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## Mutually Unbiasedness between Maximally Entangled Bases and Unextendible Maximally Entangled Bases in $\mathbb{C}^2 \otimes \mathbb{C}^{2^k}$

by


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# Mutually Unbiasedness between Maximally Entangled Bases and Unextendible Maximally Entangled Bases in $\mathbb{C}^2 \otimes \mathbb{C}^{2^k}$

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**Abstract:** We study maximally entangled bases and unextendible maximally entangled bases in bipartite systems $\mathbb{C}^2 \otimes \mathbb{C}^{2^k}$ $(k > 1)$ which are mutually unbiased. We derive the mutually unbiased conditions of two such bases, and present an approach of constructing a pair of maximally entangled basis and unextendible maximally entangled basis which are mutually unbiased. In particular, explicit examples in $\mathbb{C}^2 \otimes \mathbb{C}^4$ and $\mathbb{C}^2 \otimes \mathbb{C}^8$ are given indetail.

**Keywords:** mutually unbiased bases; maximally entangled states; unextendible maximally entangled bases

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Mutually unbiased bases (MUBs) have been extensively investigated due to their important roles played in quantum kinematics[1], quantum state tomography[2−3], quantum key distribution[4], cryptographic protocols[5−6], mean king problem[7], quantum teleportation and superdense coding[8−10], and in quantifying wave-particle duality in multipath interferometers[4]. Two orthogonal bases $\mathcal{B}_1 = \{|\phi_i\rangle\}_{i=1}^d$ and $\mathcal{B}_2 = \{|\psi_i\rangle\}_{i=1}^d$ of $\mathbb{C}^d$ are said to be mutually unbiased if

$$|\langle\phi_i|\psi_j\rangle| = \frac{1}{\sqrt{d}}, \quad \forall i, j = 1, 2, \ldots, d.$$

A set of orthonormal bases $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m$ in $\mathbb{C}^d$ is said to be a set of mutually unbiased bases if every pair of the bases in the set is mutually unbiased. The maximum number of MUBs in $\mathbb{C}^d$ is shown to be no more than $d + 1$ [3]. For prime power dimensional case and qubits systems, different constructions of MUBs have been presented in $[11 - 18]$.

For bipartite systems, there are many different kinds of bases such as product bases (PB)[19], unextendible product basis (UPB)[20], unextendible maximally entangled basis (UMEB)[21−26] and maximally entangled basis (MEB)[27] etc., according to the quantum entanglement of the related basic vectors in the bases.

The maximally entangled states play a vital role in quantum information processing tasks such as perfect teleportation[28−36]. A pure state $|\psi\rangle$ is said to be a $d \otimes d'$ $(d' > d)$ maximally entangled state

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if and only if for an arbitrary given orthonormal complete basis $\{|i_A\rangle\}$ of subsystem $A$, there exists an orthonormal basis $\{|i_B\rangle\}$ of subsystem $B$ such that $|\psi\rangle$ can be written as $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|i_A\rangle\otimes|i_B\rangle$[35]. A maximally entangled basis (MEB) is a complete set of orthogonal maximally entangled vectors. In [27], the authors provided a systematic way of constructing MEBs in arbitrary bipartite system $\mathbb{C}^d\otimes\mathbb{C}^{kd}$ $(k\in Z^+)$. Then necessary and sufficient conditions of constructing two mutually unbiased maximally entangled bases (MUMEBs) are derived, and explicit constructions of MUMEBs in $\mathbb{C}^2\otimes\mathbb{C}^4$ and $\mathbb{C}^2\otimes\mathbb{C}^6$ are presented.

An unextendible maximally entangled basis (UMEB) in $\mathbb{C}^d\otimes\mathbb{C}^d$ is a set of less than $d^2$ orthogonal maximally entangled states in $\mathbb{C}^d\otimes\mathbb{C}^d$ such that whose complementary space has no maximally entangled vectors that are othogonal to all of them. It has been proved that UMEBs do not exist for $d=2$, and explicit examples are presented for a 6-member UMEB for d=3 and a 12-member UMEB for d=4[21]. In [22], a systematic way of constructing a set of $d^2$ orthonormal maximally entangled states in $\mathbb{C}^d\bigotimes\mathbb{C}^{d'}(\frac{d'}{2}<d<d')$ was established. In [23, 24], UMEBs in $\mathbb{C}^d\bigotimes\mathbb{C}^{d'}$ with $d'=dq+r$ $(0<r<d)$ have been constructed. Also, UMEBs in $\mathbb{C}^d\bigotimes\mathbb{C}^{qd}(q\geq 2)$ have been constructed in [24]. UMEBs in $\mathbb{C}^d\bigotimes\mathbb{C}^d$ have been investigated in [25]. In [22], the authors first considered the mutually unbiased bases in which all the bases are unextendible maximally entangled ones, and presented two mutually unbiased unextendible maximally entangled bases(MUUMEBs) in $\mathbb{C}^2\otimes\mathbb{C}^3$. Necessary conditions of constructing a pair of MUUMEBs in $\mathbb{C}^2\bigotimes\mathbb{C}^3$ are derived in [26]. In [23], two MUUMEBs in $\mathbb{C}^2\otimes\mathbb{C}^5$ and $\mathbb{C}^3\otimes\mathbb{C}^4$ are established.

Instead of the investigation of mutually unbiased bases from two MEBs or from two UMEBs, in this paper we study the mutually unbiasedness between one MEB and one UMEB in arbitrary bipartite spaces $\mathbb{C}^2\otimes\mathbb{C}^{2^k}(k>1)$. We first derive the necessary and sufficient conditions of constructing mutually unbiased MEB and UMEB, then present explicit constructions of mutually unbiased MEB and UMEB in $\mathbb{C}^2\otimes\mathbb{C}^4$ and $\mathbb{C}^2\otimes\mathbb{C}^8$.

We first begin with the general forms of MEB and UMEB in $\mathbb{C}^2\otimes\mathbb{C}^{2^k}(k>1)$. Let $\{|0\rangle,\ |1\rangle\}$ and $\{|i'\rangle\}_{i=0}^{2^k-1}$ denote the orthonormal bases of $\mathbb{C}^2$ and $\mathbb{C}^{2^k}(k>1)$, respectively. An MEB in $\mathbb{C}^2\otimes\mathbb{C}^{2^k}(k\in Z^+\setminus\{1\})$ has been constructed in [27],

$$|\phi_{n,m}^{(\alpha)}\rangle=\frac{1}{\sqrt{2}}\sum_{p=0}^{1}(-1)^{np}|p\oplus_2 m\rangle|(p+2\alpha)'\rangle,\quad \alpha=0,1,\ldots,2^{k-1}-1;\quad n,m=0,1, \tag{1}$$

where $p\oplus_2 m$ denotes $(p+m)\mod 2$. And a $(2^{k+1}-2)$-member UMEB in $\mathbb{C}^2\otimes\mathbb{C}^{2^k}(k>1)$ has been constructed in [24],

$$\begin{aligned}&|\eta_{i,j}\rangle=\frac{1}{\sqrt{2}}\sum_{q=0}^{1}(-1)^{qi}|q\rangle|(q\oplus_{(2^k-1)}j)'\rangle,\ i=0,1;\quad j=0,1,\ldots,2^k-2,\\&|\eta_{i,2^k-1}\rangle=|i\rangle|(2^k-1)'\rangle,\quad i=0,1,\end{aligned}$$

where $q\oplus_{(2^k-1)}j$ denotes $(q+j)\mod(2^k-1)$.

Let $\{|a_i'\rangle\}_{i=0}^{2^k-1}$ be another orthonormal basis in $\mathbb{C}^{2^k}$ that is different from $\{|i'\rangle\}_{i=0}^{2^k-1}$. New $(2^{k+1}-2)$-member UMEB in $\mathbb{C}^2\otimes\mathbb{C}^{2^k}$ can be obtained in the following way,

$$\begin{aligned}&|\psi_{i,j}\rangle=\frac{1}{\sqrt{2}}\sum_{q=0}^{1}(-1)^{qi}|q\rangle|a'_{q\oplus_{(2^k-1)}j}\rangle,\ i=0,1;\quad j=0,1,\ldots,2^k-2,\\&|\psi_{i,2^k-1}\rangle=|i\rangle|a'_{(2^k-1)}\rangle,\quad i=0,1,\end{aligned} \tag{2}$$

The MEB (1) and the UMEB (2) in $\mathbb{C}^2\otimes\mathbb{C}^{2^k}$ are mutually unbiased if and only if they satisfy the following relations,

$$|\langle\phi_{n,m}^{(\alpha)}|\psi_{i,j}\rangle|=\frac{1}{\sqrt{2^{k+1}}},\quad \alpha=0,1,\ldots,2^{k-1}-1;\quad n,m,i=0,1;\quad j=0,1,\ldots,2^k-2. \tag{3}$$

$$|\langle \phi^{(\alpha)}_{n,m}|\psi_{i,2^k-1}\rangle| = \frac{1}{\sqrt{2^k}}, \quad i=0,1. \tag{4}$$

Let $T$ denote the transition matrix from the basis $\{|i'\rangle\}_{i=0}^{2^k-1}$ to the basis $\{|a_i'\rangle\}_{i=0}^{2^k-1}$ in $\mathbb{C}^{2^k}$,

$$\begin{pmatrix} |a_0'\rangle \\ |a_1'\rangle \\ \vdots \\ |a'_{(2^k-1)}\rangle \end{pmatrix} = T \begin{pmatrix} |0'\rangle \\ |1'\rangle \\ \vdots \\ |(2^k-1)'\rangle \end{pmatrix}, \tag{5}$$

i.e. $|a_i'\rangle = \sum_{j=0}^{2^k-1} t_{ij}|j'\rangle$, where $t_{ij}$ are entries of the matrix $T$. Then conditions (3) and (4) are valid if and only if $T$ satisfies the following relations,

$$|\sum_{p=0}^{1} \xi^p t_{p\oplus_{(2^k-1)}\beta,\ p+2\gamma}| = \frac{1}{\sqrt{2^{k-1}}}, \tag{6}$$

$$|\sum_{p=0}^{1} \xi^p t_{p\oplus_{(2^k-1)}\beta,\ 1-p+2\gamma}| = \frac{1}{\sqrt{2^{k-1}}}, \tag{7}$$

$$|t_{2^k-1,j}| = \frac{1}{\sqrt{2^k}}, \tag{8}$$

where $\xi = 1,-1;\ \beta = 0,1,\ldots,2^k-2;\ \gamma = 0,1,\ldots,2^{k-1}-1;\ \ j=0,1,\ldots,2^k-1$.

For a detailed construction of a pair of mutually unbiased MEB and UMEB, we first consider the case of $\mathbb{C}^2\otimes\mathbb{C}^4$. Let us take the second basis $\{|a_i'\rangle\}_{i=0}^3$ in $\mathbb{C}^4$ as $(|a_0'\rangle,|a_1'\rangle,|a_2'\rangle,|a_3'\rangle)^t = A\,(|0'\rangle,|1'\rangle,|2'\rangle,|3'\rangle)^t$, where $t$ denotes transposition,

$$A = \frac{1}{2}\begin{pmatrix} 1 & i & 1 & -i \\ -1 & i & 1 & i \\ 1 & i & -1 & i \\ 1 & -i & 1 & i \end{pmatrix}$$

with $i=\sqrt{-1}$. It is direct to verify that the matrix $A$ satisfies the mutually unbiased conditions (3) and (4). From (1) and (2) we have the mutually unbiased MEB and the completed 6-member UMEB in $\mathbb{C}^2\otimes\mathbb{C}^4$ respectively,

$$|\phi^{(\alpha)}_{n,m}\rangle = \frac{1}{\sqrt{2}}\sum_{p=0}^{1}(-1)^{np}|p\oplus_2 m\rangle|(p+2\alpha)'\rangle, \quad \alpha=0,1; \quad n,m=0,1, \tag{9}$$

$$|\psi_{i,j}\rangle = \frac{1}{\sqrt{2}}\sum_{q=0}^{1}(-1)^{qi}|q\rangle|a'_{q\oplus_3 j}\rangle; \qquad |\psi_{i,2^k-1}\rangle = |i\rangle|a'_{(2^k-1)}\rangle, \tag{10}$$

where $i=0,1;\ \ j=0,1,2$.

As another example, we present a detailed construction of mutually unbiased MEB and complete 14-member UMEB in $\mathbb{C}^2\otimes\mathbb{C}^8$. According to (1) we have a MEB in $\mathbb{C}^2\otimes\mathbb{C}^8$,

$$|\phi^{(\alpha)}_{n,m}\rangle = \frac{1}{\sqrt{2}}\sum_{p=0}^{1}(-1)^{np}|p\oplus_2 m\rangle|(p+2\alpha)'\rangle, \quad \alpha=0,1,2,3; \quad n,m=0,1. \tag{11}$$

For the mutually unbiased UMEB in $\mathbb{C}^2 \otimes \mathbb{C}^8$, we take the basis $\{|b'_j\rangle\}_{j=0}^7$ in $\mathbb{C}^8$ as

$$(|b'_0\rangle, |b'_1\rangle, |b'_2\rangle, |b'_3\rangle, |b'_4\rangle, |b'_5\rangle, |b'_6\rangle, |b'_7\rangle)^t = B\,(|0'\rangle, |1'\rangle, |2'\rangle, |3'\rangle, |4'\rangle, |5'\rangle, |6'\rangle, |7'\rangle)^t,$$

where

$$B = \frac{1}{\sqrt{8}} \begin{pmatrix} -i & -1 & 1 & -i & -i & -1 & 1 & -i \\ -i & -1 & -1 & i & -i & -1 & -1 & i \\ -i & 1 & -1 & -i & -i & 1 & -1 & -i \\ -i & 1 & 1 & i & -i & 1 & 1 & i \\ -i & -1 & 1 & -i & i & 1 & -1 & i \\ -i & -1 & -1 & i & i & 1 & 1 & -i \\ -i & 1 & -1 & -i & i & -1 & 1 & i \\ -i & 1 & 1 & i & i & -1 & -1 & -i \end{pmatrix}.$$

Then the corresponding complete 14-member UMEB in $\mathbb{C}^2 \otimes \mathbb{C}^8$ has the form,

$$|\psi_{i,j}\rangle = \frac{1}{\sqrt{2}} \sum_{q=0}^{1} (-1)^{qi} |q\rangle |b'_{q\oplus_7 j}\rangle; \qquad |\psi_{i,7}\rangle = |i\rangle |b'_7\rangle, \tag{12}$$

where $i = 0, 1; \quad j = 0, 1, \ldots, 6$.

It is direct to verify that the transformation matrix $B$ satisfies the relation (7) and (8), then the MEB (11) and the completed 14-member UMEB (12) in $\mathbb{C}^2 \otimes \mathbb{C}^8$ are mutually unbiased.

We have derived the sufficient and necessary conditions of mutually unbiasedness between maximally entangled basis and unextendible maximally entangled basis in $\mathbb{C}^2 \otimes \mathbb{C}^{2^k}$ $(k > 1)$. As detailed applications, we have constructed a pair of mutually unbiased bases, one of which is a maximally entangled basis and another one is an unextendible maximally entangled basis in $\mathbb{C}^2 \otimes \mathbb{C}^4$ and in $\mathbb{C}^2 \otimes \mathbb{C}^8$ respectively.

There are still many open problems related to maximally entangled bases and unextendible maximally entangled bases which are mutually unbiased, such as the case in $\mathbb{C}^d \otimes \mathbb{C}^d$ or $\mathbb{C}^d \otimes \mathbb{C}^{kd}$ $(k \in Z^+)$ for $d > 2$, as well as to the roles played by such bases in quantum information processing.

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