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Strong Variance-Based Uncertainty Relations and Uncertainty Intervals

by

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Strong Variance-Based Uncertainty Relations

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Uncertainty relations occupy a fundamental position in quantum mechanics. We propose stronger variance-based uncertainty relations for the product and sum of variances of two incompatible observables in a finite dimensional Hilbert space. It is shown that the new uncertainty relations provide near-optimal state-dependent bounds, which can be useful for quantum metrology, entanglement detection etc. in quantum information theory. It is further shown that the uncertainty relations are related to the "spreads" of the distribution of measurement outcomes caused by incompatible observables. Intuitively, this means that the ability of learning the distribution has both the upper and lower bounds. Combination of these bounds provides naturally an uncertainty interval which captures the essence of uncertainty in quantum theory. Finally, we explain how to employ entropic uncertainty relations to derive lower bounds for the product of variances of incompatible observables.

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I. INTRODUCTION

A distinguished aspect of quantum mechanics is that the uncertainty relations [1] between incompatible observables allow for succinct quantitative formulations of this revolutionary idea: it is impossible to simultaneously measure precisely two complementary variables of a particle. The uncertainty relations underlie many intrinsic differences between classical and quantum mechanics, and have direct applications for entanglement detection [2], quantum metrology [3–5], quantum cryptography [6], signal processing [7] and quantum speed limit [8] etc.

For arbitrary incompatible observables A and B with bounded spectrums, the Schrödinger uncertainty relation states that

$$V(A)V(B) \geq \mid \frac{1}{2}\langle [A,B]\rangle \mid^2 + \mid \frac{1}{2}\langle \{\overline{A},\overline{B}\}\rangle \mid^2, \quad (1)$$

where $V(A) = \langle \overline{A}^2 \rangle$ (resp. $V(B)$) denotes the variance of the observable A (resp. B), $\overline{A} = A - \langle A\rangle$, and the expectation value $\langle \ \rangle$ is over the quantum state $|\Psi\rangle$. The product form of the variance-based uncertainty relation cannot fully capture incompatibility of observables since the lower bound may become trivial even if the observables A and B do not commute with each other. It is thus necessary to formulate uncertainty relations in terms of the sum of variances (i.e. $V(A) + V(B)$).

An important question is how to improve the lower bound of the uncertainty relation, which is useful in quantum theory and quantum information theory. It appears that most strong variance-based uncertainty relations [9, 10] rely on $|\Psi\rangle$ and its mutually exclusive physical states $|\Psi^{\perp}\rangle$, which are usually given in a complicated process. It is known that the mutual exclusive physical states are harder to be determined as the dimension of the quantum state increases. It is thus necessary to seek for uncertainty relations in the absence of the mutually exclusive physical states $|\Psi^{\perp}\rangle$.

The goal of this paper is to derive tighter upper and lower bounds for both the product and sum forms of the variance-based uncertainty relations. Comparisons among recent strong bounds [11] and our new ones are given in figures. In Sec. II, we obtain stronger lower bounds to the product of variances by generalizing the Cauchy-Schwarz inequality. Sec. III derives lower bounds on the sum of variances for two incompatible observables in finite dimensional spaces. Sec. IV introduces the concept of *uncertainty intervals* on which the lower and upper bounds combine together. Moreover, our uncertainty interval reveals the intrinsic restrictions on the ability of learning the distribution of measurement outcomes caused by incompatible observables. Finally, we conclude with some typical examples, in which our lower bounds are near-optimal, in Sec. V.

II. PRODUCT FORMS OF VARIANCE-BASED UNCERTAINTY RELATIONS

Let $A = \sum_i a_i|a_i\rangle\langle a_i|$ and $B = \sum_i b_i|b_i\rangle\langle b_i|$ be the spectral decompositions of two incompatible observables, and $\overline{A} = \sum_i a'_i|a_i\rangle\langle a_i|$ (resp. $\overline{B} = \sum_i b'_i|b_i\rangle\langle b_i|$) the corresponding decomposition, where $a_i, a'_i, b_i, b'_i \in \mathbb{R}$ are the eigenvalues of $A, \overline{A}, B, \overline{B}$, respectively. By choosing any complete orthonormal basis $\{|\psi_i\rangle\}$, we can write $\overline{A}|\Psi\rangle = \sum_i \alpha_i|\psi_i\rangle$ and $\overline{B}|\Psi\rangle = \sum_i \beta_i|\psi_i\rangle$.

Then $V(A) =| \overrightarrow{x} |^2$ (resp. $V(B) =| \overrightarrow{y} |^2$) for the real vectors $\overrightarrow{x} = (x_1, x_2, \cdots, x_n)$ (resp. $\overrightarrow{y} = (y_1, y_2, \cdots, y_n)$), where $x_i = |\alpha_i| \geq 0, y_i = |\beta_i| \geq 0$. The Cauchy-Schwarz inequality implies that

$$V(A)V(B) \geq \left(\sum_i x_i y_i\right)^2 \tag{2}$$

which is one of the main lower bounds recently obtained by Mondal et al in [11]. Note that with $x_i =| a'_i | \sqrt{\langle\Psi|a_i\rangle\langle a_i|\Psi\rangle}$, $y_i =| b'_i | \sqrt{\langle\Psi|b_i\rangle\langle b_i|\Psi\rangle}$, one gets another main result in [11]. For simplicity, denote $\langle\Psi|a_i\rangle\langle a_i|\Psi\rangle$ ($\langle\Psi|b_i\rangle\langle b_i|\Psi\rangle$) by F_i^a (F_i^b), which is the fidelity between $|\Psi\rangle$ and $|a_i\rangle$ ($|b_i\rangle$).

To derive stronger bounds, we investigate the relation between the *arithmetic* and *geometric* mean inequality (AM-GM inequality) and the Cauchy-Schwarz inequality. Observe that

$$\begin{aligned}
| \overrightarrow{\alpha} |^2| \overrightarrow{\beta} |^2 &= \sum_{ij} x_i^2 y_j^2 \\
&= \sum_{i<j}(x_i^2 y_j^2 + x_j^2 y_i^2) + \sum_i x_i^2 y_i^2 \\
&\geq \sum_{i<j}(2x_i x_j y_j y_i) + \sum_i x_i^2 y_i^2 \\
&= \left(\sum_i x_i y_i\right)^2,
\end{aligned} \tag{3}$$

where the inequality is a result of $n(n-1)/2$ AM-GM inequalities of $x_i^2 y_j^2 + x_j^2 y_i^2 \geq 2x_i y_j x_j y_i$, thus the equality holds if and only if $x_i y_j = x_j y_i$ for all $i \neq j$.

For $0 \leq k \leq n$, define

$$\begin{aligned}
I_k = \sum_{1\leq i<j\leq k}(2x_i x_j y_j y_i) + &\sum_{\substack{1\leq i<j\leq n \\ k<j}}(x_i^2 y_j^2 + x_j^2 y_i^2) \\
&+ \sum_{1\leq i\leq n} x_i^2 y_i^2,
\end{aligned} \tag{4}$$

so $I_0 =| \overrightarrow{x} |^2| \overrightarrow{y} |^2= V(A)V(B)$ and $I_n = \left(\sum_i x_i y_i\right)^2$.

Theorem 1. For any n-dimensional real vectors $\overrightarrow{x}$ and $\overrightarrow{y}$ with positive components, one has that

$$I_0 \geq I_2 \geq \cdots \geq I_{n-1} \geq I_n.$$

In fact, it follow from the AM-GM inequality that

$$\begin{aligned}
I_{k+1} &= I_k + \sum_{i=1}^{k}(2x_i x_{k+1} y_i y_{k+1} - x_i^2 y_{k+1}^2 - x_{k+1}^2 y_i^2) \\
&\leq I_k
\end{aligned}$$

Geometrically, the inequality $| \overrightarrow{x} |^2| \overrightarrow{y} |^2 \geq I_k$ means the Cauchy-Schwarz inequality is only applied on the first k components locally, which can be seen as a partial Cauchy-Schwarz inequality. These apparently give $(n-2)$ tighter lower bounds for $V(A)V(B)$ than the main result of [11], which is $I_0 = V(A)V(B) \geq I_n$. Moreover, we can insert more terms in the above descending chain by selecting arbitrary $x_i^2 y_j^2 + x_j^2 y_i^2$ ($i < j$). Here we only formulate the inequalities on all i, j with $1 \leq i < j \leq k$ for simplicity.

For example, one of the new variance-based uncertainty relations $| \overrightarrow{\alpha} |^2| \overrightarrow{\beta} |^2 \geq I_{n-1}$ can be read as

$$\begin{aligned}
V(A)V(B) \geq &\frac{1}{4}\left(\sum_{i=1}^{n-1}\left|\langle[\overline{A}, \overline{B}_n]\rangle + \langle\{\overline{A}, \overline{B}_n\}\rangle\right|\right)^2 \\
&+ \left|\langle\Psi|\overline{A}|\psi_n\rangle\right|^2 \left(\sum_{i=1}^{n}\left|\langle\Psi|\overline{B}|\psi_n\rangle\right|^2\right) \\
&+ \left|\langle\Psi|\overline{B}|\psi_n\rangle\right|^2 \left(\sum_{i=1}^{n}\left|\langle\Psi|\overline{A}|\psi_n\rangle\right|^2\right) \\
&- \left|\langle\Psi|\overline{A}|\psi_n\rangle\right|^2 \left|\langle\Psi|\overline{B}|\psi_n\rangle\right|^2 := \mathcal{L}_1,
\end{aligned} \tag{5}$$

which offers a stronger bound than that of [11]:

$$\mathcal{L}_1 \geq \frac{1}{4}\left(\sum_{i=1}^{n}\left|\langle[\overline{A}, \overline{B}_n]\rangle + \langle\{\overline{A}, \overline{B}_n\}\rangle\right|\right)^2 \geq \left|\langle\overline{AB}\rangle\right|^2. \tag{6}$$

Next, we use the symmetric group to strengthen the bounds. Note that the symmetric group $\mathfrak{S}_n$ acts on the components of $\overrightarrow{\alpha}$ and $\overrightarrow{\beta}$ by permutation. For any two permutations $\pi_1, \pi_2 \in \mathfrak{S}_n$ we define

$$\begin{aligned}(\pi_1,\pi_2)I_k = &\sum_{1\le\pi_1(i)<\pi_2(j)\le k}(2x_{\pi_1(i)}x_{\pi_2(j)}y_{\pi_2(j)}y_{\pi_1(i)})\\ &+\sum_{\substack{1\le\pi_1(i)<\pi_2(j)\le n\\ k<\pi_2(j)}}(x^2_{\pi_1(i)}y^2_{\pi_2(j)}+x^2_{\pi_2(j)}y^2_{\pi_1(i)})\\ &+\sum_{\pi_1(i)=\pi_2(j)}x^2_{\pi_1(i)}y^2_{\pi_2(j)}. \end{aligned} \tag{7}$$

Clearly I_0 is stable under the action of $\mathfrak{S}_n \times \mathfrak{S}_n$.

The following *state-dependent variance-based uncertainty relations* are easy consequences of Theorem 1.

Theorem 2. *For any permutations $\pi_1, \pi_2 \in \mathfrak{S}_n$, one has that*

$$\begin{aligned}V(A)V(B) &= I_0\\ &\ge (\pi_1,\pi_2)I_2 \ge \cdots \ge (\pi_1,\pi_2)I_{n-1} \ge (\pi_1,\pi_2)I_n.\end{aligned}$$

Optimizing over the $\mathfrak{S}_n$, we get a stronger version of the *state-dependent variance-based uncertainty relations* in the following.

Theorem 3.

$$\begin{aligned}I_0 &\ge \max_{\pi_1,\pi_2\in\mathfrak{S}_n}(\pi_1,\pi_2)I_2 \ge \cdots \ge \max_{\pi_1,\pi_2\in\mathfrak{S}_n}(\pi_1,\pi_2)I_{n-1}\\ &\ge \max_{\pi_1,\pi_2\in\mathfrak{S}_n}(\pi_1,\pi_2)I_n. \end{aligned} \tag{8}$$

This new uncertainty relations are tighter than the result in Thm. 1, since $\max_{\pi_1,\pi_2\in\mathfrak{S}_n}(\pi_1,\pi_2)I_k \ge I_k$ for any $2 \le k \le n$.

III. SUM FORMS OF VARIANCE-BASED UNCERTAINTY RELATIONS

The product form of variance-based uncertainty relations cannot fully capture the incompatibility of observables, since the uncertainty can be trivial if the state is an eigenstate of A or B. It is necessary to consider other forms of the variance-based uncertainty relations, such as the sum form. Before introducing our strong sum form variance-based uncertainty relations, we recall the *rearrangement inequality* first.

Let (x_i) and (y_i) be two n-tuple of real positive numbers such that $x_i \ge x_{i+1}$ and $y_i \ge y_{i+1}$, then the *direct sum*, *random sum* and *reverse sum* between x_i and y_i are defined as

$$\begin{aligned}Di &:= x_1y_1 + x_2y_2 + \cdots + x_ny_n,\\ Ra &:= x_1y_{\pi(1)} + x_2y_{\pi(2)} + \cdots + x_ny_{\pi(n)}, \quad \pi\in\mathfrak{S}_n\\ Re &:= x_1y_n + x_2y_{n-1} + \cdots + x_ny_1.\end{aligned} \tag{9}$$

The following lemma establishes the relationship among the three sums.

Lemma.(Rearrangement inequality) *For any two descending n-tuples x and y of nonnegative numbers, one has that*

$$Di \ge Ra \ge Re. \tag{10}$$

Recall the parallelogram law

$$V(A)+V(B) = \frac{1}{2}\sum_i(x_i+y_i)^2 + \frac{1}{2}\sum_i(x_i-y_i)^2. \tag{11}$$

Combining with the rearrangement inequality we get the following result.

Theorem 4. *For any two permutations $\pi_1, \pi_2 \in \mathfrak{S}_n$ one has that*

$$\begin{aligned}V(A)+V(B) \ge &\frac{1}{2}\sum_i(x_i+y_i)(x_{\pi_1(i)}+y_{\pi_1(i)})\\ &+\frac{1}{2}\sum_i|x_i-y_i|\left|x_{\pi_2(i)}-y_{\pi_2(i)}\right|.\end{aligned} \tag{12}$$

Clearly, by setting $\pi_1 = (1)$ the new uncertainty relation outperforms the main result for the sum of variances in [11] all the time. We will also denote by $\mathcal{L}_2$ the bound of Thm. 4 corresponding to the choice of $\pi_1 = (1)$, $\pi_2 = (1\ 2\ \cdots\ n)$, $x_i = |\alpha_i|$, $y_i = |\beta_i|$, which will be used in Sect. V

IV. UNCERTAINTY INTERVALS

In [11], the authors have shown that quantum mechanics imposes restrictions on the ability of learning the

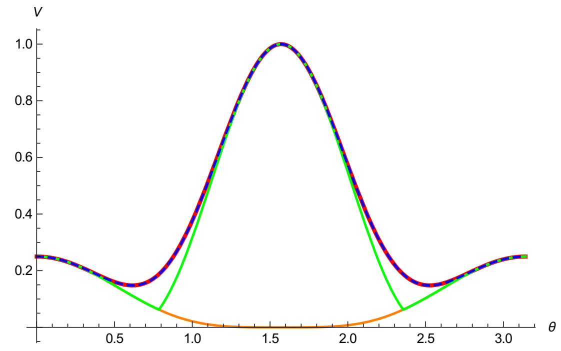

FIG. 1: Lower bounds of $V(A)V(B)$ for a family of spin-1 particles: $V(A)V(B)$, our bound $\mathcal{L}_1$ and the bound of [11] are respectively shown in red, blue and green. The orange curve denotes the Schrödinger uncertainty relation.

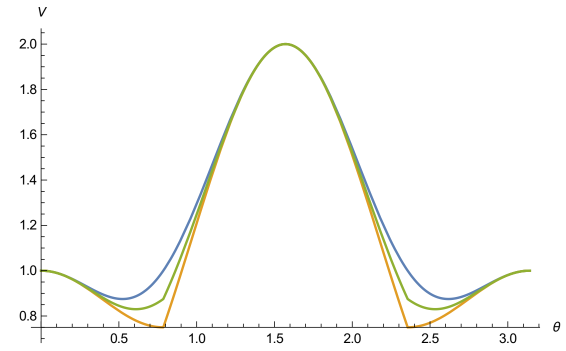

FIG. 2: Several lower bounds for the sum of variances of a family of spin-1 particles: the sum $V(A)+V(B)$, our lower bound $\mathcal{L}_2$ and the bound of [11] are in blue, green, and yellow respectively.

distribution of measurement outcomes caused by incompatible observables, these restrictions have both lower and upper bounds, or *the bound* and *reverse bound* of the uncertainty relation. Their upper bounds are far from being tight. In this section, we propose stronger upper bounds for both $V(A)V(B)$ and $V(A)+V(B)$, and introduce the concept of the uncertainty interval to characterize the restrictions on uncertainty relations.

Let $X=\max_i\{x_i\}$, $x=\min_i\{x_i\}$, $Y=\max_i\{y_i\}$ and $y=\min_i\{y_i\}$, where the extremes are taken over for $1\le i\le n$. Using the rearrangement inequality, we have

$$\frac{(xy+XY)^2}{4xyXY}\left(\sum_i x_i y_i\right)^2 \ge \frac{(xy+XY)^2}{4xyXY}\left(\sum_i x_i y_{\pi(i)}\right)^2 \ge V(A)V(B). \tag{13}$$

Therefore by taking minimum over $\pi\in\mathfrak{S}_n$ we construct a tighter upper bound for $V(A)V(B)$:

$$V(A)V(B)\le \min_{\pi\in\mathcal{S}_n}\frac{(xy+XY)^2}{4xyXY}\left(\sum_i x_i y_{\pi(i)}\right)^2 := \mathcal{U}_1. \tag{14}$$

This means that the distribution of measurement outcomes caused by incompatible observables A and B (for the product form) is restricted within the interval $[\mathcal{L}_1,\mathcal{U}_1]$, i.e. $V(A)V(B)\in[\mathcal{L}_1,\mathcal{U}_1]$. In other words, $[\mathcal{L}_1,\mathcal{U}_1]$ is an *uncertainty interval* for $V(A)V(B)$.

Next we formulate an upper bound of $V(A)+V(B)$ and construct one of its uncertainty intervals. Using the fact $V(A)=|\overrightarrow{\alpha}|^2$ and $V(B)=|\overrightarrow{\beta}|^2$, one can derive the upper bound on the sum of variances of incompatible observables A and B as

$$V(A)+V(B)=\sum_i(x_i^2+y_i^2)\le\sum_i(x_i+y_i)^2. \tag{15}$$

Recalling the definitions $x_i=|\alpha_i|$ and $y_i=|\beta_i|$, we have that

$$V(A)+V(B)\le\sum_i\left(\left|\langle\psi_n|\overline{A}|\Psi\rangle\right|+\left|\langle\psi_n|\overline{B}|\Psi\rangle\right|\right)^2. \tag{16}$$

Denote the right-hand (RHS) of (16) by $\mathcal{U}_2$. Thus we have obtained a uncertainty interval for $V(A)+V(B)$: $[\mathcal{L}_2,\mathcal{U}_2]$. We remark that $\mathcal{U}_2$ is not always better than the bound obtained by [11], but it provides a complementary one. The comparison will be discussed by examples in the next section.

V. EXAMPLES AND CONCLUSIONS

In this section we give examples to show how the new bounds obtained works compared with the recent bounds in [11], which are some of the stronger ones for variance-based uncertainty relations.

We consider first the spin-1 particle with the state $|\Psi\rangle=\cos\theta|1\rangle-\sin\theta|0\rangle$ (note that $|0\rangle$ and $|1\rangle$ are eigenstates of the angular momentum L_z). This example was also given in [11] to show their bounds. We take the incompatible observables as $A=L_x$ and $B=L_y$, which are the angular momentum operators for spin-1 particle. To calculate the bounds, we choose $x_i=|\alpha_i|$ and

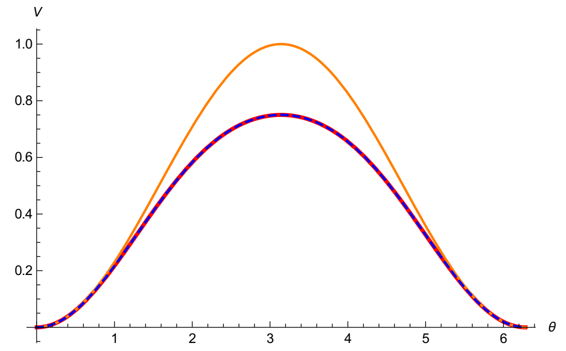

FIG. 3: Several upper bounds for the product of two variances for a family of spin-$\frac{1}{2}$ particles: The red line shows the product of variances $V(A)V(B)$, the blue points stand for our near-optimal upper bound $\mathcal{U}_1$, the orange line is the upper bound for the product of variances given in [11].

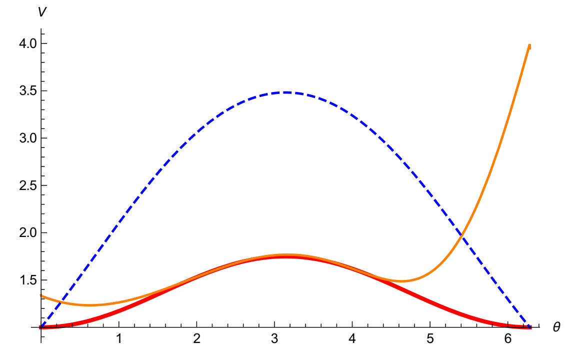

FIG. 4: Several upper bounds for the sum of variances of a family of spin-$\frac{1}{2}$ particles: The sum $V(A)+V(B)$, our bound $\mathcal{U}_2$ and that of [11] are respectively shown in red, blue, and orange.

$y_i = |\beta_i|$ (similar for $x_i =| a_i' | \sqrt{\langle\Psi|a_i\rangle\langle a_i|\Psi\rangle}$ and $y_i =| b_i' | \sqrt{\langle\Psi|b_i\rangle\langle b_i|\Psi\rangle}$).

In FIG. 1, our new bound $\mathcal{L}_1$ is compared with that of [11] in the product form for the family of spin-1 particles $|\Psi\rangle$. In the comparison, the bound $\mathcal{L}_1$ provides the best estimation and is almost optimal (thus shown by blue dots, on the red curve for the product of variances), and $\mathcal{L}_1$ is tighter than the bound of [11] everywhere. Schrödinger's uncertainty relation (in orange curve) is used as a background in the comparison，and is the worst among the three.

In FIG. 2, we plot the bounds for the sum of variances for the family of the spin-1 particles $|\Psi\rangle$. Our bound $\mathcal{L}_2$ outperforms the lower bound from [11].

From now on, let us consider the spin-$\frac{1}{2}$ particle with density matrix

$$\rho = \frac{1}{2}\left(Id + \cos\frac{\theta}{2}\sigma_x + \frac{\sqrt{3}}{2}\sin\frac{\theta}{2}\sigma_y + \frac{1}{2}\sin\frac{\theta}{2}\sigma_z\right), \quad (17)$$

where the two incompatible observables are taken as $A = \sigma_x$ and $B = \sigma_z$.

In FIG. 3 we study the upper bounds for the product of variances $V(A)V(B)$ for a family of spin-$\frac{1}{2}$ states ρ. Our upper bound $\mathcal{U}_1$ provides the best estimation for the product of two variances and typically outperforms the upper bound from [11]. Note that our bound is almost optimal, as it is shown almost identical to the optimal value.

However, our upper bound $\mathcal{U}_2$ for the sum of variances $V(A)+V(B)$ for states ρ is not always tighter than that of [11]. Nevertheless, it still provides a complementary bound for $V(A)+V(B)$ in [11], as the figure shows there are portions of the region where the bound $\mathcal{U}_2$ outperforms that of [11] markedly, see FIG. 4.

Apart from constructing stronger uncertainty relations, our method used in Sec. II also helps to fill up the gaps between the product form of variance-based uncertainty relation and the entropic uncertainty relation. Following [12], assume the sum form of variance-based uncertainty relation

$$V(A) + V(B) \geq H(A) + H(B) + c, \quad (18)$$

where $H(\cdot)$ stands for the Shannon entropy and c is a state-independent constant. Using the basic inequality in Thm. 1, we derive

$$\begin{aligned} & V(A)V(B) \\ \geq & \frac{1}{4}\left(\sum_{i=1}^{n-1} x_i y_i\right)^2 + x_n^2 V(B) + y_n^2 V(A) - x_n^2 y_n^2. \end{aligned} \quad (19)$$

On the one hand, the term $x_n^2 V(B) + y_n^2 V(A)$ forms a so-called *weighted uncertainty relation* [10]. Using the weighted uncertainty relation, we then have a lower bound. On the other hand, notice that we can always assume $x_n^2 = y_n^2$ in the numerical calculation, since $V(rA)V(B) = r^2 V(A)V(B)$. In that case, (19) can be changed to

$$\begin{aligned} & V(A)V(B) \\ \geq & \frac{1}{4}\left(\sum_{i=1}^{n-1} x_i y_i\right)^2 + x_n^2\left(H(A)+H(B)+c\right) - x_n^4. \end{aligned} \quad (20)$$

Therefore both the incompatibility between observables and mixing status of the state will affect the variance-based uncertainty relations. Moreover, any entropic uncertainty relation can be employed to construct a lower bound for $V(A)V(B)$.

In conclusion, we have proposed stronger state dependent variance-based uncertainty relations both in the sum and product forms. After obtaining tighter upper bounds, we have introduced the concept of uncertainty intervals, which restricts the ability of learning the distribution of measurement outcomes caused by incompatible observables. Our newly constructed uncertainty relations provide near-optimal approximations in some typical and nontrivial examples, which can be used in entanglement detection, quantum metrology, quantum speed limits and other related topics in quantum information theory. Finally, our method in deriving stronger state dependent variance-based uncertainty relations precisely fills the gap among the product form of variance-based uncertainty relations and entropic uncertainty relations.

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[1] W. Heisenberg, Z. Phys. **43**, 172 (1927).

[2] H. F. Hofmann and S. Takeuchi, Phys. Rev. A **68**, 032103 (2003).

[3] V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004).

[4] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. Lett. **96**, 010401 (2006).

[5] V. Giovannetti, S. Lloyd, and L. Maccone, Nat. Photon. **5**, 222 (2011).

[6] S. Wehner and A. Winter, New J. Phys. **12**, 025009 (2010).

[7] E. Candes, J. Romberg, and T. Tao, IEEE Trans. Inf. Theory **52**, 489 (2006).

[8] D. P. Pires, M. Cianciaruso, L. C. Céleri, G. Adesso, and D. O. Soares-Pinto, Phys. Rev. X **6**, 021031 (2016).

[9] L. Maccone and A. K. Pati, Phys. Rev. Lett. **113**, 260401 (2014).

[10] Y. Xiao, N. Jing, X. Li-Jost, and S.-M. Fei, Sci. Rep. **6**, 23201 (2016).

[11] D. Mondal, S. Bagchi, and A. K. Pati, arXiv:1607.06712 (2016).

[12] Y. Huang, Phys. Rev. A **86**, 024101 (2012).