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by

Xue-Na Zhu and Shao-Ming Fei

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Xue-Na Zhu¹ and Shao-Ming Fei^{2,3}

¹*School of Mathematics and Statistics Science, Ludong University, Yantai 264025, China*

²*School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*

³*Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany*

Entanglement monogamy is a fundamental property of multipartite entangled states. We investigate the monogamy relations for multiqubit generalized W -class states. Analytical monogamy inequalities are obtained for the concurrence of assistance, the entanglement of formation and the entanglement of assistance.

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I. INTRODUCTION

Quantum entanglement [1–6] is an essential feature of quantum mechanics that distinguishes the quantum from the classical world. It is one of the fundamental differences between quantum entanglement and classical correlations that a quantum system entangled with one of the other systems limits its entanglement with the remaining others. This restriction of entanglement shareability among multi-party systems is known as the monogamy of entanglement. The monogamy relations give rise to the structures of entanglement in the multipartite setting. For a tripartite system A , B , and C , the monogamy of an entanglement measure ε implies that the entanglement between A and BC satisfies $\varepsilon_{A|BC} \geq \varepsilon_{AB} + \varepsilon_{AC}$.

In Ref.[7, 8] the monogamy of entanglement for multiqubit W -class states has been investigated, and the monogamy relations for tangle and the squared concurrence have been proved. In this paper, we show the general monogamy relations for the x -power of concurrence of assistance, the entanglement of formation, and the entanglement of assistance for generalized multiqubit W -class states.

II. MONOGAMY OF CONCURRENCE OF ASSISTANCE

For a bipartite pure state $|\psi\rangle_{AB}$ in vector space $H_A \otimes H_B$, the concurrence is given by [9–11]

$$C(|\psi\rangle_{AB}) = \sqrt{2[1 - \text{Tr}(\rho_A^2)]}, \quad (1)$$

where ρ_A is reduced density matrix by tracing over the subsystem B , $\rho_A = \text{Tr}_B(|\psi\rangle_{AB}\langle\psi|)$. The concurrence is extended to mixed states $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, $p_i \geq 0$, $\sum_i p_i = 1$, by the convex roof construction,

$$C(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle), \quad (2)$$

where the minimum is taken over all possible pure state decompositions of ρ_{AB} .

For a tripartite state $|\psi\rangle_{ABC}$, the concurrence of assistance (CoA) is defined by [12]

$$C_a(|\psi\rangle_{ABC}) \equiv C_a(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle), \quad (3)$$

for all possible ensemble realizations of $\rho_{AB} = \text{Tr}_C(|\psi\rangle_{ABC}\langle\psi|) = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|$. When $\rho_{AB} = |\psi\rangle_{AB}\langle\psi|$ is a pure state, then one has $C(|\psi\rangle_{AB}) = C_a(\rho_{AB})$.

For an N -qubit state $|\psi\rangle_{AB_1 \dots B_{N-1}} \in H_A \otimes H_{B_1} \otimes \dots \otimes H_{B_{N-1}}$, the concurrence $C(|\psi\rangle_{A|B_1 \dots B_{N-1}})$ of the state $|\psi\rangle_{A|B_1 \dots B_{N-1}}$, viewed as a bipartite with partitions A and $B_1 B_2 \dots B_{N-1}$, satisfies the follow inequality[13]

$$C_{A|B_1 B_2 \dots B_{N-1}}^\alpha \geq C_{AB_1}^\alpha + C_{AB_2}^\alpha + \dots + C_{AB_{N-1}}^\alpha, \quad (4)$$

and

$$C_{A|B_1 B_2 \dots B_{N-1}}^\beta < C_{AB_1}^\beta + C_{AB_2}^\beta + \dots + C_{AB_{N-1}}^\beta, \quad (5)$$

where $\alpha \geq 2$, $\beta \leq 0$, $C_{AB_i} = C(\rho_{AB_i})$ is the concurrence of $\rho_{AB_i} = \text{Tr}_{B_1 \dots B_{i-1} B_{i+1} \dots B_{N-1}}(\rho)$, $C_{A|B_1 B_2 \dots B_{N-1}} = C(|\psi\rangle_{A|B_1 \dots B_{N-1}})$. Due to the monogamy of concurrence, the generalized monogamy relation based on the concurrence of assistance has been proved in Ref. [14],

$$C^2(|\psi\rangle_{A|B_1 \dots B_{N-1}}) \leq \sum_{i=1}^{N-1} C_a^2(\rho_{AB_i}). \quad (6)$$

In the following we study the monogamy property of the concurrence of assistance for the n -qubit generalized W-class states $|\psi\rangle \in H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_n}$ defined by

$$|\psi\rangle = a|000\dots\rangle + b_1|01\dots 0\rangle + \dots + b_n|00\dots 1\rangle, \quad (7)$$

with $|a|^2 + \sum_{i=1}^n |b_i|^2 = 1$.

Lemma 1 For n -qubit generalized W-class states (7), we have

$$C(\rho_{A_1 A_i}) = C_a(\rho_{A_1 A_i}), \quad (8)$$

where $\rho_{A_1 A_i} = \text{Tr}_{A_2 \dots A_{i-1} A_{i+1} \dots A_n}(|\psi\rangle\langle\psi|)$.

[Proof] It is direct to verify that [7], $\rho_{A_1 A_i} = |x\rangle_{A_1 A_i}\langle x| + |y\rangle_{A_1 A_i}\langle y|$, where

$$\begin{aligned} |x\rangle_{A_1 A_i} &= a|00\rangle_{A_1 A_i} + b_1|10\rangle_{A_1 A_i} + b_i|01\rangle_{A_1 A_i}, \\ |y\rangle_{A_1 A_i} &= \sqrt{\sum_{k \neq i} |b_k|^2} |00\rangle_{A_1 A_i}. \end{aligned}$$

From the Hughston- Jozsa-wootters theorem Ref.[7], for any pure-state decomposition of $\rho_{A_1 A_i} = \sum_{h=1}^r |\phi_h\rangle_{A_1 A_i}\langle\phi_h|$, one has $|\phi_h\rangle_{A_1 A_i} = u_{h1}|x\rangle_{A_1 A_i} + u_{h2}|y\rangle_{A_1 A_i}$ for some $r \times r$ unitary matrices u_{h1} and u_{h2} for each h . Consider the normalized state $|\tilde{\phi}_h\rangle_{A_1 A_i} = |\phi_h\rangle_{A_1 A_i} / \sqrt{p_h}$ with $p_h = |\langle\phi_h|\phi_h\rangle|$. One has the concurrence of each two-qubit pure $|\tilde{\phi}_h\rangle_{A_1 A_i}$,

$$C^2(|\tilde{\phi}_h\rangle_{A_1 A_i}) = \frac{4}{p_h^2} |u_{h1}|^4 |b_1|^2 |b_i|^2.$$

Then for the two-qubit state $\rho_{A_1 A_i}$, we have

$$\sum_h p_h C(|\tilde{\phi}_h\rangle_{A_1 A_i}) = \sum_h p_h \frac{2}{p_h} |u_{h1}|^2 |b_1| |b_i| = 2|b_1| |b_i|.$$

Thus we obtain

$$\begin{aligned} C(\rho_{A_1 A_i}) &= \min_{p_h, |\tilde{\phi}_h\rangle_{A_1 A_i}} \sum_h p_h C(|\tilde{\phi}_h\rangle_{A_1 A_i}) \\ &= \max_{p_h, |\tilde{\phi}_h\rangle_{A_1 A_i}} \sum_h p_h C(|\tilde{\phi}_h\rangle_{A_1 A_i}) \\ &= C_a(\rho_{A_1 A_i}). \end{aligned}$$

□

Specifically, in Ref. [8] the same result $C(\rho_{A_1 A_i}) = C_a(\rho_{A_1 A_i})$ has been proved for the generalized W-class states (7) with $a = 0$.

Theorem 1 For the n -qubit generalized W-class states $|\psi\rangle \in H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_n}$, the concurrence of assistance satisfies

$$C_a^x(\rho_{A_1 | A_{j_1} \dots A_{j_{m-1}}}) \geq \sum_{i=1}^{m-1} C_a^x(\rho_{A_1 A_{j_i}}), \quad (9)$$

where $x \geq 2$ and $\rho_{A_1 A_{j_1} \dots A_{j_{m-1}}}$ is the m -qubit, $2 \leq m \leq n$, reduced density matrix of $|\psi\rangle$.

[Proof] For the n -qubit generalized W -class state $|\psi\rangle$, according to the definitions of $C(\rho)$ and $C_a(\rho)$, one has $C_a(\rho_{A_1|A_{j_1}\dots A_{j_{m-1}}}) \geq C(\rho_{A_1|A_{j_1}\dots A_{j_{m-1}}})$. When $x \geq 2$, we have

$$\begin{aligned} C_a^x(\rho_{A_1|A_{j_1}\dots A_{j_{m-1}}}) &\geq C^x(\rho_{A_1|A_{j_1}\dots A_{j_{m-1}}}) \\ &\geq \sum_{i=1}^{m-1} C^x(\rho_{A_1 A_{j_i}}) \\ &= \sum_{i=1}^{m-1} C_a^x(\rho_{A_1 A_{j_i}}). \end{aligned}$$

Here we have used in the first inequality the inequality $a^x \geq b^x$ for $a \geq b > 0$ and $x \geq 0$. The second inequality is due to the monogamy of concurrence (4). The last equality is due to the Lemma 1.

Theorem 2 For the n -qubit generalized W -class state $|\psi\rangle \in H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_n}$ with $C(\rho_{A_1 A_{j_i}}) \neq 0$ for $1 \leq i \leq m-1$, we have

$$C_a^y(\rho_{A_1|A_{j_1}\dots A_{j_{m-1}}}) < \sum_{i=1}^{m-1} C_a^y(\rho_{A_1 A_{j_i}}), \quad (10)$$

where $y \leq 0$ and $\rho_{A_1 A_{j_1}\dots A_{j_{m-1}}}$ is the m -qubit reduced density matrix as in Theorem 1.

[Proof] For $y \leq 0$, we have

$$\begin{aligned} C_a^y(\rho_{A_1|A_{j_1}\dots A_{j_{m-1}}}) &\leq C^y(\rho_{A_1|A_{j_1}\dots A_{j_{m-1}}}) \\ &< \sum_{i=1}^{m-1} C^y(\rho_{A_1 A_{j_i}}) \\ &= \sum_{i=1}^{m-1} C_a^y(\rho_{A_1 A_{j_i}}). \end{aligned}$$

We have used in the first inequality the relation $a^x \leq b^x$ for $a \geq b > 0$ and $x \leq 0$. The second inequality is due to the monogamy of concurrence (5). The last equality is due to Lemma 1.

According to (9) and (10), we can also obtain the lower bounds of $C_a(\rho_{A_1|A_{j_1}\dots A_{j_{m-1}}})$. As an example, consider the 5-qubit generalized W -class states (7) with $a = b_2 = \frac{1}{\sqrt{10}}$, $b_1 = \frac{1}{\sqrt{15}}$, $b_3 = \sqrt{\frac{2}{15}}$, $b_4 = \sqrt{\frac{3}{5}}$. We have

$$C_a(\rho_{A_1|A_2 A_3}) \geq \frac{2}{\sqrt{15}} \sqrt[2]{\left(\frac{1}{\sqrt{10}}\right)^x + \left(\sqrt{\frac{2}{15}}\right)^x}$$

and

$$C_a(\rho_{A_1|A_2 A_3 A_4}) \geq \frac{2}{\sqrt{15}} \sqrt[3]{\left(\frac{1}{\sqrt{10}}\right)^x + \left(\sqrt{\frac{2}{15}}\right)^x + \left(\sqrt{\frac{3}{5}}\right)^x}$$

with $x \geq 2$. The optimal lower bounds can be obtained by varying the parameter x , see Fig. 1, where for comparison the upper bounds are also presented by using the formula $C_a(\rho_{AB}) \leq \sqrt{2(1 - \text{Tr}(\rho_A^2))}$ [15], namely, $C_a(\rho_{A_1|A_2 A_3}) \leq \frac{2}{\sqrt{18}}$ and $C_a(\rho_{A_1|A_2 A_3 A_4}) \leq \frac{2}{\sqrt{18}}$. From Fig.1, one gets that the optimal lower bounds of $C_a(\rho_{A_1|A_2 A_3})$ and $C_a(\rho_{A_1|A_2 A_3 A_4})$ are 0.249 and 0.471, respectively, attained at $x = 2$.

III. MONOGAMY OF ENTANGLEMENT OF FORMATION

The entanglement of formation of a pure state $|\psi\rangle \in H_A \otimes H_B$ is defined by

$$E(|\psi\rangle) = S(\rho_A), \quad (11)$$

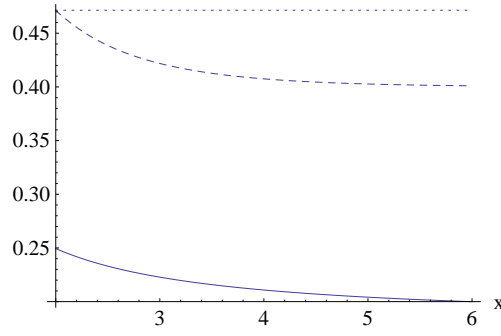


Fig. 1: Solid line is the lower bound of $C_a(\rho_{A_1|A_2A_3})$, dashed line is the lower bound of $C_a(\rho_{A_1|A_2A_3A_4})$ as functions of $x \geq 2$, and dotted line is the upper bound of $C_a(\rho_{A_1|A_2A_3})$ and $C_a(\rho_{A_1|A_2A_3A_4})$.

where $\rho_A = \text{Tr}_B(|\psi\rangle\langle\psi|)$ and $S(\rho) = \text{Tr}(\rho \log_2 \rho)$. For a bipartite mixed state $\rho_{AB} \in H_A \otimes H_B$, the entanglement of formation is given by

$$E(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle), \quad (12)$$

with the infimum taking over all possible decompositions of ρ_{AB} in a mixture of pure states $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where $p_i \geq 0$ and $\sum_i p_i = 1$.

It has been shown that the entanglement of formation does not satisfy the inequality $E_{AB} + E_{AC} \leq E_{A|BC}$ [16]. Rather it satisfies [13],

$$E_{A|B_1B_2\dots B_{N-1}}^\alpha \geq E_{AB_1}^\alpha + E_{AB_2}^\alpha + \dots + E_{AB_{N-1}}^\alpha, \quad (13)$$

where $\alpha \geq \sqrt{2}$.

The corresponding entanglement of assistance (EoA) [17] is defined in terms of the entropy of entanglement [18] for a tripartite pure state $|\psi\rangle_{ABC}$,

$$E_a(|\psi\rangle_{ABC}) \equiv E_a(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle), \quad (14)$$

which is maximized over all possible decompositions of $\rho_{AB} = \text{Tr}_C(|\psi\rangle_{ABC}) = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, with $p_i \geq 0$ and $\sum_i p_i = 1$. For any N -qubit pure state $|\psi\rangle \in H_A \otimes H_{B_1} \otimes \dots \otimes H_{B_{N-1}}$, it has been shown that the entanglement of assistance satisfies [13],

$$E(|\psi\rangle_{A|B_1B_2\dots B_{N-1}}) \leq \sum_{i=1}^{N-1} E_a(\rho_{AB_i}). \quad (15)$$

In fact, generally we can prove the following results for the n -qubit generalized W-class states about the entanglement of formation and the entanglement of assistance.

Theorem 3 For the n -qubit generalized W-class states $|\psi\rangle \in H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_n}$, we have

$$E(|\psi\rangle_{A_1|A_2\dots A_n}) \leq \sum_{i=2}^n E(\rho_{A_1A_i}), \quad (16)$$

where $\rho_{A_1A_i}$, $2 \leq i \leq n$, is the 2-qubit reduced density matrix of $|\psi\rangle$.

[Proof] For the n -qubit generalized W -class states $|\psi\rangle$, we have

$$\begin{aligned} E(|\psi\rangle_{A_1|A_2\dots A_n}) &= f(C^2(|\psi\rangle_{A_1|A_2\dots A_n})) \\ &= f\left(\sum_{i=2}^n C^2(\rho_{A_1 A_i})\right) \\ &\leq \sum_{i=2}^n f(C^2(\rho_{A_1 A_i})) \\ &= \sum_{i=2}^n E(\rho_{A_1 A_i}), \end{aligned}$$

where for simplify, we have denoted $f(x) = h(\frac{1+\sqrt{1-x}}{2})$ with $h(x) = -x \log_2(x) - (1-x) \log_2(1-x)$. We have used in the first and last equalities that the entanglement of formation obeys the relation $E(\rho) = f(C^2(\rho))$ for a bipartite $2 \otimes D$, $D \geq 2$, quantum state ρ [19]. The second equality is due to the fact that $C^2(|\psi\rangle_{A_1\dots A_n}) = \sum_{i=2}^n C^2(\rho_{A_1 A_i})$. The inequality is due to the fact $f(x+y) \leq f(x) + f(y)$. \square

As for the entanglement of assistance, we have the following conclusion.

Theorem 4 For the n -qubit generalized W -class states $|\psi\rangle \in H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_n}$, we have

$$E_a(\rho_{A_1|A_{j_1}\dots A_{j_{m-1}}}) \leq \sum_{i=1}^{m-1} E_a(\rho_{A_1 A_{j_i}}), \quad (17)$$

where $\rho_{A_1|A_{j_1}\dots A_{j_{m-1}}}$ is the m -qubit reduced density matrix of $|\psi\rangle$, $2 \leq m \leq n$.

[Proof] Let $\rho_{A_1|A_{j_1}\dots A_{j_{m-1}}} = \sum_h p_h |\psi\rangle_{A_1|A_{j_1}\dots A_{j_{m-1}}}^h \langle\psi|$ be the optimal decomposition of $E_a(\rho_{A_1|A_{j_1}\dots A_{j_{m-1}}})$. We have

$$\begin{aligned} E_a(\rho_{A_1|A_{j_1}\dots A_{j_{m-1}}}) &= \sum_h p_h E(|\psi\rangle_{A_1|A_{j_1}\dots A_{j_{m-1}}}^h) \\ &\leq \sum_h p_h \sum_{i=1}^{m-1} E(\rho_{A_1 A_{j_i}}^h) \\ &= \sum_{i=1}^{m-1} \left(\sum_h p_h E(\rho_{A_1 A_{j_i}}^h) \right) \\ &\leq \sum_{i=1}^{m-1} \sum_h p_h \sum_j q_j E(|\psi_j\rangle_{A_1 A_{j_i}}^h \langle\psi_j|) \\ &= \sum_{i=1}^{m-1} \sum_{hj} p_h q_j E(|\psi_j\rangle_{A_1 A_{j_i}}^h \langle\psi_j|). \end{aligned}$$

We obtain the first inequality by noting that $|\psi\rangle_{A_1|A_{j_1}\dots A_{j_{m-1}}}^h$ is again a generalized W class state[7]. The second inequality is due to the definition of the entanglement of formation (12) for mixed quantum states. Since $\sum_{hj} p_h q_j = 1$ and $\sum_{hj} p_h q_j |\psi_j\rangle_{A_1 A_{j_i}}^h \langle\psi_j|$ is a pure decomposition of $\rho_{A_1 A_{j_i}}^h$, we have (17). \square

IV. CONCLUSIONS AND REMARKS

Entanglement monogamy is a fundamental property of multipartite entangled states. We have shown the monogamy for the x -power of concurrence of assistance $C_a(\rho_{A_1|A_{j_1}\dots A_{j_{m-1}}})$ of the m -qubit reduced density matrices, $2 \leq m \leq n$, for the n -qubit generalized W -class states. The monogamy relations for the entanglement of formation and the entanglement of assistance the monogamy relation for the n -qubit generalized W -class states have been also

investigated. These relations give rise to the restrictions of entanglement distribution among the qubits in generalized W -class states.

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