \quad x_1 + e^{-\frac{1}{2}}x \in \partial M; \\ \widetilde{v}(x,t) = \psi(x_1 + e^{-\frac{1}{2}}x), & if \quad x_1 + e^{-\frac{1}{2}}x \in \partial M. \end{cases} \tag{2.14}$$

Moreover, we have

$$\sup_{S_1} |\nabla\widetilde{u}|^2 = e^{-1} \sup_{P^M_{e^{-1/2}}(z_1)} |\nabla u|^2 \le e^{-1} \sup_{P^M_{\rho+e^{-1/2}}(z_0)} |\nabla u|^2 \le e^{-1} \sup_{P^M_{\frac{1+\rho}{2}}(z_0)} |\nabla u|^2 \le 4$$

and

$$|\nabla\widetilde{u}|^2(0) = e^{-1}|\nabla u|^2(z_1) = 1.$$

Since $\widetilde{v}$ satisfies

$$|\Delta\widetilde{v}| \le C|\nabla\widetilde{u}||\nabla\widetilde{v}|,$$

by the standard elliptic estimates, for any $1 < p < +\infty$ we have

$$\sup_{-1\le t\le 0} \|\widetilde{v}\|_{W^{2,p}(D_{\frac{7}{8}}(0))} \le C(p)(\|\nabla\widetilde{v}\|_{L^p(D_1(0))} + \|\psi\|_{C^2(M)}). \tag{2.15}$$

Taking first $p = 2$ and using (2.15) again ($p > 2$), by Sobolev embedding for any $0 < \beta = 1 - 2/p < 1$,

$$\sup_{-1\le t\le 0} \|\widetilde{v}\|_{C^{1+\beta}(D_{\frac{3}{4}}(0))} \le C(\beta). \tag{2.16}$$

Next, we want to show that there exists a constant $C > 0$ such that

$$1 \le C \int_{S_{3/4}} |\nabla \widetilde{u}|^2 dxdt. \tag{2.17}$$

If C does not exist, we can find a sequence $\{(\widetilde{u}_i, \widetilde{v}_i)\}$ satisfying

$$\begin{cases} \partial_t \widetilde{u}_i = \Delta \widetilde{u}_i + A(\widetilde{u}_i)(\nabla \widetilde{u}_i, \nabla \widetilde{u}_i) + (\nabla^N \beta)(\widetilde{u}_i)|\nabla \widetilde{v}_i|^2, & in \quad S_1; \\ -div(\beta(\widetilde{u}_i)\nabla \widetilde{v}_i) = 0, & in \quad S_1, \end{cases} \tag{2.18}$$

with the boundary data

$$\begin{cases} \widetilde{u}_i(x,t) = \phi(x_1 + e^{-\frac{1}{2}}x), & if \quad x_1 + e^{-\frac{1}{2}}x \in \partial M; \\ \widetilde{v}_i(x,t) = \psi(x_1 + e^{-\frac{1}{2}}x), & if \quad x_1 + e^{-\frac{1}{2}}x \in \partial M. \end{cases} \tag{2.19}$$

and

$$\sup_{S_{3/4}} \left(|\nabla \widetilde{u}_i| + |\nabla \widetilde{v}_i| \right) \le C, \tag{2.20}$$

$$|\nabla \widetilde{u}_i|^2(0) = 1, \tag{2.21}$$

$$\int_{S_{3/4}} |\nabla \widetilde{u}_i|^2 dxdt \le \frac{1}{i}. \tag{2.22}$$

By a similar argument as in **Step 1** (since $(\widetilde{u}_i, \widetilde{v}_i)$ satisfy (2.18), (2.19) and (2.20)), for any $0 < \beta < 1$, we have

$$\|\nabla \widetilde{u}_i\|_{C^{\beta,\beta/2}(S_{1/2}(0))} \le C(\beta). \tag{2.23}$$

Therefore, there exist a subsequence of $\{\widetilde{u}_i\}$ (still denoted by $\{\widetilde{u}_i\}$) and a function $\overline{u} \in C^{1+\gamma,\gamma/2}(S_{1/2})$ such that

$$\nabla \widetilde{u}_i \to \nabla \overline{u} \quad in \quad C^{\gamma,\gamma/2}(S_{1/2})$$

where $0 < \gamma < \beta$. Then by (2.22), we know

$$\int_{S_{1/2}} |\nabla \overline{u}|^2 dxdt = 0 \tag{2.24}$$

which implies $\nabla \overline{u} \equiv 0$ in $S_{1/2}$. But, (2.21) tells us $|\nabla \overline{u}|(0) = 1$. This is impossible and then (2.17) must be true. Thus, we have

$$\begin{aligned} 1 \le C \int_{S_{3/4}} |\nabla \widetilde{u}|^2 dxdt &\le C \sup_{-1<t<0} \int_{B^M_{e^{\frac{1}{2}}}(x_1)} |\nabla u|^2 (t_1 + e^{-1}t) dx \\ &\le C \sup_{-1<t<0} \int_{B^M_1(z_0)} |\nabla u|^2(t) dx \le C\epsilon_1. \end{aligned}$$

By choosing $\epsilon_1 > 0$ sufficiently small, it leads to a contradiction. Therefore we must have $(1-\rho)^2 e \le 4$ and then

$$(1 - 3/4)^2 \sup_{P^M_{3/4}(z_0)} |\nabla u|^2 \le (1-\rho)^2 e \le 4.$$

Since v satisfies $|\Delta v| \leq C(N)|\nabla u||\nabla v|$, $\|\nabla u\|_{L^\infty(P^M_{3/4}(z_0))} \leq 8$, $\|\nabla v\|_{L^4(M)} \leq C$ and $v|_{\partial M} = \psi \in C^{2+\alpha}(M)$, by the elliptic estimates for the Laplace operator and Sobolev embedding, we easily get

$$\|\nabla v\|_{L^\infty(P^M_{1/2}(z_0))} \leq C.$$

Thus we obtain the inequality (2.6) and finish the proof of the lemma. □

3. SHORT-TIME EXISTENCE RESULTS

To prove the local existence for the equations (1.5), we first state some properties of the Dirichlet heat kernel when the dimension of the domain is 2. Let $G = G(x, y, t)$ be the heat kernel. We have

Lemma 3.1. *(estimates for Dirichlet heat kernel, see* [7], [20]*) For any $\alpha > 0$, there exists a constant $c(\alpha)$ such that*

$$G(x, y, t) \leq c(\alpha)t^{\alpha-1}dist(x, y)^{-2\alpha},$$
$$\|\nabla G(x, y, t)\| \leq c(\alpha)t^{\alpha-2}dist(x, y)^{1-2\alpha}.$$

By using the above lemma, we can give the proof of Theorem 1.1.

Proof of Theorem 1.1. For $\epsilon > 0$ and $u \in C^1(M \times [0, \epsilon])$ and $v \in C^1(M)$, we define the space

$$X_\epsilon = \{u|_{M\times\{t=0\}} = u|_{\partial M\times[0,\epsilon]} = \phi, u \in C^1(M) \text{ for any fixed } t \in [0, \epsilon], \|u\|_X < +\infty\},$$

where the norm of X_ϵ is defined by

$$\|u\|_X := \|u\|_{C^0(M\times[0,\epsilon])} + \sup_{t\in[0,\epsilon]} \|\nabla u(., t)\|_{C^0(M)}.$$

For a solution (u, v) of (1.5), we claim that

$$\|v\|_X < +\infty, \quad \text{if} \quad \|u\|_X < +\infty. \tag{3.1}$$

Notice that v satisfies the following equation

$$\begin{cases} -\Delta v = 2\frac{B^\top(u)}{\beta(u)}\nabla u \cdot \nabla v, \\ v|_{\partial\Omega} = \psi. \end{cases} \tag{3.2}$$

The elliptic estimates for (3.2) tell us that, for any $p > 1$, we have

$$\begin{aligned} \|v\|_{W^{1,p}} \leq C\|v\|_{W^{2,2}} &\leq C\left\|\frac{B^\top(u)}{\beta(u)}\nabla u \cdot \nabla v\right\|_2 + C\|\psi\|_{W^{2,2}} \\ &\leq C\|\nabla u\|_{C^0}\|\nabla v\|_2 + C\|\psi\|_{W^{2,2}}. \end{aligned} \tag{3.3}$$

Therefore, for any $p > 1$, we have the $W^{2,p}$ estimate

$$\begin{aligned} \|v\|_{W^{2,p}} &\leq C\|\nabla u\|_{C^0}\|\nabla v\|_p + C\|\psi\|_{W^{2,p}} \\ &\leq C\|\nabla u\|_{C^0}\{\|\nabla u\|_{C^0}\|\nabla v\|_2 + C\|\psi\|_{W^{2,2}}\} + C\|\psi\|_{W^{2,p}}. \end{aligned} \tag{3.4}$$

By (2.1), it implies that

$$\|v\|_{W^{2,p}} \quad \leq \quad C(\psi)(\|u\|_X^2 + \|u\|_X) + C(\psi).$$

Therefore, by the Sobolev embedding theorem we have, for some $\alpha \in (0,1)$,

$$\|v\|_{C^{1,\alpha}} \leq C(\psi) + C(\psi)\|u\|_X + C(\psi)\|u\|_X^2 \tag{3.5}$$

This proves the claim.

Define

$$u_0(x,t) = \int_M G(x,y,t)\phi(y)dy - \int_0^t \int_{\partial M} \frac{\partial G}{\partial n}(x-y,t-s)\phi(y)d\sigma ds.$$

We consider the operator $\mathbb{T} : X_\epsilon \to X_\epsilon$

$$\mathbb{T}u(x,t) \quad = \quad u_0(x,t) - \int_0^t \int_M G(x-y,t-s)(A(\nabla u, \nabla u) - B^\top |\nabla v|^2)(y,s)dyds.$$

For $\delta > 0$, we define

$$B_\delta := \{u \in X_\epsilon, \|u - u_0\|_X \leq \delta\}.$$

To prove the existence of a local solution, we need

i): $\mathbb{T} : B_\delta \to B_\delta$;

ii): $\mathbb{T}$ is a contraction mapping in B_δ.

Proof of i). For $u \in B_\delta$, we have

$$\mathbb{T}u - u_0 = -\int_0^t \int_M G(x-y,t-s)(A(\nabla u, \nabla u) - B^\top |\nabla v|^2)(y,s)dyds.$$

Notice that, for any $u \in B_\delta$,

$$\|u\|_X \leq \|u - u_0\|_X + \|u_0\|_X \leq C.$$

By (3.1) we know that

$$\|v\|_X \leq C.$$

Letting $\alpha \in (1, \frac{3}{2})$ in Lemma 3.1. For any $(x,t) \in M \times [0,\epsilon)$, we have

$$\begin{aligned} |\mathbb{T}u - u_0|(x,t) \quad &\leq \quad C \int_0^t \int_M G(x-y,t-s)\left(|\nabla u|^2 + |\nabla v|^2\right) dyds \\ &\leq \quad C \left\{ \sup_{t \in [0,\epsilon]} \|\nabla u\|^2_{C^0(M)} + C \right\} \cdot \int_0^t \int_\Omega G(x-y,t-s)dyds \\ &\leq \quad C \int_0^\epsilon (\epsilon - s)^{\alpha - 1} \int_M d(x,y)^{-2\alpha} dy \\ &\leq \quad C\epsilon^\alpha. \end{aligned} \tag{3.6}$$

Furthermore, we have

$$
\begin{aligned}
|\nabla(\mathbb{T}u - u_0)|\,(x,t) &\leq C\int_0^t\int_M |\nabla_x G|(x-y,t-s)\left(|\nabla u|^2 + |\nabla v|^2\right) dyds \\
&\leq C\int_0^t\int_M |\nabla_x G|(x-y,t-s) dyds \\
&\leq C\int_0^\epsilon (\epsilon - s)^{\alpha-2}\int_M d(x,y)^{1-2\alpha} dy \\
&\leq C\epsilon^{\alpha-1}, \qquad (3.7)
\end{aligned}
$$

where C is a constant which depends on the norm $\|u\|_X$. Then (3.6) and (3.7) give us that, for any $\delta > 0$, there exists $\epsilon > 0$, such that $\mathbb{T}$ is a map from B_δ in to B_δ.

Proof of ii). We need to show that there exists $\rho \in (0,1)$ such that, for any $u_1, u_2 \in B_\delta$,

$$\|\mathbb{T}u_1 - \mathbb{T}u_2\|_X \leq \rho\|u_1 - u_2\|_X.$$

We have

$$
\begin{aligned}
|\mathbb{T}u_1 - \mathbb{T}u_2|\,(x,t) \leq& \int_0^t\int_M G(x-y,t-s)|A(u_1)(\nabla u_1,\nabla u_1) - A(u_2)(\nabla u_2,\nabla u_2) \\
&+B^\top(u_1)|\nabla v_1|^2 - B^\top(u_2)|\nabla v_2|^2| dyds. \qquad (3.8)
\end{aligned}
$$

Firstly, we estimate

$$
\begin{aligned}
& |A(u_1)(\nabla u_1,\nabla u_1) - A(u_2)(\nabla u_2,\nabla u_2)| \\
\leq\;& |A(u_1)(\nabla u_1,\nabla u_1) - A(u_1)(\nabla u_2,\nabla u_2)| + |A(u_1)(\nabla u_2,\nabla u_2) - A(u_2)(\nabla u_2,\nabla u_2)| \\
\leq\;& C\{(|\nabla u_1| + |\nabla u_2|)|\nabla(u_1 - u_2)| + |\nabla u_2|^2|u_1 - u_2|\} \\
\leq\;& C\|u_1 - u_2\|_X.
\end{aligned}
$$

In the following we estimate $|B^\top(u_1)|\nabla v_1|^2 - B^\top(u_2)|\nabla v_2|^2|$. From (3.2), we have

$$div(\beta(u_1)\nabla v_1) - div(\beta(u_2)\nabla v_2) = 0.$$

Multiply by $(v_1 - v_2)$ and integrate on M. We have

$$
\begin{aligned}
0 &= \int_M div(\beta(u_1)\nabla v_1 - \beta(u_2)\nabla v_2)(v_1 - v_2) dx \\
&= \int_M \langle \beta(u_1)(\nabla v_1 - \nabla v_2) + (\beta(u_1) - \beta(u_2))\nabla v_2, \nabla(v_1 - v_2)\rangle dx, \qquad (3.9)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\lambda\int_M |\nabla v_1 - \nabla v_2|^2 dx &\leq \int_M \beta(u_1)|\nabla v_1 - \nabla v_2|^2 dx \\
&= \int_M \langle(\beta(u_1) - \beta(u_2))\nabla v_2, \nabla(v_1 - v_2)\rangle dx \\
&\leq \frac{\lambda}{2}\int_M |\nabla v_1 - \nabla v_2|^2 dx + \frac{1}{2\lambda}\int_M |\beta(u_1) - \beta(u_2)|^2|\nabla v_2|^2 dx. \qquad (3.10)
\end{aligned}
$$

From (3.10), we get

$$\begin{aligned}\int_M |\nabla v_1 - \nabla v_2|^2 dx &\le C \int_M |\beta(u_1) - \beta(u_2)|^2 |\nabla v_2|^2 dx \\ &\le C\|u_1 - u_2\|^2_{C^0} \int_M |\nabla v_2|^2 dx \\ &\le C(\psi)\|u_1 - u_2\|^2_X. \end{aligned} \tag{3.11}$$

By (3.2), we have

$$\Delta(v_1 - v_2) = \gamma(u_1)\nabla u_1 \cdot (\nabla v_1 - \nabla v_2) + 2\left(\gamma(u_1)\nabla u_1 - \gamma(u_2)\nabla u_2\right) \cdot \nabla v_2 \tag{3.12}$$

where $\gamma(u) := 2\frac{B^\top(u)}{\beta(u)}$, with the boundary condition

$$(v_1 - v_2)|_{\partial M} = 0.$$

The elliptic estimates for (3.12) tell us that

$$\begin{aligned}\|v_1 - v_2\|_{W^{2,2}} &\le C\|u_1\|_X \|\nabla v_1 - \nabla v_2\|_2 \\ &\quad + C\|\nabla(u_1 - u_2)\|_{C^0}\|\nabla v_2\|_2 + C\|u_1 - u_2\|_{C^0}\|\nabla u_2\|_{C^0}\|\nabla v_2\|_2. \end{aligned} \tag{3.13}$$

By (3.11), we have, for any $p > 1$,

$$\|v_1 - v_2\|_{W^{1,p}} \le C\|v_1 - v_2\|_{W^{2,2}} \le C\|u_1 - u_2\|_X \tag{3.14}$$

Applying this estimate and (3.3) to (3.12), we have

$$\begin{aligned}\|v_1 - v_2\|_{W^{2,p}} &\le C\|u_1\|_X \|\nabla v_1 - \nabla v_2\|_p \\ &\quad + C\|\nabla(u_1 - u_2)\|_{C^0}\|\nabla v_2\|_p + C\|u_1 - u_2\|_{C^0}\|\nabla u_2\|_{C^0}\|\nabla v_2\|_p \\ &\le C(\psi)\|u_1 - u_2\|_X. \end{aligned} \tag{3.15}$$

(3.15) implies that

$$\|v_1 - v_2\|_{C^{1+\alpha}} \le C\|u_1 - u_2\|_X. \tag{3.16}$$

Therefore, we get from (3.5) and (3.16),

$$\begin{aligned}&|B^\top(u_1)|\nabla v_1|^2 - B^\top(u_2)|\nabla v_2|^2| \\ &\le (B^\top(u_1) - B^\top(u_2))|\nabla v_1|^2 + B^\top(u_2)\langle \nabla v_1, \nabla(v_1 - v_2)\rangle \\ &\quad + B^\top(u_2)\langle \nabla v_2, \nabla(v_1 - v_2)\rangle \\ &\le C|\nabla v_1|^2|u_1 - u_2| + C|\nabla v_1||\nabla(v_1 - v_2)| + C|\nabla v_2||\nabla(v_1 - v_2)| \\ &\le C(\psi)\|u_1 - u_2\|_X. \end{aligned} \tag{3.17}$$

(3.8) and (3.17) give us

$$\left|\mathbb{T}u_1 - \mathbb{T}u_2\right|(x,t) \le C(\psi)\epsilon\|u_1 - u_2\|_X. \tag{3.18}$$

Similarly, we can also show that

$$\left|\nabla(\mathbb{T}u_1 - \mathbb{T}u_2)\right|(x,t) \le C(\psi)\epsilon\|u_1 - u_2\|_X. \tag{3.19}$$

Then we can conclude the claim that $\mathbb{T}$ is a contraction mapping in B_δ which implies immediately that there exists a unique fixed point $(u, v) \in B_\delta$ of $\mathbb{T}$ such that (u, v) solves (1.5).

Regularity: Since $\nabla u, \nabla v \in L^\infty([0,\epsilon]\times M)$, according to the classical $W_p^{2,1}$ estimates of second order parabolic equations, for any $p>1$, we have

$$\|u\|_{W_p^{2,1}([0,\epsilon]\times M)} \leq C(1+\|\phi\|_{W^{2,p}(M)})$$

which implies $\nabla u \in C^{\alpha,\frac{\alpha}{2}}([0,\epsilon]\times M)$ by Sobolev embedding. The Schauder estimates for parabolic and elliptic equations give us that the fixed point (u,v) has the desired regularity. (See Lemma 2.6 for a similar argument.)

We still need to show that if the image of the initial map $\phi(M)\subset N$, along the flow, we have $u(M\times[0,\epsilon))\subset N$. To this end, let $\pi: M_\sigma \to M$ be the smooth nearest point projection map. We now compute the evolution equation of

$$\rho(u) := |\pi(u)-u|^2.$$

$$\begin{aligned}
\frac{1}{2}(\partial_t-\Delta)\rho &= \langle \pi-u, \partial_t(\pi-u)\rangle - |\nabla(\pi-u)|^2 - \langle \pi-u, \Delta(\pi-u)\rangle \\
&= \langle \pi-u, d\pi(\partial_t u)-\partial_t u\rangle - |\nabla(\pi-u)|^2 - \langle \pi-u, \nabla\cdot(d\pi(\nabla u))-\Delta u\rangle \\
&= -|\nabla(\pi-u)|^2 + \langle \pi-u, d\pi((\partial_t-\Delta)u)\rangle \\
&\quad + \langle \pi-u, A(\nabla u,\nabla u)-(\partial_t-\Delta)u\rangle \\
&= -|\nabla(\pi-u)|^2 + \langle \pi-u, d\pi((\partial_t-\Delta)u)\rangle + \left\langle \pi-u, B^\top|\nabla v|^2\right\rangle \\
&= -|\nabla(\pi-u)|^2 \leq 0. \qquad (3.20)
\end{aligned}$$

To get the last equality of (3.20), we shall notice that, for $p\in N$, $d\pi(p)$ is an orthogonal projection to T_pN, $B^\top$ is orthogonal to $(\pi-u)$. Since $\rho(u)(\cdot,0)=\rho(u)|_{\partial M\times[0,\epsilon]}=0$, by the maximum principle, we have $\rho(u)\equiv 0$.

Finite singularities: By Lemma 2.6, we know that the maximum existence time T_1 is characterized by

$$\limsup_{x\in M, t\to T_1} E(u;B_r^M(x)) > \epsilon_1 \text{ for all } r>0.$$

We just need to prove that the singular set $S(u,T_1)$ at the singular time T_1 is a finite set.

Let $\{x_j\}_{j=1}^J$ be any finite subset of $S(u,T_1)$. Then we have

$$\limsup_{t\to T_1} E(u;B_r^M(x_j)) > \epsilon_1 \text{ for all } r>0,\ 1\leq j\leq J.$$

Therefore, we can choose $R>0$ such that $\{B_{2R}^M(x_j)\}_{j=1}^J$ are mutually disjoint. By Lemma 2.5, we get that for any $1\leq j\leq J$ and any $s\in[T_1-\frac{\epsilon_1 R^2}{2C}, T_1)$, there holds

$$\begin{aligned}
J\epsilon_1 \leq \sum_{j=1}^J \limsup_{t\to T_1} E(u(t);B_R^M(x_j)) &\leq \sum_{j=1}^J \Big(E(u(s);B_{2R}^M(x_j)) + \frac{\epsilon_1}{2}\Big) \\
&\leq E(u(s)) + \frac{J\epsilon_1}{2}
\end{aligned}$$

which implies

$$J \leq \frac{2(1+\Lambda)E(\phi,\psi)}{\epsilon_1}.$$

So, we proved the finiteness of $S(u, T_1)$.

Uniquness: Suppose $(u_1, v_1), (u_2, v_2)$ are two solutions of (1.5) and (1.6). Let $\tilde{u} = u_1 - u_2$, $\tilde{v} = v_1 - v_2$. By (3.18) and (3.19), we know that, for any $\varepsilon > 0$,

$$\|\tilde{u}\|_X \leq \varepsilon \|\tilde{u}\|_X.$$

It implies that $\tilde{u} \equiv 0$. Then by (3.16), we get that $\tilde{v} \equiv 0$. □

4. BEHAVIOR OF SINGULARITIES

In this section, we use the blow up analysis to study the behavior of singularities at the singular time of the solution derived by Theorem 1.1. We will prove Theorem 1.2 in this section.

First, we recall a removable singularity theorem which will be used in this section.

Theorem 4.1 (Theorem 3.4 in [16]). *Let (u, v) be a smooth harmonic map from the punctured disk $D \setminus \{0\}$ to $(N \times \mathbb{R}, g - \beta(d\theta)^2)$ with bounded energy $E(u, v; M) < \infty$, where $D \subset \mathbb{R}^2$ is the unit disk, then (u, v) extends to the whole disk D.*

Proof. We repeat the idea of the proof of [16] here for completeness. By a similar argument as in Lemma A.2 in [19], it is easy to see that (u, v) is a weakly harmonic map from D into $N \times \mathbb{R}$. Then the regularity Theorem 1.3 in [34] gives that (u, v) is smooth in D and hence the singularity point $\{0\}$ is removable. □

Proof of Theorem 1.2. Let (x_0, T_1) be a singularity. Without loss of generality, we assume $0 < T_1 < \infty$. The proof of $T_1 = \infty$ is similarly.

Since there are at most finitely many singular points at the singular time T_1, we may assume

$$\nabla u \in C^{\alpha,\alpha/2}_{loc}(B^M_\delta(x_0) \times [T_1 - \delta^2, T_1] \setminus \{(x_0, T_1)\})$$

for some $\delta > 0$. By Lemma 2.6, there exist sequences $t_i \nearrow T_1$, $x_i \to x_0$, $r_i \to 0$ such that

$$E(u(\cdot, t_i), B^M_{r_i}(x_i)) = \sup_{\substack{(x,t)\in B^M_\delta(x_0)\times[T_1-\delta^2, t_i] \\ B^M_r(x)\subset B^M_\delta(x_0), r\leq r_i}} E(u(\cdot, t_i), B^M_r(x)) = \frac{\epsilon_1}{2}. \tag{4.1}$$

According to Lemma 2.5, for any $T_1 - \delta^2 \leq s \leq t_i < T_1$, there holds

$$E(u(t_i); B^M_{r_i}(x_i)) \leq E(u(s); B^M_{2r_i}(x_i)) + C_1 \frac{t_i - s}{r_i^2} + C_2(t_i - s),$$

where C_1 and C_2 are the constants in Lemma 2.5. Setting $\tau = \frac{\epsilon_1}{8C_1} + \frac{\epsilon_1}{8C_2}$, then for any $s \in [t_i - \tau r_i^2, t_i]$, we get

$$E(u(s); B^M_{2r_i}(x_i)) \geq \frac{\epsilon_1}{4}. \tag{4.2}$$

We first deal with the second statement in the theorem.

Step 1: Let $x_0 \in \partial M$ and we show the statement (1) holds under the assumption

$$\limsup_{i\to\infty} \frac{dist(x_i, \partial M)}{r_i} = \infty.$$

After passing to a subsequence, we may assume $\lim_{i\to\infty} \frac{dist(x_i,\partial M)}{r_i} = \infty$. As i tends to infinity, we can assume $t_i - \frac{\delta^2}{4} > T_1 - \delta^2$. Denote

$$D_i := \{x \in \mathbb{R}^2 | x_i + r_i x \in B^M_\delta(x_0)\}$$

and

$$u_i(x,t) := u(x_i + r_i x, t_i + r_i^2 t); v_i(x,t) := v(x_i + r_i x, t_i + r_i^2 t).$$

It is easy to see that (u_i, v_i) lives in $D_i \times [-\frac{\delta^2}{4r_i^2}, 0]$ which tends to $\mathbb{R}^2 \times (-\infty, 0]$ as $i \to \infty$ and satisfies

$$\begin{cases} \partial_t u_i = \Delta u_i + A(u_i)(\nabla u_i, \nabla u_i) + B^\top(u_i)|\nabla v_i|^2, & in\ D_i \times [-\frac{\delta^2}{4r_i^2}, 0] \\ -div(\beta(u_i)\nabla v_i) = 0, & in\ D_i \times [-\frac{\delta^2}{4r_i^2}, 0] \end{cases} \tag{4.3}$$

with the boundary data

$$\begin{cases} u_i(x,t) = \phi(x_i + r_i x), & if\ x_i + r_i x \in \partial M, \\ v_i(x,t) = \psi(x_i + r_i x), & if\ x_i + r_i x \in \partial M. \end{cases} \tag{4.4}$$

By Lemma 2.2 and Corollary 2.3, for any τ we have

$$\int_{-\tau}^0 \int_{D_i} |\partial_t u_i|^2 dxdt \leq \int_{t_i - r_i^2\tau}^{t_i} \int_M |\partial_t u|^2 dMdt \to 0, \quad \text{as } i \to \infty, \tag{4.5}$$

and

$$\sup_{-\frac{\delta^2}{4r_i^2} \leq t \leq 0} \|dv_i\|_{L^2(D_i)} \leq \sup_{T_1 - \delta^2 \leq t \leq T_1} \|dv\|_{L^2(M)} \leq C, \tag{4.6}$$

$$\sup_{-\frac{\delta^2}{4r_i^2} \leq t \leq 0} \|du_i\|_{L^2(D_i)} \leq \sup_{T_1 - \delta^2 \leq t \leq T_1} \|du\|_{L^2(M)} \leq C. \tag{4.7}$$

By (4.1), we can see that

$$\sup_{-\tau \leq t \leq 0} \sup_{x \in D_i} \int_{B_1(x) \cap D_i} |\nabla u_i|^2(y,t)dy \leq \sup_{\substack{(x,t) \in B^M_\delta(x_0) \times [T_1 - \delta^2, t_i] \\ B^M_r(x) \subset B^M_\delta(x_0),\ r \leq r_i}} E(\Phi(t), B^M_r(x)) = \frac{\epsilon_1}{2}.$$

So, for any $x \in \mathbb{R}^2$, when i is sufficiently large, we have

$$\sup_{-\tau \le t \le 0} \int_{B_1(x)} |\nabla u_i|^2(y,t)dy \le \frac{\epsilon_1}{2}. \tag{4.8}$$

Combining (4.6), (4.8) with Lemma 2.6, we have

$$\sup_{-\frac{\tau}{2} \le t \le 0} \|v_i(\cdot,t)\|_{C^{2+\alpha}(B_{1/2}(x))} + \sup_{-\frac{\tau}{2} \le t \le 0} \|u_i(\cdot,t)\|_{C^{1+\alpha}(B_{1/2}(x))} \le C, \tag{4.9}$$

which tells us

$$\sup_{-\frac{\tau}{2} \le t \le 0} \|v_i(\cdot,t)\|_{C^{2+\alpha}_{loc}(\mathbb{R}^2)} + \sup_{-\frac{\tau}{2} \le t \le 0} \|u_i(\cdot,t)\|_{C^{1+\alpha}_{loc}(\mathbb{R}^2)} \le C. \tag{4.10}$$

From (4.5) and (4.10), we can find $\sigma_i \in [-\frac{\tau}{2}, 0]$ such that as $i \to \infty$, there holds

$$\int_{D_i} |\partial_t u_i|^2(x,\sigma_i)dx \to 0 \tag{4.11}$$

and

$$\|v_i(\cdot,\sigma_i)\|_{C^{2+\alpha}_{loc}(\mathbb{R}^2)} + \|u_i(\cdot,\sigma_i)\|_{C^{1+\alpha}_{loc}(\mathbb{R}^2)} \le C. \tag{4.12}$$

Therefore, there exist $\widetilde{u} \in C^1_{loc}(\mathbb{R}^2)$, $\widetilde{v} \in C^2_{loc}(\mathbb{R}^2)$ and a subsequence of $(u_i(\cdot,\sigma_i), v_i(\cdot,\sigma_i))$ such that

$$u_i(\cdot,\sigma_i) \to \widetilde{u} \quad in \quad C^1_{loc}(\mathbb{R}^2) \text{ and } v_i(\cdot,\sigma_i) \to \widetilde{v} \quad in \quad C^2_{loc}(\mathbb{R}^2).$$

Setting $t = \sigma_i$ in the equation (2.18) and letting $i \to \infty$, it is easy to see that $(\widetilde{u}, \widetilde{v})$ is a harmonic map from $\mathbb{R}^2$ into the Lorentzian manifold $N \times \mathbb{R}$ with

$$\frac{\epsilon_1}{4} \le \|\nabla \widetilde{u}\|_{L^2(\mathbb{R}^2)} \le C, \ \|\nabla \widetilde{v}\|_{L^2(\mathbb{R}^2)} \le C.$$

Here we use (4.2) and (4.7). Let $f : \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{S\}$ be the stereographic projection, where S is the south pole of the sphere. Due to the conformal invariance and removable singularity Theorem 4.1, $(\widetilde{u}(f^{-1}(x)), \widetilde{v}(f^{-1}(x)))$ is a harmonic map from $\mathbb{S}^2$ into the Lorentzian manifold $N \times \mathbb{R}$. For simplicity, we still denote $(\widetilde{u}(f^{-1}(x)), \widetilde{v}(f^{-1}(x)))$ by $(\widetilde{u}, \widetilde{v})$. It is clear that $\widetilde{v}$ satisfies the equation $div(\beta(\widetilde{u})\nabla \widetilde{v}) = 0$ in $\mathbb{S}^2$ with finite energy $\|\nabla \widetilde{v}\|_{L^2(\mathbb{S}^2)} \le C$. It follows that $\widetilde{v}$ must be a constant map. Then $\widetilde{u}$ is a nontrivial harmonic sphere.

Step 2: If $x_0 \in \partial M$, then $\limsup_{i\to\infty} \frac{dist(x_i,\partial M)}{r_i} \to \infty$.

If not, up to subsequence, we may assume $\frac{dist(x_i,\partial M)}{r_i} \to a$ as $i \to \infty$. Then

$$D_i \to \mathbb{R}^2_a := \{(x^1,x^2) | x^2 \ge -a\}.$$

Furthermore, we have that, for any $x \in \{x^2 = -a\}$ on the boundary, $x_i + r_i x \to x_0$ and

$$u_i(x,t) = \varphi(x_i + r_i x) \quad if \quad x_i + r_i x \in \partial M;$$
$$v_i(x,t) = \psi(x_i + r_i x) \quad if \quad x_i + r_i x \in \partial M;$$

By Lemma 2.6 and (4.1), for any $B_{4R}(0)\subset\mathbb{R}^2, R>0$, we have

$$\sup_{-\frac{\tau}{2}\le t\le 0}\|v_i(\cdot,t)\|_{C^{2+\alpha}(B_{4R}(0)\cap D_i)}+\sup_{-\frac{\tau}{2}\le t\le 0}\|u_i(\cdot,t)\|_{C^{1+\alpha}(B_{4R}(0)\cap D_i)}\le C. \tag{4.13}$$

From (4.5) and (4.13), we can find $\sigma_i\in[-\frac{\tau}{2},0]$ such that as $i\to\infty$, we have

$$\int_{D_i}|\partial_t u_i|^2(x,\sigma_i)dx\to 0 \tag{4.14}$$

and

$$\|v_i(\cdot,\sigma_i)\|_{C^{2+\alpha}(B_{4R}(0)\cap D_i)}+\|u_i(\cdot,\sigma_i)\|_{C^{1+\alpha}(B_{4R}(0)\cap D_i)}\le C. \tag{4.15}$$

Setting $d_i:=dist(x_i,\partial M)$ and $B_R^+(0):=\{(x_1,x_2)\in B_R(0)|x_2\ge 0\}$, for i,R sufficiently large, (4.15) implies

$$\|v_i(x-(0,\frac{d_i}{r_i}),\sigma_i)\|_{C^{2+\alpha}(B_{3R}^+(0))}+\|u_i(x-(0,\frac{d_i}{r_i}),\sigma_i)\|_{C^{1+\alpha}(B_{3R}^+(0))}\le C. \tag{4.16}$$

Then there exist a subsequence of (u_i,v_i) and a harmonic map $(\widetilde{u}^1,\widetilde{v}^1)\in C^{2+\alpha}_{loc}(\mathbb{R}^2_a,N\times\mathbb{R})$ satisfying $\widetilde{u}^1|_{\partial^0\mathbb{R}^2_0}=\phi(x_0)$, $\widetilde{v}^1|_{\partial^0\mathbb{R}^2_0}=\psi(x_0)$ such that

$$\lim_{i\to\infty}\|v_i(x-(0,\frac{d_i}{r_i}),\sigma_i)-\widetilde{v}^1\|_{C^1(B_{3R}^+(0))}=0,$$
$$\lim_{i\to\infty}\|u_i(x-(0,\frac{d_i}{r_i}),\sigma_i)-\widetilde{u}^1\|_{C^1(B_{3R}^+(0))}=0.$$

Set $\widetilde{u}(x):=\widetilde{u}^1(x+(0,a))$ and $\widetilde{v}(x):=\widetilde{v}^1(x+(0,a))$. We have

$$\lim_{i\to\infty}\|v_i(\cdot,\sigma_i)-\widetilde{v}\|_{W^{1,2}(B_{2R}(0)\cap D_i\cap\mathbb{R}^2_a)}=0,$$
$$\lim_{i\to\infty}\|u_i(\cdot,\sigma_i)-\widetilde{u}\|_{W^{1,2}(B_{2R}(0)\cap D_i\cap\mathbb{R}^2_a)}=0.$$

Combining these with (4.15) and noticing that the measure of $B_{2R}(0)\cap D_i\setminus\mathbb{R}^2_a$ and $B_{2R}(0)\cap\mathbb{R}^2_a\setminus D_i$ goes to zero, we have

$$\lim_{i\to\infty}\|v_i(\cdot,\sigma_i)\|_{W^{1,2}(B_R(0)\cap D_i)}=\|\widetilde{v}\|_{W^{1,2}(B_R(0)\cap\mathbb{R}^2_a)},$$
$$\lim_{i\to\infty}\|u_i(\cdot,\sigma_i)\|_{W^{1,2}(B_R(0)\cap D_i)}=\|\widetilde{u}\|_{W^{1,2}(B_R(0)\cap\mathbb{R}^2_a)}.$$

According to (4.2),(4.6) and (4.7), we obtain

$$\frac{\epsilon_1}{4}\le\|\nabla\widetilde{u}\|_{L^2(\mathbb{R}^2_a)}\le C;\ \|\nabla\widetilde{v}\|_{L^2(\mathbb{R}^2_a)}\le C. \tag{4.17}$$

Due to the conformal invariance, we can take $(\widetilde{u},\widetilde{v})$ as a harmonic map from the unit disk $B_1(0)$ into the Lorentzian manifold $N\times\mathbb{R}$. Since $\widetilde{v}$ satisfies

$$div(\beta(\widetilde{u})\nabla\widetilde{v})=0\ in\ B_1(0)$$

and $\widetilde{v}|_{\partial B_1(0)}\equiv\psi(x_0)$, $\widetilde{v}$ must be a constant map. Thus, $\widetilde{u}$ is a harmonic maps from $B_1(0)$ with constant boundary data $\widetilde{u}|_{\partial B_1(0)}=\phi(x_0)$ which should be a constant map [23]. This is a contradiction with (4.17) and the second statement (2) is proved.

For the first statement in the theorem, the argument is almost the same as what we have done in **Step 1** and we omit it here for brevity. □

5. LONG TIME EXISTENCE AND CONVERGENCE RESULTS

In this section, we use arguments from blow up analysis to prove some long time existence and convergence results for (1.5) and (1.6). Theorem 1.3 and Theorem 1.4 will be proved in this section.

Proof of Theorem 1.3. By the short time existence theorem, we just need to show that the solution (u, v) does not blow up at any time $t \in (0, \infty]$.

If not, we may assume that T_1 is the first singular (or blow up) time and $(x_0, T_1) \in S(u, T_1)$ is a singular point (or energy concentration point), *i.e.*

$$\limsup_{t \to T_1} E(u; B_R^M(x_0)) > \epsilon_1 \text{ for all } R > 0.$$

Let $t_i, x_i, r_i, \widetilde{u}$ be as in Theorem 1.2. Denote

$$u_i(x) := u(x, t_i).$$

By Theorem 1.1, we know that the singular set $S(u, T_1)$ at the singular time T_1 is a finite set and we may denote it by $S(u, T_1) := \{p_1, ..., p_J\}$. By Lemma 2.2, we have $E(u_i; M) \le E(\phi)$. Thus, there exists a weak limit in $W^{1,2}(M, N)$ which is denoted by $u(x, T_1)$ such that $u(x, T_1)|_{\partial M} = \phi$ and

$$u_i(x) \rightharpoonup u(x, T_1)$$

as $i \to \infty$. Moreover, by the definition of $S(u, T_1)$ and Lemma 2.6, we know

$$u_i(x) \to u(x, T_1) \ in \ C^2_{loc}(M \setminus S(u, T_1))$$

as $i \to \infty$. Therefore we have

$$\begin{aligned}
\lim_{i\to\infty} E(u_i; M) &= \lim_{\delta\to 0}\lim_{i\to\infty} (E(u_i; M \setminus \cup_{j=1}^J B_\delta^M(p_j)) + E(u_i; \cup_{j=1}^J B_\delta^M(p_j))) \\
&= E(u(x, T_1); M) + \sum_{j=1}^J \lim_{\delta\to 0}\lim_{i\to\infty} E(u_i; B_\delta^M(p_j)) \\
&\ge E(u(x, T_1); M) + \lim_{\delta\to 0}\lim_{i\to\infty} E(u_i; B_\delta^M(x_0)) \\
&\ge E(u(x, T_1); M) + \lim_{R\to\infty}\lim_{\delta\to 0}\lim_{i\to\infty} E(u_i; B_{r_i R}^M(x_i)) \\
&= E(u(x, T_1); M) + E(\widetilde{u}; S^2),
\end{aligned}$$

where $\widetilde{u}$ is a nontrivial harmonic sphere.

By the definition of $\overline{\epsilon}_1, \overline{\epsilon}_2, \overline{\epsilon}$(see Theorem 1.3) and Lemma 2.2, we have

$$\begin{aligned}
\overline{\epsilon} &\le E(u(x, T_1); M) + E(\widetilde{u}; S^2) \\
&\le \lim_{i\to\infty} E(u_i; M) \le E(\phi) + (\Lambda - \lambda)E(\psi) < \overline{\epsilon},
\end{aligned}$$

which is a contradiction.

By Corollary 2.3, we have

$$\int_0^\infty \int_M |\partial_t u|^2 dxdt \le C.$$

Then there exists a time sequence $t_i \to \infty$ such that $\partial_t u(x, t_i) \to 0,\ a.e.\ x \in M$. Since the flow dose not blow up, by Lemma 2.6, there exists a subsequence of $(u(x, t_i), v(x, t_i))$ which is still denoted by $(u(x, t_i), v(x, t_i))$ such that

$$u(x, t_i) \to u_\infty(x) \text{ and } v(x, t_i) \to v_\infty(x) \text{ in } C^2(M)$$

as $i \to \infty$, where (u_∞, v_∞) is a harmonic map from M to the Lorentzian manifold $N \times \mathbb{R}$ with the boundary data $(u_\infty, v_\infty)|_{\partial M} = (\phi, \psi)$. Then the theorem is proved. □

Now, we proceed to prove our last Theorem 1.4.

Proof of Theorem 1.4. By the proof of Theorem 1.3, we only need to prove that the solution (u, v) does not blow up at any time $t \in (0, \infty]$.

If not, then we may assume T_1 is the first singular (or blow up) time and $(x_0, T_1) \in S(u, T_1)$ is a singular point (or energy concentration point). By Theorem 1.2, we can get a nontrivial harmonic sphere $\widetilde{u} : \mathbb{S}^2 \to N$. However, under the assumptions of Theorem 1.4, the main result in [24] tells us that the manifold N cannot admit any nontrivial harmonic sphere. This is a contradiction and we have finished the proof. □

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