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by

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THE SEPARATING SEMIGROUP OF A REAL CURVE

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Abstract. We introduce the separating semigroup of a real algebraic curve of dividing type. The elements of this semigroup record the possible degrees of the covering maps obtained by restricting separating morphisms to the real part of the curve. We also introduce the hyperbolic semigroup which consists of elements of the separating semigroup arising from morphisms which are compositions of a linear projection with an embedding of the curve to some projective space.

We completely determine both semigroups in the case of maximal curves. We also prove that any embedding of a real curve to projective space of sufficiently high degree is hyperbolic. Using these semigroups we show that the hyperbolicity locus of an embedded curve is in general not connected.

1. Introduction

Here a curve will always be a non-singular projective and geometrically irreducible algebraic curve over $\mathbb{R}$. Furthermore, we always use $\mathbb{P}^n$ to denote the projective space defined over $\mathbb{R}$. For a variety V defined over $\mathbb{R}$ we denote by $V(\mathbb{R})$ and $V(\mathbb{C})$ the real and complex points of V, respectively.

A basic fact concerning the classification of real algebraic curves, or real Riemann surfaces, is the following dichotomy which goes back to Klein [Kle23, §23]: If X is a curve, then the set $X(\mathbb{C}) \smallsetminus X(\mathbb{R})$ has either one or two connected components. If the latter is the case, X is called *of dividing type*. Curves of dividing type are often called *curves of type I* or *separating* in the literature. If there exists a morphism $f\colon X \to \mathbb{P}^1$ with the property $f^{-1}(\mathbb{P}^1(\mathbb{R})) = X(\mathbb{R})$, then $X(\mathbb{C}) \smallsetminus X(\mathbb{R})$ can not be connected. This is since $\mathbb{P}^1(\mathbb{C}) \smallsetminus \mathbb{P}^1(\mathbb{R})$ has two connected components and their preimages under f yield two connected components of $X(\mathbb{C}) \smallsetminus X(\mathbb{R})$. Therefore, such a morphism is also called *separating* since it certifies that X is of dividing type. It follows from the work of Ahlfors [Ahl50, §4.2], though proved in a different context, that conversely every separating curve admits a separating morphism.

Rokhlin [Rok78] used the existence of separating morphisms given by pencils as a certification of certain real plane curves being of dividing type. In Mikhalkin's [Mik00] study of extremal topology of real curves in $(\mathbb{C}^*)^2$, he showed that the logarithmic Gauss map of a simple Harnack curve is separating. Conversely, it was shown by Passare and Risler [PR11] that if a planar curve has separating logarithmic Gauss map, then it is a Harnack curve.

The existence of separating morphisms and their properties have been considered by several authors [Hui01, Gab06, CH13, Cop13]. For example, Gabard [Gab06] showed that every separating curve X admits a separating morphism of degree at most $\frac{g+r+1}{2}$ where g is the genus and r the number of connected components of $X(\mathbb{R})$. Later Coppens [Cop13] constructed, for every value of k between r and

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$\frac{g+r+1}{2}$, a separating curve X of genus g with $X(\mathbb{R})$ having r components such that k is the smallest possible degree of a separating morphism.

In this work, we take a complementary approach. Namely, we fix a curve X of dividing type of genus g and study the set of all separating morphisms $X \to \mathbb{P}^1$. Let $X(\mathbb{R})$ consist of r connected components $X_1, \dots, X_r$. Since X is of dividing type $r+g$ must be odd. A separating morphism f is always unramified over $X(\mathbb{R})$ [KS15, Theorem 2.19]. Therefore, the restriction of f to each X_i is a covering map of $\mathbb{P}^1(\mathbb{R})$. This implies that the degree of a separating morphism is at least r. Let $\mathbb{N}$ denote the positive integers, so $0 \notin \mathbb{N}$. We denote by $d_i(f) \in \mathbb{N}$ the degree of the covering map $X_i \to \mathbb{P}^1$ and set $d(f) := (d_1(f), \dots, d_r(f)) \in \mathbb{N}^r$. For $d = (d_1, \dots, d_r) \in \mathbb{N}^r$ we let $|d| := \sum_{i=1}^r d_i$. Our first main object of interest is the set of all such degree partitions.

Definition 1.1. The set

$$\mathrm{Sep}(X) = \{d(f) \in \mathbb{N}^r \mid f\colon X \to \mathbb{P}^1 \text{ separating}\}$$

is called the *separating semigroup.*

Since we assume that X is a separating curve, the set $\mathrm{Sep}(X)$ is always non-empty. The term semigroup is justified by the fact that this set turns out to be closed under componentwise addition, see Proposition 2.1.

In Corollary 3.4, we show that $d + \mathbb{Z}^r_{\geq 0} \subset \mathrm{Sep}(X)$ for every $d \in \mathrm{Sep}(X)$ with $|d|$ sufficiently large. Our main technique is making use of *interlacing sections*: Two sections s and s' of a line bundle on X are called interlacing if they both have only simple and real zeros and if on each component X_i between each two consecutive zeros of s there is exactly one zero of s'. This notion generalizes the notion of interlacing polynomials to sections of algebraic curves. The concept of interlacing polynomials has attracted a lot of attention since Marcus, Spielman and Srivastava used it to solve the Kadison–Singer problem as well as to find bipartite Ramanujan graphs of all degrees [MSS15a, MSS15b]. We will make use of the fact that the morphism to $\mathbb{P}^1$ defined by s and s' is separating if and only if s and s' are interlacing. This is proved in Lemma 2.10.

We also study the subset $\mathrm{Sep}(X)$ consisting of all degree partitions that are realized by a separating morphism which is actually a linear projection of some embedding of X in projective space from a linear space disjoint from X. This is motivated by the following definition from [SV14].

Definition 1.2. Let $X \subset \mathbb{P}^n$ be a curve and $E \subset \mathbb{P}^n$ be a linear subspace of codimension two such that $E \cap X = \emptyset$. Then X is *hyperbolic with respect to* E if the linear projection $\pi_E : X \to \mathbb{P}^1$ from E is separating.

Following the terminology of [SV14] we will call such embedded curves *hyperbolic.* These curves are a generalization of planar curves defined by hyperbolic polynomials in three homogeneous variables. In general hyperbolic polynomials have attracted interest in different areas of mathematics like partial differential equations [Går51, Hör05], optimization [Gül97, Ren06] and combinatorics [COSW04, Brä07].

Whereas at first sight hyperbolicity might seem to be a rare phenomenon, it is actually quite ubiquitous in the case of curves as justified by the next theorem. It says that for a given separating curve X, *every* embedding of high enough degree turns out to be hyperbolic. We first remark any divisor D on a curve X with degree $k > 2g$ is very ample by [Har77, Corollary 3.2 b)], and therefore the map $X \to \mathbb{P}(\mathscr{L}(D)^\vee)$ is an embedding.

Theorem 1.3. *Let X be a curve of dividing type of genus g. There exists a $k > 2g$ with the following property: For any divisor D of degree at least k the corresponding embedding of X to $\mathbb{P}(\mathscr{L}(D)^\vee)$ is hyperbolic.*

Definition 1.4. The *hyperbolic semigroup* $\mathrm{Hyp}(X)$ is the set of all elements of $\mathrm{Sep}(X)$ where the corresponding f can be chosen to be the composition of a linear projection with an embedding of X to some $\mathbb{P}^n$, where the center of the projection is disjoint from X.

Remark 1.5. Replacing $\mathbb{P}^n$ by $\mathbb{P}^3$ in Definition 1.4 results in an equivalent condition [KS15, §2]. Also, in the definition of $\mathrm{Hyp}(X)$, one could equivalently just require f to be separating and $f^*\mathcal{O}_{\mathbb{P}^1}(1)$ to be very ample.

The set $\mathrm{Hyp}(X)$ also turns out to be a semigroup, see Proposition 2.1. In Proposition 2.12, we give an equivalent criterion for a curve to be hyperbolic in terms of the linking numbers of its components with the linear subspace from which we project. Moreover, Proposition 2.17 describes a sufficient condition for an embedded curve to be hyperbolic.

For a planar curve X of dividing type, a pencil of curves is said to be totally real with respect to X if every curve in the pencil intersects X in only real points. In [Tou13], Fiedler-Le Touzé asks if for every planar curve of dividing type there exists a totally real pencil. Using our techniques we can answer this question in the affirmative even when the base points of the pencil are not contained in the curve. In the next theorem, we let $\mathcal{V}$ denote the subvariety of $\mathbb{P}^2$ defined by a collection of homogeneous polynomials in $\mathbb{R}[x, y, z]$.

Theorem 1.6. *If $X \subset \mathbb{P}^2$ is a curve of dividing type, then there exists an integer k such that for any $k' \geq k$ there are homogeneous polynomials $f, g \in \mathbb{R}[x, y, z]$ of degree k' for which $\mathcal{V}(f, g) \cap X(\mathbb{R}) = \emptyset$ and such that $\mathcal{V}(\lambda f + \mu g)$ intersects X only in real points for every $\lambda, \mu \in \mathbb{R}$ not both zero.*

A curve X of genus g is called an *M-curve* if $X(\mathbb{R})$ has $r = g + 1$ connected components. Every M-curve is of dividing type. In the case of M-curves we give a complete description of both the separating and hyperbolic semigroups.

Theorem 1.7. *Let X be an M-curve.*
a) If $g = 0$, then $\mathrm{Hyp}(X) = \mathrm{Sep}(X) = \mathbb{N}$.
b) If $g = 1$, then $\mathrm{Sep}(X) = \mathbb{N}^2$ *and* $\mathrm{Hyp}(X) = \mathbb{N}^2 \smallsetminus \{(1, 1)\}$.
c) If $g > 1$, then $\mathrm{Sep}(X) = \mathbb{N}^{g+1}$ *and* $\mathrm{Hyp}(X) = \{d \in \mathbb{N}^{g+1} : \sum_{i=1}^{g+1} d_i \geq g + 3\}$.

Finally, in Section 5 we study the hyperbolicity locus of an embedded curve in an example. Given an embedded curve X in $\mathbb{P}^n$, Shamovich and Vinnikov asked if the subset of the Grassmannian $\mathrm{Gr}(n-1, n+1)$ corresponding to the linear spaces from which the projection of X is separating is connected [SV14]. In Example 5.1, using the hyperbolic semigroup we construct an example where the answer is negative.

2. The separating and hyperbolic semigroups

We begin by showing that the sets $\mathrm{Sep}(X)$ and $\mathrm{Hyp}(X)$ are indeed semigroups.

Proposition 2.1. *Let X be a curve of dividing type. Then both* $\mathrm{Sep}(X)$ *and* $\mathrm{Hyp}(X)$ *are semigroups.*

Proof. Let $f_1, f_2 \colon X \to \mathbb{P}^1$ be two separating morphisms. Let X_+ be one of the connected components of $X(\mathbb{C}) \smallsetminus X(\mathbb{R})$. Without loss of generality we may assume that $f_1(X_+) = f_2(X_+) = H_+$ is the upper half-plane. Identify $\mathbb{P}^1(\mathbb{C})$ with $\mathbb{C} \cup \{\infty\}$ and let $\phi \colon \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be a Möbius transformation sending the circle $|z| = 1$ to $\mathbb{P}^1(\mathbb{R})$. Define the map $g \colon X \to \mathbb{P}^1(\mathbb{C})$ by

$$g(x) = \phi^{-1}(f_1(x)) \cdot \phi^{-1}(f_2(x)).$$

The preimage $g^{-1}(z)$ of any point z with $|z| = 1$ is contained in $\mathbb{P}^1(\mathbb{R})$. Moreover, this preimage consists of exactly $d_i(f_1) + d_i(f_2)$ points on X_i for $i = 1, \ldots, r$. Then the composition $f = \phi \circ g(x)$ is a separating map which satisfies $d(f) = d(f_1)+d(f_2)$. This proves that $\mathrm{Sep}(X)$ is a semigroup.

To show that $\mathrm{Hyp}(X)$ is a semigroup, suppose that $\mathcal{L}_1 = f_1^*\mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{L}_2 = f_2^*\mathcal{O}_{\mathbb{P}^1}(1)$ are both very ample. Then the line bundle $f^*\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{L}_1 \otimes \mathcal{L}_2$, where f is defined as above, is also very ample. Therefore $\mathrm{Hyp}(X)$ is also a semigroup. $\square$

When $X = \mathbb{P}^1$ we have $1 \in \mathrm{Hyp}(\mathbb{P}^1)$. Therefore, the next corollary is an immediate consequence of Proposition 2.1.

Corollary 2.2. *We have* $\mathrm{Sep}(\mathbb{P}^1) = \mathrm{Hyp}(\mathbb{P}^1) = \mathbb{N}$.

Example 2.3. In fact, we have $\mathrm{Sep}(X) = \mathbb{N}$ if any only if $X = \mathbb{P}^1$. This is because if X is a curve of genus $g > 0$ there is no map $X \to \mathbb{P}^1_{\mathbb{C}}$ of degree 1, by the Riemann-Hurwitz formula.

Remark 2.4. For a dividing curve X of genus g, every $d \in \mathrm{Sep}(X)$ with $|d| \geq 2g+1$ is also in $\mathrm{Hyp}(X)$. This is because every line bundle $\mathscr{L}$ on X with $\deg(\mathscr{L}) \geq 2g+1$ is very ample [Har77, Corollary 3.2 b)].

Example 2.5. Let X be a hyperelliptic curve of genus $g = 2$ given by $x_2^2 = p(x_0, x_1)$ where p is a positive definite form of degree six. We consider X to be a subvariety of the weighted projective space $\mathbb{P}^2(1, 1, 3)$, so that it is non-singular. Then the canonical map is separating. Since $r + g$ must be odd and f is unramified over the real points, we conclude that $r = 1$. Thus, we have $2 \in \mathrm{Sep}(X)$. By [Ahl50, §4.2] we also have $3 \in \mathrm{Sep}(X)$ but we do not have $1 \in \mathrm{Sep}(X)$ because the curve is not rational. Thus, we have $\mathrm{Sep}(X) = \mathbb{N}_{\geq 2}$.

Remark 2.6. In general, the separating and hyperbolic semigroups do not only depend on r and g. For example, it is possible to construct a hyperelliptic curve of genus three whose canonical map is separating. In that case we have $r = 2$. But there are also separating curves of genus three with $r = 2$ that are not hyperelliptic and therefore do not admit a separating morphism of degree two.

Example 2.7. Let $g = 1$ and suppose that $r = 2$. Then there is an automorphism of X that sends X_1 to X_2. Thus, $\mathrm{Sep}(X)$ is stable under the action of the symmetric group $\mathfrak{S}_2$. From embeddings to $\mathbb{P}^2$ we obtain $(2, 1), (1, 2) \in \mathrm{Hyp}(X)$. One also has $(1, 1) \in \mathrm{Sep}(X) \smallsetminus \mathrm{Hyp}(X)$.

Example 2.8. In this example we show that the separating semigroup of a planar hyperbolic curve X of degree k is not symmetric for $k \geq 4$. This version of the argument was suggested by Erwan Brugallé following our original approach for $k = 4$. The number of connected components of $X(\mathbb{R})$ is $r = \lceil \frac{k}{2} \rceil$. A linear system of rank 2 on a curve X of genus $g \geq 3$ is unique, [Nam79, §2.3] or [ACGH85, A.18]. So we can label the connected components $X_1, \ldots, X_r$ from the innermost oval X_1 to the outermost oval X_r if k is even. If k is odd then X_{r-1} is the outermost and X_r is the unique pseudoline. By our hyperbolicity assumption $(2, 2, \ldots, 2) \in \mathrm{Hyp}(X) \subset \mathrm{Sep}(X)$ if k is even and $(2, 2, \ldots, 2, 1) \in \mathrm{Hyp}(X) \subset \mathrm{Sep}(X)$ if k is odd.

The gonality of a planar curve X of degree k is $k - 1$ and moreover every map $f\colon X \to \mathbb{P}^1$ of degree $k-1$ is induced by a projection $\mathbb{P}^2 \to \mathbb{P}^1$ whose center is a point on X, [Nam79, §2.3] or [ACGH85, A.18]. Therefore, we have $(1, 2, \ldots, 2) \in \mathrm{Sep}(X)$ if k is even and, if k is odd $(1, 2, \ldots, 2, 1) \in \mathrm{Sep}(X)$. However, no other permutation of these degree sequences is possible, since a projection whose center is not on the innermost oval of X would not be a separating morphism.

Therefore, the semigroup $\mathrm{Sep}(X)$ is in general not preserved under the action of the symmetric group. Moreover, it follows that the semigroup $\mathrm{Sep}(X)$ is an invariant of the intrinsic curve X which is capable of distinguishing which connected component of $X(\mathbb{R})$ is the innermost oval of a hyperbolic embedding to $\mathbb{P}^2$ when $g \geq 3$.

2.1. **Separating morphisms and interlacing sections.** The following definition generalizes the interlacing property for polynomials [Fis06] to sections of line bundles.

Definition 2.9. Let $\mathscr{L}$ be a line bundle on X. Let $s_0, s_1 \in \Gamma(X, \mathscr{L})$ be two global sections that both have only simple and real zeros. We say that s_0 and s_1 *interlace* if each connected component of $X(\mathbb{R}) \setminus \{P \mid s_0(P) = 0\}$ contains exactly one zero of s_1 and vice versa.

Lemma 2.10. *Let $\mathscr{L}$ be a line bundle on X and let $s_0, s_1 \in \Gamma(X, \mathscr{L})$ be global sections that generate $\mathscr{L}$ and that both have only simple and real zeros. Then the morphism $X \to \mathbb{P}^1$ given by $x \mapsto (s_0(x) : s_1(x))$ is separating if and only if s_0 and s_1 are interlacing.*

Proof. Assume that s_0 and s_1 interlace. If the map is not separating, then there is a $\lambda \in \mathbb{R}$ such that $s_0 + \lambda s_1$ has a double zero on $X(\mathbb{R})$. Since s_0 and s_1 generate $\mathscr{L}$, the section $s_0 + \lambda s_1$ does not vanish on any zero of s_1 for any λ. Thus, because s_0 has exactly one zero on each connected component of $X(\mathbb{R}) \setminus \{P \mid s_1(P) = 0\}$, a double root of $s_0 + \lambda s_1$ is impossible. Therefore, interlacing implies separating.

Conversely, assume that the morphism under consideration is separating. It follows immediately that all zeros of s_0 and s_1 are real. The other properties of interlacing sections follow from the fact that the restriction of separating morphisms to the real part is unramified [KS15, Theorem 2.19]. □

To verify the interlacing property of a pair of sections we have the following sufficient criterion which we will use later on.

Proposition 2.11. *Let $\mathscr{L}$ be a line bundle on X and let $s_0, s_1 \in \Gamma(X, \mathscr{L})$ be global sections that generate $\mathscr{L}$. Let s_0 have only simple and real zeros. Let I be the set of indices i such that s_0 has more than one zero on X_i. Assume that for each $i \in I$ there is exactly one zero of s_1 on each connected component of $X_i \setminus \{P \mid s_0(P) = 0\}$. Then s_0 and s_1 are interlacing and the morphism $X \to \mathbb{P}^1$ given by $x \mapsto (s_0(x) : s_1(x))$ is separating.*

Proof. As in the proof of the preceding lemma we show that there is no $\lambda \in \mathbb{R}$ such that $s_0 + \lambda s_1$ has a double zero on $X(\mathbb{R})$. Indeed, as shown in the proof of the preceding lemma, two zeros on one of the X_i for $i \in I$ cannot come together as λ varies. Since there is only one zero of s_0 on the other components, the claim is also true for those. □

2.2. **Conditions for hyperbolic morphisms.** We first point out that a curve being hyperbolic with respect to some linear space is a purely topological property.

We begin by recalling linking numbers of spheres embedded in the n-dimensional sphere S^n. For the general definition of linking numbers and more detailed information we refer to [Pra07]. Suppose that X and Y are disjoint embedded oriented spheres in S^n of dimensions p and q respectively where $n = p + q + 1$. Consider the fundamental cycles $[X]$ and $[Y]$ as cycles in the integral homology of S^n. There exists a chain W whose boundary is $[X]$. The *linking number* $\operatorname{lk}(X, Y)$ is defined to be the intersection number of W and $[Y]$.

Now let $K \subset \mathbb{P}^n(\mathbb{R})$ be the image of an embedding of S^1 and $L \subset \mathbb{P}^n(\mathbb{R})$ be a linear subspace of codimension 2. Let $\pi\colon S^n \to \mathbb{P}^n(\mathbb{R})$ be an unramified 2 to 1 covering map. Notice that $\pi^{-1}(L)$ is a sphere of dimension $n-2$ in S^n and $\pi^{-1}(K)$ is either an embedded circle or two embedded circles. Define the linking number of K and L in $\mathbb{P}^n(\mathbb{R})$ to be the linking number of $\pi^{-1}(K)$ and $\pi^{-1}(L)$ in S^n if $\pi^{-1}(K)$ is a single connected component and define it to be the sum of the linking numbers of K_1 with L and K_2 with L if $K_1 \cup K_2 = \pi^{-1}(K)$. Now we are able to give a topological characterization of hyperbolic curves in terms of linking numbers.

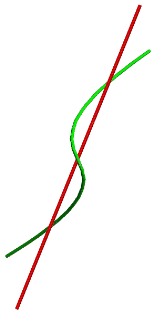

FIGURE 1. The twisted cubic drawn in green is hyperbolic with respect to the red line.

Proposition 2.12. *Let $X \subset \mathbb{P}^n$ be a curve and $E \subset \mathbb{P}^n$ be a linear subspace of dimension $n-2$ with $X \cap E = \emptyset$. Then X is hyperbolic with respect to E if and only if* $\deg(X) = \sum_{i=1}^{r} |\operatorname{lk}(X_i, E(\mathbb{R}))|$. *When X is hyperbolic with respect to E, then the tuple*

$$(|\operatorname{lk}(X_1, E(\mathbb{R}))|, \ldots, |\operatorname{lk}(X_r, E(\mathbb{R}))|)$$

is the element in $\operatorname{Hyp}(X)$ *corresponding to the projection from E.*

Proof. The curve X is hyperbolic with respect to E if and only if every hyperplane $H \subset \mathbb{P}^n$ that contains E intersects X in $\deg(X)$ many distinct real points. Let $\pi \colon S^n \to \mathbb{P}^n(\mathbb{R})$ be an unramified 2 to 1 covering map. For any choice of a hyperplane $H \subset \mathbb{P}^n$ that contains E, the preimage $X = \pi^{-1}(E)$ is a sphere of dimension $n-2$ inside $\pi^{-1}(H)$ which is a sphere of dimension $n-1$. Let $W \subset \pi^{-1}(H)$ be a hemisphere whose boundary is $\pi^{-1}(E)$.

If X is hyperbolic with respect to E, then the absolute values of the linking numbers $\operatorname{lk}(X_i, E(\mathbb{R}))$, which are the intersection numbers of the $\pi^{-1}(X_i)$ with W, sum up to $\deg(X)$. Conversely, if the intersection number of W with the preimage of $X(\mathbb{R})$ is $\deg(X)$, then H has (at least) $\deg(X)$ many real intersection points with X. The final statement about the element of $\operatorname{Hyp}(X)$ arising from the projection from E is immediate. $\square$

Example 2.13. Let $Q \subset \mathbb{P}^3$ be the quadratic surface defined by the equation $x^2 + y^2 = z^2 + w^2$. Its real part $Q(\mathbb{R})$ is the hyperboloid. For a curve X contained in Q, we can describe a topological condition for X to be hyperbolic with respect to the line E given by $x = y = 0$.

The hyperboloid $Q(\mathbb{R})$ is homeomorphic to the torus $S^1 \times S^1$ and taking a real line from each of the two rulings of Q gives a pair of generators of $H_1(Q(\mathbb{R})) \cong \mathbb{Z} \oplus \mathbb{Z}$. We will assume that these lines are oriented in the upwards z direction in the affine chart $w = 1$. For each hyperplane H containing the line E we have $[H \cap Q(\mathbb{R})] = (1,1) \in H_1(Q(\mathbb{R}))$, up to switching the orientation of $H \cap Q(\mathbb{R})$.

If $X \subset Q$, then a connected component of $X(\mathbb{R})$ realizes either the trivial class in $H_1(Q(\mathbb{R}))$ or the class (p, q) for p, q coprime integers. Otherwise, the connected components of $X(\mathbb{R})$ would have non-trivial intersections contradicting the fact that X is non-singular. In order for X to be hyperbolic with respect to E, no component of $X(\mathbb{R})$ can realize the trivial class. This is because if a connected component X_i is trivial in homology, it is the boundary of a disc contained in Q which would not intersect E and so $d_i = |\operatorname{lk}(X_i; E(\mathbb{R}))| = 0$. If furthermore $\deg(X) = r \cdot (p+q)$, then X must be hyperbolic with respect to E.

For example, we can construct a curve X of degree $2k$ in $\mathbb{P}^3$ which is hyperbolic with respect to E in the following way. Let X be the complete intersection of Q with a hypersurface S which is a small perturbation of the union of hyperplanes H_i with equations of the form $z + a_i w$ for $i = 1, \ldots, k$. Then $X(\mathbb{R})$ consists of k connected components, and each one realizes the class $(1, -1) \in H_1(Q(\mathbb{R}))$ up to changing the orientations of the connected components of $X(\mathbb{R})$. By the above remark, the curve X is hyperbolic with respect to E and thus $(2, \ldots, 2) \in \operatorname{Hyp}(X)$. Let E' be any real line on Q. By our choice of generators of $H_1(Q(\mathbb{R}))$, we can suppose that $[E'(\mathbb{R})] = (1, 0)$, up to a change in orientation. Therefore, the real points $E'(\mathbb{R})$ intersect $X(\mathbb{R})$ in at least k points. The projection of X from E' is a separating map of degree k with degree partition $(1, \ldots, 1) \in \mathbb{N}^k$. The curve X has genus $(k-1)^2$ and $X(\mathbb{R})$ has k connected components. In particular, if $k \geq 3$, then X is not an M-curve, see Section 4.

Example 2.14. We can carry out the construction from the previous example with Q now being the quadratic surface defined by $x^2 + y^2 + z^2 = w^2$. Then $Q(\mathbb{R})$ is the sphere. Again let S be the hypersurface which is a small perturbation of the union of hyperplanes H_i with equations of the form $z + a_i w$ with $-1 < a_i < 1$ for $i = 1, \ldots, k$.

Let X be the complete intersection of S and Q. Then the resulting curve X is again hyperbolic with respect to the line E defined by $x = y = 0$ and we have $(2, \ldots, 2) \in \operatorname{Hyp}(X)$. As in Example 2.13, the curve X has genus $(k-1)^2$ and $X(\mathbb{R})$ has k connected components. But we will show that unlike in the preceding example, we have $(1, \ldots, 1) \notin \operatorname{Sep}(X)$ if $k > 2$. In fact, there is no real morphism $X \to \mathbb{P}^1$ of degree k. Any complex line on Q intersects X in k points and the projection from this line gives a map to $\mathbb{P}^1$ of degree k. The gonality of the complexified curve is k and every morphism $X \to \mathbb{P}^1$ of degree k comes from the projection from a line by [Bas96] and [CL84]. However, since the surface Q does not contain any real lines, there are no real lines intersecting $X(\mathbb{R})$ in more than two points proving the claim that there are no real morphisms of degree k to $\mathbb{P}^1$.

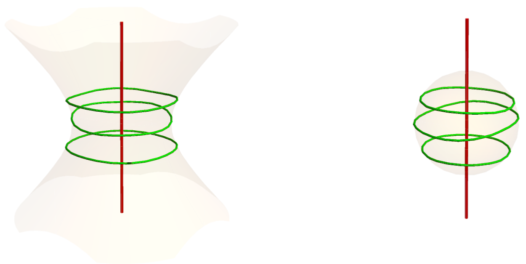

FIGURE 2. Two canonical curves of genus three with real part having three connected components that are hyperbolic with respect to the red line. The curve on the left has $(1,1,1)$ in its separating semigroup whereas the one on the right does not.

Example 2.15. Let X be a curve of genus g with $X(\mathbb{R})$ having r connected components. Assume that there is a separating morphism $f \colon X \to \mathbb{P}^1$ with the property that $f^*\mathcal{O}_{\mathbb{P}^1}(1)$ is the canonical line bundle. The degree of a divisor of a non-zero holomorphic differential form on X restricted to any connected component of $X(\mathbb{R})$ must be even, see for example [GH81, Proposition 4.2]. Therefore, we have that $d(f) = 2d'$ for some $d' \in \mathbb{N}^r$. This implies that $r \leq g-1$. Examples 2.5, 2.8, and 2.13 show that for $g = 2, 3, 4$ such a morphism exists for a curve with $r = g-1$. For planar curves $X \subset \mathbb{P}^2$ of degree $d \geq 4$ the canonical bundle is given by $\mathcal{O}_X(d-3)$. Furthermore, it was shown in [Tou13] that for some planar sextic curves with 9 ovals there exists a pencil of cubics without base points on X that gives rise to a separating morphism. Thus for $g = 10$ and $r = 9$ we can also find such a morphism. For curves of genus different from $2, 3, 4$ and 10 we do not know if this is the case.

We will now prove a sufficient criterion for a curve embedded in projective space to be hyperbolic. First we need a lemma from linear algebra.

Lemma 2.16. *Suppose that $\mathcal{P}$ is a collection of points on a hyperplane $H \subset \mathbb{R}^n$. Let*

$$\cup_{i=1}^{r}(\mathcal{P}_i^+ \cup \mathcal{P}_i^-)$$

be a partition of $\mathcal{P}$ such that the collection $\mathcal{P}_i^+ \cup \mathcal{P}_i^-$ is non-empty and consists of affinely linearly independent elements for each i. If $|\mathcal{P}| - r \leq n-1$, then there exists a linear polynomial $l \in \mathbb{R}[x_1, \dots, x_n]$ such that

$$\begin{aligned} l(p)l(q) > 0 \quad &\text{if} \quad p, q \in \mathcal{P}_i^\epsilon \text{ and} \\ l(p)l(q) < 0 \quad &\text{if} \quad p \in \mathcal{P}_i^\epsilon \text{ and } q \in \mathcal{P}_i^{\epsilon'}, \end{aligned}$$

where $\epsilon, \epsilon' \in \{+, -\}$ and $\epsilon \neq \epsilon'$.

Proof. Let $p_i := |\mathcal{P}_i^+| + |\mathcal{P}_i^-|$. By our assumption on the collection $\mathcal{P}_i^+ \cup \mathcal{P}_i^-$, the affine spans of $\mathcal{P}_i^+$ and $\mathcal{P}_i^-$ are disjoint. Therefore, we can find an affine hyperplane in the affine span of $\mathcal{P}_i^+ \cup \mathcal{P}_i^-$ which separates the points in $\mathcal{P}_i^+$ and $\mathcal{P}_i^-$. This affine subspace is of dimension $p_i - 2$.

The affine span L of all such subspaces has dimension $|\mathcal{P}| - r - 1$. If this is strictly less than $n-1$ we can find a hyperplane H' distinct from H containing L. A defining equation of H' gives the linear polynomial l. This completes the proof. □

Proposition 2.17. *Let $X \subset \mathbb{P}^n$ be a real curve of degree k which is not contained in a hyperplane. Suppose that there exists a hyperplane $H \subset \mathbb{P}^n$ such that H intersects $X(\mathbb{R})$ transversally in k points and intersects each connected component of $X(\mathbb{R})$. If $k - r \leq n-1$, then there exists a codimension two linear space L such that X is hyperbolic with respect to L. Moreover, the restriction of the projection from L to the component X_i has degree $k_i = X_i \cap H$.*

Proof. Without loss of generality we will assume that $X_1, \dots, X_{r'}$ are the connected components of $X(\mathbb{R})$ for which $k_i > 1$. If $k_i = 1$ for a component X_i, then any hyperplane H' must intersect that component in at least one point. Thus we can ignore such components.

We will partition the points in $H \cap X_i = \{P_1^i, \dots, P_{k_i}^i\}$ into two sets $\mathcal{P}_i^+$ and $\mathcal{P}_i^-$ for $i = 1, \dots, r'$. Choose an affine chart of $\mathbb{P}^n$ so that the hyperplane at infinity H_∞ intersects $X(\mathbb{R})$ transversally and $X(\mathbb{R}) \cap H \cap H_\infty = \emptyset$. For each i we have $|X_i \cap H_\infty| \equiv k_i \mod 2$. Assume that $P_1^i, \dots, P_{k_i}^i$ are cyclically arranged on X_i and set $P_1^i \in \mathcal{P}_i^+$. Then partition the rest of the set of points on X_i by setting, $P_{j-1}^i, P_j^i \in \mathcal{P}_i^\epsilon$ if P_{j-1}^i and P_j^i are in different connected components of $X_i \setminus H_\infty$. When P_{j-1}^i and P_j^i are in the same connected component of $X_i \setminus H_\infty$, then set $P_j^i \in \mathcal{P}_i^{\epsilon'}$, where $P_{j-1}^i \in \mathcal{P}_i^\epsilon$ for $\epsilon' \neq \epsilon$.

Let $k' = \sum_{i=1}^{r'} k_i$ and notice that $k' - r' = k - r$. By assumption $k' - r' \leq n-1$, so we can apply Lemma 2.16 to find a linear polynomial $l \in \mathbb{R}[x_1, \dots, x_n]$ such that the hyperplane H' it defines separates the points in $\mathcal{P}_i^+$ and $\mathcal{P}_i^-$ for $i = 1, \ldots, r'$. Let L be the intersection of H and H'. The projection from L is given by $x \mapsto (s(x) : s'(x))$ where s, s' are the sections of $\mathcal{O}_X(1)$ corresponding to H and H' respectively. By construction, the sections s and s' satisfy the assumptions of Proposition 2.11 and so X is hyperbolic with respect to L.

The final claim about the degree of the projection follows immediately from the construction. This proves the proposition. □

Remark 2.18. In order for the condition $k - r \leq n - 1$ in Proposition 2.17 to be satisfied either X must be a M-curve in the sense that $r = g + 1$ (see Section 4) or the divisor D on X arising from the embedding $\mathbb{P}^n$ is special. This is a simple consequence of the Riemann–Roch theorem.

The next two lemmas exclude specific elements from the hyperbolic semigroup.

Lemma 2.19. *Let X be a curve of dividing type. If $d = (1, \ldots, 1) \in \mathrm{Hyp}(X)$, then X is a rational curve, and hence $d = (1)$.*

Proof. Assume that there is a hyperbolic embedding of X to $\mathbb{P}^3$ with each component having degree 1. The linear projection of X to the plane from any point not on X will send each X_i to a pseudoline. In $\mathbb{P}^2$ each two pseudolines intersect, which implies that the image of X will have at least $\frac{1}{2}r(r-1)$ simple nodes given that the center of projection was chosen generally enough. On the other hand, the degree is r. By the genus–degree formula this implies that

$$g \leq \frac{1}{2}(r-1)(r-2) - \frac{1}{2}r(r-1) = 1 - r.$$

Therefore, if $r = 1$ then X is a rational curve and if $r > 1$ then X is cannot be irreducible. This proves the lemma. □

Lemma 2.20. *Let X be a curve of dividing type such that $r > 2$. Then no permutation of $(2, 1, \ldots, 1)$ is in $\mathrm{Hyp}(X)$.*

Proof. Assume that $(2, 1, \ldots, 1) \in \mathrm{Hyp}(X)$ and let $X \subset \mathbb{P}^3$ be a realizing embedding. Let X be hyperbolic with respect to a line L and let $H \subset \mathbb{P}^3$ be any hyperplane containing L. Let $H \cap X_1 = \{P_1, P_2\}$ and $H \cap X_i = \{Q_i\}$ for all $i = 2, \ldots, r$. Note that X is not contained in any plane since $r > 2$. Thus, we can assume that there is a Q_{i_0} that is not on the line spanned by P_1 and P_2. The set A of all real lines in H that do not pass through P_1 or P_2 has two connected components. Let $L' \subset H$ be a line through Q_{i_0} that is in the same connected component of A as L. Every hyperplane containing L' intersects X_1 in at least two real points because of Proposition 2.12 and $|\operatorname{lk}(X_1, L(\mathbb{R}))| = |\operatorname{lk}(X_1, L'(\mathbb{R})|$. Also it intersects every X_i with $i \geq 2$ in at least one real point. Now any hyperplane H' spanned by L' and a point $Q' \neq Q_{i_0}$ on X_{i_0} would intersect X in more than degree many points. □

3. Some results for the general case

In general it is not easy to determine the separating and the hyperbolic semigroup for a given curve. In this section we provide a method that allows us under some reasonable assumptions to construct from a separating morphism another separating morphism of one degree higher. The main result of this section is that any embedding of a separating curve of high enough degree is hyperbolic. Therefore, hyperbolic embeddings are the rule rather than the exception.

Lemma 3.1. *Let $\mathscr{L}$ be a line bundle on X. Let $s_1, s_2 \in \Gamma(X, \mathscr{L})$ be two global sections. Let $(s_1)_0 = P_0 + \ldots + P_n$ with pairwise distinct $P_j \in X(\mathbb{R})$ and $s_2(P_0) \neq 0$. Let $U_j \subset X(\mathbb{R})$ be an open neighbourhood of P_j for all $j = 1, \ldots, n$. There is an* *open neighbourhood* $U_0 \subset X(\mathbb{R})$ *of* P_0 *such that for every* $Q_0 \in U_0$ *there are* $Q_j \in U_j$ *for* $j = 1, \ldots, n$ *and* $s_3 \in \Gamma(X, \mathscr{L})$ *with* $(s_3)_0 = Q_0 + \ldots + Q_n$.

Proof. It suffices to show the claim in the case where the U_j are pairwise disjoint and do not contain P_0 for $j = 1, \ldots, n$. Let f be a rational function on X with the property that $(f) = (s_1)_0 - (s_2)_0$. Since f has only simple and real zeros, there is a $c > 0$ such that for all $\epsilon \in]-c, c[$ the rational function $f + \epsilon$ has a zero in each U_j for all j with P_j a zero of f. Now let $U_0 \subset X(\mathbb{R})$ be an open neighbourhood of P_0 which is disjoint from each of the U_j and is contained in $f^{-1}(]-c, c[)$. Then for every $Q_0 \in U_0$ there is an $\epsilon \in]-c, c[$ such that Q_0 is a zero of $f + \epsilon$. Note that $f + \epsilon$ has the same poles of the same orders as f. Thus, the effective divisor $(f + \epsilon) + (s_2)_0$ is of the form $Q_0 + \ldots + Q_n$ where $Q_j \in U_j$. □

Proposition 3.2. *Let* $\mathscr{L}$ *be a line bundle on* X. *Let* $s_0, s_1 \in \Gamma(X, \mathscr{L})$ *be two global sections that interlace. Let* $D = (s_0)_0$ *and let* $P \in X(\mathbb{R})$ *with* $s_0(P) \neq 0$ *such that* $\ell(D + P) > \ell(D)$. *Then there are interlacing sections* $s_0', s_1' \in \Gamma(X, \mathscr{L}')$ *such that* $(s_0')_0 = D + P$ *where* $\mathscr{L}'$ *is the line bundle corresponding to* $D + P$.

Proof. Without loss of generality, we can assume that $s_1(P) \neq 0$. Let $(s_1)_0 = D' = P_1 + \cdots + P_n$. Since $\ell(D+P) > \ell(D)$, there is a rational function $g \in \mathscr{L}(D+P)$ that has a pole at P. Then $D'' = D + P + (g)$ is an effective divisor with $P \notin \operatorname{Supp} D''$. The effective divisors $D+P$, $D'+P$, and D'' correspond to global sections s_0', f_1, and f_2 of $\mathscr{L}'$, respectively. Let U_j be the connected component of $X(\mathbb{R}) \smallsetminus \operatorname{Supp}(D+P)$ that contains P_j for all $j = 1, \ldots, n$. Applying Lemma 3.1 to f_1 and f_2 shows that there is a global section s_1' of $\mathscr{L}'$ such that s_0' and s_1' interlace. □

Remark 3.3. The condition $\ell(D + P) > \ell(D)$ is for example satisfied when D is non-special.

Corollary 3.4. *Let* X *be a curve of dividing type. Let* $d \in \operatorname{Sep}(X)$ *and* l *be the number of indices where* d_i *is odd. If* $|d| + l \geq 2g - 1$, *then* $d + \mathbb{Z}^r_{\geq 0} \subset \operatorname{Sep}(X)$.

Proof. By [Hui03, Theorem 2.5] the divisor corresponding to a separating morphism realizing d is non-special. By Remark 3.3 the statement then follows from Proposition 3.2. □

Proof of Theorem 1.3. By Ahlfors' theorem [Ahl50, §4.2] there is a separating morphism $f\colon X \to \mathbb{P}^1$. By Proposition 2.1 we can furthermore assume that the degree of f is more than $2g - 2$. Let $\mathscr{L} = f^*\mathcal{O}_{\mathbb{P}^1}(1)$ and $s_0, s_1 \in \Gamma(X, \mathscr{L})$ be two global sections that interlace. Then $D_0 = (s_0)_0$ is non-special. By [Sch00, Cor. 2.10, Rem. 2.14], there is an integer $n > 0$ such that every divisor on X of degree at least n is linearly equivalent to a sum of distinct points from $X(\mathbb{R}) \smallsetminus \operatorname{Supp} D_0$. We will show that the claim holds for $k = \max(2g + 1, n + \deg D_0)$. Indeed, let D be a divisor with $\deg D \geq k$. Then $D - D_0$ is linearly equivalent to a sum of distinct points from $X(\mathbb{R}) \smallsetminus \operatorname{Supp} D_0$ since $\deg(D - D_0) \geq n$. An iterated application of the previous proposition shows that the corresponding embedding of X to $\mathbb{P}(\mathscr{L}(D)^\vee)$ is hyperbolic. □

Proof of Theorem 1.6. The existence of a totally real pencil of curves of degree k satisfying our assumptions follows immediately by applying Theorem 1.3 to the line bundle $\mathcal{O}_X(k)$. □

Remark 3.5. If we allow base points on the curve, then the existence of a totally real pencil simply follows from Ahlfors' theorem [Ahl50, §4.2]. Indeed, let $X \subset \mathbb{P}^2$ be a curve of dividing type and $f\colon X \to \mathbb{P}^1$ be a separating morphism defined by two sections s_0, s_1 of a suitable line bundle. The rational function $\frac{s_0}{s_1}$ can be expressed in the coordinates x, y, z of $\mathbb{P}^2$ as the fraction of two homogeneous forms

f, g in x, y, z of the same degree. Then $\mathcal{V}(\lambda f + \mu g)$ intersects X only in real points for every $\lambda, \mu \in \mathbb{R}$ not both zero. However, the intersection $\mathcal{V}(f, g) \cap X$ is in general not empty, so the statement of Theorem 1.6 does not follow from this argument.

Question 1. *What are the smallest possible values for k in Theorem 1.3 and Theorem 1.6?*

Remark 3.6. Since our proof of Theorem 1.6 is not constructive, it further motivates the desire for a bound on k in Question 1. In [Tou13], it was shown that for plane sextic curves of type $\langle 2 \amalg 1\langle 6 \rangle\rangle$ and $\langle 6 \amalg 1\langle 2 \rangle\rangle$ one can choose $k = 3$ in Theorem 1.6.

Example 3.7. We consider the *Vinnikov quartic* $X \subset \mathbb{P}^2$ from [PSV11, Exp. 4.1] which is the planar curve defined by

$$2x^4 + y^4 + z^4 - 3x^2y^2 - 3x^2z^2 + y^2z^2 = 0.$$

We have $g = 3$ and $r = 2$. Moreover, the curve X is of dividing type since it is hyperbolic with respect to $(0 : 0 : 1)$. Let X_1 be the inner oval and X_2 be the outer oval. We have $(1, 2), (2, 2) \in \mathrm{Sep}(X)$, but Example 2.8 shows that $(1, 1), (2, 1) \notin \mathrm{Sep}(X)$. We can also realize $(3, 2)$ and $(1, 3)$ with pencils of conics having 3 and 4 base points on $X(\mathbb{R})$ respectively, see Figure 3. In coordinates, the separating morphisms are given by the rational functions $\frac{(2z+\sqrt{2}x-2y)(-2z+\sqrt{2}x-4y)}{xy}$ and $\frac{(2z+\sqrt{2}x-2y)(z+x+y)}{xy}$, respectively. By Corollary 3.4, we have that $(1, 2) + \mathbb{Z}_{\geq 0} \subset \mathrm{Sep}(X)$. This determines the separating semigroup of the Vinnikov curve except for some cases when $d_2 = 1$. In fact, we do not know whether or not $(n, 1)$ is in $\mathrm{Sep}(X)$ for $n \in \mathbb{N} \setminus \{1, 2\}$.

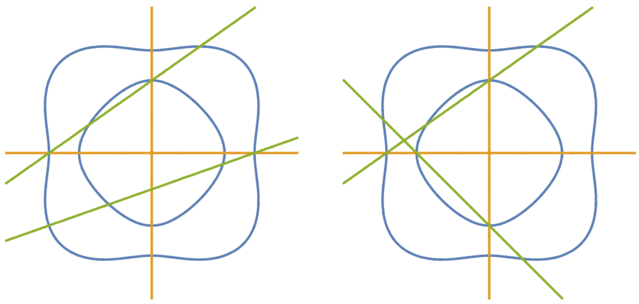

FIGURE 3. Realizing $(3, 2)$ and $(1, 3)$ with pencils of conics having 3 and 4 base points on $X(\mathbb{R})$.

4. M-CURVES

Recall that an M-curve X has exactly $g + 1$ connected components in $X(\mathbb{R})$ and that every M-curve is of dividing type. Here we prove Theorem 1.7, which completely determines the separating and hyperbolic semigroups of M-curves.

Proposition 4.1. *If X is an M-curve, then the all ones vector is in* $\mathrm{Sep}(X)$.

Proof. By [Gab06, Proposition 4.1] or [Ahl50, §4.2] there is a separating morphism of degree $g + 1$. Since the degree on each connected component is at least one, the claim follows. $\square$

Proposition 4.2. *Let X be an M-curve, then $(d_1, \ldots, d_r) \in \mathrm{Sep}(X)$ whenever at most one of the d_i is not equal to* 1.

Proof. Without loss of generality let $(d_1, \dots, d_r) = (n, 1, \dots, 1)$ with $n \geq 1$. Let $P_1, \dots, P_n \in X_1$ be n distinct points and let $Q_i \in X_i$ for $i = 2, \dots, n$. Furthermore, let $P_1', \dots, P_n'$ be points chosen from each connected component of $X_1 \smallsetminus \{P_1, \dots, P_n\}$. We consider the divisor $D = \sum_{i=1}^n (P_i - P_i') + \sum_{j=2}^r Q_i$. It follows from the Riemann–Roch formula that

$$\ell(D) \geq \deg D + 1 - g = r - g = 1.$$

Thus, there is an $0 \neq f \in \mathscr{L}(D)$ and by Proposition 2.11 it follows that the corresponding map $f \colon X \to \mathbb{P}^1$ is separating with degree partition $(n, 1, \dots, 1)$. $\square$

The next proposition is an application of Proposition 2.17 to the case of M-curves.

Proposition 4.3. *Let X be an M-curve, then $d = (d_1, \dots, d_r) \in \mathrm{Hyp}(X)$ whenever $|d| \geq g + 3$.*

Proof. Let $l = |d|$. Then since $l \geq g + 3$, by Halphen's theorem there exists a very ample non-special divisor D of degree l. Moreover, the space of such divisors which are not of this type is of codimension at least one [Har77, Proof of Proposition 6.1]. Therefore, we can suppose that $D = \sum_{i=1}^r \sum_{j=1}^{d_i} P_j^i$ where the points $P_1^i, \dots, P_{d_i}^i$ are on the connected component X_i of $X(\mathbb{R})$.

Since D is non-special, we have $\ell(D) = l - g + 1$. Therefore, we have an embedding $\iota \colon X \to \mathbb{P}^{l-g}$ and a hyperplane $H \subset \mathbb{P}^{l-g}$ so that $H \cap X_i = \{P_1^i, \dots, P_l^i\}$. Moreover, $l - r = l - g - 1$ so we can apply Proposition 2.17 to see that $\iota(X)$ is hyperbolic with respect to some codimension 2 subspace L in $\mathbb{P}^{l-g}$. The projection of X from L is a map to $\mathbb{P}^1$ that is of degree d_i restricted to X_i, so that $(d_1, \dots, d_r) \in \mathrm{Hyp}(X)$. $\square$

We conclude this section with the proof of Theorem 1.7 which completely describes the separating and hyperbolic semigroups of M-curves.

Proof of Theorem 1.7. The claim for $\mathrm{Sep}(X)$ follows in all cases from Propositions 4.1, 4.2 and 4.3. The rest of part *a*) follows from Corollary 2.2.

For the hyperbolic semigroup in part *b*) note that $(1, 1) \notin \mathrm{Hyp}(X)$ by Lemma 2.19 and that $(1, 2), (2, 1) \in \mathrm{Hyp}(X)$ by Example 2.7. Proposition 4.3 implies that every other element of $\mathbb{N}^2$ is in $\mathrm{Hyp}(X)$. The same proposition also implies one inclusion of part *c*). The other inclusion follows from Lemmas 2.19 and 2.20. $\square$

Remark 4.4. It is also possible to use a result of Huisman combined with Proposition 3.2 to find the separating semigroup of an M-curve X. If X is an M-curve, then by [Hui01, Theorem 2.4] every effective divisor whose support intersects each connected component of $X(\mathbb{R})$ is non-special. Thus starting with a separating morphism of type $(1, \dots, 1)$ by iterated application of Proposition 3.2 and Corollary 3.3 we find that $\mathrm{Sep}(X) = \mathbb{N}^r$. It is also possible to deduce Proposition 4.3 from this argument together with Halphen's theorem.

5. Domain of hyperbolicity

Given a curve $X \subset \mathbb{P}^n$ not contained in a hyperplane, we study the hyperbolicity locus $\mathcal{H}(X)$ of X. Define

$$\mathcal{H}(X) := \{L \mid X \text{ is hyperbolic with respect to } L\} \subset \mathrm{Gr}(n-1, n+1),$$

where $\mathrm{Gr}(n-1, n+1)$ is the space of codimension 2 subspaces of $\mathbb{P}^n$. In [SV14, Question 3.13] the authors asked whether $\mathcal{H}(X)$ is connected. This question is motivated by the case of planar curves where the answer is yes [HV07, Theorem 5.2]. Further evidence towards a positive answer was given by [SV14, Theorem 7.2] which states that $\mathcal{H}(X)$ is the intersection of $\mathrm{Gr}(n-1, n+1)$ with a convex cone in $\mathbb{P}^{N-1}(\mathbb{R})$ where $N = \binom{n}{n-2}$. Here we consider $\mathrm{Gr}(n-1, n+1)$ as a subset of $\mathbb{P}^{N-1}$ via the Plücker embedding. However, the next example shows that the hyperbolicity locus is in general not connected.

Example 5.1. Let $X \subset \mathbb{P}^2$ be a planar elliptic curve with $X(\mathbb{R})$ having two connected components X_1 and X_2. We can find two quadrics $q_1, q_2 \in \mathbb{R}[x, y, z]_2$ that intersect X transversally and only in real points. Furthermore, we can assume that q_i intersects X_i in exactly 4 points. This can be done by simply taking the quadric passing through 4 fixed points on X_i and a single point on X_j.

Consider the image of X in $\mathbb{P}^5$ under the second Veronese embedding. The quadric q_i determines a hyperplane H_i in $\mathbb{P}^5$ which intersects X transversally in only real points and intersects X_i in exactly 4 points. Together the curve X and the hyperplane H_i satisfy the hypotheses of Proposition 2.17. Therefore, there are two 3-planes L_1 and L_2 with respect to which X is hyperbolic. The projection maps produce the elements $(2, 4)$ and $(4, 2)$ of the semigroup $\mathrm{Hyp}(X)$. Therefore, the 3-planes L_1 and L_2 can not lie in the same connected component of $\mathcal{H}(X)$.

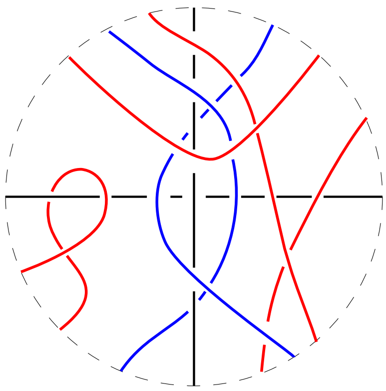

FIGURE 4. The link diagram of the curve from Example 5.2 together with two lines of hyperbolicity giving rise to different degree partitions.

Example 5.2. One can even obtain an example of a hyperbolic curve $X \subset \mathbb{P}^3$ with $\mathcal{H}(X)$ not connected. For this we proceed as is Example 5.1 to construct two totally real pencils of conics, $\lambda q_0 + \mu q_1$ and $\lambda p_0 + \mu p_1$, that give rise to different degree partitions on a plane elliptic curve. Then X is obtained as the image the plane elliptic curve under the map

$$x \mapsto (q_0(x) : q_1(x) : p_0(x) : p_1(x)).$$

For a precise example, take the image of the plane curve defined by $-z^3 + 2xz^2 - x^3 + y^2z = 0$ under the above map with $q_0 = xy$, $q_1 = x^2 - y^2$, $p_0 = y(x - 4z)$ and $p_1 = (3x - 4z - y)(2x + y)$.

Figure 4 shows the link diagram, up to isotopy, following a linear projection of this embedding back to $\mathbb{P}^2$. The two components of the links are depicted in red and blue, and the image of the two lines of hyperbolicity are shown in black. The real projective plane is depicted as a disk with antipodal boundary points identified.

Question 2. *Is there an example of a non-singular curve $X \subset \mathbb{P}^n$ where two connected components of $\mathcal{H}(X)$ give rise to the same element in* $\mathrm{Hyp}(X)$*?*

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