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matrices and the nullity condition for
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by

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Abstract

We study the von Neumann measurement on an m -dimensional quantum system. The properties of the matrices related to von Neumann measurement are investigated. It is shown that such $(m^2 - 1) \times (m^2 - 1)$ matrices are idempotent and have rank $m - 1$. These properties give rise to necessary conditions for the nullity of quantum correlations in bipartite systems. Finally, as an example we discuss quantum correlation in Bell diagonal states.

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INTRODUCTION

Measurement is an “irreversible” and “non-deterministic” process, since once the measurement is completed and the information is extracted, the initial state is collapsed. Before the measurement, one can not predict what the measurement result and the post-measurement state will be. Von Neumann measurement is one, but not the only kinds of measurements one can make on a quantum state. But one of its importance is that one can derive all other measurements from such von Neumann measurements.

In Ref. [1], it has been shown that driving a known initial quantum state onto a known pure state without resorting to unitary transformations can be achieved by means of a discrete sequence of von Neumann measurement only. Von Neumann measurement on a subsystem of a composite quantum system could also unavoidably create distillable entanglement between the measurement apparatus and the system if the state has nonzero quantum discord [2], which is used to quantify the quantum correlation [3, 4].

Based on von Neumann measurement, Refs. [5, 6] evaluate the geometric measure of quantum discord for quantum state in Bloch representation, in which any quantum state are expanded by the basis of orthonormal traceless Hermitian matrix, or the generators of special unitary group. The Bloch representation for two dimensional quantum state is already well known [7]. However, for higher dimension quantum system, the geometry and boundary of the Bloch sphere, or the determination of the Bloch vector are still unclear [8, 9].

The Bloch representation has many other applications in quantum theory. For example, Refs. [10, 11] obtain a necessary condition for separability in multipartite system in terms of Bloch representation. Ref. [12] presents a multipartite entanglement measure for multiqubit states by using Bloch representation. This representation can also be used to evaluate entanglement [13] and characterize the minimum error probability of discriminating two single qubit quantum operations by separable input states [14]. Moreover, Ref. [15] investigates theoretically the convex set of the steady states and the invariant set of the Lindblad master equation, and Ref. [16] establishes a closed formula of quantum system following a Lindblad evolution with self-adjoint Lindblad operators based on Bloch representation.

In this paper, we study the action of von Neumann measurement on the basis of orthonormal traceless Hermitian matrices, which improves our understanding of its action on

quantum states in Bloch representations. Suppose the von Neumann measurement acts on the m -dimensional quantum system. Then its action on the basis composed by $m^2 - 1$ orthonormal traceless Hermitian matrices corresponds to an $(m^2 - 1) \times (m^2 - 1)$ matrix. We show this matrix corresponding to the von Neumann measurement is rank $m - 1$. As applications, a necessary condition for the existence of quantum correlations in bipartite systems is derived.

THE VON NEUMANN MEASUREMENT-RELATED MATRICES

Let \mathcal{M} be an arbitrary von Neumann measurement acting on an m -dimensional quantum system, $\mathcal{M} = \{|\phi_i\rangle\langle\phi_i|\}_{i=0}^{m-1}$, where $|\phi_i\rangle = \sum_{j=0}^{m-1} a_{ij}|j\rangle$, $A = (a_{ij})$ is an $m \times m$ unitary matrix due to the orthogonality and completeness of \mathcal{M} . Under the von Neumann measurement, a quantum state ρ is mapped to $\mathcal{M}(\rho) \equiv \mathcal{M}\rho\mathcal{M}^\dagger = \sum_{i=0}^{m-1} |\phi_i\rangle\langle\phi_i|\rho|\phi_i\rangle\langle\phi_i|$. Suppose $\{\mu_i\}_{i=1}^{m^2-1}$ is an orthonormal basis of $m \times m$ traceless Hermitian matrices. Then corresponding to \mathcal{M} , there exists an $(m^2 - 1) \times (m^2 - 1)$ matrix M such that

$$\begin{aligned} & \mathcal{M}(\mu_1, \mu_2, \dots, \mu_{m^2-1}) \\ & \equiv (\mathcal{M}\mu_1\mathcal{M}^\dagger, \mathcal{M}\mu_2\mathcal{M}^\dagger, \dots, \mathcal{M}\mu_{m^2-1}\mathcal{M}^\dagger) \\ & = (\mu_1, \mu_2, \dots, \mu_{m^2-1})M, \end{aligned} \tag{1}$$

where M is given with the entries M_{ij} defined by $\mathcal{M}\mu_i\mathcal{M}^\dagger = \sum_{j=1}^{m^2-1} \mu_j M_{ij}$, $i = 1, \dots, m^2-1$. It is easy to prove that the matrix M corresponding to the von Neumann measurement \mathcal{M} satisfies $M^2 = M$. Hence M is idempotent. In the following we study the properties of the matrix M and show that M is also singular. Before we present our main results, we need the following lemma.

Lemma Let A be an arbitrary $m \times m$ unitary matrix with the i -th row and j -th column entry given by $a_{i-1, j-1}$. Let C_1 be an $m \times (m - 1)$ matrix with the s -th row and p -column entry given by $\sqrt{\frac{1}{p(p+1)}}[\sum_{j=0}^{p-1} |a_{s-1, j}|^2 - p|a_{s-1, p+1}|^2]$, $s = 1, \dots, m$; $p = 1, \dots, m - 1$, C_2 be an $m \times \frac{m(m-1)}{2}$ matrix with the s -th row and kl -column entry given by $\frac{1}{\sqrt{2}}(a_{s-1, k}^* a_{s-1, l} + a_{s-1, l}^* a_{s-1, k})$, $s = 1, \dots, m$; $k, l = 1, \dots, m - 1$ and $k \neq l$, and C_3 be an $m \times \frac{m(m-1)}{2}$ matrix with the s -th row and kl -column entry given by $\frac{i}{\sqrt{2}}(a_{s-1, k}^* a_{s-1, l} - a_{s-1, l}^* a_{s-1, k})$, $s = 1, \dots, m$; $k, l = 1, \dots, m - 1$ and $k \neq l$. Then the $m \times (m^2 - 1)$ matrix C given by the block matrices C_1, C_2 and C_3 , $C = (C_1, C_2, C_3)$ has rank $m - 1$.

The proof of the Lemma is given in Appendix. From the Lemma we can prove the following result:

Theorem 1 Suppose \mathcal{M} is an arbitrary von Neumann measurement acting on an m -dimensional quantum system, then the $(m^2 - 1) \times (m^2 - 1)$ matrix M defined in Eq. (1) is rank $m - 1$.

[Proof]. The generalized Gell Mann basis for m -dimensional quantum systems can be constructed as follows. For $1 \leq p \leq m - 1$,

$$w_p = \sqrt{\frac{1}{p(p+1)}} \left(\sum_{a=0}^{p-1} |a\rangle\langle a| - p|p\rangle\langle p| \right), \quad (2)$$

and

$$\begin{aligned} w_{kl} &= \frac{1}{\sqrt{2}} (|k\rangle\langle l| + |l\rangle\langle k|), \\ w'_{kl} &= \frac{i}{\sqrt{2}} (|k\rangle\langle l| - |l\rangle\langle k|) \end{aligned} \quad (3)$$

with $k < l$, $k, l = 0, 1, \dots, m - 1$. We denote $w_p = w_{kl}$ for $p = m, \dots, m + \frac{m(m-1)}{2}$, and $w_p = w'_{kl}$ for $p = m + 1 + \frac{m(m-1)}{2}, \dots, m^2 - 1$ in the following for simplicity. Therefore, $\{w_p\}_{p=1}^{m-1} \cup \{w_{kl}, w'_{kl}\}_{k,l=0,k<l}^{m-1} = \{w_p\}_{p=1}^{m^2-1}$ is an orthonormal basis of $m \times m$ traceless Hermitian operators.

First, we show that for the orthonormal basis of $m \times m$ traceless Hermitian matrix in Eq. (2) and Eq. (3), the matrix M corresponding to a von Neumann measurement \mathcal{M} has rank $m - 1$. In fact, by straightforward calculation we have

$$\begin{aligned} \mathcal{M}(w_p) &= \sqrt{\frac{1}{p(p+1)}} \left[\left(\sum_{j=0}^{p-1} |a_{0j}|^2 - p|a_{0,p+1}|^2 \right) |\phi_0\rangle\langle\phi_0| + \left(\sum_{j=0}^{p-1} |a_{1j}|^2 - p|a_{1,p+1}|^2 \right) |\phi_1\rangle\langle\phi_1| \right. \\ &\quad \left. + \dots + \left(\sum_{j=0}^{p-1} |a_{m-1,j}|^2 - p|a_{m-1,p+1}|^2 \right) |\phi_{m-1}\rangle\langle\phi_{m-1}| \right], \\ \mathcal{M}(w_{kl}) &= \frac{1}{\sqrt{2}} \left[(a_{0k}^* a_{0l} + a_{0l}^* a_{0k}) |\phi_0\rangle\langle\phi_0| + (a_{1k}^* a_{1l} + a_{1l}^* a_{1k}) |\phi_1\rangle\langle\phi_1| \right. \\ &\quad \left. + \dots + (a_{m-1,k}^* a_{m-1,l} + a_{m-1,l}^* a_{m-1,k}) |\phi_{m-1}\rangle\langle\phi_{m-1}| \right], \\ \mathcal{M}(w'_{kl}) &= \frac{i}{\sqrt{2}} \left[(a_{0k}^* a_{0l} - a_{0l}^* a_{0k}) |\phi_0\rangle\langle\phi_0| + (a_{1k}^* a_{1l} - a_{1l}^* a_{1k}) |\phi_1\rangle\langle\phi_1| \right. \\ &\quad \left. + \dots + (a_{m-1,k}^* a_{m-1,l} - a_{m-1,l}^* a_{m-1,k}) |\phi_{m-1}\rangle\langle\phi_{m-1}| \right], \end{aligned}$$

where $p = 1, \dots, m-1$; $k, l = 0, \dots, m-1$ and $k \neq l$. Therefore

$$\begin{aligned}
& \mathcal{M}(w_1, \dots, w_{m-1}, w_m, \dots, w_{m+\frac{m(m-1)}{2}}, w_{m+1+\frac{m(m-1)}{2}}, \dots, w_{m^2-1}) \\
&= (|\phi_0\rangle\langle\phi_0|, |\phi_1\rangle\langle\phi_1|, \dots, |\phi_{m-1}\rangle\langle\phi_{m-1}|)C \\
&= (w_1, \dots, w_{m-1}, w_m, \dots, w_{m+\frac{m(m-1)}{2}}, w_{m+1+\frac{m(m-1)}{2}}, \dots, w_{m^2-1})M',
\end{aligned} \tag{4}$$

where the $m \times (m^2 - 1)$ block matrix $C = (C_1, C_2, C_3)$ is defined in Lemma.

By the Lemma, the rank of matrix C is $m-1$. Hence the maximum of linear independent column vectors in Eq. (4) is $m-1$. Therefore, the rank of M' is also $m-1$, i.e. the rank of the matrix M' corresponding to \mathcal{M} under the basis Eq. (2) and Eq. (3) is $m-1$.

An arbitrary orthonormal basis of $m \times m$ traceless Hermitian matrix $\{\mu_i\}_{i=1}^{m^2-1}$ can be obtained from the basis Eq. (2) and Eq. (3), $(\mu_1, \dots, \mu_{m^2-1}) = (w_1, \dots, w_{m^2-1})O$, where O is an orthogonal matrix. If M is the matrix corresponding to \mathcal{M} under the basis $\{\mu_i\}_{i=1}^{m^2-1}$, then the matrix corresponds to \mathcal{M} under the basis $\{w_i\}_{i=1}^{m^2-1}$ is OMO^T . This shows that the rank of matrix M corresponding to \mathcal{M} is invariant under the transformation of the orthonormal matrix basis. Hence the rank of matrix M in Eq. (1) is $m-1$. \square

We remark that if as the basis $\{\mu_i\}_{i=1}^{m^2-1}$ of $m \times m$ traceless Hermitian matrix is not necessarily orthonormal, the matrix M corresponding to the von Neumann measurement \mathcal{M} given in Eq. (1) is still of rank $m-1$. This can be seen from the proof of the Theorem, as replacing the orthogonal matrix by reversible matrix does not change the rank.

Moreover, for any $(m^2 - 1) \times (m^2 - 1)$ idempotent matrix Y with rank $m-1$, there must exist a basis of $m \times m$ traceless Hermitian matrix such that Y corresponds to the von Neumann measurement \mathcal{M} . Since for any rank $m-1$ matrix Y , there exists a reversible matrix Q such that $Y = QMQ^{-1}$. Hence if M is the matrix corresponding to the von Neumann measurement \mathcal{M} with the basis $\{\mu_i\}_{i=1}^{m^2-1}$, then Y is the matrix corresponding to the von Neumann measurement \mathcal{M} with the basis $\{\mu'_i\}_{i=1}^{m^2-1}$ satisfying $(\mu'_1, \dots, \mu'_{m^2-1}) = (\mu_1, \dots, \mu_{m^2-1})Q^{-1}$.

The quantum correlations among the subsystems of a multipartite system play significant roles in many information processing tasks and physical processes. Physically, to get information from a system one needs to measure the system. For a bipartite systems, generally a minimal amount of information will be left after measuring one of the subsystems. This amount is described by the so called quantum discord, as introduced by Oliver and

Zurek [17–19]. Nevertheless, it is a challenging problem to compute the quantum correlation and the analytically formulae are available only for some special quantum states like Bell-diagonal states, X-type states [20] under the projective measurements. It has been shown that the quantum discord is required for some information processing like assisted optimal state discrimination [21]. Quantum discord exists for states without quantum entanglement. According to quantum discord quantum states can be classified as classical-classical correlated, quantum-classical and classical-quantum correlated, and quantum-quantum correlated. Our Theorem 1 can be used to detect quantum correlations.

For an $m \otimes n$ bipartite quantum state ρ , its Bloch representation has the form,

$$\rho = \frac{1}{mn} (I_m \otimes I_n + \sum_{i=1}^{m^2-1} r_i \mu_i \otimes I_n + \sum_{j=1}^{n^2-1} s_j I_m \otimes \nu_j + \sum_{i=1}^{m^2-1} \sum_{j=1}^{n^2-1} t_{ij} \mu_i \otimes \nu_j), \quad (5)$$

where I_m is the $m \times m$ identity matrix, $\{\mu_i\}_{i=1}^{m^2-1}$ and $\{\nu_j\}_{j=1}^{n^2-1}$ are the orthonormal bases of $m \times m$ and $n \times n$ traceless Hermitian matrices respectively. If one measures the first subsystem by von Neumann measurement \mathcal{M} , then by Eq. (1) we get

$$\mathcal{M} \otimes I \rho \mathcal{M}^\dagger \otimes I = \frac{1}{mn} [I \otimes I + \sum_{i=1}^{m^2-1} (MR)_i \mu_i \otimes I + \sum_{j=1}^{n^2-1} s_j I \otimes \nu_j + \sum_{i=1}^{m^2-1} \sum_{j=1}^{n^2-1} (MT)_{ij} \mu_i \otimes \nu_j], \quad (6)$$

where $R = (r_i)$ and $S = (s_j)$ are the vectors with components r_i and s_j respectively, $T = (t_{ij})$ is the matrix with entries t_{ij} . Therefore, ρ is classical-quantum correlated, i.e. $\mathcal{M} \otimes I \rho \mathcal{M}^\dagger \otimes I = \rho$, if and only if there exists von Neumann measurement \mathcal{M} acting on the first subsystem such that $MR = R$ and $MT = T$. Similarly, ρ is quantum-classical correlated, i.e. $I \otimes \mathcal{M}' \rho I \otimes \mathcal{M}'^\dagger = \rho$, if and only if there exists von Neumann measurement \mathcal{M}' acting on the second subsystem such that $M'S = S$ and $T(M')^T = T$, which is equivalent to $M'S = S$ and $M'T^T = T^T$. And ρ is classical-classical correlated, i.e. $\mathcal{M} \otimes \mathcal{M}' \rho \mathcal{M} \otimes \mathcal{M}'^\dagger = \rho$, if and only if there exist von Neumann measurement \mathcal{M} acting on the first subsystem and von Neumann measurement \mathcal{M}' acting on the second subsystem such that $MR = R$, $M'S = S$ and $MT(M')^T = T$. From the ranks of M and M' , we get the following necessary conditions for the classical-quantum, quantum-classical and classical-classical correlated states.

Theorem 2 For $m \otimes n$ ($m \leq n$) quantum state ρ , if it is classical-quantum correlated, then the rank of (R, T) is no more than $m - 1$. If it is quantum-classical correlated, then

the rank of (R, T) is no more than $n - 1$. If it is classical-classical correlated, then the rank of (R, S, T) is no more than $m - 1$.

As one of the applications of the property of von Neumann measurement related matrix, Theorem 2 gives a simple detection of the correlations in a bipartite system. Let us consider some examples.

Example. We consider the two-qubit states with the maximally mixed marginal. Such state is locally equivalent to

$$\rho_1 = \frac{1}{4} \left(I + \sum_{i=1}^3 t_i \sigma_i \otimes \sigma_i \right),$$

where σ_i are Pauli matrices, $\vec{t} = (t_1, t_2, t_3)$ belongs to the tetrahedron defined by the set of vertices $(-1, -1, -1)$, $(-1, 1, 1)$, $(1, -1, 1)$ and $(1, 1, -1)$. ρ_1 is separable if \vec{t} belongs to the octahedron defined by the set of vertices $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$. If there are more than one nonzero elements in $\{t_1, t_2, t_3\}$, then ρ_1 is quantum-quantum correlated. That is, the points except for the vertices and the central point in the octahedron are all quantum-quantum correlated states. While the six vertices in the octahedron and the central point are all classical-classical correlated states. Therefore, the necessary and sufficient conditions for ρ_1 having quantum correlations on both subsystems are that the rank of $T = \text{diag}(t_1, t_2, t_3)$ is no more than 1.

More generally, consider $m \otimes m$ quantum state

$$\rho_2 = \frac{1}{m^2} \left(I_m \otimes I_m + \sum_{i=1}^{m^2-1} t_i \mu_i \otimes \mu_i \right),$$

with $\{\mu_i\}$ the orthonormal basis of $m \times m$ traceless Hermitian matrix. By Theorem 2, we have that if there are more than $m - 1$ nonzero elements in $\{t_i\}$, then ρ_2 is a quantum-quantum correlated state.

CONCLUSIONS

We have investigated the properties of the matrix related to the von Neumann measurement. General results of the matrix have been derived. It has been shown that such $(m^2 - 1) \times (m^2 - 1)$ matrix is idempotent and has rank $m - 1$. As an application, our results give rise to necessary conditions for detecting the quantum correlations in bipartite systems.

As von Neumann measurement is a fundamental and important operation in quantum theory, our results may highlight the researches in quantum mechanics and quantum information processing.

APPENDIX

Proof of Lemma By some elementary matrix transformations, the rank of matrix C is equal to the rank of matrix C_0 ,

$$C_0 = \begin{pmatrix} \cdots & |a_{00}|^2 - |a_{0i}|^2 & \cdots & a_{0k}^* a_{0l} & \cdots \\ \cdots & |a_{10}|^2 - |a_{1i}|^2 & \cdots & a_{1k}^* a_{1l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & |a_{m-1,0}|^2 - |a_{m-1,i}|^2 & \cdots & a_{m-1,k}^* a_{m-1,l} & \cdots \end{pmatrix},$$

which is composed by the column vectors

$$\vec{\alpha}_i = (|a_{00}|^2 - |a_{0i}|^2, |a_{10}|^2 - |a_{1i}|^2, \cdots, |a_{m-1,0}|^2 - |a_{m-1,i}|^2)^T$$

for $i = 1, \cdots, m-1$, and

$$\vec{\beta}_{kl} = (a_{0k}^* a_{0l}, a_{1k}^* a_{1l}, \cdots, a_{m-1,k}^* a_{m-1,l})^T$$

for $k, l = 0, \cdots, m-1$ and $k \neq l$.

To show that the rank of C_0 is $m-1$, let us first consider the case of $m=2$,

$$C_0 = \begin{pmatrix} |a_{00}|^2 - |a_{01}|^2 & a_{00}^* a_{01} & a_{01}^* a_{00} \\ |a_{10}|^2 - |a_{11}|^2 & a_{10}^* a_{11} & a_{11}^* a_{10} \end{pmatrix}.$$

C_0 is rank 1 because its two rows are linearly dependent due to that the matrix A is unitary. Hence the conclusion is true for $m=2$. Now suppose the conclusion is also true for all $k \times k$ unitary matrix A with $k < m$. Noting that the sum of m rows of the matrix C_0 is zero, therefore the rank of C_0 is no more than $m-1$. Next, we show that there are $m-1$ linearly independent column vectors in C_0 .

We first consider the first column of the matrix A . Without loss of generality we suppose that the first p_0 elements of the first column of A are all nonzero and the rest are all 0, then $1 \leq p_0 \leq m$. If $p_0 = 1$, then A is a direct sum of 1 and an $(m-1) \times (m-1)$ unitary matrix A' . By assumption, the matrix C'_0 deduced in the same way from A' as that C_0

deduced from A is rank $m - 2$ and there exist $m - 2$ linearly independent column vectors $\vec{\gamma}'_1, \vec{\gamma}'_2, \dots, \vec{\gamma}'_{m-2}$ in C'_0 . Therefore, $(0, \vec{\gamma}'_1)^t, (0, \vec{\gamma}'_2)^t, \dots, (0, \vec{\gamma}'_{m-2})^t$, and the first column vector $\vec{\alpha}_1 = (1, -|a_{11}|^2, \dots, -|a_{m-1,1}|^2)^t$ in C_0 , are the $m - 1$ linearly independent vectors of C_0 .

If $p_0 > 1$, consider the column vectors $\vec{\beta}_{01}, \vec{\beta}_{02}, \dots, \vec{\beta}_{0,m-1}$. Let $x_1\vec{\beta}_{01} + x_2\vec{\beta}_{02} + \dots + x_{m-1}\vec{\beta}_{0,m-1} = 0$. One has a set of linear equations,

$$\begin{aligned} x_1 a_{01} + x_2 a_{02} + \dots + x_{m-1} a_{0,m-1} &= 0, \\ x_1 a_{11} + x_2 a_{12} + \dots + x_{m-1} a_{1,m-1} &= 0, \\ &\vdots \\ x_1 a_{p_0-1,1} + x_2 a_{p_0-1,2} + \dots + x_{m-1} a_{p_0-1,m-1} &= 0. \end{aligned} \tag{7}$$

Taking into account that the rank of the following $p_0 \times m$ matrix

$$F_1 = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \dots & a_{0,m-1} \\ a_{10} & a_{11} & a_{12} & \dots & a_{1,m-1} \\ \dots & \dots & \dots & \dots & \dots \\ a_{p_0-1,0} & a_{p_0-1,1} & a_{p_0-1,2} & \dots & a_{p_0-1,m-1} \end{pmatrix}$$

is p_0 because of the unitarity of A , we have that the maximum of linearly independent column vectors of F_1 is p_0 . Since the first column vector of A is linearly independent from the others, the rank of the coefficient matrix of Eqs. (7) is $p_0 - 1$. This demonstrates that the maximum of linearly independent column vectors $\{\vec{\beta}_{01}, \vec{\beta}_{02}, \dots, \vec{\beta}_{0,m-1}\}$ is $p_0 - 1$. Without loss of generality, suppose

$$\begin{aligned} \vec{\gamma}_1 &:= (a_{00}^* a_{0,i_1}, \dots, a_{p_0-1,0}^* a_{p_0-1,i_1}, 0, \dots, 0)^t, \\ \vec{\gamma}_2 &:= (a_{00}^* a_{0,i_2}, \dots, a_{p_0-1,0}^* a_{p_0-1,i_2}, 0, \dots, 0)^t, \\ &\vdots \\ \vec{\gamma}_{p_0-1} &:= (a_{00}^* a_{0,i_{p_0-1}}, \dots, a_{p_0-1,0}^* a_{p_0-1,i_{p_0-1}}, 0, \dots, 0)^t \end{aligned} \tag{8}$$

are these linearly independent vectors.

Now we consider the column vectors of A except for first column vector, and pick out the column vector that has the maximal nonzero entries from the p_0 -th entry to the m -th entry. Without loss of generality, we suppose that the second column vector of A is such a vector and has p_1 zeros from the p_0 -th entry to the $p_0 + p_1 - 1$ -th entry with $1 \leq p_1 \leq m - p_0$. If $p_1 = 1$, then $\vec{\alpha}_2 = (|a_{00}|^2 - |a_{01}|^2, \dots, |a_{p_0-1,0}|^2 - |a_{p_0-1,1}|^2, -|a_{p_0,1}|^2, 0, \dots, 0)^t$ is linearly independent from the vectors $\vec{\gamma}_1, \vec{\gamma}_2, \dots, \vec{\gamma}_{p_0-1}$.

If $p_1 > 1$, then we check the column vectors $\vec{\beta}_{10}, \vec{\beta}_{12}, \dots, \vec{\beta}_{1,m-1}$ and $\vec{\alpha}_1$. Let $x_0\vec{\beta}_{10} + x_1\vec{\alpha}_1 + x_2\vec{\beta}_{12} + \dots + x_{m-1}\vec{\beta}_{1,m-1} = 0$. We have linear equations,

$$\begin{aligned} x_0a_{p_0,0} - x_1a_{p_0,1} + x_2a_{p_0,2} + x_3a_{p_0,3} + \dots + x_{m-1}a_{p_0,m-1} &= 0, \\ x_0a_{p_0+1,0} - x_1a_{p_0+1,1} + x_2a_{p_0+1,2} + x_3a_{p_0+1,3} + \dots + x_{m-1}a_{p_0+1,m-1} &= 0, \\ &\vdots \\ x_0a_{p_0+p_1-1,0} - x_1a_{p_0+p_1-1,1} + x_2a_{p_0+p_1-1,2} + x_3a_{p_0+p_1-1,3} + \dots + x_{m-1}a_{p_0+p_1-1,m-1} &= 0. \end{aligned} \tag{9}$$

Taking into account that the rank of coefficient matrix of Eqs. (9)

$$F_2 = \begin{pmatrix} a_{p_0,0} & -a_{p_0,1} & a_{p_0,2} & \dots & a_{p_0,m-1} \\ a_{p_0+1,0} & -a_{p_0+1,1} & a_{p_0+1,2} & \dots & a_{p_0+1,m-1} \\ \dots & \dots & \dots & \dots & \dots \\ a_{p_0+p_1-1,0} & -a_{p_0+p_1-1,1} & a_{p_0+p_1-1,2} & \dots & a_{p_0+p_1-1,m-1} \end{pmatrix}$$

is p_1 because of the unitarity of A , there exist p_1 linearly independent vectors in $\vec{\beta}_{10}, \vec{\beta}_{12}, \dots, \vec{\beta}_{1,m-1}$ and $\vec{\alpha}_1$. Without loss of generality, we assume

$$\begin{aligned} \vec{\gamma}_{p_0} &= (y_{1,p_0}, \dots, y_{p_0+p_1-1,p_0}, 0, \dots, 0)^t, \\ \vec{\gamma}_{p_0+1} &= (y_{1,p_1}, \dots, y_{p_0+p_1-1,p_1}, 0, \dots, 0)^t, \\ &\vdots \\ \vec{\gamma}_{p_0+p_1-1} &= (y_{1,p_0+p_1-1}, \dots, y_{p_0+p_1-1,p_0+p_1-1}, 0, \dots, 0)^t \end{aligned} \tag{10}$$

are linearly independent with $(y_{p_0,j}, \dots, y_{p_0+p_1-1,j}) \neq 0$ for $j = p_0, \dots, p_0 + p_1 - 1$. It is obvious that these p_1 vectors $\vec{\gamma}_{p_0}, \dots, \vec{\gamma}_{p_0+p_1-1}$ are linearly independent from the vectors $\vec{\gamma}_1, \dots, \vec{\gamma}_{p_0-1}$. Therefore we have now $p_0 + p_1 - 1$ linearly independent column vectors in matrix C_0 .

At last, we check the column vectors of A except for the first two column vectors, and pick out the column vector that has the maximal nonzero entries from the $(p_0 + p_1)$ -th entry to the m -th entry. Continuing the procedure above, we can find $m - 1$ linearly independent vectors. \square

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