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Superforms and Tropical Cohomology

by

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SUPERFORMS AND TROPICAL COHOMOLOGY

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1. Introduction

These notes are a summary of my talk given at the Simons Symposium on Non-Archimedean and Tropical Geometry in May 2017. In this talk I presented joint work with Philipp Jell and Jascha Smacka relating the Dolbeault cohomology of superforms and tropical cohomology for polyhedral spaces [JSS15]. My talk was preceded by the talk of Ilia Itenberg on Tropical homology and Betti numbers of real algebraic varieties and followed by a talk of Yifeng Liu on Tropical Dolbeault cohomology of non-Archimedean spaces.

I would like thank the organizers, Matt Baker and Sam Payne, for inviting me to speak at this meeting, as well as the Simons Foundation for their support of this event.

2. Superforms

Superforms on $\mathbb{R}^r$ are bigraded real-valued differential forms introduced by Lagerberg [Lag12].

Definition 2.1. Let $U \subset \mathbb{R}^r$ be an open subset. Denote by $\mathcal{A}^q(U)$ the space of differential forms of degree q on U. The space of (p,q)-*superforms* on U is defined as

$$\mathcal{A}^{p,q}(U) := \mathcal{A}^p(U) \otimes_{C^\infty(U)} \mathcal{A}^q(U) = \bigwedge^p \mathbb{R}^{r*} \otimes_{\mathbb{R}} \mathcal{A}^q(U),$$

where $\bigwedge^p$ denotes the p-th exterior power.

In choosing a basis $x_1, \dots, x_r$ of $\mathbb{R}^r$, it is convenient to formally write a superform $\alpha \in \mathcal{A}^{p,q}(U)$ as

$$\alpha = \sum_{|K|=p,|L|=q} \alpha_{KL} d'x_K \wedge d''x_L$$

where $K = \{i_1, \dots i_p\}$ and $L = \{j_1, \dots j_q\}$ are ordered subsets of $\{1, \dots, r\}$, the coefficients $\alpha_{KL} \in C^\infty(U)$ are smooth functions and

$$d'x_K \wedge d''x_L := (dx_{i_1} \wedge \dots \wedge dx_{i_p}) \otimes_{\mathbb{R}} (dx_{j_1} \wedge \dots \wedge dx_{j_q}).$$

This conventional abuse of notation follows [CLD12, Gub16].

There differential operators d', d'', and $d = d' + d''$ on superforms. These operators are analogous to the differential operators ∂, $\overline{\partial}$, and d on complex differential forms. Here we will be most interested in the differential operator d'', which acts as,

$$d'' \colon \mathcal{A}^{p,q}(U) = \bigwedge^p \mathbb{R}^{r*} \otimes_{\mathbb{R}} \mathcal{A}^q(U) \to \mathcal{A}^{p,q+1}(U) = \bigwedge^p \mathbb{R}^{r*} \otimes_{\mathbb{R}} \mathcal{A}^{q+1}(U),$$

given by $(-1)^p \mathrm{id} \otimes D$, where D is the usual differential operator on forms.

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In the above mentioned coordinates d'' behaves in the following way,

$$d''\left(\sum_{K,L}\alpha_{KL}d'x_K\wedge d''x_L\right)=(-1)^p\sum_{K,L}\sum_{i=1}^{r}\frac{\partial\alpha_{KL}}{\partial x_i}d'x_K\wedge d''x_i\wedge d''x_L.$$

In tropical geometry, the tropical affine line is $\mathbb{T}=[-\infty,\infty)$. We equip $\mathbb{T}$ with the topology so that it is homeomorphic to a half open interval. Then $\mathbb{T}^r$ is equipped with the product topology. The space $\mathbb{T}^r$ is stratified for $I\subset\{1,\dots,r\}$, let $\mathbb{R}^r_I:=\{x\in\mathbb{T}^r \mid x_i=-\infty$ if and only if $i\in I\}$. We call $\mathbb{R}^r_I$ the points of *sedentarity* I. Notice that $\mathbb{R}^r_I\cong\mathbb{R}^{r-|I|}$.

Superforms can be defined on *polyhedral subspaces* of $\mathbb{R}^r$ and $\mathbb{T}^r$. A polyhedral subspace in $\mathbb{R}^r$ is a subset which is the support of some polyhedral complex $\mathcal{C}$. A polyhedral subspace in $\mathbb{T}^r$ is the closure in $\mathbb{T}^r$ of polyhedral spaces contained in the strata $\mathbb{T}^r_I\cong\mathbb{R}^{r-|I|}$. Lagerberg defines superforms on open subsets $\mathbb{R}^r$ [Lag12]. Later restrictions of superforms to polyhedral subspaces of $\mathbb{R}^r$ were considered in [CLD12, Gub16]. The restrictions of two distinct forms on $\mathbb{R}^r$ may become indistinguishable upon restriction to a polyhedral subspace, therefore forms on a polyhedral subspace are given by equivalence classes. In [JSS15], we define a superform on a polyhedral space $X\subset\mathbb{T}^r$ as a collection of superforms $\alpha=(\alpha_I)$ such that α_I and α_J satisfy a compatibility condition whenever $I\subset J$. To avoid technical details, I refer the reader to Section 2.1 of [JSS15] for the precise definition.

The main point is that this extension of the definition of superforms to a polyhedral space X yields sheaves of superforms $\mathcal{A}^{p,q}_X$ which are fine and acyclic [JSS15, Lemma 2.15]. The differential operator d'' produces a complex of sheaves for each p,

$$0\to\mathcal{A}^{p,0}_X\xrightarrow{d''}\mathcal{A}^{p,1}_X\xrightarrow{d''}\mathcal{A}^{p,2}_X\to\dots.$$

This extends the definition and the properties of the sheaves of superforms on polyhedral subspaces of $\mathbb{R}^r$ from [CLD12] to polyhedral subspaces of $\mathbb{T}^r$ and also to abstract polyhedral spaces. It is important to point out that the main interest of Chambert-Loir and Ducros' work is to study superforms Berkovich analytic spaces. They construct these forms via their polyhedral counterparts which we treat here. They were also the first to define the cohomologies of superforms with respect to the operators d', d'', and $d:=d'+d''$ in both the tropical and Berkovich setting.

Our Theorem 3.4 in the next section answers the question posed in [CLD12, Section 0.4.3] about whether there is a relation between tropical homology and the cohomology of superforms.

3. The Dolbeault cohomology of superforms

Following the work of Chambert-Loir and Ducros [CLD12], the consideration of the cohomology of superforms on polyhedral subspaces in $\mathbb{R}^r$ with respect to differential operators d' and d'' continued with a Poincaré lemma proved by Jell [Jel16] proved in both the tropical and Berkovich setting. In [JSS15] we extend the polyhedral side of this lemma to superforms on polyhedral subspaces of $\mathbb{T}^r$ and also to superforms on *polyhedral spaces* [JSS15, Theorem 3.16]. These are topological spaces equipped with an atlas of charts to polyhedral subspaces of $\mathbb{T}^r$. Included in this class of spaces are abstract tropical varieties and tropical manifolds, see for example [IKMZ16, MZ14, Sha15].

Definition 3.1. For X a polyhedral space and $p\in\mathbb{N}$ we define the sheaf

$$\mathcal{L}^p_X:=\ker(d''\colon\ \mathcal{A}^{p,0}_X\to\mathcal{A}^{p,1}_X).$$

The Poincaré lemma relates the cohomology of the sheaf $\mathcal{L}^p_X$ with the Dolbeault cohomology of superforms.

Theorem 3.2. [JSS15, Corollary ???] *For a polyhedral space X and all $p \in \mathbb{N}$, the complex*

$$0 \to \mathcal{L}^p \to \mathcal{A}^{p,0} \xrightarrow{d''} \mathcal{A}^{p,1} \xrightarrow{d''} \mathcal{A}^{p,2} \to \dots$$

of sheaves on X is exact. Furthermore it is an acyclic resolution, we thus have canonical isomorphisms

$$H^q(X, \mathcal{L}^p) \cong H^{p,q}_{d''}(X) \qquad \textit{and} \qquad H^q_c(X, \mathcal{L}^p) \cong H^{p,q}_{d'',c}(X).$$

We call the above cohomology groups the *Dolbeault cohomology of superforms*, since the operator d'' behaves analogously to the operator $\overline{\partial}$ for complex differential forms.

Tropical cohomology, as introduced by Itenberg, Katzarkov, Mikhalkin, and Zharkov [IKMZ16], is the cohomology of singular cochains of a polyhedral complex with non-constant coefficients. The coefficient systems, which are denoted by $\mathbf{F}^p$, are determined by the geometry of the complex.

Definition 3.3. Let $\mathcal{C}$ be a polyhedral complex in $\mathbb{T}^r$. For $\sigma \in \mathcal{C}$, let $x \in \operatorname{int}(\sigma)$ and $I \subset [r]$ be such that $\operatorname{int}(\sigma) \subset \mathbb{R}^r_I$. The p-th *multi-cotangent space* of $\mathcal{C}$ at σ are the vector subspaces

$$\mathbf{F}^p_X(\sigma) = \left(\sum_{\tau \in \mathcal{C}_I : \sigma \prec \tau} \bigwedge^p \mathbb{L}(\tau) \right)^*$$

Via the tropicalization procedure, this cohomology theory is capable of determining Hodge theoretic information of projective varieties. For example, under suitable conditions on the tropicalization of a family of non-singular complex projective varieties, the dimensions of the tropical cohomology groups are equal to the Hodge numbers of a generic member of the family [IKMZ16, Corollary 2].

The coefficient systems from Definition 3.3 can be turned into sheaves $\mathcal{F}^p_X$ with respect to the inherent topology on a polyhedral space X. Moreover, the tropical cohomology groups are isomorphic to the cohomology of the sheaves $\mathcal{F}^p_X$. These groups will be denoted by $H^{p,q}_{\mathrm{trop}}(X) := H^q(X; \mathcal{F}^p_X)$. There is the following theorem relating the Dolbeault cohomology of superforms and the tropical cohomology of Itenberg, Katzarkov, Mikhalkin, and Zharkov by comparing the sheaves $\mathcal{L}^p_X$.

Theorem 3.4. [JSS15, Theorem 1] Let X be a polyhedral space equipped with a face structure. Then there are canonical isomorphisms

$$H^{p,q}_{\mathrm{trop}}(X) \cong H^{p,q}_{d''}(X) \qquad \text{and} \qquad H^{p,q}_{\mathrm{trop},c}(X) \cong H^{p,q}_{d'',c}(X),$$

where $H^{p,q}_{\bullet,c}(X)$ denotes cohomology with compact support.

Following this theorem we will simply use $H^{p,q}(X)$ to denote the tropical or Dolbeault cohomology of X. The main upshot of this theorem is that tropical cohomology is in principal much simpler to compute. Under some mild assumptions on the polyhedral space, tropical cohomology is isomorphic to either the cohomology of the sheafs $\mathcal{F}^p_X$ or the singular or cellular cohomology of the coefficient systems (or cellular sheaves) $\mathbf{F}^p_X$. The cellular version of tropical cohomology is the cohomology of finite dimensional chain complexes, and it has been implemented in polymake [KSW16].

Question 3.5. Under what circumstances is the tropical cohomology of a tropicalization isomorphic to the cohomology of superforms on Berkovich analytic spaces?

Jell and Wanner calculated the Dolbeault cohomology of superforms for $\mathbb{P}^1$ and Mumford curves [JW16]. The Betti numbers that they obtain coincide with the Betti numbers of the tropical cohomology of a non-singular tropical curve of the same genus [BIMS15, Section 7.8]. Using different techniques than in [JW16], Jell computed more examples of the Dolbeault cohomology of superforms for non-archimedean curves and also provided a necessary and sufficient condition for Poincaré duality in this situation [Jel17]. He also provides examples of curves for which $H^{1,1}_{d''}$ is not finite dimensional.

However, there are tropicalizations of embedded curves $\iota : \mathcal{X} \to \mathbb{P}^2$ for which $H^{p,q}_{d''}(\mathcal{X}^{an})$ is not isomorphic to the tropical cohomology of $\mathrm{Trop}(\iota\mathcal{X})$. These tropicalizations are even faithful in the sense of [BPR16]. I optimisitically expect that requiring a tropicalization to be matroidal should be sufficient to ensure that the tropical Dolbeault of superforms of $\mathcal{X}^{an}$ is isomorphic to the tropical cohomology of $\mathrm{Trop}(\mathcal{X})$.

4. Poincaré duality for tropical manifolds

In the last sections, superforms, their cohomology, and also tropical cohomology were defined for general polyhedral spaces. Notice that the *balancing condition*, which is ubiquitous in tropical geometry, made no appearance in the first two sections of these notes. To see how the balancing condition influences tropical cohomology and the theory of superforms we can look to the integration of superforms [Gub16, CLD12]. The reader is also pointed to [JSS15] for the integration of superforms on polyhedral spaces with points of sedentarity.

A tropical space is a polyhedral space equipped with a weight function such that the image in each chart is a tropical cycle and that the weight function on X is consistent in the different charts [JSS15, Definition 4.8]. The following is a version of Stokes' theorem for superforms.

Theorem 4.1 ([CLD12, Gub16, JSS15])**.** *Let X be an n-dimensional weighted rational polyhedral space. Then X is a tropical space if and only if for all $\beta \in \mathcal{A}^{n,n-1}_c(X)$ we have*

$$\int\limits_X d''\beta = 0.$$

For X an n-dimensional tropical space there is a map

$$\mathrm{PD}\colon H^{p,q}_{d''}(X) \to H^{n-p,n-q}_{d'',c}(X)^*,$$

called the Poincaré duality map. This map is obtained by integrating the wedge product of forms, thus it is similar to the integration pairing on the cohomology of a manifold. The fact that the Poincaré duality map on superforms descends to cohomology when X is a tropical space follows from the analogue of Stokes' theorem above.

We also point out that when X is a tropical space, it is quite simple to construct a fundamental class $[X] \in H^{BM}_{p,q}(X)$, where $H^{BM}_{\bullet,\bullet}(X)$ denotes the tropical Borel Moore homology of X. The recipe for the fundamental class is a simple sum over the top dimensional faces of X, assuming X has a *face structure* [JSS15, Definition 3.2]. More precisely, if X is n dimensional

$$[X] = \sum_{|\sigma|=n} w_\sigma \beta_\sigma \sigma,$$

where w_σ is the weight associated to points in σ and $\beta_\sigma = v_1 \wedge \cdots \wedge v_n \in \mathbf{F}_p(\sigma)$ where $v_1, \ldots, v_n$ is an ordered basis of the lattice parallel to the image of the face σ in a chart. This basis must be ordered so it induces the same orientation as chosen on σ. Here $|\sigma|$ denotes the dimensional of a face σ of X.

It is a fact that $\partial[X] = 0$ is equivalent to X being a polyhedral space which satisfies the balancing condition [MZ14, Sha15]. Two advantages of the tropical fundamental class over the situation for manifolds is that this construction produces a cycle regardless of whether X is orientable or even tropically non-singular.

Tropical manifolds are tropical spaces with the extra condition that they are locally modeled on matroidal tropical cycles [MR, Sha11]. A matroidal tropical cycle is supported on the Bergman fan of a matroid and equipped with weight one. Some matroidal cycles arise as tropicalizations of linear spaces, however they are much more general and may even have no algebraic counterpart [Stu02]. Despite perhaps being far from smooth objects in the algebraic or differentiable sense, tropical manifolds exhibit many properties analogous to smooth spaces [Sha11].

Theorem 4.2. [JSS15, Theorem 2] If X is an n-dimensional tropical manifold then there are isomorphisms

$$H^{p,q}(X) \cong H_c^{n-p,n-q}(X)^*$$

for all p and q induced by the Poincaré duality map.

As with Poincaré duality for differentiable manifolds, we deduce this statement by first proving it when X is an open subsets of Bergman fans of matroids. Then for general tropical manifolds the theorem follows from methods in algebraic topology.

I should point out that it is fairly easy to construct examples of balanced weighted rational polyhedral fans which do not satisfy the above duality theorem. Therefore, this duality does not hold for general tropical spaces. Yet still the condition that a space be locally matroidal is not necessary. There are also integral tropical homology and cohomology groups and a Poincaré duality relating tropical homology and cohomology should also hold over the integers. Taking the fundamental class of a tropical variety, as described above, produces an integral tropical homology class for any tropical cycle. The remaining ingredient in defining a cycle map to integral tropical cohomology is this Poincaré duality over the integers.

We end with some open questions about the properties of these cohomology groups for tropical manifolds and tropical spaces. For a compact tropical manifold X of dimension n, Theorem 4.2 implies a symmetry amongst the tropical cohomology groups, namely that,

$$H^{p,q}(X) \cong H^{n-p,n-q}(X).$$

It is relevant to ask in what situations do other properties of the Hodge decomposition of complex projective varieties transfer over to the tropical setting.

Question 4.3. Under what assumptions do we have $H^{p,q}(X) \cong H^{q,p}(X)$?

In general there are tropical models of non-Kähler complex surfaces for which do not exhibit this symmetry in their tropical cohomology groups [RS17]. Another appropriate question to ask is, what is a good analogue of a Kähler condition in tropical geometry?

When X is the tropicalization of a $\mathbb{Q}$-projective variety these isomorphisms hold by [IKMZ16], [MZ14]. A canonical isomorphism is offered by the eigenwave map in [MZ14]. Liu defines a monodromy map on superforms on Berkovich analytic spaces which is also well defined in the tropical setting [Liu17]. Liu also conjectures that this monodromy map and the eigenwave descend to the same action on the cohomology of polyhedral spaces via the isomorphism provided in Theorem 3.4.

Question 4.4. Under what assumptions do analogues of the Hard Lefschetz theorem and the Hodge-Riemann bilnear relations hold for tropical cohomology?

In the recent work of Adiprasito, Huh, and Katz, they showed that analogues of both theorems hold for a commutative ring associated to a matroid [AHK15]. For matroids representable in characteristic zero, this ring is the cohomology ring of the wonderful compactification of the complement of a hyperplane arrangement over $\mathbb{C}$ and is related to the tropical cohomology of a compactification of a matroidal fan. It is important to point out that the direct translation of the Hodge Index theorem fails for tropical surfaces [Sha13]. These examples show that the kernel of the monodromy/eigenwave map is important in determining the signature of the intersection pairings on tropical cohomology.

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