Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

The local indistinguishability of multipartite product states

by

Yan-Ling Wang, Mao-Sheng Li, Zhu-Jun Zheng, and Shao-Ming Fei

Preprint no.: 18 2017

The local indistinguishability of multipartite product states

Yan-Ling Wang[1], Mao-Sheng Li[2], Zhu-Jun Zheng[1,*] & Shao-Ming Fei[3,4]

[1]Department of Mathematics, South China University of Technology, Guangzhou 510640, China

[2]Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

[3]School of Mathematical Sciences, Capital Normal University, Beijing 100048, China

[4]Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany

[*] Correspondence to zhengzj@scut.edu.cn

Abstract

We study the local indistinguishability of multipartite product states. Firstly, we follow the method of Z.-C. Zhang *et al* [Phys. Rev. A 93, 012314(2016)] to give another more concise set of $2n-1$ orthogonal product states in $\mathbb{C}^m \otimes \mathbb{C}^n$ $(4 \leq m \leq n)$ which can not be distinguished by local operations and classical communication (LOCC). Then we use the 3-dimensional cubes to present some product states which give us an intuitive view on how to construct locally indistinguishable product states in tripartite quantum systems. At last, we give an explicit construction of locally indistinguishable orthogonal product states for general multipartite systems.

In quantum information theory, the problem that distinguishing the quantum states using local operations and classical communication (LOCC) has been extensively studied in the past 20 years, and numerous important results have been reported. The local distinguishability of a given set of states is an important problem connected with the LOCC. In spite of these considerable efforts, the local indistinguishability of orthogonal multipartite states is still not completely solved.

Since the problem was considered, the maximally entangled states and the product states are concerned by most of the researchers. References [1–9] are an incomplete list of the results about the local distinguishability of maximally entangled states. Meanwhile, many people considered the local distinguishability of product states [10–23]. The case of product states was first considered by Bennett. et al. who presented nine LOCC indistinguishable product states in $\mathbb{C}^3 \otimes \mathbb{C}^3$ [10]. Since then, the locally indistinguishability of orthogonal product states in the bipartite system have attracted much attention in recent years and have many advances [14,16,19–23]. But for the multipartite systems, there are only a few papers which considered the locally indistinguishability of orthogonal product states [15, 17, 18]. And in [19, 21] the authors give a full base of LOCC indistinguishable product states in tripartite systems. Therefore, the study of LOCC indistinguishability of multipartite orthogonal product states is still of significance.

In this paper, we focus on finding the locally perfect indistinguishable multipartite orthogonal product states. We separate the problem into two cases: the even partites and the odd partites. For the even case, we first construct the locally indistinguishable orthogonal product states for bipartite systems, then extend it to any even partites. For the odd case, we first solve the problem for tripartite case by considering the 3-dimensional cubic representation of the states, which give an intuitive view on how to construct locally indistinguishable product states in tripartite systems. Then combining the results for the even cases, we obtain the results for any odd-partite systems.

We present our results in the following framework: In sec II, we present $2(n_2+n_4+\ldots+n_{2k}-k)+1$ LOCC indistinguishable orthogonal product states in even partite system $\mathbb{C}^{n_1}\otimes\mathbb{C}^{n_2}\otimes\cdots\otimes\mathbb{C}^{n_{2k}}$. In sec III, we first construct $2(n_1+n_3)-3$ orthogonal product states in tripartite system $\mathbb{C}^{n_1}\otimes\mathbb{C}^{n_2}\otimes\mathbb{C}^{n_3}$, which cannot be perfectly distinguished by LOCC. Then we give $2(n_1+n_3+\ldots+n_{2k+1}-k)+1$ orthogonal product states which are LOCC indistinguishable in $\mathbb{C}^{n_1}\otimes\mathbb{C}^{n_2}\otimes\cdots\otimes\mathbb{C}^{n_{2k+1}}$.

Results

In this paper, we use $\{|i\rangle\}_{i=1}^{n}$ to denote the orthonormal base of an n-dimensional quantum system. In a k-partite quantum system $\mathcal{H}=\mathbb{C}^{n_1}\otimes\mathbb{C}^{n_2}\otimes\cdots\otimes\mathbb{C}^{n_k}$, a state $|\psi\rangle$ is called a stopper state in $\mathcal{H}$ if it can be written as the following form under the assumed bases,

$$|\psi\rangle=(\sum_{i_1=1}^{n_1}|i_1\rangle)(\sum_{i_2=1}^{n_2}|i_2\rangle)...(\sum_{i_k=1}^{n_k}|i_k\rangle). \tag{1}$$

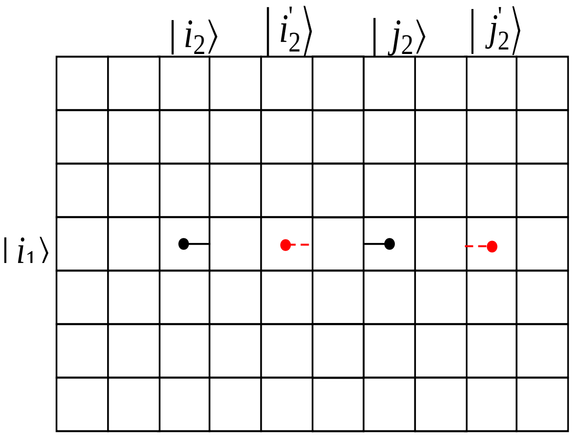
We consider the following special form of product states,

$$|\psi\rangle=|i_1\rangle|i_2\rangle...|i_{l-1}\rangle|i_l-j_l\rangle|i_{l+1}\rangle...|i_k\rangle,\ 1\leq i_s\leq n_s,\ 1\leq i_l<j_l\leq n_l, \tag{2}$$

where $|i_l-j_l\rangle\equiv|i_l\rangle-|j_l\rangle$.

Graph representation of special product states in bipartite system Under the above assumptions, a special product state in bipartite quantum system $\mathbb{C}^{n_1}\otimes\mathbb{C}^{n_2}$ is either $|i_1\rangle|i_2-j_2\rangle$ or $|i_1-j_1\rangle|i_2\rangle$. Firstly, we draw an $n_1\times n_2$ table, see the left graph in Fig. 1. In order to represent the state $|i_1\rangle|i_2-j_2\rangle$, we draw a horizontal line with a dot at the left hand side at coordinate (i_1,i_2), then we draw a horizontal line with a dot at the right hand side at the coordinate (i_1,j_2). The pair of these two horizontal lines of the same color in the same row represent the product state $|i_1\rangle|i_2-j_2\rangle$. The product state $|i_1\rangle|i_2'-j_2'\rangle$ is represented by the pair of red lines in the i-th row, see the graphical representations of $|i_1\rangle|i_2-j_2\rangle$ and $|i_1\rangle|i_2'-j_2'\rangle$ in Fig.1. We call these states horizontal ones. Similarly, to represent the state

$|i_1 - j_1\rangle|i_2\rangle$, we draw a vertical line with a dot at its top at (i_1, i_2) and a vertical line with a dot at its bottom at (j_1, i_2). The pair of these two lines with the same color represent $|i_1 - j_1\rangle|i_2\rangle$. We call these vertical ones.



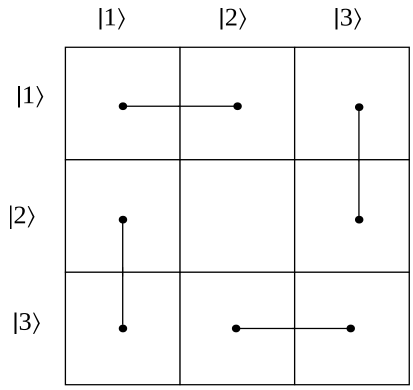


Figure 1: The graph at the left hand side is the graphical representations of the product state $|i_1\rangle|i_2 - j_2\rangle$ and $|i_1\rangle|i_2' - j_2'\rangle$ in bipartite quantum system: the pair of black horizontal lines represent $|i_1\rangle|i_2 - j_2\rangle$, while the pair of red horizontal lines represent $|i_1\rangle|i_2' - j_2'\rangle$. The right hand side graph is the graphical representations of the four product states $|1\rangle|1-2\rangle, |1-2\rangle|3\rangle, |3\rangle|2-3\rangle, |2-3\rangle|1\rangle$ in $\mathbb{C}^3 \otimes \mathbb{C}^3$.

Constructions for even-partite case

Firstly, we give the following construction:

Theorem 1. In $\mathbb{C}^m \otimes \mathbb{C}^n$ $(4 \leq m \leq n)$, the following $2n-1$ orthogonal product states are LOCC indistinguishable,

$$
\begin{aligned}
&|\phi_1\rangle = (|1\rangle + |2\rangle + \cdots + |m\rangle)(|1\rangle + |2\rangle + \cdots + |n\rangle),\\
&|\phi_i\rangle = |i\rangle|1-i\rangle, \quad i = 2, 3, \ldots, m,\\
&|\phi_{m+1}\rangle = |1-n\rangle|2\rangle,\\
&|\phi_{m+j-1}\rangle = |1-(j-1)\rangle|j\rangle, \quad j = 3, 4, \ldots, m,\\
&|\phi_{m+l-1}\rangle = |1-2\rangle|l\rangle, \quad l = m+1, m+2, \ldots, n,\\
&|\phi_{m+n}\rangle = |m\rangle|3-(m+1)\rangle,\\
&|\phi_{n+s}\rangle = |m-1\rangle|s-(s+1)\rangle,\\
&|\phi_{n+t}\rangle = |m\rangle|t-(t+1)\rangle,\\
&s = m+2k-1,\ t = m+2k,\ k = 1, 2, \ldots, \lfloor \tfrac{n-m}{2} \rfloor.
\end{aligned}
\tag{3}
$$

See Methods for the proof of Theorem 1.

These states in Theorem 1 are different from the states in [23] for $\mathbb{C}^m \otimes \mathbb{C}^n$ $(4 \leq m \leq n)$. Actually, we can show the structure of these states by graphs. As for two examples, Fig.2 and Fig.3 are the graph structures for $m = 4$, $n = 7$ and $m = 6$, $n = 12$.

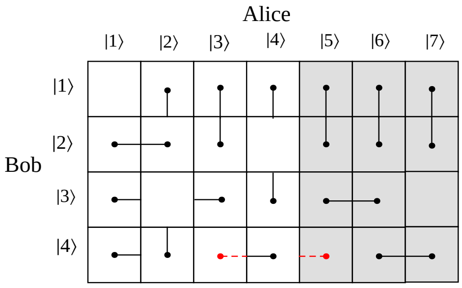


Figure 2: LOCC indistinguishable product states in $\mathbb{C}^4 \otimes \mathbb{C}^7$.

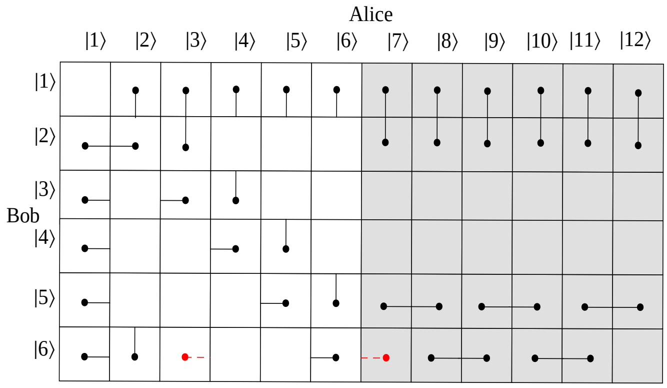


Figure 3: LOCC indistinguishable product states in $\mathbb{C}^6 \otimes \mathbb{C}^{12}$.

Now we consider the construction of small set of LOCC indistinguishable product states in general even-partite case. Let us denote the product states (3) in Theorem 1 by $|\phi_{i_s}\rangle_s$ for the bipartite system $\mathbb{C}^{n_{2s-1}} \otimes \mathbb{C}^{n_{2s}}$, $i_s = 1, 2, \ldots, 2n_{2s} - 2$, $s = 1, 2, \ldots, k$.

Theorem 2. In $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_{2k-1}} \otimes \mathbb{C}^{n_{2k}}$ $(3 \le n_1 \le n_2 \le \cdots \le n_{2k-1} \le n_{2k}, k \ge 2)$ the $2(n_2+n_4+\cdots+n_{2k}-k)+1$ orthogonal product states given by the union of the following k sets: $\{|\phi_{i_1}\rangle_1 |11\rangle_2 \cdots |11\rangle_k\}$, $\{|11\rangle_1 |\phi_{i_2}\rangle_2 |11\rangle_3 \cdots |11\rangle_k\}$, $\cdots$, $\{|11\rangle_1 \cdots |11\rangle_{k-1} |\phi_{i_k}\rangle_k\}$, $i_s = 1, 2 \ldots 2n_{2s} - 2$, together with a stopper state

$$|\phi\rangle = \sum_{i_1=1}^{n_1} |i_1\rangle \sum_{j_1=1}^{n_2} |j_1\rangle \cdots \sum_{i_k=1}^{n_{2k}} |i_k\rangle \sum_{j_k=1}^{n_{2k}} |j_k\rangle, \tag{4}$$

are LOCC indistinguishable.

See Methods for the proof of Theorem 2.

Constructions for odd-partite case The odd-partite cases are different from the even-partite ones. We first consider the tripartite system. Then we combine the tripartite case with the even-partite cases to tackle with the general odd-partite cases. We first give a concrete example to show how to construct a set of LOCC indistinguishable product states in $\mathbb{C}^4 \otimes \mathbb{C}^5 \otimes \mathbb{C}^6$, and then generalize it to the general tripartite system $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$ $(4 \le n_1 \le n_2 \le n_3)$.

In $\mathbb{C}^4 \otimes \mathbb{C}^5 \otimes \mathbb{C}^6$, there are 17 orthogonal product states that are LOCC indistinguishable. To show the construction of these 17 states, we set

$$|\phi_1\rangle = (|1\rangle + |2\rangle + |3\rangle + |4\rangle)(|1\rangle + |2\rangle + |3\rangle + |4\rangle + |5\rangle)(|1\rangle + |2\rangle + |3\rangle + |4\rangle + |5\rangle + |6\rangle). \tag{5}$$

By Theorem 1, we have the following 11 LOCC indistinguishable product states in $\mathbb{C}^5 \otimes \mathbb{C}^6$,

$|\psi_1\rangle = (\sum_{i=1}^{5}|i\rangle)(\sum_{j=1}^{6}|j\rangle),$
$|\psi_2\rangle = |2\rangle|1-2\rangle, \quad |\psi_3\rangle = |3\rangle|1-3\rangle,$
$|\psi_4\rangle = |4\rangle|1-4\rangle, \quad |\psi_5\rangle = |5\rangle|1-5\rangle,$
$|\psi_6\rangle = |1-5\rangle|2\rangle, \quad |\psi_7\rangle = |1-2\rangle|3\rangle,$
$|\psi_8\rangle = |1-3\rangle|4\rangle, \quad |\psi_9\rangle = |1-4\rangle|5\rangle,$
$|\psi_{10}\rangle = |5\rangle|3-6\rangle, \quad |\psi_{11}\rangle = |1-2\rangle|6\rangle.$

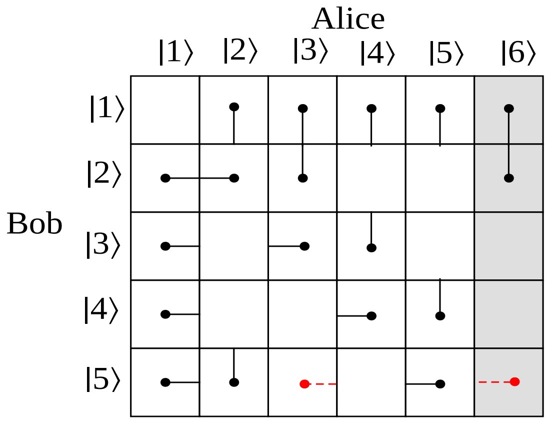


Figure 4: The graphic representation of the states $|\psi_i\rangle$, $i = 2, ..., 10$.

Setting $|\phi_i\rangle = |4\rangle|\psi_i\rangle$ for $i = 2, 3, \ldots, 11$, we obtain 11 tripartite product states. If Alice does the first measurement $\{M_s^A\}$ that preserves the orthogonality of these 11 states, then

$$\langle\phi_j|I_C \otimes I_B \otimes (M_s^A)^\dagger(M_s^A)|\phi_i\rangle = \lambda\langle\psi_j|I_B \otimes (M_s^A)^\dagger(M_s^A)|\psi_i\rangle, \tag{6}$$

where λ is either $\langle 4|I_C|4\rangle = 1$, or $\langle 4|I_C\sum_{i=1}^{4}|i\rangle) = 1$ or $(\sum_{i=1}^{4}\langle i|)I_C(\sum_{i=1}^{4}|i\rangle) = 4$. In any cases we have the following relations:

$$\langle\phi_j|I_C \otimes I_B \otimes (M_s^A)^\dagger(M_s^A)|\phi_i\rangle = 0 \quad \Leftrightarrow \quad \langle\psi_j|I_B \otimes (M_s^A)^\dagger(M_s^A)|\psi_i\rangle = 0. \tag{7}$$

However, from the proof of Theorem 1, Alice cannot apply any nontrivial measurements. Otherwise, the orthogonality of the states $\{|\psi_i\rangle\}_{i=1}^{11}$ can not be preserved. The same arguments apply to Bob too. Therefore Alice and Bob can not be the first one to do the measurement.

Next, we construct more states such that Charlie can only do the trivial measurement as well,

$|\xi_{11}\rangle = |4\rangle|1-2\rangle, \quad |\xi_{12}\rangle = |3\rangle|1-2\rangle,$
$|\xi_{13}\rangle = |2\rangle|1-2\rangle, \quad |\xi_{14}\rangle = |1\rangle|1-2\rangle,$
$|\xi_{15}\rangle = |1-2\rangle|4\rangle, \quad |\xi_{16}\rangle = |2-3\rangle|5\rangle,$
$|\xi_{17}\rangle = |3-4\rangle|4\rangle.$

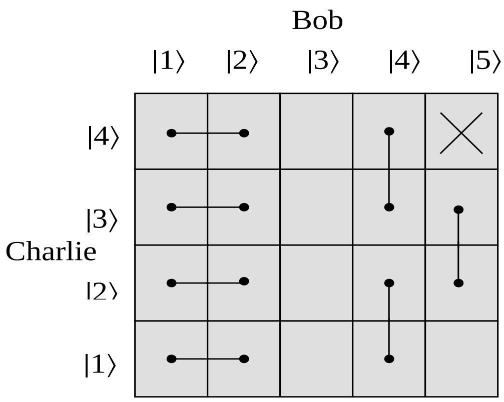


Figure 5: The graphic representation of the states $|\xi_i\rangle$, $i = 11, ..., 17$.

Now we set $|\phi_i\rangle = |\xi_i\rangle|6\rangle$, $i = 11, 12, ..., 17$. Here $|\phi_{11}\rangle$ is the same as the one defined by $|\phi_{11}\rangle = |4\rangle|\psi_{11}\rangle$ from Fig. 4. If Charlie does the first measurement $\{M_s^C\}$ that preserves the orthogonality of the states $|\phi_i\rangle = |\xi_i\rangle|6\rangle$, $i = 11, 12, ..., 17$, and the stopper state, then

$$\langle\phi_j|(M_s^C)^\dagger(M_s^C) \otimes I_B \otimes I_A|\phi_i\rangle = \lambda\langle\psi_j|(M_s^C)^\dagger(M_s^C) \otimes I_B|\psi_i\rangle, \quad (8)$$

where $\lambda = \langle 6|I_C|6\rangle = 1$, or $\lambda = \langle 6|I_C \sum_{i=1}^{6} |i\rangle) = 1$ or $\lambda = (\sum_{i=1}^{6}\langle i|)I_C(\sum_{i=1}^{6} |i\rangle) = 6$, which lead to the following relations:

$$\langle\phi_j|(M_s^C)^\dagger(M_s^C) \otimes I_B \otimes I_A|\phi_i\rangle = 0 \quad \Leftrightarrow \quad \langle\psi_j|(M_s^C)^\dagger(M_s^C) \otimes I_B|\psi_i\rangle = 0. \quad (9)$$

It is easy to derive that in order to preserve the orthogonality of the horizontal states in Charlie and Bob's system, the nondiagonal elements of $(M_s^C)^\dagger(M_s^C)$ should be zero. And in order to preserve the orthogonality of the vertical states and the stopper state, the diagonal elements of $(M_s^C)^\dagger(M_s^C)$ should be equal. Hence, Charlie can only do the trivial measurement if he does the first measurement. Fig. 6 gives a sketch proof of the LOCC indistinguishability of the product states we constructed above.

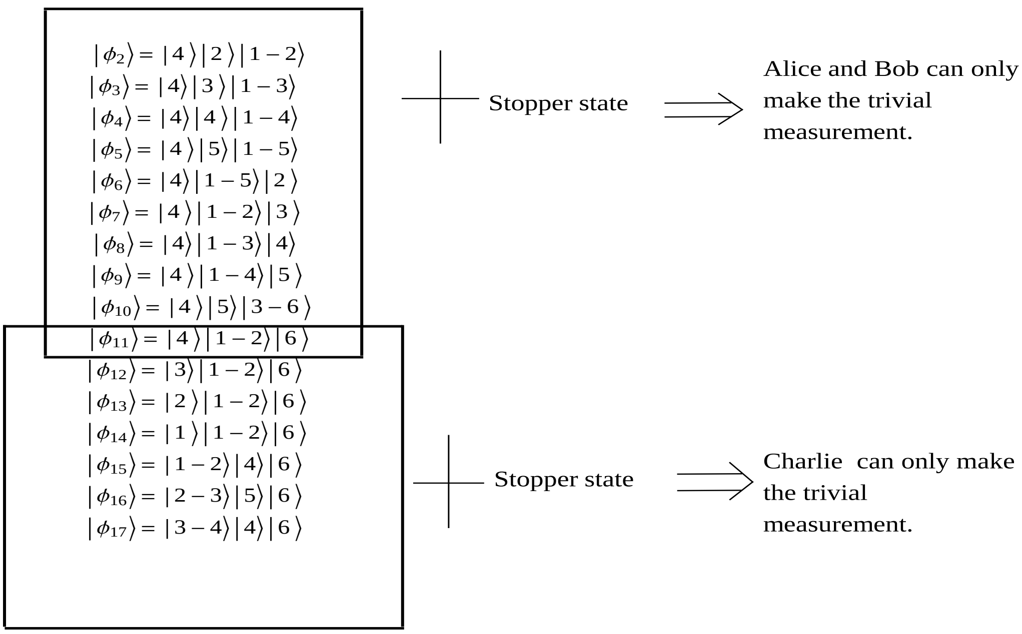


Figure 6: A sketch proof of the LOCC indistinguishability of the above states.

Similar to the bipartite case, we can intuitively image a special product state in tripartite quantum system as some cube in three dimensional space. Fig.7 is the graphical representation of the product states we constructed above. Any special product state we considered here can be represented by two cubes. All the states we constructed above lie either on the top surface or the right surface. The states on the top surface can be viewed as the tensors of the bipartite states associated with Alice and Bob and the state $|4\rangle$ associated with Charlie. Similarly, The states on the right surface can be viewed as the tensors of the bipartite states associated with Bob and Charlie and the state $|6\rangle$ associated with Alice.

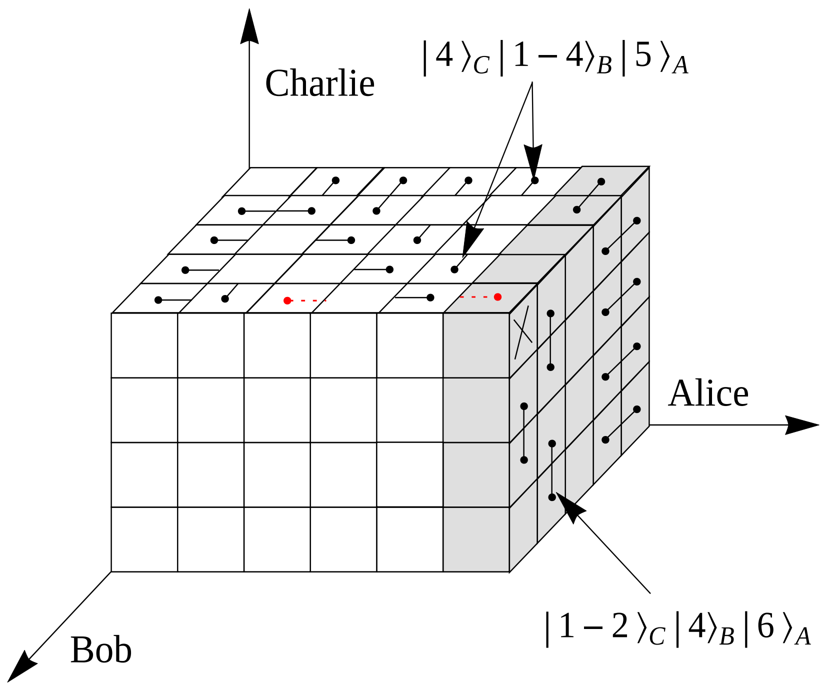


Figure 7: The graphic representation of states by 3-dimensional cubes.

For general tripartite system $\mathbb{C}^{n_1}\otimes\mathbb{C}^{n_2}\otimes\mathbb{C}^{n_3}$ ($4 \leq n_1 \leq n_2 \leq n_3$), the orthogonal product states we want to construct can also be described by two parts: the **Top surface part** ($\mathcal{T}$) and the **Right surface part** ($\mathcal{R}$). As we will see, the **Top surface part** leads to the conclusion that both Alice and Bob can only make trivial measurements if they are the first one to do the measurement. Meanwhile, the construction of the **Right surface part** aims at that Charlie can only do the trivial measurement as well.

Top surface part: We denote the $2n_3-1$ states, except for the stopper state, in $\mathbb{C}^{n_2}\otimes\mathbb{C}^{n_3}$ constructed from Theorem 1 by $\{|\psi_i\rangle\}_{i=1}^{2n_3-2}$. Then denote $\mathcal{T} = \{|\phi_i\rangle = |n_1\rangle \otimes |\psi_i\rangle \mid i = 1, 2, ..., 2n_3 - 2\}$. The number of the states in $\mathcal{T}$ is $2n_3 - 2$.

There are three possible cases: (a) $n_2 < n_3$ and $n_3 - n_2$ is odd, (b) $n_2 < n_3$ and $n_3 - n_2$ is even, (c) $n_2 = n_3$. Correspondingly the right column of the top surface and the top row of the right surface have the forms as showed in Fig.8. Now our aim is to add some states lying in the right surface which might imply that Charlie can only make trivial measurements if he measures first. In the following we only consider the case (a) (other cases can be dealt with similarly).

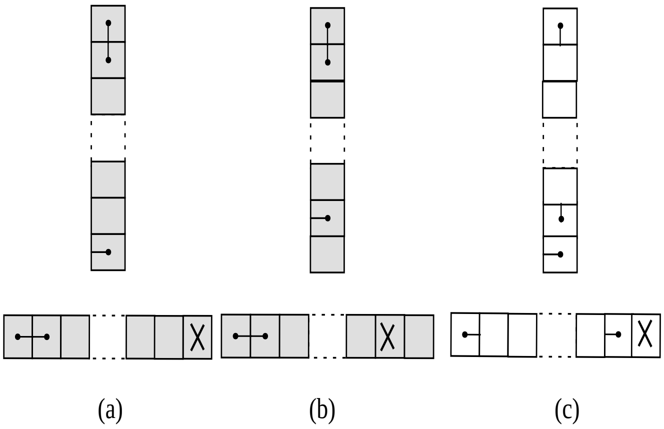


Figure 8: Right column of the top surface and the top row of the right surface with respect to the cases of (a), (b) and (c).

The **Right surface part** $\mathcal{R}$ is the union of some horizontal states and some vertical states. The horizontal part H is given by

$$H = \{|i\rangle|1-2\rangle|n_3\rangle \bigm| 1 \leq i \leq n_1\}. \tag{10}$$

The vertical part V is given by

$$V = \{|i-(i+1)\rangle|(n_2 - \delta_{(n_1-i)})\rangle|n_3\rangle \bigm| 1 \ \leq i \leq n_1 - 1\}, \tag{11}$$

where $\delta_i = \frac{1}{2}(1+(-1)^i)$, that is, $\delta_i = 0$ if i is odd and $\delta_i = 1$ if i is even. Then $\mathcal{R} = H \cup V$ has $2n_1 - 1$ states.

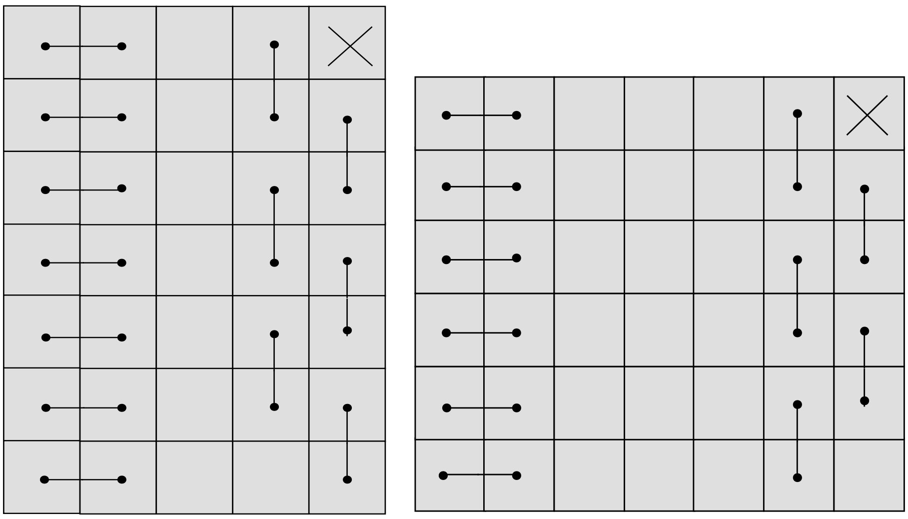


Figure 9: Examples of the left view of the **Right surface part** we have constructed.

Note that $\mathcal{R} \cap \mathcal{T} = \{|n_1\rangle|1-2\rangle|n_3\rangle\}$. Thus to construct a set of orthogonal product states one needs to include the stopper state $|\psi\rangle = |\psi_1\rangle_C|\psi_2\rangle_B|\psi_3\rangle_A$, where $|\psi_1\rangle_C = |1\rangle + |2\rangle + \ldots +$

$|n_1\rangle$, $|\psi_2\rangle_B = |1\rangle + |2\rangle + \ldots + |n_2\rangle$, $|\psi_3\rangle_A = |1\rangle + |2\rangle + \ldots + |n_3\rangle$. Let $\mathcal{S}$ denote the set of states $\mathcal{R} \cup \mathcal{T}$ plus the stopper state. Then the number of states in $\mathcal{S}$ is $(2n_3 - 2) + (2n_1 - 1) - 1 + 1 = 2(n_1 + n_3) - 3$.

Theorem 3. In tripartite system $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$ $(4 \leq n_1 \leq n_2 \leq n_3)$, the $2(n_1 + n_3) - 3$ orthogonal product states in $\mathcal{S}$ are LOCC indistinguishable.

See Methods for the proof of Theorem 3.

Having solved the problem for tripartite systems, we have the following results for general odd-partite quantum systems.

Theorem 4. In $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_{2k}} \otimes \mathbb{C}^{n_{2k+1}}$ $(4 \leq n_1 \leq n_2 \leq \cdots \leq n_{2k+1})$, there are $2(n_1 + n_3 + \cdots + n_{2k+1} - k) + 1$ product states that are LOCC indistinguishable.

See Methods for the proof of Theorem 4.

Conclusion

We have presented a new construction of LOCC indistinguishable product states in $\mathbb{C}^m \otimes \mathbb{C}^n (4 \leq m \leq n)$. Then we obtained $2(n_2 + n_4 + \ldots + n_{2k} - k) + 1$ LOCC indistinguishable orthogonal product states in even-partite systems $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_{2k}}$ $(3 \leq n_1 \leq n_2 \leq \cdots \leq n_{2k})$. For odd-partite systems, we first used 3-dimensional cubes to give an intuitive view in constructing $2(n_1 + n_3) - 3$ orthogonal product states in tripartite systems $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$ that can not be distinguished by LOCC. At last, we presented $2(n_1 + n_3 + \ldots + n_{2k+1} - k) + 1$ orthogonal product states that are LOCC indistinguishable in $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_{2k+1}}$ $(4 \leq n_1 \leq n_2 \leq \cdots \leq n_{2k+1})$.

We notice that the simplest lower bound on the size of unextendible product base (UPB) [24–26] in $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_k}$ is $1 + \sum_{i=1}^{k}(n_i - 1)$. For the even-partite cases with $n_1 = n_2 = \ldots = n_{2k} = d$, the bound $1 + \sum_{i=1}^{2k}(n_i - 1)$ is the same as the number of $2(n_2 + n_4 + \ldots + n_{2k} - k) + 1$ LOCC indistinguishable orthogonal product states we obtained. However, the lower bound of the size of UPB may not be reached, in which cases the local indistinguishability of product states we presented is different from the local indistinguishability of UPB. It would be also interesting to compare the local indistinguishability of UPB with that of orthogonal product states.

Note added: — After finishing this work, we noticed that the authors in [27] also considered the local indistinguishable multipartite product states.

Methods

The proofs of the Theorems are based on the following two facts: if a set of quantum states is LOCC distinguishable, then some partite has to start with a nontrivial measurement, i.e. not all measurements $M^\dagger M$ are proportional to the identity. And the measurements must preserve the orthogonality relations of the quantum states. Hence for a given set of states, if the measurement preserving the orthogonality relations of the given states must be trivial, then one can conclude that the given set of states are LOCC indistinguishable.

Proof of Theorem 1 First, suppose Alice does the POVM measurement $M_m^\dagger M_m$, which is of the following matrix form in the basis $\{|1\rangle, |2\rangle, \ldots, |m\rangle\}_A$,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix}. \tag{12}$$

Since the postmeasurement states $(M_m \otimes I)|\phi_i\rangle$ are orthogonal with each other, we have $\langle\phi_j|M_m^\dagger M_m \otimes I|\phi_i\rangle = \langle j|\langle 1-j|M_m^\dagger M_m \otimes I|i\rangle|1-i\rangle = \langle j|M_m^\dagger M_m|i\rangle = a_{ji} = 0$ for $i \neq j$, $i, j = 2, 3, \ldots, m$. In the same way, consider the orthogonality between $|\phi_i\rangle$ and $|\phi_{m+i}\rangle$ for $i = 3, 4, \ldots, m$, and between $|\phi_{m+1}\rangle$ and $|\phi_2\rangle$, we have that $a_{1j} = 0, j = 2, 3, \ldots, m$. Consider further that between the states $|\phi_1\rangle$ and $|\phi_{m+j}\rangle$, $j = 1, 2, \ldots, m$, we get $a_{11} = a_{22} = \cdots = a_{mm}$. Thus $M_m^\dagger M_m$ is proportional to the identity.

When Bob starts with the measurements denoted by the matrix $M_n^\dagger M_n$ with entries given by b_{ij}, $i, j = 1, ..., n$, in the basis $\{|1\rangle, |2\rangle, \ldots, |n\rangle\}_B$, one has the postmeasurement states $(I \otimes M_n)|\phi_i\rangle$ which are mutually orthogonal. Consider the states $|\phi_{m+i}\rangle$ and $|\phi_{m+j}\rangle$, $i, j = 1, 2, \ldots, m+n-1$, we can get $b_{ij} = 0$, $i \neq j = 2, 3, \ldots, n$. And from the orthogonality between $|\phi_i\rangle$, $i = 2, 3, \ldots, m$, and $|\phi_{m+j-1}\rangle$, $j = 2, 3, \ldots, n$, we have $b_{1j} = b_{ji} = 0$, $j = 2, 3, \ldots, n$. Finally, from the orthogonality between states $|\phi_1\rangle$ and $|\phi_i\rangle$, $i = 2, 3, \ldots, m, m+n, m+n+1, \ldots, 2n-1$, we obtain $b_{11} = b_{22} = \cdots = b_{nn}$. Therefore the measurements $M_n^\dagger M_n$ are proportional to the identity. Thus, these $2n-1$ product states are LOCC indistinguishable.
■

Proof of Theorem 2 For simplicity we call the $\mathbb{C}^{n_{2s}-1}$ system the s-th Bob system, and $\mathbb{C}^{n_{2s}}$ system the s-th Alice system. We only need to show all the partites could only do the trivial measurement. If the s-th Alice or Bob do the first measurement, it must preserve the orthogonality of all the quantum states. In particular, it preserves the orthogonality of the following $2n_{2s}-1$ quantum states.

We denote the states $|\psi_{i_s}\rangle = |11\rangle_1 \cdots |11\rangle_{s-1}|\phi_{i_s}\rangle_s|11\rangle_{s+1}\cdots|11\rangle_k$, $i_s = 1, 2, \ldots, 2n_{2s}-2$, and $|\psi_{2n_{2s}-1}\rangle = |\phi\rangle$. Then the states $I \otimes \cdots \otimes I \otimes M_s^A \otimes I \cdots \otimes I|\psi_i\rangle$ and $I \otimes \cdots \otimes I \otimes M_s^A \otimes I \cdots \otimes I|\psi_j\rangle$ are orthogonal. By a direct calculation, we obtain

$${}_s\langle\phi_j|I \otimes (M_s^A)^\dagger (M_s^A)|\phi_i\rangle_s = \delta_{ij}. \tag{13}$$

From the proof of Theorem 1, any measurement which preserves the orthogonality of $|\phi_i\rangle_s$ and the corresponding stopper state must be trivial. Then from (13) we deduce that $M_s^{A\dagger} M_s^A \propto I_s$. Hence, in order to preserve the orthogonality of the above quantum states the only measurements of the s-th Alice can do is the trivial measurement.

Similarly, if we consider the s-th Bob system, we have ${}_s\langle\phi_j|(M_s^B)^\dagger(M_s^B) \otimes I|\phi_i\rangle_s = \delta_{ij}$ too. Hence the measurement of the s-th Bob's system is also a trivial one. ■

Proof of Theorem 3 Suppose Alice or Bob apply the first measurement that preserves the orthogonality of the states from the **Top surface part** and the stopper state. Similar to our example for tripartite system, the problem is equivalent to that the corresponding measurement should preserve the orthogonality of the states from the top surface and the stopper state of the reduced system without Charlie. Then the measurement must be a trivial one since the **Top surface part** states are derived from the Theorem 1.

If Charlie first applies a measurement $\{M_s^C\}$, then the measurement has to preserve the orthogonality of the states from the **Right surface part** and the stopper state. In order to preserve the orthogonality of the horizontal states H and the stopper state, the nondiagonal elements of $(M_s^C)^\dagger(M_s^C)$ are required to be zero, and in order to preserve the orthogonality of the vertical states V and the stopper state, all the diagonal elements of $(M_s^C)^\dagger(M_s^C)$ should be equal. Hence, Charlie can only do the trivial measurement if he measures first.

Hence, none of the three people can apply a nontrivial measurement first. ■

Proof of Theorem 4 We give the proof by two steps: construction of the states and proof of their LOCC indistinguishability.

Step 1: Construction of the states

Let $|\psi_{i_1}\rangle_1$, $i_1 = 1, 2, \ldots, 2(n_1 + n_3) - 4$, denote the product states except for the stopper states we constructed in Theorem 3 in $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$. We call the corresponding system the 1-th Charles 1-th Bob and 1-th Alice system.

Let $|\psi_{i_s}\rangle_s$, $i_s = 1, 2, \ldots, 2n_{2s+1} - 2$, denote the product states except for the stopper states we constructed in Theorem 1 in $\mathbb{C}^{n_{2s}} \otimes \mathbb{C}^{n_{2s+1}}$. We call the corresponding system the s-th Bob and s-th Alice system for all integer $2 \leq s \leq k$.

Then the set $\mathcal{S}$ of product states in multipartite quantum system we want to construct is the union of the following sets and a stopper state $|\psi\rangle$,

$$\begin{aligned} \mathcal{S}_1 &= \{|\psi_{i_1}\rangle_1 |11\rangle_2 \cdots |11\rangle_k \mid i_1 = 1, 2, \ldots, 2(n_1 + n_3) - 4\}, \\ \mathcal{S}_2 &= \{|111\rangle_1 |\psi_{i_2}\rangle_2 |11\rangle_3 \cdots |11\rangle_k \mid i_2 = 1, 2, \ldots, 2(n_5) - 2\}, \\ &\vdots \\ \mathcal{S}_k &= \{|111\rangle_1 \cdots |11\rangle_{k-1} |\psi_{i_k}\rangle_k \mid i_k = 1, 2, \ldots, 2(n_{2k+1}) - 2\}. \end{aligned} \tag{14}$$

Here the stopper state $|\psi\rangle = |\psi_1\rangle|\psi_2\rangle \ldots |\psi_{2k+1}\rangle$, where $|\psi_1\rangle = |1\rangle + |2\rangle + \cdots + |n_1\rangle, |\psi_2\rangle = |1\rangle + |2\rangle + \cdots + |n_2\rangle, \cdots, |\psi_{2k+1}\rangle = |1\rangle + |2\rangle + \cdots + |n_{2k+1}\rangle$. Then the number of elements in the set $\mathcal{S}$ is $2(n_1 + n_3 + \cdots + n_{2k+1} - k) + 1$.

Now we need to check the orthogonality of the states in the set $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup ... \cup \mathcal{S}_k \cup \{|\psi\rangle\}$. The orthogonality of the states in $\mathcal{S}_s \cup \{|\psi\rangle\}$ is obvious by our construction for all $s = 1, 2, ..., k$. Hence we only need to check that the states in $\mathcal{S}_i$ are orthogonal to the states in $\mathcal{S}_j$ whenever $i \neq j$. We observe that $|111\rangle_1$ is orthogonal to $|\psi_{i_1}\rangle_1$ for all $i_1 = 1, 2, \ldots, 2(n_1 + n_3) - 4$ and $|11\rangle_s$ is orthogonal to $|\psi_{i_s}\rangle_s$ for all $i_s = 1, 2, ..., 2n_{2s+1}$ when $2 \leq s \leq k$. All these facts give the orthogonality of the states between $\mathcal{S}_i$ and $\mathcal{S}_j$ for $i \neq j$.

Step 2: proof of the LOCC indistinguishability

If the 1-th Charles applies the first measurement, then the measurement should preserve the orthogonal of the set of states $|\psi_{i_1}\rangle_1|11\cdots 11\rangle$ and the stopper state. An easy calculation shows the measurement must be trivial.

If the s-th Alice or Bob apply the first measurement, then his measurement should preserve the orthogonal of the set of states $|11...11\rangle|\psi_{i_s}\rangle_s|11\cdots 11\rangle$ and the stopper state, which gives rise to that the measurement is trivial. ■

References

[1] S. Ghosh, G. Kar, A. Roy, A.Sen(De), and U. Sen, Distinguishability of Bell States, Phys. Rev. Lett. **87**, 277902 (2001).

[2] J. Walgate and L. Hardy, Nonlocality, Asymmetry, and Distinguishing Bipartite States, Phys. Rev. Lett. **89**, 147901 (2002).

[3] H. Fan, Distinguishability and Indistinguishability by Local Operations and Classical Communication, Phys. Rev. Lett. **92**, 177905 (2004).

[4] H. Fan, Distinguishing bipartite states by local operations and classical communication, Phys. Rev. A **75**,014305 (2007).

[5] M. Nathanson, Distinguishing bipartitite orthogonal states using LOCC: Best and worst cases, J. Math. Phys. **46**, 062103 (2005).

[6] N. Yu, R. Duan, and M. Ying, Four Locally Indistinguishable Ququad-Ququad Orthogonal Maximally Entangled States, Phys. Rev. Lett. **109**, 020506 (2012).

[7] A. Cosentino, Positive partial transpose indistinguishable states via semidefinite programming, Phys. Rev. A **87**, 012321 (2013).

[8] N. Yu, R. Duan, and M. Ying, Distinguishability of quantum states by positive operator-valued measures with positive partial transpose, IEEE Trans. Inf. Theory **60**, 2069 (2014).

[9] S. X. Yu and C. H. Oh, Detecting the local indistinguishability of maximally entangled states, arXiv:1502.01274v1.

[10] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor,E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, Quantum nonlocality without entanglement, Phys. Rev. A **59,** 1070 (1999).

[11] C. H. Bennett, D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Unextendible Product Bases and Bound Entanglement, Phys. Rev. Lett. **82**, 5385 (1999).

[12] M. Horodecki, A. Sen(De), U. Sen, and K. Horodecki, Local Indistinguishability: More Nonlocality with Less Entanglement, Phys. Rev. Lett. **90**, 047902 (2003).

[13] S. De Rinaldis, Distinguishability of complete and unextendible product bases, Phys. Rev. A **70**, 022309(2004).

[14] P.-X. Chen and C.-Z. Li, Distinguishing the elements of a full product basis set needs only projective measurements and classical communication, Phys. Rev. A **70**, 022306 (2004).

[15] J. Niset and N. J. Cerf, Multipartite nonlocality without entanglement in many dimensions, Phys. Rev. A **74**, 052103 (2006).

[16] Y. Feng and Y.-Y. Shi, Characterizing locally indistinguishable orthogonal product states, IEEE Trans. Inf. Theory **55**, 2799 (2009).

[17] R. Y. Duan, Y. Xin, and M. S. Ying, Locally indistinguishable subspaces spanned by three-qubit unextendible product bases, Phys. Rev. A **81**, 032329 (2010).

[18] Y.-H. Yang, F. Gao, G.-J. Tian, T.-Q. Cao, and Q.-Y. Wen, Local distinguishability of orthogonal quantum states in a $2 \otimes 2 \otimes 2$ system, Phys. Rev. A **88**, 024301 (2013).

[19] Z.-C. Zhang, F. Gao, G.-J. Tian, T.-Q. Cao and Q.-Y. Wen, Nonlocality of orthogonal product basis quantum states, Phys. Rev. A **90**, 022313 (2014).

[20] T. Ma, M.-J. Zhao, Y.-K. Wang and S.-M. Fei, Non-commutativity and Local Indistinguishability of Quantum States, Sci. Rep. **4**, 6336 (2014).

[21] Z.-C. Zhang, F. Gao, S.J. Qin, Y.-H. Yang and Q.-Y.Wen, Nonlocality of orthogonal product states, *Phys. Rev. A.* **91**, 012332 (2015).

[22] Y.-L. Wang, M.-S. Li, Z.-J. Zheng and S.-M. Fei, Nonlocality of orthogonal product-basis quantum states, *Phys. Rev. A.* **92**, 032313 (2015).

[23] Z.-C. Zhang, F. Gao, Y. Cao, S.J. Qin and Q.-Y. Wen, Local indistinguishability of orthogonal product states, *Phys. Rev. A.* **93**, 012314 (2016).

[24] D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Unextendible product bases, uncompletable product bases and bound entanglement, Comm. Math. Phys. **238**, 379 (2003).

[25] N. Alon and L. Lovsz, unextendible product bases, J. Comb. Theor. Ser A **95**, 169 (2001).

[26] J. Chen, N. Johnston, The Minimum Size of Unextendible Product Bases in the Bipartite Case (and Some Multipartite Cases), Commun. Math. Phys. **333**, 351 (2015).

[27] G.-B. Xu, Q.-Y. Wen, S.-J. Qin, Y.-H. Yang, and F. Gao, Quantum nonlocality of multipartite orthogonal product states, Phys. Rev. A **93**, 032341 (2016).

Acknowledgements

This work is supported by the NSFC 11475178, NSFC 11571119 and NSFC 11275131.

Author contributions

Yan-Ling Wang and Mao-Sheng Li wrote the main manuscript text. All the authors reviewed the manuscript.

Additional Information

Competing Financial Interests: The authors declare no competing financial interests.