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Bidding with Coordination Risks

by

Timo Ehrig, Jürgen Jost, and Rosemarie Nagel

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Timo Ehrig [*†] Jürgen Jost[‡§] Rosemarie Nagel [¶]

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Abstract

We propose a game to study the interaction of bidding and coordination uncertainty. Two players bid under strategic market game rules over shares of a joint project after obtaining noisy signals about its value. Each player has the alternative option to refrain from bidding and to get a safe outside payoff. Ex ante there is no motive to be the only investor, since this is associated with a low project value. We prove the existence of a unique perfect Bayesian equilibrium by using techniques from the calculus of variations. Because of the coordination risk a joint threshold emerges above which it is optimal for agents to invest. Furthermore, this risk dampens competition and therefore, compared to a standard strategic market game with incomplete information, bids are substantially lower, especially when the expected project value is low. From a technical point of view, we are the first to study a global game with continuous strategies containing strategic complements as well as strategic substitutes.

Keywords: Coordination Uncertainty, Global Games, Strategic Market Games, Strategic Uncertainty, Price Formation, Common Projects, Joint Investments

1 Introduction

Many economic interactions are characterized by both competition and coordination. Take the example of two venture capitalists [7] investing in a new start-up. In such joint projects [22] not only the financial investment is important, but also the human capital that both investors provide.

[*]Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany

[†]University of Southern Denmark

[‡]Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany

[§]Santa Fe Institute, New Mexico, USA

[¶], Institucia Catalana de Recerca i Estudis Avanats (ICREA), Department of Economics, Universitat Pompeu Fabra, Spain

The relative amounts of invested capital determines the share of profits for each investor which creates a competitive situation. This competition arises in a bidding process which mimics the investment in the financial dimension and in the effort of providing human capital. Nevertheless, both players also need their complementary know-how for the success of the start-up [27]. Due to this complementarity, especially in the human capital component, there can be a potential decrease of the value of the start-up if only one venture capitalist invests. Thus, the interaction of the venture capitalists is also characterized by coordination aspects.

Competition expressed as strategic substitutes and coordination as strategic complements [25] have been extensively studied in the economics literature, but their intertwining has received much less attention. Regarding complements, a fundamental problem for making predictions arises due to the multiplicity of equilibria that typically exist in coordination games, given common knowledge and solution concepts such as Nash or rationalizability. In such games, adding even a small degree of payoff uncertainty can, however, restore uniqueness of the equilbrium [8, 20, 6, 2, 11]. Less is known, however, how competition helps to resolve, or further intensifies, the coordination risk. Moreover, the competition arising in the bidding process is likely to be influenced by the coordination risk. But how exactly? Will coordination risk dampen the competition and affect the amount players bid? Will they bid less, given coordination uncertainty, as they would do without it?

To answer these questions, we propose a new game. The novelty of our approach consists in coupling a strategic market game and a coordination game, combined with incomplete information modelled as in global games [8]. Two agents submit bids in a strategic market game [23, 9], after individually obtaining imprecise signals about an a priori unknown fundamental value of a risky common project. They have the option not to bid. In the latter case they would receive a safe outside option. However, in addition to the competitive nature of bidding there is a cooperative component since the fundamental value of the project also depends on the joint effort of both players. It is always in a player's interest that both participants invest, since then, by construction, the value of the project is higher than if one invests alone. When only one invests, the project value will be decreased, resulting in a payoff which we call a 'side-payoff'. While it would have been better to invest together, as the corresponding 'in-payoff' is higher than the side-payoff, the side-payoff can still dominate the outside payoff. This is the case when the player observed a very high signal.

We construct two variants of the game which distinguish whether there is bid revelation or not. In the first variant both firms bid simultaneously and outcomes are realized as described above which we model as a one stage game. Second, as can be exemplified in the venture capital case, there is, however, also the possibility to observe the other firms bids with a possible decision to withdraw the bid conditional on the other firms' bid. We model this as a two stage game of first bidding and then bid-revelation and coordination to stay invested or not. The two stage variant has the interesting feature to mimic market or firms's activities before any coordination is taking place and those market activities are revealed to the participants in the coordination stage. The fact that players can anticipate this later revelation of the bids already at the stage when they determine their bids affects the value of those bids, that is, the market price. It is one of the main achievements of this paper that we can predict this effect and model it in quantitative terms.

While this game poses some non-trivial challenges – bidding functions are best responses to bidding functions which are discontinuous at an emerging threshold – a simple and intuitive solution emerges. We show that indeed, coordination uncertainty dampens competition, especially when the project value is low. Agents bid less when coordination risks are present than in the case when they are absent. We refer to this effect as the *coordinational discount effect*. The intuition behind this is simple. From the game structure, a threshold emerges above which it is optimal to bid, and below which it is optimal to refrain from doing so. When agents optimize their bids, they need to integrate over all possible signals that the other player may have observed. For signals below the threshold, they expect losses. The expected value of investment decreases, and agents bid less than they would for the case without coordination uncertainty. Moreover, as in global games, our game possesses multiple equilibria under common knowledge, while adding Gaussian noise to the payoff observation yields a unique perfect Bayesian equilibrium. Agents face the risk of miscoordination, also in this equilibrium. An agent who observed a signal just above the threshold expects higher payoffs from bidding than from not bidding, as, in expectation, potential gains outweigh potential losses. However, this agent can still be unlucky – the other agent may have seen a signal below the threshold. The agents are aware of this and bid less. Thus, competition is dampened and prices for the joint project decrease. While these effects are intuitive, we are the first to demonstrate them in a closed form model and provide a precise quantitative characterisation of them. The two variants predict different thresholds and bidding functions. In the one stage variant thresholds are higher and bidding near the thresholds are lower than in the two stage variant. The reason is that coordination risks are higher without the possibity to withdrawal. The two variants predict different thresholds and bidding functions. In the one stage variant thresholds are higher and bidding near the thresholds are lower than in the two stage variant. The reason is that coordination risks are higher without the possibility to withdraw later.

Also from a more technical point of view, the study of this game yields a number of important insights. First, we demonstrate that the theoretical phenomenon that noisy payoff observations reinstall uniqueness of equilibria (as observed in global games) carries over to a game with continuous strategies, as in [11]. Different from [11], our game is characterised by both strategic substitutes and complementarities while their game contains continuous strategies with complentaritites.

Second, the coordinational discount effect is a novel prediction, which is easily testable by economic experiments. The coordinational discount effect also emerges in the one stage variant of our game. It arises from the uncertainty regarding the ex ante decision to invest of the other player, while bids are chosen.

Third and finally, modeling strategic substitutes and complementarities, continuous bids and incomplete information poses a number of challenging technical problems. We outline these challenges in the next subsection, discussing the related literature, in order to explain the need of novel mathematical tools.

1.1 Related Literature

Our game combines strategic substitutes and complementarities, continuous bids and incomplete information. This combination has not been addressed in prior literature. [11] et al [11] allow for continuous actions, but do not allow for substitutes in the payoffs. Those substitutes are present in our game, as it contains a strategic market game in the case that both agents invest. Karp et al. [15] model a global game with substitutes and complements, but only for binary actions, while we allow for continuous *action spaces*. Admati and Perry [1] are concerned with joint projects, like us, albeit with a different focus. They consider multiple time periods, but only binary actions. Mathevet and Steiner [19] relax assumptions of complementarity, but likewise only allow for binary actions, while we allow for continuous actions.

Angeletos and Werning [4] study the role of prices as signalling devices for later coordination in the setting of a noisy rational-expectation, that is, a macroeconomic model. In such models (e.g., [12]) there is a fixed-point relation between realized prices, and demand conditioned on those realized prices, and in a setting with noise, such prices implicitly average over the signals of the agents. They are then concerned with the multiplicity of such rational expectation equilibria and its dependence on the noise structure, more precisely the relation between private and public noise. In the present paper, we model a situation with only two players. In this setting, there is no such REE price, and so agents cannot use prices to better predict one another's actions. In fact, in a one stage variant of our game, bids are not revealed at any point during the game, and we still obtain a unique perfect Bayesian equilibrium in the class of pure strategies. In this equilibrium, bids are discounted just as in the two stage game we study. The bid discount and indeed also the uniqueness of equilibrium is a result of the uncertainty about actions of the other player. In [4, 3], because of the REE construction, this uncertainty is largely reduced, as information about the later actions of the players is flowing through prices that the agents observe before they choose demand.

Moreover, although in some part of our game, when both bid, the payoff structure is as in Tullock type games [24, 16, 5, 26], the uncertainty structure is fundamentally different from the ones in the Tullock literature. In our model, the agents can opt for an outside option. If one agent chooses to opt for the outside option, the value of the project decreases for the remaining agent. As we will discuss, this introduces additional strategic uncertainty into the bidding process and, as a result, a threshold emerges below which the players do not engage in bidding. Near that threshold, the optimal bid quickly rises to signigicantly positive values, and for that reason, we cannot control the derivative of the bidding functions at the threshold. Therefore, we cannot apply compactness arguments (Schauder fixed point theorem). Nor can we apply contraction mapping arguments as in [17] or [26] and other related papers, because in our game a variation of the opponent's bid does not necessarily lead to a response that is smaller in magnitude. Also, the general methods of [11] do not apply here, because if one player increases her bid, the optimal response of the opponent could be a decrease, rather than an increase, of the bid. In fact, when the opponent's bid is increased above the equilibrium value, the best response entails to decrease the own bid, and when it is decreased below the equilibrium value, one should also decrease. We shall therefore need to use schemes of implicit differentiation to derive precise formulae for the change of the optimal bid in response to changes of the signal, the other player's bid, or the noise parameter, in the context of our variational problem. This will enable us to derive the monotonicity of optimal response functions, which together with an upper bound (as a function of the signal received) on optimal bids, will ensure convergence of an iterated best response.

The paper is structured as follows. We first introduce the one stage variant of our game and then the two stage variant, proving uniqueness and existence of a perfect Bayesian equilibrium in both cases. We compare the two variants and conclude.

2 One stage strategic market game under coordination risk

We introduce and analyze both our variants of the game in several steps as they contain strategics substitutes in form of a strategic market game and strategic complementarities in form of coordination risk. In this section we discuss the one stage variant of bidding and coordination risk. We first introduce the strategic market game under complete and incomplete information and then add in the third subsection the coordination risk.

2.1 Strategic market game under complete information

We start with a standard strategic market game under complete information [23, 18]. In a different context, the same game structure is also known as a Tullock game [24]. Henceforth, the game has two players that bid over a single project. The true value of the project is denoted by θ. This value is measured in units of a currency, and so will be the players' bids. The players are labelled by i and $-i$. The game is symmetric. The bids that players place for the project are denoted by σ_i and σ_{-i}. Negative bids are not allowed, and bidding zero means abstention. The payoffs are

$$\frac{\sigma_i}{\sigma_i + \sigma_{-i}}\theta - \sigma_i \quad \text{for player } i \text{ and analogously} \tag{1}$$

$$\frac{\sigma_{-i}}{\sigma_i + \sigma_{-i}}\theta - \sigma_{-i} \quad \text{for player } -i. \tag{2}$$

The market price of the project is $\sigma_i + \sigma_{-i}$.

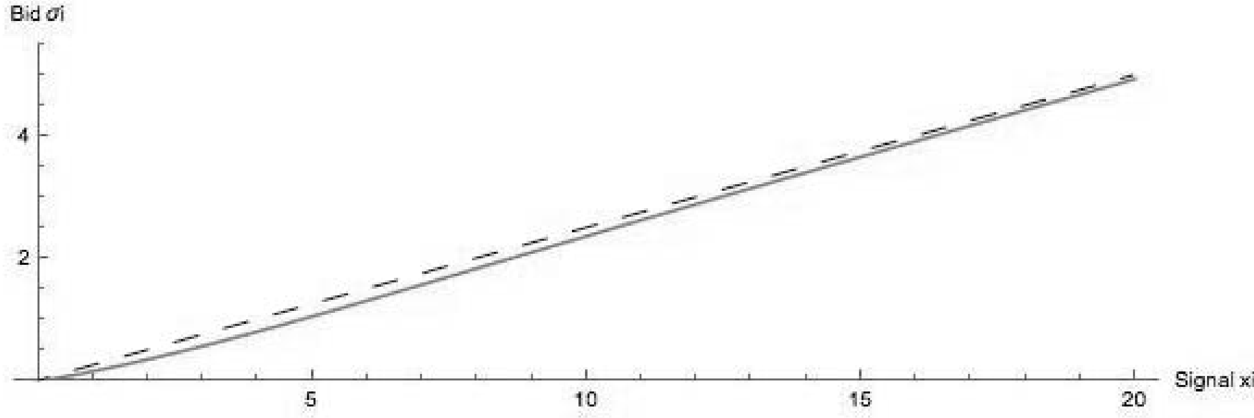
The analysis of this game provides a baseline upon which we can subsequently build. For $\theta \leq 0$, the players will bid 0, and so, we may assume $\theta > 0$. When $-i$ bids σ_{-i}, the optimal bid of i is

$$\sigma_i^* := \operatorname{argmax}_{\sigma_i} \theta \frac{\sigma_i}{\sigma_i + \sigma_{-i}} - \sigma_i = \sqrt{\theta\sigma_{-i}} - \sigma_{-i}. \tag{3}$$

In particular,

$$\frac{d\sigma_i^*}{d\sigma_{-i}} = \frac{\sqrt{\theta}}{2\sqrt{\sigma_{-i}}} - 1 \quad \begin{cases} > 0 & \text{for } \sigma_{-i} < \frac{\theta}{4} \\ < 0 & \text{for } \sigma_{-i} > \frac{\theta}{4} \end{cases}. \tag{4}$$

Figure 1: Baseline: Strategic Market Game with Noise ($\gamma = 3a$)



Thus, σ_i^* becomes maximal when $\sigma_i = \frac{\theta}{4}$. Consequently the equilibrium bid is [18]

$$\sigma^* = \frac{\theta}{4}, \tag{5}$$

and it is never rational to bid more than that, because of the sunk costs σ_i, σ_{-i} in (1), (2). Since the two players bid over a project of fixed supply, competition is imperfect and the market price is only half of the projects fundamental value [18]. Thus, the two investors can appropriate profits and will receive positive payoffs.

The payoff of each player at the equilibrium (5) is

$$\theta \frac{\frac{\theta}{4}}{\frac{\theta}{4} + \frac{\theta}{4}} - \frac{\theta}{4} = \frac{\theta}{4}. \tag{6}$$

The market price for the project is therefore $\frac{\theta}{2}$. Later, when we add coordination and incomplete information, two discount effects will lower that market price.

2.2 Strategic market game under incomplete information

We next add noise to the signals that the players receive. We assume that the fundamental value θ is unknown to the players. Investor i receives a private signal

$$x_i = \theta + \epsilon_i, \tag{7}$$

about θ, and $-i$ analogously receives x_{-i}. ϵ_i and ϵ_{-i} are identically and independently drawn from $N(0, \gamma)$. This is common knowledge. The assumption that the noise is Gaussian will make some subsequent computations possible.

Given this information structure, i does not know $-i$'s signal, but only its distribution determined by the own signal x_i and the common knowledge parameter γ. So she has to respond to the opponent's entire bidding function $\sigma_{-i}(x_{-i})$ for all possible values x_{-i}, by integrating over those values on the basis of her own signal x_i.

Figure (1) depicts a numerical simulation of the bidding function in a standard strategic market game , but with noisy observations of θ in the way we just described. Details will be provided in subsection (3.6). As we can see in figure (1), noisy observations of θ do not have dramatic effects on bidding. The bidding function still roughly corresponds to $\frac{\theta}{4}$. There are some smaller effects of the payoff uncertainty in standard strategic market games which we will discuss in detail in subsection (3.6).

In this version of the game, there are some undesirable effects, however. According to the rules, a player who bids zero gets nothing. And when the signal is negative, one should not bid. If $\theta > 0$, and the opponent does not bid, for instance because he has received a negative signal, player i gets the full share $\theta \frac{\sigma_i}{\sigma_i + \sigma_{-i}} - \sigma_i = \theta - \sigma_i$ as soon as σ_i is positive. The player should then bid an arbitrarily small positive amount. This introduces a discontinuity.

Moreover, central for studying the questions we raised in the introduction, we wish to superimpose the bidding process with coordination uncertainty, to see how the former and the latter interact. We shall therefore change the rules of the game so that a player who does not bid gets a safe positive return, and a player who bids alone gets only a small fraction of the total value θ. That will introduce the issue of coordination and leads to the emergence of a positive bidding threshold below which neither player will place a bid.

2.3 Strategic market game with coordination risk under incomplete information

We now add coordination risk to the strategic market game by introducing out and side payoffs into the game. In contrast to the previous version, a player who does not bid receives a secure payoff a. Furthermore, we assume that when only one player bids, she will only receive a fraction $\frac{\theta}{b}$ of the project value for some large b. This assumption reflects that the project needs two bidding parties to fully succeed. A player who abstains from bidding will obtain a safe positive return a.

The normal form of that game is depicted in table (1).

Without noise, that is, with complete information the equilibria are the same as in the two stage game with complete information which we discuss in more detail in the next section. This means that in equilibrium players learn nothing new from the bid revelation. For a certain range of parameters, the equilibrium is not unique, as bidding alone is suboptimal. Either both players bid as in the game under complete information treated in Section 2.1 and get their profits of the project as in that game, or both could abstain and receive a. Noise will have the same effect as in global games [11]: The multiplicity of equilibria gives way to a unique equilibrium. However, we shall also see a novel effect: due to the coordination uncertainty bids are discounted.

Under incomplete information, in this game, a threshold, that is, a signal t emerges endoge-

The analysis of this game provides a baseline

Table 1: Normal form of the one stage variant of our game

Investor i \ Investor $-i$	NO BID	BID STAY
NO BID	a, a	a, $\frac{\theta}{b} - \sigma_{-i}$
BID STAY	$\frac{\theta}{b} - \sigma_i$, a	$\frac{\sigma_i}{\sigma_i + \sigma_{-i}}\theta - \sigma_i$, $\frac{\sigma_{-i}}{\sigma_i + \sigma_{-i}}\theta - \sigma_{-i}$

nously from the game definition, so that agents do not invest if their signal is below t. Both agents know that for some threshold t, if $x_i < t$, payoff a is better than the payoff resulting from bidding. On the other hand, they also know that there is some signal for which the expected value of θ is so high that the in and side payoffs are both better than a. Thus, there is a lower dominance region of signals for which not bidding is a dominant strategy, and an upper dominance region for which investing is better than abstaining. As in global games [8], there is a threshold t for the signal above which players invest and below which they abstain.

Since we assume that $b > 8a$, agents have no incentives to invest unilaterally, unless the signal is very high. Therefore, we can assume that the agents agree on the threshold.

When $-i$ employs the bidding function $\sigma_{-i}(x_{-i})$ in response to his signal x_{-i}, i's optimal response is given by

$$\begin{aligned}
\bar{\sigma}_i(x_i) = \operatorname*{argmax}_{\sigma_i(x_i)} \int_{-\infty}^{\infty} \int_{x_k=t}^{\infty} \frac{\sigma_i(x_i)}{\sigma_i(x_i) + \sigma_{-i}(x_k)} \theta p(x_k|\theta) dx_k \\
+ \int_{-\infty}^{x_k=t} \frac{\theta}{b} p(x_k|\theta) dx_k] p(\theta|x_i) d\theta - \sigma(x_i) \\
= \operatorname*{argmax}_{\sigma_i(x_i)} \int_{x_k=t}^{\infty} (\frac{1}{\gamma\sqrt{4\pi}}) e^{-\frac{(x_i-x_k)^2}{4\gamma^2}} \frac{\sigma_i(x_i)}{\sigma_i(x_i) + \sigma_{-i}(x_k)} \int_{-\infty}^{\infty} \theta \frac{1}{\gamma\sqrt{\pi}} e^{-(\frac{\theta - (\frac{x_i+x_k}{2})}{\gamma})^2} d\theta dx_k \\
+ \int_{-\infty}^{x_k=t} \frac{\theta}{b} p(x_k|\theta) dx_k] p(\theta|x_i) d\theta - \sigma(x_i) \\
= \operatorname*{argmax}_{\sigma_i(x_i)} \int_{x_k=t}^{\infty} (\frac{1}{\gamma\sqrt{4\pi}}) e^{-\frac{(x_i-x_k)^2}{4\gamma^2}} (\frac{x_k + x_i}{2}) \frac{\sigma_i(x_i)}{\sigma_i(x_i) + \sigma_{-i}(x_k)} dx_k - \sigma_i(x_i).
\end{aligned} \tag{8}$$

Here, we have used some simple computations for Gaussians and also the assumption that when i observes x_i and assumes that $-i$ observes x_k, the best estimate for θ is $\frac{x_i+x_k}{2}$. Also, when only i bids, the revenue $\frac{\theta}{b}$ does not depend on the value of σ_i, and therefore the corresponding integral can be omitted from the optimization.

The scheme (8) leads to the condition

$$\int_{x_k=t}^{\infty} \frac{1}{\gamma\sqrt{4\pi}} e^{-\frac{(x_i-x_k)^2}{4\gamma^2}} \frac{x_k + x_i}{2} \frac{\sigma_{-i}(x_k)}{(\sigma_{-i}(x_k) + \sigma^*(x_i))^2} dx_k - 1 = 0. \tag{9}$$

Evaluating equation (9) allows us to prove that bids are discounted, that is, that they, or more precisely, their expected values are lower as the complete information strategic market game bid $\frac{\theta}{4}$. A closer look at (9) shows us that there are in fact two effects that lead to discounted bids. We refer to the first one as an *informational discount effect.* Player i no longer responds to the optimal bid of $-i$ at the same signal that i is observing, but to a bidding function in response to a Gaussian distribution of signals. Since at other signals than x_i, the optimal bid of $-i$ is not the equilibrium value for i and therefore suboptimal, i will post lower bids. Formally, $\frac{s}{(s+\sigma)^2}$ for a given σ is maximal when $s = \sigma$. And for $s = \sigma$, we have $\frac{s}{(s+\sigma)^2} = \frac{1}{4\sigma}$. Using this in (9) yields

Theorem 1 (Upper bound/ discount effect)**.** *The optimal bid of player i is upper bounded from above byby*

$$\sigma^*(x_i) < \frac{1}{4} \int_{x_k=t}^{\infty} \frac{1}{\gamma\sqrt{4\pi}} e^{-\frac{(x_i-x_k)^2}{4\gamma^2}} \frac{x_k + x_i}{2} dx_k = \frac{1}{4}\left(\frac{\gamma}{\sqrt{4\pi}} e^{-\frac{(x_i-t)^2}{4\gamma^2}} + x_i \int_{x_k=t}^{\infty} \frac{1}{\gamma\sqrt{4\pi}} e^{-\frac{(x_i-x_k)^2}{4\gamma^2}} dx_k \right). \tag{10}$$

which for large x_i approaches $\frac{x_i}{4}$.

Second, the upper bound (10) can be significantly smaller than $\frac{\theta}{4}$, and the large decrease has its origin in coordination uncertainty. This becomes intuitively clear when we evaluate (10) at $x_i := t$. Here, the upper bound is approximately $\frac{\theta}{8}$ for small variances. This effect is induced by the coordination uncertainty. Thus, we call this effect *coordinational discount effect*. In subsection 3.6 we will discuss informational and coordinational discount effects in detail, and distinguish the former from the latter.

2.4 Existence and uniqueness of the equilibrium in the strategic market game with coordination risk under incomplete information

We now prove that the game has a unique Bayesian Nash equilibrium.

Theorem 2. *In the specified game with incomplete information, there is a unique Bayesian Nash equilibrium.*

Proof. The strategy of the proof consists in an iteration of optimal responses of the players to each other's current strategy and a proof that this iteration converges to a unique limit. In view of our main result in the next section, a corresponding existence and uniqueness theorem for the game with an additional coordination stage, we shall develop a proof that depends on variational formulas extracted from implicit differentiation of the optimal response function.

We first verify that it is rational for players to bid positively in response to signals above the threshold.

We note that we have strict inequality in (10), because equality can only hold for $\sigma_{-i}(x_k) \equiv \sigma^*(x_i)$, that is, for an opponent's strategy that does not depend on x_k, but rather on x_i, which, however, is not possible as the opponent is assumed to act on the basis of his signal x_k, but does not know x_i.

Since the situation is symmetric, the opponent can also be assumed to satisfy the bound (30). From this, we deduce the following lemma

Lemma 1. *The optimal response $\sigma^*(x_i)$ is positive for $x_i \geq t$ when the opponent satisfies the bound (10).*

Proof. Indeed, if we had $\sigma^*(x_i) < 0$, then inserting the bound (10) for $-i$, that is,

$$\sigma_{-i}(x_k) < \frac{1}{4}\int_{x_\ell=t}^{\infty} \frac{1}{\gamma\sqrt{4\pi}} e^{-\frac{(x_k-x_\ell)^2}{4\gamma^2}} \frac{x_\ell + x_k}{2} dx_\ell$$

into (12) would yield a value > 1 for the integral, which is not compatible with (12). This completes the proof of Lemma 1.

Note that we do not allow negative bids anyway, but for the consistency of the scheme, we needed to show that this is a consistent assumption, and more strongly, that each player can assume that the opponent's bids are positive and bounded by (10) when he receives a signal above t.

We now return to the *proof of the Theorem*. Given $-i$'s strategy σ_{-i}, i selects her strategy according to (8), that is,

$$\sigma^*(x_i) = \underset{\sigma_i(x_i)}{\operatorname{argmax}} \int_{x_k=t}^{\infty} \frac{1}{\gamma\sqrt{4\pi}} e^{-\frac{(x_i-x_k)^2}{4\gamma^2}} \frac{x_k + x_i}{2} \frac{\sigma_i(x_i)}{\sigma_i(x_i) + \sigma_{-i}(x_k)} dx_k - \sigma_i(x_i). \tag{11}$$

From Theorem 1 we know that the bidding function of a rational player i is always bounded from above by a function that asymptotically behaves like $\frac{x_i}{4}$ where x_i is the signal seen by i. That is, rational players do not make abnormally high bids. We can then show (see Lemma 1 below and Lemma 2 in the Appendix) that the optimal response to such an upper bounded bidding functions is positive above threshold, differentiable, and monotonically increasing in the signal.
A necessary condition for a maximum in (11) is that the first variation of the right hand side of (11) vanishes, that is,

$$\int_{x_k=t}^{\infty} \frac{1}{\gamma\sqrt{4\pi}} e^{-\frac{(x_i-x_k)^2}{4\gamma^2}} \frac{x_k + x_i}{2} \frac{\sigma_{-i}(x_k)}{(\sigma_{-i}(x_k) + \sigma^*(x_i))^2} dx_k - 1 = 0. \tag{12}$$

Since the second derivative of that right hand side, that is, the first derivative of (12) w.r.t. $\sigma^*(x_i)$ is negative, for every x_i, there is at most one solution, and this solution is a maximum. Also, since that derivative is negative, (12) is a decreasing function of $\sigma^*(x_i)$, and since because of the asymptotics $\frac{x}{4}$, the left hand side of (12) is positive for small values of $\sigma^*(x_i)$ and negative for large ones, there has to be a unique solution. But this is not yet the equilibrium.

We now come to the main step of the proof. The main tool will be implicit differentiation of the identity (12) that determines $\sigma^*(x_i)$. The purpose is to analyze the effects of variations of the opponent's strategy σ_{-i}, of the parameter γ, and the value of the threshold t on $\sigma^*(x_i)$, and also compute the derivative of $\sigma^*(x_i)$ with respect to x_i. To see the scheme, we write (12) in the schematic form

$$F(\sigma^*(x_i); \sigma_{-i}, x_i, \gamma) = 0. \tag{13}$$

By the implicit function theorem (see e.g. [13]), $\sigma^*(x_i)$ will be a differentiable function of the various other quantities $(\sigma_{-i}, x_i, \gamma)$ as soon as $\frac{\partial F(\sigma^*(x_i);\sigma_{-i},x_i,\gamma)}{\partial \sigma^*(x_i)} \neq 0$, which will be verified in the Appendix, see Lemma 2. In particular, $\sigma^*(x_i)$ is a differentiable function of x_i. We then take total derivatives of the identity (47).

In particular, we can use this to compute the variation of $\sigma^*(x_i)$ with respect to a variation $\delta\sigma_{-i}$ of σ_{-i}. This gives

$$\frac{\partial F(\sigma^*(x_i); \sigma_{-i}, x_i, \gamma, t)}{\partial \sigma^*(x_i)} \frac{d\sigma^*(x_i)}{d\sigma_{-i}(x_k)} + \frac{\partial F(\sigma^*(x_i); \sigma_{-i}, x_i, \gamma, t)}{\partial \sigma_{-i}(x_k)} = 0. \tag{14}$$

We obtain

$$\delta\sigma^*(x_i) := \frac{d\sigma^*(x_i)}{d\sigma_{-i}} = \frac{\int_{x_k=t}^{\infty} \frac{1}{\gamma\sqrt{4\pi}} e^{-\frac{(x_i-x_k)^2}{4\gamma^2}} \frac{x_k+x_i}{2} \frac{\sigma^*(x_i)-\sigma_{-i}(x_k)}{(\sigma_{-i}(x_k)+\sigma^*(x_i))^3} \delta\sigma_{-i}(x_k) dx_k}{\int_{x_k=t}^{\infty} \frac{1}{\gamma\sqrt{4\pi}} e^{-\frac{(x_i-x_k)^2}{4\gamma^2}} \frac{x_k+x_i}{2} \frac{2\sigma_{-i}(x_k)}{(\sigma_{-i}(x_k)+\sigma^*(x_i))^3} dx_k}. \tag{15}$$

Here σ_{-i} is positive for a signal above 0. This formula has some interesting consequences. In all regions where $\sigma^*(x_i) > \sigma_{-i}(x_k)$, an increase of $\sigma_{-i}(x_k)$ induces also an increase of $\sigma^*(x_i)$ whereas when $\sigma^*(x_i) < \sigma_{-i}(x_k)$, an increase of $\sigma_{-i}(x_k)$ induces a decrease of $\sigma^*(x_i)$. (This, in fact, is the same phenomenon as we also observe in the game without uncertainty, see (4). When the opponent's bid increases above the equilibrium, the own bid decreases below the equilibrium, and conversely.) In particular, we can insert a Dirac delta function for $\delta\sigma_{-i}$ and see how a change of σ_{-i} at some point x_0 induces a change of $\sigma^*(x_i)$. The strength of the effect depends on the variance γ and the distance between x_0 and x_i.

We shall now show that we can apply a fixed point argument to show the existence and uniqueness of an equilibrium for each such γ. The numerator and the denominator in (15) differ by the term

$$\frac{\sigma^*(x_i) - \sigma_{-i}(x_k)}{2\sigma_{-i}(x_k)}, \tag{16}$$

This term is either > 0 or bounded in absolute value by $\frac{1}{2}$. We can now show the convergence of an iteration scheme. We let $\sigma_{n,*}$ be the optimal reaction to the opponent's action $\sigma_{n-1,*}$. Thus, for instance, the actions $\sigma_{2m,*}$ are those of player i, whereas $\sigma_{2m-1,*}$ are those of $-i$. The starting bid $\sigma_{0,*}$ is supposed to be positive and satisfy the bound (30). From our preceding reasoning, all subsequent bids then will also satisfy these bounds. The quantity $\delta\sigma^*(x_i)$ in (15) then becomes $\sigma_{2m+2,*} - \sigma_{2m,*}$, the change of i's bid in reaction to the change $d\sigma_{-i}$ by $-i$. The analogous formula obtains for $-i$, of course. Because of the absolute value bound $\frac{1}{2}$, the negative parts of $\sigma_{2m+2,*}-\sigma_{2m,*}$ form a Cauchy sequence that is controlled by a geometric series as in the proof of the Banach fixed point theorem (see for instance [13]). Thus, we need to consider only the positive part of the difference. Since we have the upper bound (30), the positive part then also needs to converge to 0 eventually, and the iteration scheme needs to converge towards some fixed point which then also has to be unique by the preceding reasoning.

This completes the proof of Theorem 2.

3 The strategic market game with coordination risk and the option to opt out after bid revelation

We shall now turn to the two stage game where investors place their bids in the first stage and then, after the bids are revealed, in the second stage decide whether to stay in or drop out. Since they know in the beginning that this second stage will occur, this will influence their bidding behavior in the first stage already. The resulting equilibrium will show some interesting effects. Compared to the one stage strategic market game with coordination under incomplete information, the threshold is lower, and bids are higher. In the two stage game, there are still large efficiency losses that stem from the coordination uncertainty, but they are weaker, compared to the one stage game.

The main technical step, the proof of the existence of a unique equilibrium, can be taken over from the one stage game, when we restrict the analysis to monotonic bidding strategies. This is needed only in order to establish the existence of a consistent bidding threshold. we start with the complete information version in order to clarify the structure of the game.

3.1 Two stage game with complete information

We now prepare for our final model by adding a coordination stage to the strategic market game. Again, we start with the complete information case.

3.1.1 Setup

The two players i and $-i$ now engage in a game that consists of two stages. The rules are common knowledge.

A version of the previous game now constitutes **Stage** 1 (Bidding). Each player (i and $-i$)

observes an exogenous fundamental value θ and then makes a non-negative bid, σ_i and σ_{-i}, respectively.

Stage 2: (Bid Relevation and Coordination and payoff consequences). Investors observe each others' bids. Investors then decide to stay invested (stay) or to withdraw their investment (not stay) with payoffs given by the following payoff matrix:

		Investor -i	
		not stay	stay
Investor i	not stay	$a - \sigma_{-i}$ $a - \sigma_i$	$\frac{\theta}{b} - \sigma_{-i}$ $a - \sigma_i$
	stay	$a - \sigma_{-i}$ $\frac{\theta}{b} - \sigma_i$	$\frac{\sigma_{-i}}{\sigma_i + \sigma_{-i}}\theta - \sigma_{-i}$ $\frac{\sigma_i}{\sigma_i + \sigma_{-i}}\theta - \sigma_i$

Table 2: Payoff matrix

The payoff matrix is identical to the payoff matrix of the one stage game. However, now the bids are determined in the first period and the second stage has a binary decision variable. Thus, bids enter as sunk costs. As before, when both players stay invested, the fundamental value θ is divided among both players in proportion to their bids as before, that is, the respective gains are $\theta\frac{\sigma_i}{\sigma_i+\sigma_{-i}}$ for player i and $\theta\frac{\sigma_{-i}}{\sigma_i+\sigma_{-i}}$ for player $-i$. If only one player stays, she receives a side-payoff of $\frac{\theta}{b}$. A player who does not stay receives an outside option payoff a, irrespectively of what the other player does.

We assume again $b > 8a$, to ensure that investing together always yields higher individual payoffs than investing alone.

3.1.2 Equilibrium analysis under complete information

When the players have complete information about θ, the game can be easily solved by backward induction. If both players chose to invest, then placed a bid and decided to stay invested, they receive the payoff from the right lower quadrant of table 2, as in subsection 2.1. In stage 1, the players compare the payoff from investing with the outside option a. If the former is bigger than the latter, agents choose to invest. As discussed earlier, by our construction agents have no incentive to unilaterally aim for the side-payoff, as it is dominated by the in-payoff. If $\frac{\theta}{4} < a$ the out-payoff dominates the in-payoff. For large θ, both the in and the side-payoff dominate the out-payoff. Thus, there has to exist a threshold point t above which agents choose to invest and below which they don't.

When $-i$ bids σ_{-i}, the threshold t for the signal above which i bids then is given by

$$t\frac{\sigma_i}{\sigma_i^* + \sigma_{-i}} - \sigma_i^* = a, \tag{17}$$

and inserting (3) yields

$$t = (\sqrt{\sigma_{-i}} + \sqrt{a})^2, \tag{18}$$

and the optimal bid (3) at this threshold then is

$$\sigma_i^*(t) = \sqrt{a\sigma_{-i}}. \tag{19}$$

Thus, when σ_{-i} increases, so does the bidding threshold (18), because the profit of i from her bid is decreased. However, her bid at that new threshold also increases, although at a smaller rate than that threshold itself. Analogously, when $-i$'s bid is decreased. When the players mutually optimize, that is, choose (5), from (17) the threshold is

$$t = 4a. \tag{20}$$

The isolated coordination game with known θ has two dominance regions: for $\theta\frac{\sigma_i}{\sigma_i+\sigma_{-i}} < a$ and $\frac{\theta}{b} < a$, with the $(notstay, notstay)$ equilibrium and for $\theta\frac{\sigma_i}{\sigma_i+\sigma_{-i}} > a$ and $\frac{\theta}{b} > a$, the $(stay, stay)$ equilibrium, and a middle region of parameters in which it is a standard stag hunt game with the two pure equilibria $(stay, stay)$ and $(notstay, notstay)$.

Dominance regions and multiplicity of equilibria under complete information

If we combine the coordination stage with the strategic market game, the two players will bid zero and choose $notstay$ which is a dominant strategy, whenever $\theta < 4a$, the threshold for positive bidding. When θ is larger than $4a$, there are two Nash equilibria for $\frac{\theta}{b} < a$ as in the coordination games with strategic complementarities: One is that both players bid zero and do not invest as in the previous result, and one is that both players bid $\frac{\theta}{4}$, as in the isolated bidding game with the same payoffs, and $stay$. In the second equilbrium, the market price is $\frac{\theta}{2}$, as in the standard strategic market game under complete information.

We should point out, however, that while $(notstay, notstay)$ is an equilibrium of the subgame defined by the coordination game, the strategy to first bid and then drop out is not an equilibrium of the full game, because it is dominated by not bidding in the first stage.

In the sequel, we shall see that adding uncertainty about the payoffs by introducing noisy private observations of θ removes the above multiplicity and leads to the emergence of a unique perfect Bayesian equilibrium in the class of monotone strategies, a phenomenon known in the setting of global games [8, 21].

3.2 Two stage game with incomplete information

We shall now discuss the two stage game of bidding and coordination with incomplete information. This is the main game we study in this paper. This brings all components together: strategic market game, coordination, and noise. We will see that introducing noise to the two stage game of bidding and coordination has profound effects: The emergence of a unique equilibrium, and the coordinational discount effect for the bids.

3.2.1 Setup

In contrast to the complete information benchmark, stage 1 now consists of a strategic market game with noisy private signals x_i and x_{-i} about an unknown fundamental value θ. In stage 2, as before, the bids σ_i and σ_{-i} are revealed, and agents decide whether to stay invested or withdraw (stay or not stay). The resulting payoffs are given in Table 2, and the rules and payoffs are common knowledge.

Stage 1 (Signals and bidding). As before, nature draws θ, and each investor independently receives a private signal about θ,

$$x_i = \theta + \epsilon_i, \quad x_{-i} = \theta + \epsilon_{-i} \tag{21}$$

where ϵ_i and ϵ_{-i} are independently drawn from the normal distribution $N(0, \gamma)$, with γ being common knowledge. Then investors make non-negative bids σ_i and σ_{-i} on the basis of their own signals and their common knowledge about the structure of the game.

Stage 2: (Bid relevation and coordination and payoff consequences). Investors observe each others bids. Investors then decide to stay invested (stay) or to withdraw their investment (not stay) with payoffs as before.

For technical reasons, we assume $\gamma < 8a\sqrt{\pi}$. This assumption does not really restrict our game, as it still allows for rather large variances.

3.2.2 Discussion of numerical plots of the bidding function

Before proceeding with the formal proof of the unique perfect Bayesian equilibrium for monotone strategies and the discount effect, we present some numerical results about the shape of the bidding function to shed light on some of the questions raised in the introduction.

In the standard strategic market game, adding payoff uncertainty by noisy observations of the fundamental value does not have dramatic effects. The bidding function is still close to the complete information benchmark (see figure 1), and there is only a small informational discount in bids. We will discuss the origin of this small informational discount effect in subsection 3.6.

In contrast, when we add the noise to the full two stage model, we see a clear and distinct effect of the noise on bidding behavior. Figure 2 shows a numerical plot of the bidding function with parameters $a = 1$ and $\gamma = 1$, with the dashed line indicating the equilibrium bidding strategy of interest for the two stage games under complete information.[1]

Three effects can be read from figure 2: First, the equilibrium bidding function approximates $\frac{x_i}{4}$ asymptotically for large signals. Second, near the threshold, we see a significant discount effect, with a bid close to $\frac{1}{2}\frac{x_1}{4}$. We refer to this as the coordinational discount effect. Third, the threshold of investing shifts to the right[2]. We will here provide an intuition for these three effects. Effect two

[1] Figures 2 and 3 were generated by a program that plays iterated best responses implemented in Mathematica©.

[2] This effect holds in general for $\gamma < 3a$

Figure 2: Plot of Bidding Function (γ = 1 and a = 1)

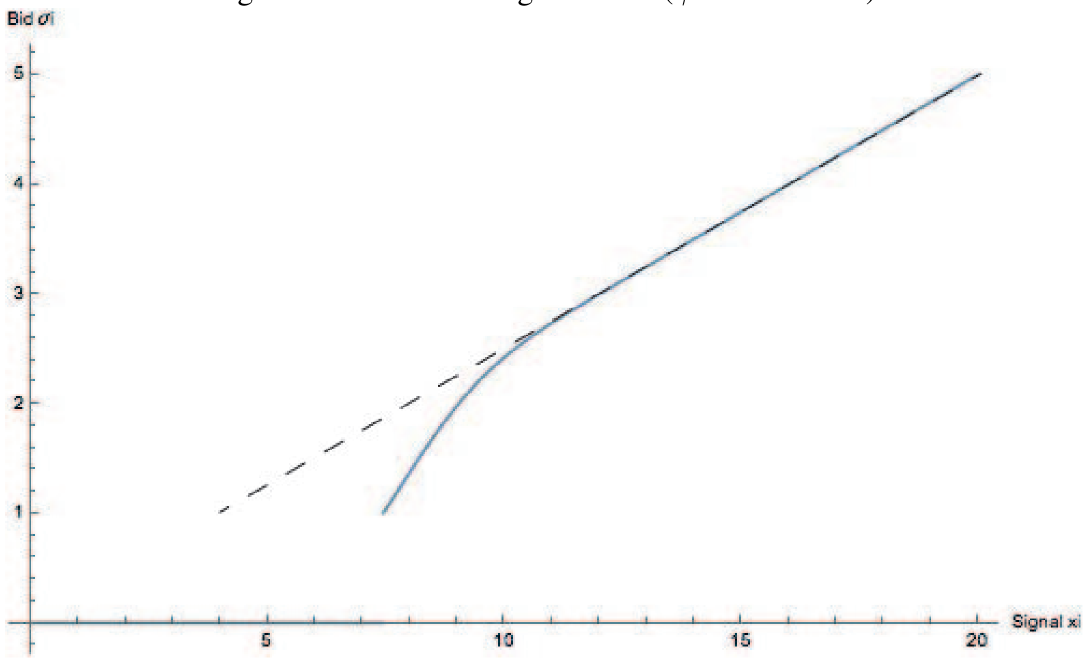
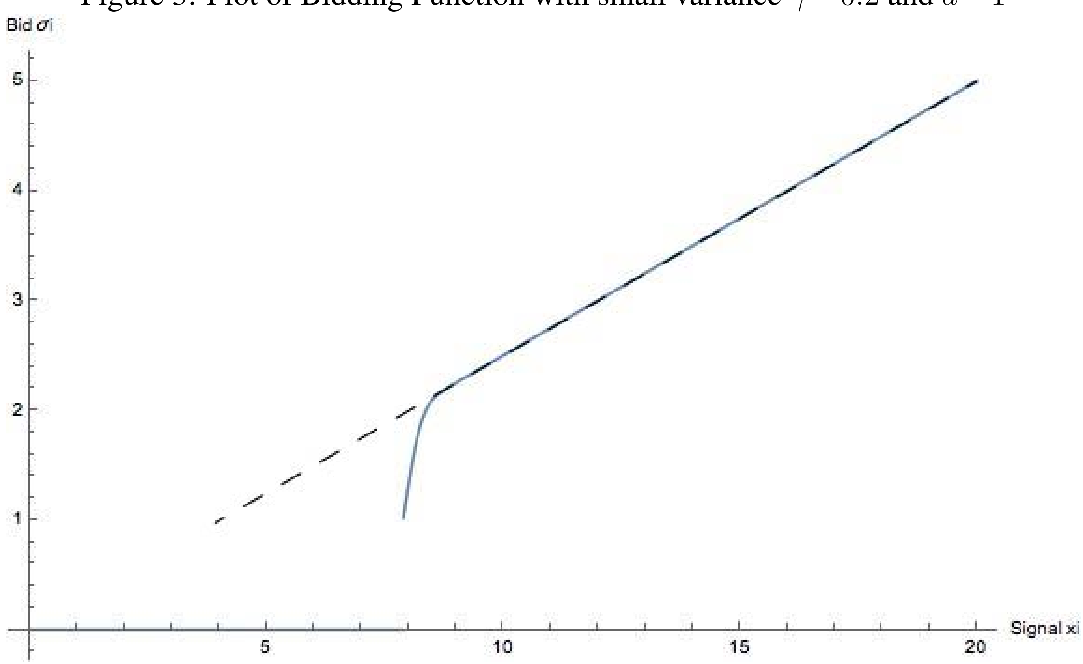


Figure 3: Plot of Bidding Function with small variance γ = 0.2 and a = 1



(discount of bids near the threshold) and effect three (the threshold shifts to the right) are related. As in global games, given a signal equal to the threshold, the probability that the other player observed a signal at or above the threshold is exactly $\frac{1}{2}$. Thus, the agent who receives a signal at or above but near the threshold (which by definition implies that she will invest) needs to account for the possibility that the other agent will not join the investment. The expected value of the investment thus decreases. This is reflected in lower bids. Instead of $\frac{x_i}{4}$, agents bid less. The first intuition would be that agents should bid $\frac{x_i}{8}$ at the threshold, but that is only approximately correct, as we shall show below. The fact that the isolated bidding game with noisy signals produces bids similar to the complete information game (see plot (1)) suggests that the significant coordinational discount and the threshold shift must have a different origin than mere incomplete information in strategic market games. Indeed, the origin comes from a combination of drastic loss of value of the project (from θ to $\frac{\theta}{b}$), once only one investor bids, and payoff uncertainty. Especially near the threshold, the investors face the uncertainty that the project looses value, if the other investor observes a signal below the threshold and therefore does not invest. Thus, the expected total value is smaller and therefore optimal bids are lower. For a specific x_i, the expected value of an investment decreases compared to the expected value of the investment given $\theta = x_i$ in the complete information game. At the threshold, as we shall discuss, the expected value of investing equals the outside option a. Thus, in the game with incomplete information, the threshold increases, to compensate for this decrease, so that the expected gain from investment again equals the safe outside option a.

The first effect (that the bidding function in the game with incomplete information is asymptotically equivalent to the bidding function in the game with complete information) is a result of the assumption of observations of θ with Gaussian noise. If a player receives a signal that is much larger than the threshold, the exponential decline of the Gaussian noise distribution implies that it is very unlikely that the other player received a signal below the threshold. In effect, the bids almost correspond to the complete information benchmark. This is illustrated in figure (3). The dashed line again indicates the equilibrium bidding strategy for the complete information game. With small variance, as predicted, the bidding function climbs very steeply from the threshold to the almost linear curve.

3.3 Main theorem

Theorem 3. *In the specified game with incomplete information, there is a unique perfect Bayesian equilibrium in the class of strictly monotone bidding strategies.*

The main step of the *proof* is the same as that of Theorem 2. We need to confine ourselves to *monotonic* strategies, that is, where the bids are strictly monotonically increasing functions of the signals received. This assumption is consistent at equilibrium. The assumption of strictly monotone strategies was not necessary for the one stage variant of our game, because there, monotone strategies can be deduced from the game structure.

As a consequence of this monotonicity, bids fully reveal signals. More precisely, after bids are placed, each agent can read the signal of the other agent from the market price, as the equilibrium bid function is an injective mapping from signals to bids.

We need to establish some general properties of the game. First of all, ex ante, before bidding, some strategies are simply ruled out by the perfect Bayesian equilibrium concept, because they are dominated already ex ante, before bidding. For instance, the strategy to bid and withdraw after observing the other agent does *also* bid is dominated by the strategy to not bid and not stay. Thus, even though there are two pure equilibria in the subgame consisting only of the second stage, for the case that both agents had bid upon signals just above the threshold, one of them is not part of an equilibrium for the whole game.

3.4 The emergence of a bidding threshold

We shall now discuss that also in the two stage game, a threshold, that is, a signal t emerges endogenously from the game definition, so that agents do not invest if their signal is below t. Ex ante (before bidding), both agents reason about what happens at period 2 (coordination). Both agents know that for some threshold t, if $x_i < t$, payoff a is better than the 'stay/stay' or 'stay/not stay' payoff. On the other hand, they also know that there is some signal for which the expected value of θ is so high that the in and side payoffs are both better than a. Thus, there is a lower dominance region of signals for which staying out is the best strategy, and an upper dominance region for which investing (a positive bid) and staying in is better than bidding zero. Moreover, due to the symmetry of the game, we can assume that expected payoffs grow strictly monotonically in the signals. Thus, there must be a threshold t above which non zero bids are played.

Since $b > 8a$, agents have no incentives to invest unilaterally, unless the signal is very high. Therefore, we can assume that the agents ex ante agree on the threshold.

If both agents invest, they will both stay invested. If agent i observes a higher bid $\sigma_{-i} > \sigma_i$ of her opponent, the proportional share of θ, that is, $\frac{\sigma_i}{\sigma_i+\sigma_{-i}}$ decreases compared to the ex ante expectation ($\frac{1}{2}$), but the expected value of θ increases at the same time, and this compensates for the smaller share of that value. Thus, i will stay invested, and so will $-i$.

The interesting case for stage 2 arises if only one agent, say i, invests, and the other agent bids zero. Agent i can compute m, the value of x_i ($m > t$) below which it is better to withdraw (the out-payoff with sunk cost, that is $a - \sigma_i$, is better than the side-payoff). Below m, i withdraws, above, i stays. For the subsequent analysis, we do not need to know m explicitly (although it can be easily computed numerically), but only need to know that such a point m exists. Nevertheless, we shall now specify the condition for the switching point m.

If agent i solely invests and observes that $-i$ did not invest, i knows that $-i$ observed a signal below the threshold. Thus, i can compute a posterior belief about the value of θ by giving equal weight to her own observation and that of her opponent:

$$E(\theta|x_i, x_{-i} < t) = \frac{1}{2}x_i + \frac{1}{2}\int_{x=-\infty}^{t} x(\frac{1}{\gamma\sqrt{4\pi}})e^{-\frac{(x_i-x)^2}{4(\gamma)^2}}\,dx \tag{22}$$

To determine the switching point m, the player compares the side with the out-payoff, given her posterior regarding the value of θ. The point m of indifference then is determined by

$$\frac{E(\theta|m, x_{-i} < t)}{b} = a \tag{23}$$

The difference to the one stage version is that an agent can withdraw after having seen the opponent's bid, and so can the opponent. Agent i, however, need not worry about $-i$'s withdrawal. If i did not bid in the first place, then $-i$ might withdraw upon learning that, but i wouldn't care. When $-i$ sees that i had bid more than him, then while his share of the profit decreases, the value of the profit increases at the same time. Although this is more subtle in the full game, assume for simplicity that each agent bids one quarter of her/his signal. Before learning about i's signal and therefore also about her bid, $-i$ expects a value of x_{-i}, bids $\frac{x_{-i}}{4}$ and expects i to bid the same, and therefore he expects the profit $\frac{x_{-i}}{4}$. After inferring x_i from i's bid $\frac{x_i}{4}$, the expected value then is $\frac{x_i+x_{-i}}{2}$, and the expected profit is $\frac{x_i+x_{-i}}{2}\frac{x_{-i}}{x_i+x_{-i}} - \frac{x_{-i}}{4} = \frac{x_{-i}}{4}$, that is, the same as before. Therefore, $-i$ has no reason to withdraw after learning about i's bid. The same applies when $-i$ realizes that i has bid less than him. Therefore, i need not worry about $-i$ withdrawing after learning about her bid. The only difference for i now is that she herself can withdraw if she learns that $-i$ did not bid in the first round. More precisely, when $-i$ did not bid, but i did on the basis of her signal x_i, the signal of $-i$ should have been lower than the threshold, and therefore also the expected profit should be smaller than $\frac{x_i+t}{2b}$ minus the sunk costs, and if that former number is smaller than a, i should better withdraw. Thus, i in the two stage version is in a better position than in the one stage game, and this lowers the bidding threshold.

As we have already argued, if both agents had forgotten the history of playing and considerations made ex ante, before bidding, there would be multiplicity in the second stage if both agents invested upon observations of signals between t and m. As $a - \sigma_i$ then dominates $\frac{\theta}{b} - \sigma_i$ (and the analog holds for player $-i$) the payoffs of the second stage have the structure of a coordination game. Both signals x_i and x_{-i} are now common belief. Thus, there would be multiplicity in this subgame. However, this strategy is ruled out by the Perfect Bayesian Equilibrium as it is dominated by the strategy not to bid and then not to stay, because in that case there would be no sunk costs.

3.5 Optimal responses

With the threshold established, we consider optimal responses to an opponent's bidding function as before, see (8).

$$\begin{aligned}\bar{\sigma}_i(x_i) &= \underset{\sigma_i(x_i)}{\operatorname{argmax}} \int_{-\infty}^{\infty} [\int_{x_k=t}^{\infty} \frac{\sigma_i(x_i)}{\sigma_i(x_i) + \sigma_{-i}(x_k)} \theta p(x_k|\theta) dx_k + \int_{-\infty}^{x_k=t} a p(x_k|\theta) dx_k] p(\theta|x_i) d\theta - \sigma_i(x_i) \\ &= \underset{\sigma_i(x_i)}{\operatorname{argmax}} \int_{-\infty}^{\infty} \int_{x_k=t}^{\infty} \frac{\sigma_i(x_i)}{\sigma_i(x_i) + \sigma_{-i}(x_k)} \theta p(x_k|\theta) dx_k p(\theta|x_i) d\theta - \sigma_i(x_i)\end{aligned} \tag{24}$$

is the same as

$$\bar{\sigma}_i(x_i) = \underset{\sigma_i(x_i)}{\operatorname{argmax}} \int_{-\infty}^{\infty} [\int_{x_k=t}^{\infty} \frac{\sigma(x_i)}{\sigma_i(x_i) + \sigma_{-i}(x_k)} \theta p(x_k|\theta) dx_k + \int_{-\infty}^{x_k=t} \frac{\theta}{b} p(x_k|\theta) dx_k] p(\theta|x_i) d\theta - \sigma(x_i) \tag{25}$$

because whichever pay-off results from cases when the opponent has a signal below threshold, this no longer depends on the own bid. This therefore leads to

$$\bar{\sigma}_i(x_i) = \underset{\sigma_i(x_i)}{\operatorname{argmax}} \int_{-\infty}^{\infty} \int_{x_k=t}^{\infty} \frac{\sigma_i(x_i)}{\sigma_i(x_i) + \sigma_{-i}(x_k)} \theta p(x_k|\theta) dx_k p(\theta|x_i) d\theta - \sigma_i(x_i) \tag{26}$$

Using the assumption about the distribution of the signals we made above (see (21)) we can simplify the equation for the best reply condition, see (8).

$$\begin{aligned}
\bar{\sigma}_i(x_i) &= \underset{\sigma_i(x_i)}{\operatorname{argmax}} \int_{-\infty}^{\infty} \int_{x_k=t}^{\infty} \frac{\sigma_i(x_i)}{\sigma_i(x_i) + \sigma_{-i}(x_k)} \theta p(x_k|\theta) dx_k p(\theta|x_i) d\theta - \sigma_i(x_i) \\
&= \underset{\sigma_i(x_i)}{\operatorname{argmax}} \int_{x_k=t}^{\infty} \frac{\sigma_i(x_i)}{\sigma_i(x_i) + \sigma_{-i}(x_k)} \int_{-\infty}^{\infty} \theta p(x_k|\theta) p(\theta|x_i) d\theta dx_k - \sigma_i(x_i) \\
&= \underset{\sigma_i(x_i)}{\operatorname{argmax}} \int_{x_k=t}^{\infty} \frac{\sigma_i(x_i)}{\sigma_i(x_i) + \sigma_{-i}(x_k)} \int_{-\infty}^{\infty} \theta (\frac{1}{\gamma\sqrt{2\pi}})^2 e^{-\frac{1}{2}[(\frac{x_k-\theta}{\gamma})^2 + (\frac{\theta-x_i}{\gamma})^2]} d\theta dx_k - \sigma_i(x_i) \\
&= \underset{\sigma_i(x_i)}{\operatorname{argmax}} \int_{x_k=t}^{\infty} (\frac{1}{\gamma\sqrt{4\pi}}) e^{-\frac{(x_i-x_k)^2}{4\gamma^2}} \frac{\sigma_i(x_i)}{\sigma_i(x_i) + \sigma_{-i}(x_k)} \int_{-\infty}^{\infty} \theta \frac{1}{\gamma\sqrt{\pi}} e^{-(\frac{\theta - (\frac{x_i+x_k}{2})}{\gamma})^2} d\theta dx_k - \sigma_i(x_i) \\
&= \underset{\sigma_i(x_i)}{\operatorname{argmax}} \int_{x_k=t}^{\infty} (\frac{1}{\gamma\sqrt{4\pi}}) e^{-\frac{(x_i-x_k)^2}{4(\gamma)^2}} (\frac{x_k+x_i}{2}) \frac{\sigma_i(x_i)}{\sigma_i(x_i) + \sigma_{-i}(x_k)} dx_k - \sigma_i(x_i)
\end{aligned} \tag{27}$$

As the agents do not know the bid of the other player when they bid, they must integrate over the possible bids of the opponent, given his possible signals. Equation (27) gives us the mutual optimisation constraints for the bidding function (a symmetric equation for $-i$ is simply given by replacing all i's with $-i$'s and vice versa).

A necessary condition for a maximum in (27) is that the first variation of the right hand side of (27) vanishes, that is, we again get the condition (9),

$$\int_{x_k=t}^{\infty} \frac{1}{\gamma\sqrt{4\pi}} e^{-\frac{(x_i-x_k)^2}{4\gamma^2}} \frac{x_k+x_i}{2} \frac{\sigma_{-i}(x_k)}{(\sigma_{-i}(x_k) + \sigma^*(x_i))^2} dx_k - 1 = 0. \tag{28}$$

Since the second derivative of that right hand side, that is, the first derivative of (28) w.r.t. $\sigma^*(x_i)$ is negative, for every x_i, there is at most one solution, and this solution is a maximum. Also, since that derivative is negative, (28) is a decreasing function of $\sigma^*(x_i)$, and since because of the asymptotics $\frac{x}{4}$, the left hand side of (28) is positive for small values of $\sigma^*(x_i)$ (in case $x_i > t$) and negative for large ones, there has to be a unique solution.

3.6 Upper bound: discount effect

We now observe that this solution, that is, the optimal bid of each player is always bounded, regardless of the opponent's strategy against which the player is optimizing, as in the case of the single stage game. We only need to replace the lower integration bound 0 by the threshold t. As for Theorem 1, $\frac{s}{(s+\sigma)^2}$ for a given σ is maximal when $s = \sigma$. And for $s = \sigma$, we have $\frac{s}{(s+\sigma)^2} = \frac{1}{4\sigma}$. Using this in (28) yields

$$\int_{x_k=t}^{\infty} \frac{1}{\gamma\sqrt{4\pi}} e^{-\frac{(x_i-x_k)^2}{4\gamma^2}} \frac{x_k + x_i}{2} dx_k > 4\sigma^*(x_i), \tag{29}$$

hence we achieve the following upper bound:

Theorem 4 (Upper bound/ discount effect)**.** *The optimal bid of player i is bounded from above by*

$$\sigma^*(x_i) < \frac{1}{4} \int_{x_k=t}^{\infty} \frac{1}{\gamma\sqrt{4\pi}} e^{-\frac{(x_i-x_k)^2}{4\gamma^2}} \frac{x_k + x_i}{2} dx_k = \frac{1}{4} \left(\frac{\gamma}{\sqrt{4\pi}} e^{-\frac{(x_i-t)^2}{4\gamma^2}} + x_i \int_{x_k=t}^{\infty} \frac{1}{\gamma\sqrt{4\pi}} e^{-\frac{(x_i-x_k)^2}{4\gamma^2}} dx_k \right). \tag{30}$$

For $\gamma \to 0$ and $x_i > t$, this converges to $\frac{x_i}{4}$, which is the value in the game without uncertainty. And for any $\gamma > 0$, for sufficiently large x_i, this bound behaves like $\frac{x_i}{4}$. This makes the above asymptotic upper bound self-consistent.

The first term in the bracket on the right hand side is largest for $x_i = t$, in fact equal to $\frac{\gamma}{\sqrt{4\pi}}$ whereas the integral takes the value $\frac{1}{2}$ at $x_i = t$. Therefore, we get an upper bound of the bidding function by $\frac{x_i}{4}$ as long as $\gamma < \sqrt{\pi}t$, that is, for variances that are not too large. Since the inequality is strict, Theorem 4 formally proves the discussed discount effect in bids.

To further explore Theorem 4, we shall now construct a simplified game in which that upper bound for the equilibrium bid turns into the equilibrium bid itself. According to the derivation of (29), we simply need to construct a situation where the bid of the opponent equals the own bid while the total value is still distributed as a Gaussian with variance $2\gamma^2$ above a threshold t. We shall now proceed to do so. We will call this game 'simplified game' for convenience. To construct the simplified game, we simply move the bids of players i and $-i$ out of the integral in the best reply condition. That is, the players still asume that θ is distributed according to a Gaussian above a cut-off point t, but that the opponent assumes for the bid the same value x_i as i herself. That is, both players assume the same noisy signal $x_{-i} = x_i$. Then we achieve the following optimization condition for player i:

$$\bar{\sigma}_i(x_i) = \underset{\sigma_i(x_i)}{\operatorname{argmax}} \int_{x_k=t}^{\infty} (\frac{1}{\gamma\sqrt{4\pi}}) e^{-\frac{(x_i-x_k)^2}{4\gamma^2}} (\frac{x_k + x_i}{2}) dx_k \frac{\sigma_i(x_i)}{\sigma_i(x_i) + \sigma_{-i}(x_i)} - \sigma_i(x_i)$$

For player $-i$, the corresponding best reply condition is:

$$\bar{\sigma}_{-i}(x_{-i}) = \underset{\sigma_{-i}(x_{-i})}{\operatorname{argmax}} \int_{x_k=t}^{\infty} (\frac{1}{\gamma\sqrt{4\pi}}) e^{-\frac{(x_{-i}-x_k)^2}{4\gamma^2}} (\frac{x_k + x_{-i}}{2}) dx_k \frac{\sigma_{-i}(x_{-i})}{\sigma_{-i}(x_i) + \sigma_i(x_{-i})} - \sigma_{-i}(x_{-i})$$

If we now simply assume that $x_i = x_{-i}$, and that this value is common knowledge, the best reply conditions are equivalent to best reply conditions in a standard complete information strategic market game with common knowledge of the total value to be distributed among the players according to their bids being

$$W(x_i) = \int_{x_k=t}^{\infty} (\frac{1}{\gamma\sqrt{4\pi}}) e^{-\frac{(x_i-x_k)^2}{4\gamma^2}} (\frac{x_k + x_i}{2}) dx_k. \tag{31}$$

That is, instead of private signals, the players now observe a noisy public signal.

This game has a unique equilibrium, which is

$$\frac{W(x_i)}{4} \tag{32}$$

.

This is, as we have discussed for the complete information benchmark of the two stage game above, a standard result (Makowski and Ostroy, 1992).

In this simplified game, we can explicitly compute the threshold t above which the two players start to bid. If we simply equate the expected payoff while bidding at the threshold with the outside option a, we obtain:

$$a = \frac{\int_{x_k=t}^{\infty} (\frac{1}{\gamma\sqrt{4\pi}}) e^{-\frac{(t-x_k)^2}{4(\gamma)^2}} (\frac{x_k+t}{2}) dx_k}{4}$$

which yields $t = 8a - \frac{\gamma}{\sqrt{\pi}}$.

We shall now visualize the total value $W(x_i)$, as this will help us to understand the origin of the coordinational discount effect in bids. The integral (see equation (31)) starts at t, while the normalisation factor $\gamma\sqrt{4\pi}$ equals the integral of the Gaussian over the entire real line. If the variance is small, in approximation, agents play a complete information strategic market game with total value $\int_{x_k=t}^{\infty} (\frac{1}{\gamma\sqrt{4\pi}}) e^{-\frac{(x_i-x_k)^2}{4\gamma^2}} (\frac{x_k+x_i}{2}) dx_k$. We now look at the resulting geometry.

Looking at figure (4), x_k is integrated over a Gaussian density normalized relative to the entire real line. Observe that in the above case, we *loose* support of positive x_k's: *Agents bid less than in the complete information reference case.*

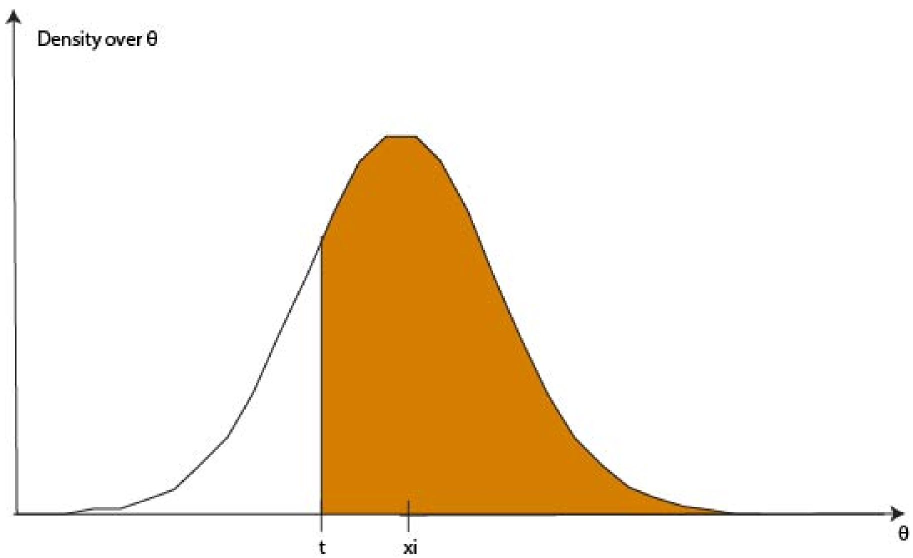


Figure 4: Coordinational discount effect

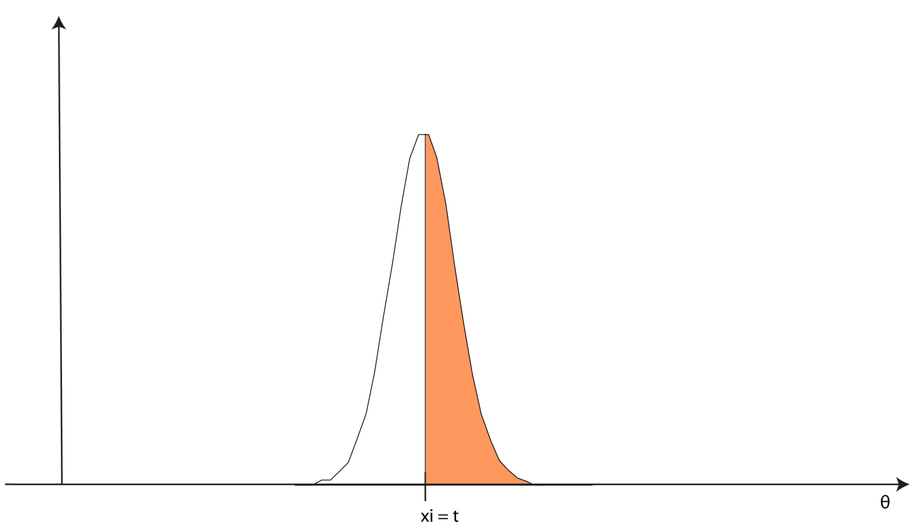


Figure 5: Strong coordinational discount effect at threshold

From figure (5) we see that this effect is most extreme at $x_i = t$. x_k is integrated over a Gaussian density normalised relative to the full real line. Here, we *loose* support of positive x_k's: Agents bid roughly half of what they would in the complete information reference case.

For large γ, however, the bound (30), which, to recall, is equal to the equilibrium bid in the simplified game, is not necessarily smaller than this value $\frac{x_i}{4}$. The intuitive reason is that i is considering the possibility that $-i$ has received a much larger signal x_k than her own signal x_i, and that therefore the true value is also significantly larger than x_i, and in this situation, she should then also place a higher bid. The larger γ, the more pronounced this effect can become, and this is reflected by the first term on the right hand side of (30).

However, note that Theorem 4 turns the equilibrium bid of the simplified game into an upper bound for the two stage game under incomplete information. Thus, the coordinational discount effect for the simplified game implies such an effect for the two stage game under incomplete information as well.

By comparing equations (30), (32) and (31) we can isolate and disentangle informational and coordinational discount effects. The equilibrium bid in the two stage game, according to (30), is lower than in the simplified game. We discussed the conceptual origin of the coordinational discount effect above, a decrease 'in size of the value pie'. In the two stage game, this effect is still present, but combined with the informational discount effect. Compared to the coordinational discount effect near the threshold, the informational discount effect is small unless γ is large.

This can be seen by making yet another comparison. In a standard strategic market game without coordination uncertainty, but with incomplete information, we can observe the informational discount effect isolated from the coordinational discount effect. In Figure 1, where we have computed the equilibrium bidding strategy in this standard strategic market game with noisy observations of θ with a standard deviation of $3a$.[3] For comparison, the dashed line indicates the equilibrium bidding strategy $\frac{\theta}{4}$ under complete information. The informational discount arises because i when observing x_i does not know which signal $-i$ has observed, but can only assume that it is Gaussian distributed around x_i. And according to (3) when the opponent does not bid what is optimal from the perspective of i, given her estimate x_i of θ, the optimal response of i is lower than $\frac{x_i}{4}$. Thus, the discount arises from the uncertainty about the other's private signal.

3.7 Proof of the existence and uniqueness of an equilibrium

Having established the properties of the threshold t, we can prove the existence and uniqueness of a Bayesian equilibrium for pure strategies in the two stage game under incomplete information, that is, our main result Theorem 3, as in the corresponding one stage game, see Theorem 2. We only need to use Theorem 4 in place of Theorem 1.

[3] The plot was generated by a program that plays iterated best replies implemented in Mathematica©.

3.8 Asymptotic behavior of the bidding function

In this section, we analyze the shape of the equilibrium curve to understand the influence of the other parameters. We also quantify the threshold effect.

We first consider the **asymptotics**: With the substitution

$$y = \frac{x_k - x_i}{\gamma}, \text{ i.e., } x_k = x_i + \gamma y, \tag{33}$$

(28) becomes

$$\int_{y=\frac{t-x_i}{\gamma}}^{\infty} \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}} \frac{2x_i + \gamma y}{2} \frac{\sigma_{-i}(x_i + \gamma y)}{(\sigma_{-i}(x_i + \gamma y) + \sigma^*(x_i))^2} dy = 1. \tag{34}$$

With the assumption

$$\sigma(x) \to \lambda_\infty x \text{ for } x \to \infty \tag{35}$$

we obtain for $x_i \to \infty$

$$\lambda_\infty = \frac{1}{4}. \tag{36}$$

Thus, asymptotically, we get the same bidding function $\sigma(x) = \frac{x}{4}$ as for the game without uncertainty.

For $\gamma \to 0$, for $x_i > t$ the limiting equation of (34) is

$$\int_{y=-\infty}^{\infty} \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}} x_i \frac{\sigma_{-i}(x_i)}{(\sigma_{-i}(x_i) + \sigma^*(x_i))^2} dy = 1,$$

which we can solve for

$$\sigma^*(x_i) = \sqrt{x_i \sigma_{-i}(x_i)} - \sigma_{-i}(x_i)$$

as in (3). The equilibrium condition $\sigma(x_i) := \sigma^*(x_i) = \sigma_{-i}(x_i)$ then leads to

$$\sigma(x_i) = \frac{x_i}{4}, \tag{37}$$

which is the solution (5) of the game without uncertainty. Actually, as we show in the Appendix (see Section 5.1), the situation at the threshold is somewhat subtler.

In fact, since we show in the Appendix that $\lim_{\gamma\to 0} \frac{d\sigma^*(x_i)}{d\gamma} \neq 0$ for $x_i > t$ (see (57)), there is no bifurcation at $\gamma = 0$, and the equilibrium solution (37) of the limiting equation is the limit of the solutions of (34) for $\gamma \to 0$ (see for instance [14]), except at the threshold itself.

3.9 Comparison of the one and two stage variants

To isolate the origin of the novel effect predicted by our model, the coordinational discount effect, we compare our full game with the one-stage variant. In that variant, bids are not revealed during the game, and so, as we have explained, the agents only specify an action at one point in time, by placing a bid, or refraining from doing so, opting for the secure outside option a. As we have shown, in this one stage game agents will use the same implicit functional form of the bidding function (given a fixed threshold t) as in the full two stage game that we specified. The coordinational discount effect in bidding stems from the coordination uncertainty that exists while agents bid: each agent knows that the other agent may not engage in bidding, which would lead to an unfavourable situation for them. While agents bid, they face strategic uncertainty; and this creates the coordinational discount effect in bids. The main difference is that there are efficiency losses in such a one stage game since a player does not know whether the other player has placed a bid and cannot withdraw a bid. Nevertheless we show that the functional form of the best reply condition and therefore the qualitative properties of the bidding function are the same as in the two stage game. However, since agents do not opt out conditional on whether the other player has bid or not, this game has only one threshold, ie. when to start with positive bidding. In our two stage game we show there is also an upper threshold from which there is a dominance regions where one stays invested independent of whether the other player has invested. Moreover, the threshold in the one shot variant of our game can be higher than in the two stage game, as expected losses may increase without the option to opt out if the other player does not invest. The reduced game resembles [11] as it is a global game, i.e. a noisy game with dominance regions, with a continuous bidding strategy. However, while in [11], the game contains only features of strategic complementarity, our game has both, strategic substitutes because of the strategic market game bidding outcome as well as strategic complements.

4 Discussion and Conclusion

We construct a simple game that combines bidding in a strategic market game with a coordination risk under incomplete information resulting in a global game with continuous bidding strategies. Therefore our game contains strategic complements and substitutes at the same time. We introduced a one stage and a two stage version of the game distinguished through a bid revelation and the possibility to withdraw bids in the two stage game. The combination of these features creates technical difficulties. The coordination part introduces a bidding threshold that prevents us from directly applying the fixed point arguments that are usually employed in the Tullock game literature. Because of the combination of these effects, before we can apply a fixed point argument, we need to develop a new scheme and work with variational arguments for proving the existence and uniqueness of the equilibrium.

Our modelling provides two major advances. On an applied level, it yields new predictions about bidding behaviour given coordination in the same market. We demonstrate that coordination uncertainty dampens competition in the bidding process. Second, on a technical level, we propose mathematical tools that allow us to tackle games with continuous strategies, strategic payoff complements *and* strategic substitutes, while payoff uncertainty is modelled by Gaussian noise.

New predictions about bidding behaviour given simultaneous coordination Our model predicts a distinct effect on bidding that can easily be tested experimentally. It predicts a threshold below which one does not bid, which does not occur in a strategic market game without the coordination risk. Furthermore, bids become lower than when coordination risks are absent, in particular for signals near the threshold, that is, when the expected project value is low. This is because agents account for the loss of expected gains due to the coordination uncertainty – whether the other agent will invest or not. These thresholds are higher and the bids are lower in the one stage variant as compared to the two stage variant. In the one stage variant, a player cannot withdraw her bid in case the other player has not placed a bid. Thus, introducing bid observation and a possibility to then withdraw increases efficiency measured by the lower threshold of bidding and increases competition through higher bidding.

Our contribution from a mathematical point of view: We show that noisy private signals with a commonly known distribution can restore the *uniqueness* of the equilibrium in coordination games with strategic substitutes. Previously, this had been known only in games with strategic complements [11]. Conversely, we show the existence and uniqueness of an equilibrium in a strategic market game with a superimposed noisy coordination game in the absence of a contraction or a compactness property that would enable us to directly apply a fixed point theorem. Again, this is new when compared to the existing literature, e.g. [10, 26]. For that purpose, we first derive an upper bound for the optimal responses to the opponent's bids. With that bound at hand, variational methods based on implicit differentiation of the optimal response condition will yield us enough constraints on optimal responses, like smooth and monotonic dependence on the signal, and a quantitative assessment of the optimal reaction to changes in the opponent's bidding function, to then derive the existence of a unique equilibrium. On the way, we also observe a novel effect, as mentioned, a coordinational discount for the optimal bids in the vicinity of the bidding threshold.

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References

[1] Admati, A.R. and Perry, M., 1991. Joint projects without commitment. The Review of Eco-

nomic Studies, 58(2), pp.259-276.

[2] Angeletos, G.M., Hellwig, C. and Pavan, A., 2007. Dynamic global games of regime change: Learning, multiplicity, and the timing of attacks. Econometrica, 75(3), pp.711-756.

[3] Angeletos, G. M. , C.Lian, Incomplete Information in Macroeconomics: Accommodating Frictions in Coordination, in: J.Taylor, H.Uhlig (eds.), Handbook of Macroeconomics, Vol. 2. , pp.1065–1240

[4] Angeletos, G. M. , Werning, I., 2006. Crises and Prices: Information Aggregation, Multiplicity, and Volatility. American Economic Review, 96(5), 1720-1736.

[5] Athey, S., 2001. Single crossing properties and the existence of pure strategy equilibria in games of incomplete information. Econometrica, 69(4), pp.861-889.

[6] Atkeson, A., 2000. Discussion of Morris and Shin's 'Rethinking multiple equilibria in macroeconomic modelling'. NBER Macroeconomics Annual, 15, pp.162-171.

[7] Bottazzi, L., Da Rin, Marco, Hellmann, T. (2016). The importance of trust for investment: Evidence from venture capital. The Review of Financial Studies, 29 (9), 2283-2318.

[8] Carlsson, H., van Damme, E. 1993. Global Games and Equilibrium Selection. Econometrica, Vol.61, pp.989-1018

[9] Dubey, P. Geanakoplos, J. and Shubik, M. 1987 The revelation of information in strategic market games. Journal of Mathematical Economics 16, pp 105-137.

[10] Fey, M., 2008. Rent-seeking contests with incomplete information. Public Choice, 135(3-4), pp.225-236.

[11] Frankel, D.M., Morris, S. and Pauzner, A., 2003. Equilibrium selection in global games with strategic complementarities. Journal of Economic Theory, 108(1), pp.1-44.

[12] Grossman, S.J. and Stiglitz, J.E., 1980. On the impossibility of informationally efficient markets. The American economic review, 70(3), pp.393-408.

[13] J.Jost, Postmodern analysis, 3rd edition, Springer, 2005

[14] J.Jost, X.Li-Jost, Calculus of variations, Cambridge Univ.Press, 1998

[15] Karp, L. ; Lee, I. ; Mason, R. 2007. A global game with strategic substitutes and complements. Games and Economic Behavior, 2007, Vol.60(1), pp.155-175

[16] Konrad, K.A. 2009: Strategy and Dynamics in Contests. Oxford: Oxford University Press

[17] Mason, R. and Valentinyi, A., 2010. The Existence and Uniqueness of Monotone Pure Strategy Equilibrium in Bayesian Games.

[18] Makowski, L., Ostroy, J. 1995. A revision of the first theorem of welfare economics, The American Economic Review, 85, 808-827

[19] Mathevet, L. and Steiner, J., 2013. Tractable dynamic global games and applications. Journal of Economic Theory, 148(6), pp.2583-2619.

[20] Morris, Stephen, and Hyun Song Shin. ”Unique equilibrium in a model of self-fulfilling currency attacks.” American Economic Review (1998): 587-597.

[21] Morris, S., Shin, H. 2006. Global Games: Theory and Applications. Chapter 3 (pp.57-114) in: Cambridge Collections Online, Cambr.Univ.Press

[22] Sakovics, J. and Steiner, J., 2012. Who matters in coordination problems?. The American Economic Review, 102(7), pp.3439-3461.

[23] Shapley, L., Shubik, M., 1977. Trade using one commodity as a means of payment. Journal of Political Economy 85, 937-968.

[24] Tullock, G.: Efficient rent seeking. In: Buchanan, J., Tollison, R., Tullock, G. (eds.) Toward a Theory of the Rent-Seeking Society, College Station: Texas A & M University Press (1980)

[25] Vives, X., 2010. Information and learning in markets: The impact of market microstructure, Princeton University Press, Princeton and Oxford.

[26] Wasser, C., 2013. Incomplete information in rent-seeking contests. Economic Theory, 53(1), pp.239-268.

[27] Zarutskie, Rebecca (2010). ”The Role of Top Management Team Human Capital in Venture Capital Markets: Evidence from First-Time Funds,” Journal of Business Venturing, vol. 25, pp. 155-172.

5 Appendix

5.1 Analysis at the threshold

We shall derive estimates at the threshold $x_i = t$. There, (34) becomes

$$\int_0^\infty \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}} \frac{2t+\gamma y}{2} \frac{\sigma_{-i}(t+\gamma y)}{(\sigma_{-i}(t+\gamma y)+\sigma^*(t))^2} dy = 1. \tag{38}$$

By the monotonicity assumption at equilibrium

$$\sigma_{-i}(t+\gamma y) \geq \sigma^*(t) \text{ for } y \geq 0, \tag{39}$$

and since the function $g(y) = \frac{y}{(y+y_0)^2}$ is monotonically decreasing for $y \geq y_0$, we then have $\frac{2t+\gamma y}{2} \frac{\sigma_{-i}(t+\gamma y)}{(\sigma_{-i}(t+\gamma y)+\sigma^*(t))^2} \leq \frac{1}{4\sigma^*(t)}$, which yields

$$\sigma^*(t) \leq \frac{1}{4} \int_0^\infty \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}} \frac{2t+\gamma y}{2} dy = \frac{t}{8} + \frac{\gamma}{8\sqrt{\pi}}. \tag{40}$$

In order to also get a lower bound, we assume that our equilibrium functions satisfy

$$\sigma(x) \leq \frac{x}{4}, \tag{41}$$

that is, the asymptotics (36) also yields an upper bound. We should point out that Theorem 4 yields such a bound only for sufficiently small γ. Since the inequality in that Theorem is an upper bound which need not be sharp, the assumption (41) is still plausible for all γ.

We then have

$$\begin{aligned} 1 &= \int_0^\infty \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}} \frac{2t+\gamma y}{2} \frac{\sigma_{-i}(t+\gamma y)}{(\sigma_{-i}(t+\gamma y)+\sigma^*(t))^2} dy \\ &\geq \int_0^\infty \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}} \frac{2t+\gamma y}{2} \frac{\frac{t+\gamma y}{4}}{(\frac{t+\gamma y}{4}+\sigma^*(t))^2} dy, \end{aligned}$$

and since this a monotonically increasing function of γ, we can further control this from below by its value for $\gamma = 0$,

$$\begin{aligned} &\geq \int_0^\infty \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}} \frac{\frac{t^2}{4}}{(\frac{t}{4}+\sigma^*(t))^2} dy \\ &= \frac{\frac{t^2}{8}}{(\frac{t}{4}+\sigma^*(t))^2} \end{aligned}$$

whence

$$\sigma^*(t) \geq \frac{(\sqrt{2}-1)}{4} t. \tag{42}$$

For $\gamma \to 0$, the preceding inequalities become equalities, and therefore, the limiting solution is

$$\sigma^*(t) = \frac{(\sqrt{2}-1)}{4} t \text{ for } \gamma = 0. \tag{43}$$

We should point out, however, that this is not the equilibrium at $x_i = t$ for $\gamma = 0$. In fact, the preceding analysis was working with the equilibrium solutions of $-i$ for $x > t$. When we choose $x_i = t$ and directly let $\gamma \to 0$ in (34), we get the conditon

$$\frac{1}{2} \frac{t\sigma_{-i}(t)}{(\sigma_{-i}(t)+\sigma^*(t))^2} = 1$$

whose solution is

$$\sigma^*(t) = \sqrt{\frac{t\sigma_{-i}(t)}{2}} - \sigma_{-i}(t)$$

and the fix point equation $\sigma(t) \coloneqq \sigma^*(t) = \sigma_{-i}(t)$ yields

$$\sigma = \frac{t}{8} \tag{44}$$

which is larger than (43). For getting (43), we had assumed that the player also takes the opponent's bids for $x_i > t$ into account, which for $\gamma \to 0$ converge to $\frac{x_i}{4}$, and hence for $x_i \to t$ converge to a value that is larger than the equilibrium value $\frac{t}{8}$. Since those values are above the equilibrium, the player's reaction therefore is below the equilibrium, that is, at (43) rather than at (44). This is the informational discount effect arising from the uncertainty about the value of θ.

In any case, in the limit $\gamma \to 0$, the solution becomes discontinuous at the threshold $x_i = t$. More generally, if $x_i \geq t$, when we assume (41) as well as

$$\sigma_{-i}(x) \geq \sigma^*(x_i) \text{ for } x \geq x_i, \tag{45}$$

and put $\sigma^*(x_i) = \lambda x_i$, we get

$$\begin{aligned} 1 &= \int_{\frac{t-x_i}{\gamma}}^{\infty} \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}} \frac{2x_i + \gamma y}{2} \frac{\sigma_{-i}(x_i + \gamma y)}{(\sigma_{-i}(x_i + \gamma y) + \sigma^*(x_i))^2} dy \\ &\geq \int_0^{\infty} \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}} \frac{2x_i + \gamma y}{2} \frac{\frac{x_i + \gamma y}{4}}{(\frac{x_i + \gamma y}{4} + \lambda x_i)^2} dy \\ &> 1 \text{ for } \lambda < \frac{\sqrt{2} - 1}{4}, \end{aligned}$$

that is, we arrive at a contradiction unless

$$\sigma^*(x_i) \geq \frac{\sqrt{2} - 1}{4} x_i. \tag{46}$$

Actually, keeping the integral from $y = \frac{t-x_i}{\gamma}$ instead of only from $y = 0$ in the preceding inequalities, we can improve (46) for $x_i > t$.

5.2 Confinement, monotonicity and selfconsistency

By implicit differentiation, we can analyze the effects of variations of the opponent's strategy σ_{-i}, of the parameter γ, and the value of the threshold t on $\sigma^*(x_i)$, and also compute the derivative of $\sigma^*(x_i)$ with respect to x_i. We recall the scheme and write (28) in the schematic form

$$F(\sigma^*(x_i); \sigma_{-i}, x_i, \gamma, t) = 0. \tag{47}$$

We then take total derivatives of this identity. We first take the derivative with respect to x_i to get

$$\frac{\partial F(\sigma^*(x_i);\sigma_{-i},x_i,\gamma,t)}{\partial\sigma^*(x_i)}\frac{d\sigma^*(x_i)}{dx_i}+\frac{\partial F(\sigma^*(x_i);\sigma_{-i},x_i,\gamma,t)}{\partial x_i}=0. \tag{48}$$

Applying this to (28) yields

$$\begin{aligned}0= & \frac{d\sigma^*(x_i)}{dx_i}\Big(-\int_{x_k=t}^{\infty}\frac{1}{\gamma\sqrt{4\pi}}e^{-\frac{(x_i-x_k)^2}{4\gamma^2}}(x_k+x_i)\frac{\sigma_{-i}(x_k)}{(\sigma_{-i}(x_k)+\sigma^*(x_i))^3}dx_k\Big)\\ & +\int_{x_k=t}^{\infty}\frac{1}{\gamma\sqrt{4\pi}}e^{-\frac{(x_i-x_k)^2}{4\gamma^2}}\Big(\frac{(x_k-x_i)(x_k+x_i)}{2\gamma^2}+\frac{1}{2}\Big)\frac{\sigma_{-i}(x_k)}{(\sigma_{-i}(x_k)+\sigma^*(x_i))^2}dx_k\end{aligned}$$

hence

$$\frac{d\sigma^*(x_i)}{dx_i}=\frac{\int_{x_k=t}^{\infty}\frac{1}{\gamma\sqrt{4\pi}}e^{-\frac{(x_i-x_k)(x_k+x_i)}{4\gamma^2}}\Big(\frac{x_k^2-x_i^2}{2\gamma^2}+\frac{1}{2}\Big)\frac{\sigma_{-i}(x_k)}{(\sigma_{-i}(x_k)+\sigma^*(x_i))^2}dx_k}{\int_{x_k=t}^{\infty}\frac{1}{\gamma\sqrt{4\pi}}e^{-\frac{(x_i-x_k)^2}{4\gamma^2}}(x_k+x_i)\frac{\sigma_{-i}(x_k)}{(\sigma_{-i}(x_k)+\sigma^*(x_i))^3}dx_k}. \tag{49}$$

From this equation, we can now deduce

Lemma 2. *$\sigma^*(x_i)$ is a monotonically increasing function of x_i.*

Proof. Since the denominator of (49) is clearly positive, the expression in (49) is finite, and therefore $\sigma^*(x_i)$ is a differentiable function of x_i. In order to verify monotonicity, we need to show that the numerator is positive as well. While this is more complicated, essentially it can be seen as follows. For $x_k^2>x_i^2+\gamma^2$, that is, in particular near the peak of the exponential factor, the factor $\frac{x_k^2-x_i^2}{2\gamma^2}+\frac{1}{2}$ is positive. Likewise for x_i at or near the threshold, the negative part of that factor is below the cut-off t of the integral. Finally, for large $|x_k|$, for large x_k, we can argue as follows in order to balance the possibly negative part by the positive contribution. We split $x_k^2-x_i^2$ as $(x_k+x_i)(x_k-x_i)$. The function $e^{-\frac{(x_i-x_k)(x_k+x_i)}{4\gamma^2}}(x_k-x_i)$ is an odd function of x_k-x_i, and positive for $x_k-x_i>0$. By the upper bound (30) of Theorem 4, the remaining factor $\frac{(x_k+x_i)\sigma_{-i}(x_k)}{(\sigma_{-i}(x_k)+\sigma^*(x_i))^2}$ is $\geq\frac{x_k+x_i}{4\sigma_{-i}(x_k)}\geq\frac{x_k+x_i}{x_k}\geq 1$, and therefore it can balance what comes from $t\leq x_k<x_i$, because, as we have also observed as a consequence of Theorem 4, all bids satisfy a positive lower bound. Therefore, altogether,

$$\int_{x_k=t}^{\infty}\frac{1}{\gamma\sqrt{4\pi}}e^{-\frac{(x_i-x_k)^2}{4\gamma^2}}\frac{(x_k-x_i)(x_k+x_i)}{2\gamma^2}\frac{\sigma_{-i}(x_k)}{(\sigma_{-i}(x_k)+\sigma^*(x_i))^2}dx_k\geq 0.$$

Also, clearly

$$\int_{x_k=t}^{\infty}\frac{1}{\gamma\sqrt{4\pi}}e^{-\frac{(x_i-x_k)^2}{4\gamma^2}}\frac{1}{2}\frac{\sigma_{-i}(x_k)}{(\sigma_{-i}(x_k)+\sigma^*(x_i))^2}dx_k>0,$$

and therefore from (49)

$$\frac{d\sigma^*(x_i)}{dx_i}>0. \tag{50}$$

$\square$

Repeating the key argument in words, under the assumption that the players behave rationally, they obey the upper bound (30) for their bidding function. Therefore, they do not make abnormally high bids that would force the opponent to lower his bid at higher signal values. Consequently, the bidding function is a monotonically increasing function of the signal.

In order to consider the asymptotics for $x_i \to \infty$, we first observe that the integrals $\int_{x_k=t}^{\infty} \frac{1}{\gamma\sqrt{4\pi}} e^{-\frac{(x_i-x_k)(x_k+x_i)}{4\gamma^2}} \frac{1}{2} \frac{\sigma_{-i}(x_k)}{(\sigma_{-i}(x_k)+\sigma^*(x_i))^2} dx_k$ and $\int_{x_k=t}^{\infty} \frac{1}{\gamma\sqrt{4\pi}} e^{-\frac{(x_i-x_k)^2}{4\gamma^2}} (x_k+x_i) \frac{\sigma_{-i}(x_k)}{(\sigma_{-i}(x_k)+\sigma^*(x_i))^3} dx_k$ occurring in (49) both behave asymptotically like $\frac{1}{x_i}$ under our confinement assumption. For the integral $\int_{x_k=t}^{\infty} \frac{1}{\gamma\sqrt{4\pi}} e^{-\frac{(x_i-x_k)(x_k+x_i)}{4\gamma^2}} \frac{x_k^2-x_i^2}{2\gamma^2} \frac{\sigma_{-i}(x_k)}{(\sigma_{-i}(x_k)+\sigma^*(x_i))^2} dx_k$, we perform again the substitution (33) to convert it into

$$\int_{y=\frac{t-x_i}{\gamma}}^{\infty} \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}} \left(\frac{x_i y}{\gamma} + \frac{y^2}{2}\right) \frac{\sigma_{-i}(x_i+\gamma y)}{(\sigma_{-i}(x_i+\gamma y)+\sigma^*(x_i))^2} dy.$$

Under the confinement assumption, the leading term behaves like

$$\int_{y=\frac{t-x_i}{\gamma}}^{\infty} \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}} \frac{y}{(2+\frac{\gamma y}{x_i})^2} dy$$

Since $\frac{y}{(2+\frac{\gamma y}{x_i})^2} = \frac{y}{4} - \frac{\gamma}{2x_i} y^2 + O\big((\frac{2\gamma}{x_i})^2\big)$ by Taylor expansion w.r.t. $\epsilon = \frac{\gamma}{x_i}$ and since $\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}} y = 0$, the asymptotics of this latter integral is again $\frac{1}{x_i}$. Thus, assuming our growth conditions

$$\frac{\sqrt{2}-1}{4} x \le \sigma(x) \le \frac{1}{4} x, \tag{51}$$

and since $\sigma^*(x_i)$ is monotonically increasing, we also get an upper bound

$$\frac{d\sigma^*(x_i)}{dx_i} \le K \tag{52}$$

for some constant K (except near $x_i = t$ for $\gamma \to 0$, but there we can control the situation anyway by the above analysis, and for every fixed positive γ, we do get (52)).

5.3 The influence of the variance

We next turn to the dependence on the variance γ. Analogously to (48)

$$\frac{\partial F(\sigma^*(x_i);\sigma_{-i},x_i,\gamma,t)}{\partial \sigma^*(x_i)} \frac{d\sigma^*(x_i)}{d\gamma} + \frac{\partial F(\sigma^*(x_i);\sigma_{-i},x_i,\gamma,t)}{\partial \gamma} = 0. \tag{53}$$

Since

$$\frac{\partial F(\sigma^*(x_i);\sigma_{-i},x_i,\gamma,t)}{\partial\gamma} = \int_{x_k=t}^{\infty}(\frac{-1}{\gamma^2\sqrt{4\pi}}+\frac{1}{\gamma\sqrt{4\pi}}\frac{(x_k-x_i)^2}{2\gamma^3})e^{-\frac{(x_i-x_k)^2}{4\gamma^2}}\frac{x_k+x_i}{2}\frac{\sigma_{-i}(x_k)}{(\sigma_{-i}(x_k)+\sigma^*(x_i))^2}dx_k, \tag{54}$$

we have

$$\frac{d\sigma^*(x_i)}{d\gamma} = \frac{\int_{x_k=t}^{\infty}(\frac{-1}{\gamma^2\sqrt{4\pi}}+\frac{1}{\gamma\sqrt{4\pi}}\frac{(x_k-x_i)^2}{2\gamma^3})e^{-\frac{(x_i-x_k)^2}{4\gamma^2}}\frac{x_k+x_i}{2}\frac{\sigma_{-i}(x_k)}{(\sigma_{-i}(x_k)+\sigma^*(x_i))^2}dx_k}{\int_{x_k=t}^{\infty}\frac{1}{\gamma\sqrt{4\pi}}e^{-\frac{(x_i-x_k)^2}{4\gamma^2}}(x_k+x_i)\frac{\sigma_{-i}(x_k)}{(\sigma_{-i}(x_k)+\sigma^*(x_i))^3}dx_k}. \tag{55}$$

Since

$$\int_{x_k=-\infty}^{\infty}\frac{1}{\gamma\sqrt{4\pi}}e^{-\frac{(x_i-x_k)^2}{4\gamma^2}}dx_k = 1$$

for all γ, we have

$$0 = \frac{d}{d\gamma}\int_{x_k=-\infty}^{\infty}\frac{1}{\gamma\sqrt{4\pi}}e^{-\frac{(x_i-x_k)^2}{4\gamma^2}}dx_k = \int_{x_k=-\infty}^{\infty}(\frac{-1}{\gamma^2\sqrt{4\pi}}+\frac{1}{\gamma\sqrt{4\pi}}\frac{(x_k-x_i)^2}{2\gamma^3})e^{-\frac{(x_i-x_k)^2}{4\gamma^2}}dx_k,$$

and hence

$$\int_{x_k=t}^{\infty}(\frac{-1}{\gamma^2\sqrt{4\pi}}+\frac{1}{\gamma\sqrt{4\pi}}\frac{(x_k-x_i)^2}{2\gamma^3})e^{-\frac{(x_i-x_k)^2}{4\gamma^2}}dx_k < 0$$

for $x_i > t$. Moreover, for small $\gamma > 0$,

$$\frac{x_k+x_i}{2}\frac{\sigma_{-i}(x_k)}{(\sigma_{-i}(x_k)+\sigma^*(x_i))^2} \approx \frac{x_k}{2(x_k+x_i)}$$

and also

$$\frac{1}{\gamma\sqrt{4\pi}}e^{-\frac{(x_i-x_k)^2}{4\gamma^2}}\frac{x_k}{2(x_k+x_i)} \approx \frac{1}{8\gamma\sqrt{\pi}}e^{-\frac{(x_i-x_k)^2}{4\gamma^2}}$$

since the exponential weight is concentrated near $x_k \approx x_i$. Therefore, under our confinement assumptions, we also get for small $\gamma > 0$

$$\frac{d\sigma^*(x_i)}{d\gamma} < 0 \text{ for } x_i > t. \tag{56}$$

In fact, also

$$\lim_{\gamma\to 0}\frac{d\sigma^*(x_i)}{d\gamma} < 0 \text{ for } x_i > t. \tag{57}$$

(For large γ, however, the fact that $\frac{x_k}{2(x_k+x_i)}$ is an increasing function of x_k should make $\frac{d\sigma^*(x_i)}{d\gamma}$ positive, in particular for x_i near t, because $\int_{x_k=t}^{\infty}(\frac{-1}{\gamma^2\sqrt{4\pi}}+\frac{1}{\gamma\sqrt{4\pi}}\frac{(x_k-t)^2}{2\gamma^3})e^{-\frac{(t-x_k)^2}{4\gamma^2}}dx_k = 0$.) Thus, we see the interesting effect that when we are in the range of small variances, an increase of the variance decreases the equilibrium bid, because it increases the players' uncertainty. In contrast, in the large variance regime, an increase of the variance might conceivably increase the equilibrium bid, because it increases the probability that the opponent receives a significantly higher signal and, consequently , also the estimate for the true value θ and therefore of the total value to be distributed among the players increases. That, in turn, justifies a higher bid.

Also, since at $\gamma = 0$, the equilibrium is $\sigma(x_i) = \frac{x_i}{4}$ for $x_i > t$, see (37), we conclude that the confinement $\frac{\sqrt{2}-1}{4} x_i \leq \sigma(x_i) \leq \frac{1}{4} x_i$ holds for small $\gamma > 0$.