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by

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Abstract

We present a new uncertainty relation by defining a measure of uncertainty based on skew information. For bipartite systems, we establish uncertainty relations with the existence of a quantum memory. A general relation between quantum correlations and tight bounds of uncertainty has been presented.

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Uncertainty principle is one of the most fascinating features of the quantum world. It asserts a fundamental limit on the precision with which certain pairs of physical properties of a particle, such as position and momentum, can be simultaneously known. The uncertainty principle has attracted considerable attention since the innovation of quantum mechanics and has been investigated in terms of various types of uncertainty inequalities. The uncertainty of measurement outcomes has been quantified in terms of the noise and disturbance [3, 4], according to successive measurements [5, 6], as informational recourses [7], in entropic terms [8–13], by means of majorization technique [14–18], and based on sum of variances and standard deviations [19–21].

For a pair of observables R and S, the well-known Heisenberg-Robertson uncertainty relation [1, 22] says that,$V_\rho(R).V_\rho(S) \geqslant \frac{1}{4}|\mathrm{Tr}\rho\,[R,S]|^2$, where $[R,S] = RS - SR$ is the commutator, $V_\rho(R)$ is the standard deviation: $V_\rho(R) = \mathrm{Tr}[\rho\, R^2] - (\mathrm{Tr}[\rho\, R])^2$.

Since the Von Neumann entropy serves as an appropriate measure of the information content of a state, it is also used to quantify the quantum uncertainty. The entropy uncertainty relation says that [8, 9]: $H(R) + H(S) \geqslant \log_2 \frac{1}{c}$, where $H(R)$ is the Shannon entropy of the probability distribution of the measurement outcomes of R, $c = \max_{j,k} |\langle\psi_j|\phi_k\rangle|^2$, $|\psi_j\rangle$ and $|\phi_k\rangle$ are the eigenvectors of observables R and S, respectively. The term $1/c$ quantifies the compatibility or complementarity of the two observables R and S. It has been proved that the entropy uncertainty relations do imply the Heisenberg's uncertainty relation.

Concerning bipartite systems, the uncertainty relations become more interesting and may depend on the correlations between the subsystems in general.In [10], the authors proved the following remarkable result. For any bipartite density matrix ρ_{AB} in tensor space $H_A \otimes H_B$, the following uncertainty relation holds:

$$S(R|B) + S(S|B) \geqslant \log_2 \frac{1}{c} + S(A|B) \tag{1}$$

where $S(R|B)$, $S(S|B)$ and $S(A|B)$ are the conditional entropies. Eq.(1) is further improved and the lower bound is connected to quantum discord [11]:

$$\begin{aligned} S(R|B) + S(S|B) \geqslant \log_2 \frac{1}{c} + S(A|B) \\ + \max\{0, D_A(\rho_{AB}) - C_A(\rho_{AB})\}, \end{aligned} \tag{2}$$

where $D_A(\rho_{AB})$ is the quantum discord, $C_A(\rho_{AB})$ is the classical correlation with respect to the measurement on subsystem A.

The quantum uncertainty relations can be also described in terms of skew information. In [23] Wigner and Yanase introduced the following quantity to quantify the degree of non-commutativity of a state ρ and an observable H, $I(\rho, H) = -\frac{1}{2}\mathrm{Tr}\left[\sqrt{\rho}, H\right]^2$. When ρ is a pure state, $I(\rho, H)$ is reduced to the variance $V_\rho(H)$. Here $I(\rho, H)$ may be interpreted as some kind of quantum uncertainty of H in ρ.

In [25], Luo introduced another quantity,$J_\rho(H) = -\frac{1}{2}\mathrm{Tr}[(\{\sqrt{\rho}, H_0\})^2]$, where $\{X, Y\} = XY + YX$ is the anti-commutator, $H_0 = H - \mathrm{Tr}(\rho H)I$ with I the identity operator. It is shown that for arbitrary two observables R and S, the following inequality holds [25],

$$\sqrt{I(\rho, R)\, J_\rho(R)}\sqrt{I(\rho, S)\, J_\rho(S)} \geqslant \frac{1}{4}|\mathrm{Tr}(\rho[R, S])|^2. \tag{3}$$

Inequality (3) can be also rewritten as $\sqrt{I(\rho, R)\, I(\rho, S)} \geqslant L_\rho(R, S)$, where $L_\rho(R, S)$ is defined by $L_\rho(R, S) := \frac{1}{4}\frac{|\mathrm{Tr}(\rho[R,S])|^2}{\sqrt{J_\rho(R) J_\rho(S)}}$. Here when $J_\rho(R)J_\rho(S) = 0$, $L_\rho(R, S)$ is defined to be zero. $\sqrt{I(\rho, R)\, J_\rho(R)}$ can be regarded as a kind of measure for quantum uncertainty, so we define $\mathcal{UN}(\rho, R) := \sqrt{I(\rho, R)\, J_\rho(R)}$.

In the following, we consider uncertainty relations based on skew information. Let R and S be two non-degenerate observables, with eigenvectors $|\phi_k\rangle$ and $|\psi_k\rangle$, respectively. Denote $\phi_k = |\phi_k\rangle\langle\phi_k|$, $\psi_k = |\psi_k\rangle\langle\psi_k|$, which are the rank one spectral projectors of R and S. We define the uncertainty of ρ associated to the projective measurement $\{\phi_k\}$ as: $\mathcal{UN}(\rho)_{\{\phi_k\}} = \sum_k \mathcal{UN}(\rho, \phi_k) = \sum_k \sqrt{I(\rho, \phi_k)\, J_\rho(\phi_k)}$.

Now we consider the case of bipartite state ρ_{AB} in tensor space $H_A \otimes H_B$. Recall that, quantum discord[30, 31], is a quantum correlation that are different from entanglement, and has found many novel applications[32]. Quantum discord was defined as the minimal difference of mutual information, before and after local projective measurement on H_A, and a bipartite state ρ_{AB} is with zero discord if and only if it is classical-quantum correlated state (CQ state) $\rho^{AB} = \sum\limits_k \lambda_k \phi_k \otimes \rho_k^B$. Besides original discord, there are some other discord-like measures(see [33, 34]), they share the same properties as discord, e.g., their values are zero iff the state is a CQ state.

In the following, we define another discord-like measure. Let $\mathcal{O}$ denote any orthogonal basis in Hilbert space H_A. Let $|\phi_k\rangle$ be an orthogonal basis of H_A and $\phi_k = |\phi_k\rangle\langle\phi_k|$ the orthogonal projections on H_A. We define the quantum correlation of ρ_{AB} as:

$$\mathcal{Q}\left(\rho^{AB}\right) = \min_{\mathcal{O}} \sum_k [I(\rho^{AB}, \phi_k \otimes I_B) - I(\rho^A, \phi_k)], \tag{4}$$

where the minimum is taken over all the orthogonal basis in H_A, ρ^A is the reduced state of system A.

From [26], one has that for any bipartite state ρ^{AB} and any observable X on H^A, $I(\rho^{AB}, X \otimes I) \geqslant I(\rho_A, X)$. It follows that for any set of rank one orthogonal projections $\{\phi_k\}$ on H_A: $\sum_k [I(\rho_{AB}, \phi_k \otimes I^B) - I(\rho_A, \phi_k)] \geqslant 0$. Therefore we have $\mathcal{Q}\left(\rho^{AB}\right) \geqslant 0$. Moreover, $\mathcal{Q}\left(\rho^{AB}\right) = 0$ if and only if ρ_{AB} is a CQ state, so it is a discord-like measure. This can be seen easily by using the method in proving the theorem 1, property (1) of [27].

The definition of the quantum correlation $\mathcal{Q}\left(\rho^{AB}\right)$ looks similar to the quantum correlation measures introduced in [28]. However, they are quite different. We added a term of measurement on the subsystem A, which gives an explicit physical meaning: the quantum correlation $\mathcal{Q}\left(\rho^{AB}\right)$ is the minimal difference of incompatibility of the projective measurements on the bipartite state ρ^{AB} and on the local reduced state ρ^A. It quantifies the quantum correlations between the subsystems A and B.

Theorem 1. Let ρ^{AB} be a quantum state on $H_A \otimes H_B$. Denote $\{\phi_k\}$ and $\{\psi_k\}$ two sets of rank one projective measurements on H^A. Then the following uncertainty relation holds:

$$\begin{aligned} &\mathcal{UN}\left(\rho^{AB}\right)_{\{\phi_k \otimes I^B\}} + \mathcal{UN}\left(\rho^{AB}\right)_{\{\psi_k \otimes I^B\}} \\ &\geqslant \sum_k 2L_{\rho^A}(\phi_k, \psi_k) + 2\mathcal{Q}\left(\rho^{AB}\right). \end{aligned} \tag{5}$$

where $L_{\rho^A}(\phi_k, \psi_k) := \frac{1}{4} \frac{|\mathrm{Tr}(\rho^A[\phi_k, \psi_k])|^2}{\sqrt{J_{\rho^A}(\phi_k) J_{\rho^A}(\psi_k)}}$.

Proof. By definition we have

$$\begin{aligned}
&\mathcal{UN}\left(\rho^{AB}\right)_{\{\phi_k\otimes I^B\}}+\mathcal{UN}\left(\rho^{AB}\right)_{\{\psi_k\otimes I^B\}}\\
&\geqslant \sum_k I\left(\rho^{AB},\phi_k\otimes I^B\right)+\sum_k I\left(\rho^{AB},\psi_k\otimes I^B\right)\\
&=\sum_k I\left(\rho^{A},\phi_k\right)+\sum_k I\left(\rho^{A},\psi_k\right)\\
&\quad+\sum_k\left[I\left(\rho^{AB},\phi_k\otimes I^B\right)-I\left(\rho^{A},\phi_k\right)\right]\\
&\quad+\sum_k\left[I\left(\rho^{AB},\psi_k\otimes I^B\right)-I\left(\rho^{A},\psi_k\right)\right]\\
&\geqslant \sum_k 2\sqrt{I\left(\rho^{A},\phi_k\right)I\left(\rho^{A},\psi_k\right)}\\
&\quad+\sum_k\left[I\left(\rho^{AB},\phi_k\otimes I^B\right)-I\left(\rho^{A},\phi_k\right)\right]\\
&\quad+\sum_k\left[I\left(\rho^{AB},\psi_k\otimes I^B\right)-I\left(\rho^{A},\psi_k\right)\right]\\
&\geqslant \sum_k 2L_{\rho^A}(\phi_k,\psi_k)+2\mathcal{Q}\left(\rho^{AB}\right).
\end{aligned} \tag{6}$$

The first inequality holds, since $I\left(\rho,R\right)\leqslant J_\rho(R)$(see [35]).The second inequality holds from the Cauchy-Schwarz inequality. The final inequality holds because the optimal measurement for $\mathcal{Q}\left(\rho^{AB}\right)$ may not be ϕ_k or ψ_k.

From theorem 1, in fact we obtain an uncertainty relation in the form of sum of skew information, which is in some sense like that of recent works[19–21]. Second, our result is quite different from that of [19–21], the results of [19–21] only deal with single partite case, in our work, we treat the bi-partite case, that is, with a quantum memory B. Last, the lower bound is interesting, since it contain two terms, one term is quantum correlation $\mathcal{Q}\left(\rho^{AB}\right)$, and another term is $\sum_k L_{\rho^A}(\phi_k,\psi_k)$, which is a degree of compatible of two measurements, just like the meaning of $\log_2\frac{1}{c}$ in the entropy uncertainty relation (that is, Eq.(1)). So our result can be seen as a analogue of bi-partite entropy uncertainty relation.

In summary, we have established a new uncertainty relation based on the skew information. We studied the bipartite case, which is the case of uncertainty relations with the existence of a quantum memory. Our result shows that quantum correlations can be used to obtain a tight bound of uncertainty.

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[1] W. Heisenberg, *Z. Phys.* 43, 172 (1927).

[2] P. Busch, T. Heinonen, P. Lahti, *Phys. Rep.* 452, 155 (2007).

[3] M. Ozawa, *Ann. Phys.* 311, 350 (2004).

[4] P. Busch, P. Lahti, R. F. Werner, *Phys. Rev. Lett.* 111, 160405 (2013).

[5] J. Distler, S. Paban, *Phys. Rev. A* 87, 062112 (2013).

[6] K. Baek, T. Farrow, W. Son, *Phys. Rev. A* 89, 032108 (2014).

[7] S. Wehner, A. Winter, *New J. Phys.* 12, 025009 (2010).

[8] Deutsch, D. *Phys. Rev. Lett.* 50, 631 (1983).

[9] H. Maassen, J. B. M. Uffink, *Phys. Rev. Lett.* 60, 1103 (1988).

[10] M. Berta, et al, Nature Physics 6, 659(2010).

[11] A. K. Pati, A. R. Usha Devi, A. K. Rajagopal, Sudha, Phys. Rev. A 86, 042105 (2012).

[12] J. Zhang, Y. Zhang, and C.- S. Yu, Scientific Reports **5**, 11701 (2015).

[13] S. Liu, L.-Z. Mu, and H. Fan, Phys. Rev. A **91**, 042133 (2015).

[14] I. Bialynicki-Birula, Ł. Rudnicki, *Statistical Complexity*, 1-34 (Springer, Netherlands, 2011).

[15] Z. Puchała, Ł. Rudnicki, K. Zyczkowski, *J. Phys. A: Math. Theor.* 46, 272002 (2013).

[16] S. Friedland, V. Gheorghiu, G. Gour, *Phys. Rev. Lett.* 111, 230401 (2013).

[17] Ł. Rudnicki, Z. Puchała, K. Życzkowski, Phys. Rev. A. 89, 052115(2014).

[18] S. Friedland, V. Gheorghiu, G. Gour, Phys. Rev. Lett. 111, 230401(2013).

[19] L. Maccone, A. K. Pati, Phys. Rev. Lett. 113, 260401(2014).

[20] Y. Yao, X. Xiao, X. Wang, C. P. Sun, Phys. Rev. A 91, 062113(2015).

[21] B. Chen and S. M. Fei, *Scientific Reports* **5**, 14238(2015).

[22] H. P. Robertson, *Phys. Rev.* 34, 163(1929).

[23] E. P. Wigner and M. M. Yanase, Proc. Natl. Acad. Sci. U.S.A. 49, 910 (1963).

[24] Z. Chen Phys. Rev. A 71, 052302 (2005).

[25] S. Luo, Phys. Rev. A, 72,042110 (2005).

[26] E. H. Lieb, Adv. Math 11, 267 (1973).

[27] S. Luo, S. Fu, C. H. Oh, Phys. Rev. A 85, 032117 (2012).

[28] C. Yu, S. Wu, X. Wang, X. Yi, H. Song, Euro. Phys. Lett. 107, 10007 (2014).

[29] D. Girolami, T. Tufarelli, G. Adesso, Phys. Rev. Lett. 110, 240402 (2013).

[30] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. 88, 017901(2001).

[31] L. Henderson and V. Vedral, J. Phys. A: Math. Gen. 34, 6899(2001).

[32] K. Modi, A. Brodutch, H. Cable, T. Paterek, V. Vedral, Rev. Mod. Phys. 84, 1655-1707 (2012).

[33] B.Dakic, V. Vedral, C. Brukner, Phys. Rev. Lett. 105, 190502 (2010).

[34] Z. Ma, Z. Chen, F. F. Fanchini, S. Fei, Scientific Reports 5, 10262 (2015).

[35] S. Furuichi, Phys. Rev. A 82, 034101(2010).