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by

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# Monogamy relations of concurrence for any dimensional quantum systems

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We study monogamy relations for arbitrary dimensional multipartite systems. Monogamy relations based on concurrence and concurrence of assistance for any dimensional  $m_1 \otimes m_2 \otimes \dots \otimes m_N$  quantum states are derived, which give rise to the restrictions on the entanglement distributions among the subsystems. Besides, we give the lower bound of concurrence for four-partite mixed states. The approach can be readily generalized to arbitrary multipartite systems.

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## I. INTRODUCTION

Quantum entanglement [1–6] is an essential feature of quantum mechanics, which distinguishes the quantum from classical world. As one of the fundamental differences between quantum entanglement and classical correlations, a key property of entanglement is that a quantum system entangled with one of other systems limits its entanglement with the remaining others. In multipartite quantum systems, there can be several inequivalent types of entanglement among the subsystems. The amount of entanglement for different types might not be directly comparable each other. The monogamy relation of entanglement is a way to characterize the different types of entanglement distribution. The monogamy relations give rise to the structures of entanglement in the multipartite setting. Monogamy is also an essential feature allowing for security in quantum key distribution [7]. Monogamy relations are not always satisfied by any entanglement measures. Although the concurrence and entanglement of formation do not satisfy such monogamy inequalities themselves, it has been shown that the  $\alpha$ th ( $\alpha \geq 2$ ) power of concurrence and  $\alpha$ th ( $\alpha \geq \sqrt{2}$ ) power entanglement of formation for  $N$ -qubit states do satisfy the monogamy relations [8].

Nevertheless, the monogamy relations have been established for qubit systems. For high dimensional systems, it has been shown that the monogamy inequalities can be violated [9, 10]. In this paper, toward the open problem of monogamy properties in higher dimensional systems, we study the monogamy relations in any  $m_1 \otimes m_2 \otimes \dots \otimes m_N$  dimensional systems. Based on concurrence and concurrence of assistance we present the monogamy inequalities for pure states and the restrictions on entanglement distribution for any dimensional mixed quantum states.

## II. MONOGAMY RELATIONS OF CONCURRENCE

The concurrence for a bipartite  $2 \otimes d$  pure state  $|\psi\rangle_{AB}$  is given by [11–13]

$$C(|\psi\rangle_{AB}) = \sqrt{2(1 - \text{Tr}(\rho_A^2))}, \quad (1)$$

where  $\rho_A$  is the reduced density matrix by tracing over the subsystem  $B$ ,  $\rho_A = \text{Tr}_B(|\psi\rangle_{AB}\langle\psi|)$ . The concurrence is extended to mixed states  $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ ,  $0 \leq p_i \leq 1$ ,  $\sum_i p_i = 1$ , by the convex roof extension,

$$C(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle), \quad (2)$$

where the minimum is taken over all possible pure state decompositions of  $\rho_{AB}$ .

For a tripartite state  $|\psi\rangle_{ABC}$ , the concurrence of assistance is defined by [14]

$$C_a(|\psi\rangle_{ABC}) \equiv C_a(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle), \quad (3)$$

for all possible ensemble realizations of  $\rho_{AB} = \text{Tr}_C(|\psi\rangle_{ABC}\langle\psi|) = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|$ . If  $\rho_{AB} = |\psi\rangle_{AB}\langle\psi|$  is a pure state, one has  $C(|\psi\rangle_{AB}) = C_a(\rho_{AB})$ .

For an  $m \otimes n \otimes l$  quantum state  $|\psi\rangle_{ABC}$ , the concurrence  $C(|\psi\rangle_{A|BC})$  of the state  $|\psi\rangle_{ABC}$ , viewed as a bipartite state with partitions  $A$  and  $BC$ , satisfies the Coffman-Kundu-Wootters inequality when  $m = n = l = 2$  [15]. In fact, for qubit systems, in [8, 16] it has been shown that the concurrence of a state  $\rho_{A|B_1 \dots B_{N-1}}$  satisfies a more general monogamy inequality,

$$C_{A|B_1 B_2 \dots B_{N-1}}^\alpha \geq C_{AB_1}^\alpha + C_{AB_2}^\alpha + \dots + C_{AB_{N-1}}^\alpha,$$

where  $\rho_{AB_i} = \text{Tr}_{B_1 \dots B_{i-1} B_{i+1} \dots B_{N-1}}(\rho_{A|B_1 \dots B_{N-1}})$ ,  $C_{A|B_1 B_2 \dots B_{N-1}} = C(\rho_{A|B_1 \dots B_{N-1}})$ ,  $C_{AB_i} = C(\rho_{AB_i})$ ,  $i = 1, \dots, N-1$ , and  $\alpha \geq 2$ . The dual inequality in terms of the concurrence of assistance for  $N$ -qubit pure state  $|\psi\rangle_{A|B_1 \dots B_{N-1}}$  has the form [17],

$$C^2(|\psi\rangle_{A|B_1 \dots B_{N-1}}) \leq \sum_{i=1}^{N-1} C_a^2(\rho_{AB_i}).$$

For higher dimensional systems, such relations are no longer satisfied in general.

First, for any dimensional case, we have the following Lemma.

**Lemma 1** For any quantum states  $\rho_{AB}$ , we have

$$C_a(\rho_{AB}) \leq \min \left\{ \sqrt{2[1 - \text{Tr}(\rho_A^2)]}, \sqrt{2[1 - \text{Tr}(\rho_B^2)]} \right\}, \quad (4)$$

where  $\rho_A = \text{Tr}_B(\rho_{AB})$  and  $\rho_B = \text{Tr}_A(\rho_{AB})$ .

[Proof] Assume that  $\rho_{AB} = \sum_i p_i |\psi\rangle_i \langle \psi|$  is the optimal decompositions of  $C_a(\rho_{AB})$ . Then

$$\begin{aligned} C_a^2(\rho_{AB}) &= \left( \sum_i p_i C(|\psi\rangle_i \langle \psi|) \right)^2 \\ &\leq \sum_i p_i C^2(|\psi\rangle_i \langle \psi|) \\ &= \sum_i p_i 2[1 - \text{Tr}(\rho_t^i)^2] \\ &\leq 2[1 - \text{Tr}(\rho_t)^2], \end{aligned}$$

where  $t \in \{A, B\}$ , the first inequality is due to the Cauchy-Schwarz inequality  $\sum_i x_i y_i \leq \sqrt{\sum_i x_i^2} \sqrt{\sum_i y_i^2}$  with  $x_i = \sqrt{p_i}$  and  $y_i = \sqrt{p_i} C(|\psi\rangle_i \langle \psi|)$ .  $\square$

From the Lemma we have the following monogamy like relations satisfied by the concurrence and the concurrence of assistance.

**Theorem 1** For any  $m \otimes n \otimes l$  pure quantum state  $|\psi\rangle_{ABC}$ , we have

$$C^2(|\psi\rangle_{A|BC}) \geq x C_a^2(\rho_{AB}) + (1-x) C_a^2(\rho_{AC}), \quad (5)$$

where  $\rho_{AB} = \text{Tr}_C(\rho_{ABC})$ ,  $\rho_{AC} = \text{Tr}_B(\rho_{ABC})$ ,  $\rho_{ABC} = |\psi\rangle_{ABC} \langle \psi|$ , and  $x \in [0, 1]$ .

[Proof] For any  $m \otimes n \otimes l$  pure state  $|\psi\rangle_{ABC}$ , one has,  $\rho_{AB} = \text{Tr}_C(\rho_{ABC})$  and  $\rho_{AC} = \text{Tr}_B(\rho_{ABC})$ . Therefore we have

$$\begin{aligned} &C^2(|\psi\rangle_{A|BC}) \\ &= 2(1 - \text{Tr}(\rho_A^2)) \\ &= 2x(1 - \text{Tr}(\rho_A^2)) + (2-2x)(1 - \text{Tr}(\rho_A^2)) \\ &\geq x C_a^2(\rho_{AB}) + (1-x) C_a^2(\rho_{AC}), \end{aligned}$$

where the first inequality is due to the inequality  $C_a(\rho_{AB}) \leq \sqrt{2(1 - \text{Tr}(\rho_A^2))}$  for any bipartite quantum state  $\rho_{AB}$ .  $\square$

As an example, let us consider the  $2 \otimes 2 \otimes 3$  pure state

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |\varphi^+\rangle|2\rangle), \quad (6)$$

where  $|\varphi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$  is one of the Bell states. One has  $C_a(\rho_{AB}) = 1$  and  $C_a(\rho_{AC}) = \frac{2\sqrt{2}}{3}$ . According to theorem 1, we have  $C(|\psi\rangle_{A|BC}) \geq \frac{\sqrt{8+x}}{3}$  with  $0 \leq x \leq 1$ . The state  $|\psi\rangle_{ABC}$  saturates the Eq.(5) with  $x = 1$ .

Theorem 1 shows that the entanglement contained in the pure quantum states  $|\psi\rangle_{A|BC}$  is related to the sum of the concurrence of assistance for bipartite states  $\rho_{AB}$  and  $\rho_{AC}$ . Similarly, we have also the follow conclusion

**Theorem 2** For any  $m \otimes n \otimes l$  mixed quantum state  $\rho_{A|B_1B_2}$ , we have

$$C^2(\rho_{A|B_1B_2}) \geq xC^2(\rho_{AB_1}) + (1-x)C^2(\rho_{AB_2}), \quad (7)$$

where  $\rho_{AB_i} = \text{Tr}_{AB_j}(\rho_{AB_1B_2})$ ,  $i \neq j$ ,  $i, j \in \{1, 2\}$ , and  $x \in [0, 1]$ .

[Proof] For any mixed states  $\rho_{A|B_1B_2}$ ,

$$\begin{aligned} & C(\rho_{A|B_1B_2}) \\ & \geq \sum_i p_i C(|\psi\rangle_{A|B_1B_2}^i \langle \psi|) \\ & \geq \sum_i p_i \sqrt{x C^2(\rho_{AB_1}^i) + (1-x) C^2(\rho_{AB_2}^i)} \\ & \geq x \left( \sum_i p_i C(\rho_{AB_1}^i) \right)^2 + (1-x) \left( \sum_i p_i C(\rho_{AB_2}^i) \right)^2 \\ & \geq x C^2(\rho_{AB_1}) + (1-x) C^2(\rho_{AB_2}). \end{aligned} \quad (8)$$

□

Theorem 2 gives the monogamy relation of concurrence for any dimensional quantum systems. To show how the monogamy inequality (7) works, let us consider the following example. Consider the pure totally antisymmetric state on a three-qutrit system Ref.[9]:  $|\psi\rangle = \frac{1}{\sqrt{6}}(|123\rangle - |132\rangle + |231\rangle - |213\rangle + |312\rangle - |321\rangle)$ . One has  $C(|\psi\rangle_{1|23}) = \frac{2\sqrt{3}}{3}$  and  $C(\rho_{12}) = C(\rho_{13}) = 1$ . Therefore one gets that  $C^2(|\psi\rangle_{1|23}) = \frac{4}{3} < 2 = C^2(\rho_{12}) + C^2(\rho_{13})$ , namely, the usual relation  $C^2(|\psi\rangle_{1|23}) \geq C^2(\rho_{12}) + C^2(\rho_{13})$  is no longer satisfied. However, our monogamy relation (7) is valid,  $C^2(|\psi\rangle_{1|23}) = \frac{4}{3} \geq 1 = x C^2(\rho_{12}) + (1-x) C^2(\rho_{13})$  for any  $x \in [0, 1]$ .

Theorem 1 and Theorem 2 can be readily generalized to arbitrary dimensional multipartite systems, and we have the following corollaries.

**Corollary 1** For any  $m_1 \otimes m_2 \otimes \dots \otimes m_N$  pure quantum state  $|\psi\rangle_{A|B_1 \dots B_{N-1}}$ , we have

$$C^2(|\psi\rangle_{A|B_1 \dots B_{N-1}}) \geq \sum_{i=1}^{N-1} p_i C_a^2(\rho_{AB_i}), \quad (9)$$

where  $\rho_{AB_i} = \text{Tr}_{B_1 \dots B_{i-1} B_{i+1} \dots B_{N-1}}(|\psi\rangle_{AB_1 \dots B_{N-1}} \langle \psi|)$ ,  $p_i \geq 0$  and  $\sum_i p_i = 1$ .

**Corollary 2** For any  $m_1 \otimes m_2 \otimes \dots \otimes m_N$  mixed quantum state  $\rho_{A|B_1 \dots B_{N-1}}$ , we have

$$C^2(\rho_{A|B_1 \dots B_{N-1}}) \geq \sum_{i=1}^{N-1} p_i C^2(\rho_{AB_i}), \quad (10)$$

where  $\rho_{AB_i} = \text{Tr}_{B_1 \dots B_{i-1} B_{i+1} \dots B_{N-1}}(\rho_{AB_1 \dots B_{N-1}})$ ,  $p_i \geq 0$  and  $\sum_i p_i = 1$ .

### III. LOWER BOUND OF CONCURRENCE FOR 4-PARTITE QUANTUM SYSTEMS

In this section, we study the concurrence of 4-partite quantum states based on (7). We present analytical expression of lower bound of concurrence based on the monogamy inequality (7).

**Theorem 3** For any  $m \otimes n \otimes p \otimes q$  pure quantum state  $|\psi\rangle_{A_1A_2|B_1B_2}$ , we have

$$C^2(|\psi\rangle_{A_1A_2|B_1B_2}) \geq \sum_{i=1,2;j=1,2} T_{ij} C^2(\rho_{A_iB_j}), \quad (11)$$

where  $\rho_{A_iB_j} = \text{Tr}_{\{A_1A_2B_1B_2\}/\{A_iB_j\}}(\rho_{A_1A_2B_1B_2})$  with  $i, j \in \{1, 2\}$ ,  $T_{11} = x_1y_{11} + x_3y_{31}$ ,  $T_{12} = x_2y_{21} + x_3y_{32}$ ,  $T_{21} = x_1y_{12} + x_4y_{41}$ ,  $T_{22} = x_2y_{22} + x_4y_{42}$ , with  $\sum_{i=1}^4 x_i = 1$  and  $\sum_{i=1}^2 y_{ti} = 1$ ,  $t \in \{1, 2, 3, 4\}$ .

[Proof] For any  $m \otimes n \otimes p \otimes q$  quantum pure state  $|\varphi\rangle\langle\varphi|_{A_1A_2|B_1B_2}$ , one has

$$\begin{aligned} & C^2(|\varphi\rangle\langle\varphi|_{A_1A_2|B_1B_2}) \\ &= 2x_1(1 - \text{Tr}(\rho_{A_1A_2}^2)) + 2x_2(1 - \text{Tr}(\rho_{A_1A_2}^2)) \\ &+ (2x_3)(1 - \text{Tr}(\rho_{B_1B_2}^2)) + (2 - 2\sum_{i=1}^3 x_i)(1 - \text{Tr}(\rho_{B_1B_2}^2)) \\ &\geq x_1 C_a^2(\rho_{A_1A_2|B_1}) + x_2 C_a^2(\rho_{A_1A_2|B_2}) \\ &+ x_3 C_a^2(\rho_{B_1B_2|A_1}) + (1 - \sum_{i=1}^3 x_i) C_a^2(\rho_{B_1B_2|A_2}), \end{aligned}$$

where the first inequality is due to Lamme 1. By using  $C_a^2(\rho_{AB|C}) \geq C^2(\rho_{AB|C})$  and Theorem 1, we obtain the theorem 3.  $\square$

For example, taking  $x_i = \frac{1}{4}$ ,  $i = 1, 2, 3, 4$ , and  $y_{ti} = \frac{1}{2}$ ,  $i = 1, 2$  for all  $t = 1, 2, 3, 4$ , we have

$$C^2(|\psi\rangle_{A_1A_2|B_1B_2}) \geq \sum_{i=1,2;j=1,2} \frac{1}{4} C^2(\rho_{A_iB_j}).$$

In particular, for the  $2 \otimes 2 \otimes 2 \otimes 2$  pure quantum states, we have  $T_{11} = x_1 + x_3$ ,  $T_{12} = x_2 + x_3$ ,  $T_{21} = x_1 + x_4$  and  $T_{22} = x_2 + x_4$  due to  $C^2(\rho_{1|23}) \geq C^2(\rho_{12}) + C^2(\rho_{13})$ .

**Theorem 4** For any  $m \otimes n \otimes p \otimes q$  mixed quantum state  $\rho_{1234}$ , we have

$$\begin{aligned} & C^2(\rho_{1234}) \\ &\geq \frac{1}{2} [C^2(\rho_{12}) + C^2(\rho_{13}) + C^2(\rho_{14}) + C^2(\rho_{23}) + C^2(\rho_{24}) + C^2(\rho_{34})] \\ &+ \frac{1}{4} \left[ \sum_{\{\{i=1,2\},\{j=3,4\}\}} T_{ij}^1 C^2(\rho_{ij}) + \sum_{\{\{i=1,3\},\{j=2,4\}\}} T_{ij}^2 C^2(\rho_{ij}) + \sum_{\{\{i=1,4\},\{j=2,3\}\}} T_{ij}^3 C^2(\rho_{ij}) \right], \end{aligned} \quad (12)$$

where  $\rho_{ij} = \text{Tr}_{\{1,2,3,4\}/\{i,j\}}(\rho_{1234})$ ,  $T_{ij}^m$ ,  $m = 1, 2, 3$ , are defined similarly as in (11).

[Proof] For the pure  $m \otimes n \otimes p \otimes q$  quantum state  $|\psi\rangle$ , the concurrence of  $|\psi\rangle$  is given by,

$$\begin{aligned} & C^2(|\psi\rangle_{1234}) \\ &= \frac{1}{4} (C_{1|234}^2(|\psi\rangle) + C_{2|134}^2(|\psi\rangle) + C_{3|124}^2(|\psi\rangle) + C_{4|123}^2(|\psi\rangle)) \\ &+ (C_{12|34}^2(|\psi\rangle) + C_{13|24}^2(|\psi\rangle) + C_{14|23}^2(|\psi\rangle)). \end{aligned}$$

Since  $C_{i|j_1j_2j_3}^2 \geq \sum_{t=1}^3 p_{it} C^2(\rho_{ij_t})$  and  $C^2(|\psi\rangle_{i_1i_2|j_1j_2}) \geq T_{i_1j_1} C^2(\rho_{i_1j_1}) + T_{i_1j_2} C^2(\rho_{i_1j_2}) + T_{i_2j_1} C^2(\rho_{i_2j_1}) + T_{i_2j_2} C^2(\rho_{i_2j_2})$ , we obtain (12).  $\square$

*Example:* We consider the  $2 \otimes 2 \otimes 2 \otimes 3$  mixed quantum state  $\rho_{1234} = \frac{1-t}{16}I_{16} + t|\psi\rangle\langle\psi|$ , where  $|\psi\rangle\langle\psi| = \frac{1}{2}(|0000\rangle + |0012\rangle + |1100\rangle + |1112\rangle)$ . We have  $C(\rho_{12}) = \max\{0, \frac{3t-1}{2}\}$ . By our theorem 4, the lower bound of concurrence is  $C(\rho_{1234}) \geq C(\rho_{12})$ , where we have taken into account that

$$\begin{aligned} C^2(\rho_{12|34}) &\geq \frac{1}{2}C^2(\rho_{14|2}) + \frac{1}{2}C^2(\rho_{12|3}) \\ &\geq C^2(\rho_{12}) \end{aligned}$$

and

$$\begin{aligned} C^2(\rho_{13|24}) &\geq \frac{1}{2}C^2(\rho_{13|2}) + \frac{1}{2}C^2(\rho_{1|24}) \\ &\geq C^2(\rho_{12}). \end{aligned}$$

Fig. 1 shows that this lower bound can detect the entanglement of  $\rho_{1234}$  for  $t > \frac{1}{3}$ .

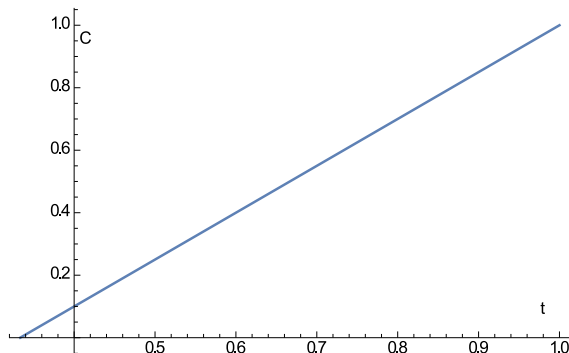


Fig. 1: Lower bound of  $\rho_{1234}$  for  $\frac{1}{3} < t \leq 1$ .

#### IV. CONCLUSION AND REMARK

Entanglement monogamy is a fundamental property of multipartite entanglement. We have presented a kind of monogamy relations satisfied by the concurrence and the concurrence of assistance for any pure quantum states, and monogamy relations of concurrence for arbitrary dimensional quantum systems. Moreover, we have obtained the lower bound of concurrence for four-partite quantum systems based on the monogamy relations. This approach for lower bound of concurrence can be readily generalized to arbitrary multipartite systems.

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