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by

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In Coles-Piani's recent remarkable version of the entropic uncertainty principle, the entropic sum is controlled by the first and second maximum overlaps between the two projective measurements. We generalize the entropic uncertainty relation and find the exact dependence on all first d largest overlaps between two measurements on any d-dimensional Hilbert space. The corresponding entropic uncertainty principle in the presence of quantum memory is also derived. Our bounds are strictly tighter than previous entropic bounds.

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The uncertainty principle, first introduced by Heisenberg [1], plays a central role in physics and marks a distinguished characteristic of quantum mechanics. The principle bounds the uncertainties of measurement outcomes of two observables, such as the position and momentum of a particle. This shows the underlying difference of quantum mechanics from classical mechanics where any properties of a physical object can be quantified exactly at the same time. In Robertson's formulation [2], the product of the standard deviations (denoted by $\Delta(R)$ for the observable R) of the measurement of two observables R and S is controlled by their commutator:

$$\Delta R \Delta S \geqslant \frac{1}{2}|\langle [R,S]\rangle|, \qquad (1)$$

where $\langle \ \rangle$ is the expectation value. The relation implies that it is impossible to simultaneously measure exactly a pair of imcompatible (noncommutative) observables.

In the context of both classical and quantum information sciences, it is more natural to use entropy to quantify uncertainties [3, 4]. The first entropic uncertainty relation for position and momentum was given in [5] (which can be shown to be equivalent to Heisenberg's original relation). Later Deutsch [6] found an entropic uncertainty relation for any pair of observables. An improvement of Deutsch's entropic uncertainty relation was subsequently conjectured by Kraus [7] and later proved by Maassen and Uffink [8] (we use base 2 log throughout this paper),

$$H(R) + H(S) \geqslant \log \frac{1}{c_1}, \qquad (2)$$

where $R = \{|u_j\rangle\}$ and $S = \{|v_k\rangle\}$ are two orthonormal bases on d-dimensional Hilbert space $\mathcal{H}_A$, and $H(R) = -\sum_j p_j \log p_j$ is the Shannon entropy of the probability distribution $\{p_j = \langle u_j|\rho_A|u_j\rangle\}$ for state ρ_A of $\mathcal{H}_A$ (similarly for $H(S)$ and $\{q_k = \langle v_k|\rho_A|v_k\rangle\}$). The number c_1 is the largest overlap among all $c_{jk} = |\langle u_j|v_k\rangle|^2$ ($\leqslant 1$) between the two projective measurements R and S.

The Maassen-Uffink bound has recently been upgraded by Coles and Piani [9], who have shown a remarkable state-independent bound

$$H(R) + H(S) \geqslant \log \frac{1}{c_1} + \frac{1-\sqrt{c_1}}{2} \log \frac{c_1}{c_2}, \qquad (3)$$

where c_2 is the second largest overlap among all c_{jk} (counting multiplicity) and other notations are the same as in Eq.(2). As $1 \geqslant c_1 \geqslant c_2$, the second term in Eq.(3) shows that the uncertainties depend on more detailed information of the transition matrix or overlaps between the two bases. The Coles-Piani bound offers a strictly tighter bound than the Maassen-Uffink bound as long as $1 > c_1 > c_2$. The goal of this letter is to report a more general and tighter bound for the entropic uncertainty relation.

To state our result, we first recall the majorization relation between two probability distributions $P =$

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$(p_1,\cdots,p_d)$, $Q=(q_1,\cdots,q_d)$. The partial order $P\prec Q$ means that $\sum_{j=1}^i p_j^{\downarrow}\leqslant\sum_{j=1}^i q_j^{\downarrow}$ for all $i=1,\cdots,d$. Here $^{\downarrow}$ denotes rearranging the components of p or q in descending order. Any probability distribution vector P is bounded by $(\frac{1}{d},\cdots,\frac{1}{d})\prec P\prec(1,0,\cdots,0)=\{1\}$. For any two probability distributions $P=(p_j)$ and $Q=(q_k)$ corresponding to measurements R and S of the state ρ, there is a state-independent bound of direct-sum majorization [10]: $P\oplus Q\prec\{1\}\oplus W$, where $P\oplus Q=(p_1\cdots,p_d,q_1,\cdots,q_d)$ and $W=(s_1,s_2-s_1,\cdots,s_d-s_{d-1})$ is a special probability distribution vector defined exclusively by the overlap matrix related to R and S. Let $U=(\langle u_j|v_k\rangle)_{jk}$ be the overlap matrix between the two bases given by R and S, and define the subset $\mathrm{Sub}(U,k)$ to be the collection of all size $r\times s$ submatrices M such that $r+s=k+1$. Following [10] we define $s_k=\max\{\|M\|: M\in\mathrm{Sub}(U,k)\}$, where $\|M\|$ is the maximal singular value of M. Denote the sum of the largest k terms in $\{1\}\oplus W$ as $\Omega_k=1+s_{k-1}$, while $s_0=0$, $s_1=\sqrt{c_1}$ and $s_d=1$. It is clear that

$$1=\Omega_1\leqslant\Omega_2\leqslant\cdots\leqslant\Omega_{d+1}=\cdots=\Omega_{2d}=2,$$

where we already noted that $\Omega_2=1+\sqrt{c_1}$.

Our first result is the following stronger state-independent bound of the quantum system $\mathcal{H}_A$ without quantum memory.

Theorem 1. *Let $R=\{|u_j\rangle\}$ and $S=\{|v_k\rangle\}$ be any two orthonormal bases on d-dimensional Hilbert space $\mathcal{H}_A$. Then for any state ρ_A over $\mathcal{H}_A$, we have the following inequality,*

$$\begin{aligned}&H(R)+H(S)\\&\geqslant\log\frac{1}{c_1}+\frac{1-\sqrt{c_1}}{2}\log\frac{c_1}{c_2}+\frac{2-\Omega_4}{2}\log\frac{c_2}{c_3}\\&+\frac{2-\Omega_6}{2}\log\frac{c_3}{c_4}+\cdots+\frac{2-\Omega_{2(d-1)}}{2}\log\frac{c_{d-1}}{c_d},\end{aligned}\tag{4}$$

where $\Omega_k=1+s_{k-1}\leqslant 2$ and c_i is the i-th largest overlap among c_{jk}: $c_1\geqslant c_2\geqslant c_3\geqslant\cdots\geqslant c_{d^2}$.

We remark that due to $\Omega_{d+1}=\cdots\Omega_{2d}=2$, the last (non-zero) term of formula (4) can be fine-tuned according to parity of d. If $d=2n$, it is $\frac{2-\Omega_d}{2}\log\frac{c_n}{c_{n+1}}$; if $d=2n+1$, it is $\frac{2-\Omega_{d-1}}{2}\log\frac{c_n}{c_{n+1}}$.

For simplicity, we leave the proof of Theorem 1 after that of Theorem 2. Let us consider the following example:

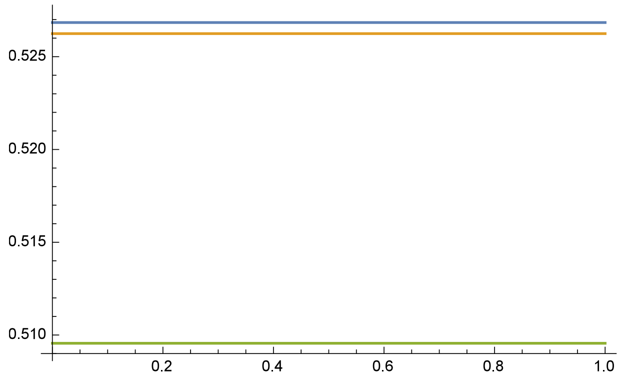

FIG. 1: Comparison of Maassen-Uffink's, Coles-Piani's and the new bounds in absence of quantum memory. They are respectively in green, yellow and blue colors.

$$\rho_A=\frac{1}{1+7p}\begin{pmatrix}\frac{1}{2}+\frac{3p}{2} & 0 & 0 & \frac{\sqrt{1-p^2}}{2}\\ 0 & 2p & 0 & 0\\ 0 & 0 & 2p & 0\\ \frac{\sqrt{1-p^2}}{2} & 0 & 0 & \frac{1}{2}+\frac{3p}{2}\end{pmatrix}.\tag{5}$$

Consider the following two projective measurements: $\{|v_k\rangle\}$ are the standard orthonormal basis on $\mathcal{H}_A$ and $\{|u_j\rangle\}$ are given by

$$\begin{aligned}|u_1\rangle&=(\frac{12}{\sqrt{205}},\frac{6}{\sqrt{205}},\frac{4}{\sqrt{205}},\frac{3}{\sqrt{205}})^T,\\|u_2\rangle&=(-\frac{66}{29\sqrt{205}},\frac{172}{29\sqrt{205}},\frac{183}{29\sqrt{205}},-\frac{324}{29\sqrt{205}})^T,\\|u_3\rangle&=(-\frac{11}{29\sqrt{298}},\frac{309}{29\sqrt{298}},-\frac{195\sqrt{\frac{2}{149}}}{29},-\frac{27\sqrt{\frac{2}{149}}}{29})^T,\\|u_4\rangle&=(\frac{9}{\sqrt{298}},-\frac{9}{\sqrt{298}},-3\sqrt{\frac{2}{149}},-5\sqrt{\frac{2}{149}})^T.\end{aligned}$$

Then the overlap matrix has the form

$$\begin{pmatrix}\frac{144}{205} & \frac{36}{205} & \frac{16}{205} & \frac{9}{205}\\ \frac{4356}{172405} & \frac{29584}{172405} & \frac{33489}{172405} & \frac{104976}{172405}\\ \frac{121}{250618} & \frac{95481}{250618} & \frac{76050}{125309} & \frac{1458}{125309}\\ \frac{81}{298} & \frac{81}{298} & \frac{18}{149} & \frac{50}{149}\end{pmatrix}.\tag{6}$$

Then $\Omega_4\neq 2$ and $c_2\neq c_3$. FIG. 1. illustrates the difference among Eq. (4), Coles-Piani's bound and Maassen-Uffink's bound in this situation. From the diagram it is clear that the new bound is tighter everywhere for each $p\in(0,1)$.

Our stronger bound for the entropic uncertainty relation can be also generalized to bipartite states in the presence of quantum memory [11]. For a bipartite quantum state ρ_{AB} on Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, without confusion, we still use H to denote the von Neumann entropy, $H(\rho_{AB}) = -\mathrm{Tr}(\rho_{AB} \log \rho_{AB})$.

Theorem 2. *Let $R = \{|u_j\rangle\}$ and $S = \{|v_k\rangle\}$ be arbitrary orthonormal bases of the subsystem A of a bipartite state ρ_{AB}. Then we have that*

$$\begin{aligned} &H(R|B) + H(S|B) \\ &\geqslant \log \frac{1}{c_1} + \frac{1-\sqrt{c_1}}{2} \log \frac{c_1}{c_2} + \frac{2-\Omega_4}{2} \log \frac{c_2}{c_3} + \\ &+ \cdots + \frac{2-\Omega_{2(d-1)}}{2} \log \frac{c_{d-1}}{c_d} + H(A|B), \end{aligned} \tag{7}$$

where $H(R|B) = H(\rho_{RB}) - H(\rho_B)$ is the conditional entropy with $\rho_{RB} = \sum_j (|u_j\rangle\langle u_j| \otimes I)(\rho_{AB})(|u_j\rangle\langle u_j| \otimes I)$ (similarly for $H(S|B)$), and d is the dimension of the subsystem A. The term $H(A|B) = H(\rho_{AB}) - H(\rho_B)$ appearing on the right-hand side is related to the entanglement between the measured particle A and the quantum memory B. Notations Ω_k and c_i are the same as in Eq.(4).

Proof. For completeness we start from the derivation of the Coles-Piani inequality. Observe that the quantum channel $\rho \to \rho_{SB}$ is in fact $\rho_{SB} = \sum_k |v_k\rangle\langle v_k| \otimes \mathrm{Tr}_A((|v_k\rangle\langle v_k| \otimes I)\rho_{AB})$. As the relative entropy $D(\rho\|\sigma) = \mathrm{Tr}(\rho \log \rho) - \mathrm{Tr}(\rho \log \sigma)$ is monotonic under a quantum channel it follows that

$$\begin{aligned} &H(S|B) - H(A|B) \\ =&D(\rho_{AB} \| \sum_k (|v_k\rangle\langle v_k| \otimes I)|\rho_{AB}(|v_k\rangle\langle v_k| \otimes I)) \\ \geqslant&D(\rho_{RB} \| \sum_{j,k} c_{jk} |u_j\rangle\langle u_j| \otimes \mathrm{Tr}_A((|v_k\rangle\langle v_k| \otimes I)\rho_{AB})) \\ \geqslant&D(\rho_{RB} \| \sum_j \max_k c_{jk} |u_j\rangle\langle u_j| \otimes \rho_B) \\ =&-H(R|B) - \sum_j p_j \log \max_k c_{jk}, \end{aligned} \tag{8}$$

where the first equation is a basic identity of the quantum relative entropy (cf. [12, 13]). So the state-dependent bound under a quantum memory follows:

$$H(R|B) + H(S|B) \geqslant H(A|B) - \sum_j p_j \log \max_k c_{jk}. \tag{9}$$

Interchanging R and S we also have

$$H(R|B) + H(S|B) \geqslant H(A|B) - \sum_k q_k \log \max_j c_{jk}. \tag{10}$$

We arrange the numbers $\max_k c_{jk}$, $j = 1, \cdots, d$, in descending order:

$$\max_k c_{j_1 k} \geqslant \max_k c_{j_2 k} \geqslant \cdots \geqslant \max_k c_{j_d k}, \tag{11}$$

where $j_1 j_2 \cdots j_d$ is a permutation of $12 \cdots d$. Clearly $c_1 = \max_k c_{j_1 k}$ and in general $c_i \geqslant \max_k c_{j_i k}$ for all i. Therefore

$$\begin{aligned} &-\sum_{j=1}^d p_j \log \max_k c_{jk} = -\sum_{i=1}^d p_{j_i} \log \max_k c_{j_i k} \\ \geqslant&-p_{j_1} \log c_1 - p_{j_2} \log c_2 - \cdots - p_{j_d} \log c_d \\ =&-(1 - p_{j_2} - \cdots - p_{j_d}) \log c_1 \\ &- p_{j_2} \log c_2 - \cdots - p_{j_d} \log c_d \\ =&-\log c_1 + p_{j_2} \log \frac{c_1}{c_2} + \cdots + p_{j_d} \log \frac{c_1}{c_d}. \end{aligned} \tag{12}$$

Similarly we also have

$$\begin{aligned} &-\sum_k q_k \log \max_j c_{jk} \\ \geqslant&-\log c_1 + q_{k_2} \log \frac{c_1}{c_2} + \cdots + q_{k_d} \log \frac{c_1}{c_d}, \end{aligned} \tag{13}$$

for some permutation $k_1 k_2, \cdots k_d$ of $12 \cdots d$. Taking the average of Eq. (9) and Eq. (10) and plugging in Eq. (12-13) we have that

$$\begin{aligned} &H(R|B) + H(S|B) \\ \geqslant&H(A|B) + \log \frac{1}{c_1} + \frac{p_{j_2} + q_{k_2}}{2} \log \frac{c_1}{c_2} \\ &+ \cdots + \frac{p_{j_d} + q_{k_d}}{2} \log \frac{c_1}{c_d}. \end{aligned} \tag{14}$$

Using $p_{j_2} + q_{k_2} = \sum_{i=2}^d (p_{j_i} + q_{k_i}) - \sum_{i=3}^d (p_{j_i} + q_{k_i})$ we see that Eq. (14) can be written equivalently as

$$\begin{aligned} &H(R|B) + H(S|B) \\ &\geqslant H(A|B) + \log \frac{1}{c_1} + \frac{1}{2} \sum_{i=2}^d (p_{j_i} + q_{k_i}) \log \frac{c_1}{c_2} \\ &+ \frac{p_{j_3} + q_{k_3}}{2} \log \frac{c_2}{c_3} + \cdots + \frac{p_{j_d} + q_{k_d}}{2} \log \frac{c_2}{c_d}. \end{aligned} \tag{15}$$

The above transformation from Eq.(14) to Eq.(15) adds all later coefficients of $\log \frac{c_1}{c_3}, \cdots, \log \frac{c_1}{c_d}$ into that of $\log \frac{c_1}{c_2}$ and modify the argument of each log to $\log \frac{c_2}{c_3}, \cdots, \log \frac{c_2}{c_d}$.

Continuing in this way, we can write Eq.(15) equivalently as

$$\begin{aligned} &H(R|B)+H(S|B)\\ =&H(A|B)-\log c_1+\frac{2-(p_{j_1}+q_{k_1})}{2}\log\frac{c_1}{c_2}\\ &+\frac{2-(p_{j_1}+q_{k_1}+p_{j_2}+q_{k_2})}{2}\log\frac{c_2}{c_3}\\ &+\cdots+\frac{2-\sum_{i=1}^{d-1}(p_{j_i}+q_{k_i})}{2}\log\frac{c_{d-1}}{c_d}. \end{aligned} \quad (16)$$

Since $P\oplus Q\prec\{1\}\oplus W$, we have $p_{j_1}+q_{k_1}\leqslant\Omega_2,\cdots,p_{j_1}+q_{k_1}+\cdots+p_{j_{d-1}}+q_{k_{d-1}}\leqslant\Omega_{2(d-1)}$. Plugging these into Eq.(14) completes the proof. ■

Theorem 1 can be similarly proved due to the following simple observation. When measurements are performed on system A, $H(R)+H(S)\geqslant-\sum_j p_j\log\sum_k q_k c_{jk}+H(A)\geqslant-\sum_j p_j\log\max_k c_{jk}+H(A)$. Then Theorem 1 follows directly from the proof of Theorem 2.

We remark that the most possible condition which can force our new bound Eq.(7) degenerates to Eq.(2) is when two orthonormal bases are mutually unbiased.

As an example, consider the following 2×4 bipartite state,

$$\rho_{AB}=\frac{1}{1+7p}\begin{pmatrix} p&0&0&0&0&p&0&0\\ 0&p&0&0&0&0&p&0\\ 0&0&p&0&0&0&0&p\\ 0&0&0&p&0&0&0&0\\ 0&0&0&0&\frac{1+p}{2}&0&0&\frac{\sqrt{1-p^2}}{2}\\ p&0&0&0&0&p&0&0\\ 0&p&0&0&0&0&p&0\\ 0&0&p&0&\frac{\sqrt{1-p^2}}{2}&0&0&\frac{1+p}{2} \end{pmatrix}, \quad (17)$$

which is known to be entangled for $0<p<1$. We take system A as the quantum memory, and consider the same measurement sets $\{|u_j\rangle\}$ and $\{|v_k\rangle\}$ as in the example of Theorem 1. $\{|u_j\rangle\}$ and $\{|v_k\rangle\}$ are now the measurement bases on space $\mathcal{H}_B$, with the overlap matrix (6). The comparison between Coles-Piani's bound and Eq. (7) in the presence of quantum memory is displayed in FIG. 2, which shows that our new bound is strictly tighter for all $p\in(0,1)$.

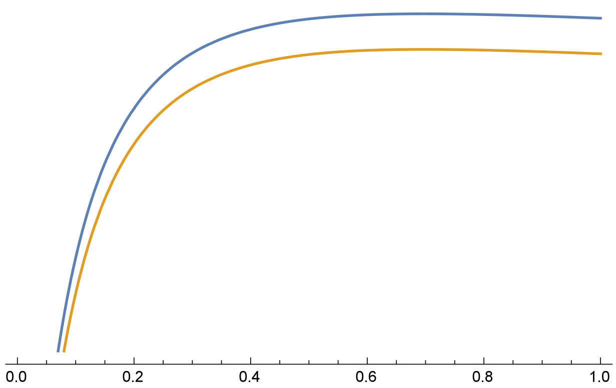

FIG. 2: Comparison of bounds for entangled ρ_{AB}. The blue curve is the new bound Eq. (7) and the yellow curve is Coles-Piani's bound.

Conclusion. We have found new lower bounds for the sum of the entropic uncertainties both with and without quantum memory. Our new bounds have formulated the complete dependence on all d largest entries in the overlap matrix between two measurements on a d-dimensional Hilbert space, while the previously best-known bound depends on the first two largest entries. We have shown that the new bounds are strictly tighter than previously known entropic uncertainty bounds by formulas and examples.

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[1] W. Heisenberg, Z. Phys. **43**, 172 (1927).
[2] H. P. Robertson, Phys. Rev. **34**, 163 (1929).
[3] I. Białynicki-Birula and L. Rudnicki, Statistical Complexity, ed. by K. Sen, Springer Netherlands, Dordrecht pp. 1-34 (2011).
[4] P. J. Coles, M. Berta, M. Tomamichel and S. Wehner, arXiv: 1511.04857v1.
[5] I. Bialynicki-Birula and J. Mycielski, Comm. Math. Phys. 44, 129 (1975).
[6] D. Deutsch, Phys. Rev. Lett. **50**, 631 (1983).
[7] K. Kraus, Phys. Rev. D. **35**, 3070 (1987).
[8] H. Maassen and J. B. M. Uffink, Phys. Rev. Lett. **60**, 1103 (1988).
[9] P. J. Coles and M. Piani, Phys. Rev. A **89**, 022112 (2014).

[10] Ł. Rudnicki, Z. Puchała, and K. Życzkowski, Phys. Rev. A **89**, 052115 (2014).

[11] M. Berta, M. Christandl, R. Colbeck, J. M. Renes, and R. Renner, Nat. Phys. **6**, 659 (2010).

[12] P. J. Coles, L. Yu, V. Gheorghiu, and R. B. Griffiths, Phys. Rev. A **83**, 062338 (2011).

[13] P. J. Coles, Phys. Rev. A **85**, 042103 (2012).