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Brakkes inequality for the thresholding
scheme

by

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Brakke's inequality for the thresholding scheme

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Abstract

We continue our analysis of the thresholding scheme from the variational viewpoint and prove a conditional convergence result towards Brakke's notion of mean curvature flow. Our proof is based on a localized version of the minimizing movements interpretation of Esedoğlu and the second author. We apply De Giorgi's variational interpolation to the thresholding scheme and pass to the limit in the resulting energy-dissipation inequality. The result is conditional in the sense that we assume the time-integrated energies of the approximations to converge to those of the limit.

Keywords: Mean curvature flow, thresholding, MBO, diffusion generated motion

Mathematical Subject Classification: 35A15, 65M12, 74N20

Contents

1 Introduction

The thresholding scheme is a time discretization for mean curvature flow. Its structural simplicity is intriguing to both applied and theoretical scientists. Merriman, Bence and Osher [25] introduced the algorithm in 1992 to overcome the numerical difficulty of multiple scales in phase-field models. Their idea is based on an operator splitting for the Allen-Cahn Equation, alternating between linear diffusion and thresholding. The latter replaces the fast reaction coming from the nonlinearity, i.e., the reaction-term, in the Allen-Cahn Equation. We refer to Algorithm 1.1 below for a precise description of the scheme. The convolution can be implemented efficiently on a uniform grid using

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the discrete Fourier Transform and the thresholding step is a simple pointwise operation. Because of its simplicity and efficiency, thresholding gained a lot of attention in the last decades. Large-scale simulations [10, 11, 12] demonstrate the efficiency of a slight modification of the scheme. For applications in materials science and image segmentation it is desirable to design algorithms that are efficient enough to handle large numbers of phases but flexible enough to incorporate external forces, variable surface tensions and even anisotropies. Not long ago, the natural extension to the multi-phase case [26] was generalized to arbitrary surface tensions by Esedoğlu and the second author [14]. They realized that thresholding preserves the gradient-flow structure of (multi-phase) mean-curvature flow in the sense that it can be viewed as a minimizing movements scheme for an energy that Γ-converges to the total interfacial area. This viewpoint allowed them to incorporate a wide class of surface tensions including the well-known Read-Shockley formulas for small-angle grain boundaries [29]. The development of thresholding schemes for anisotropic motions started with the work [18] of Ishii, Pires and Souganidis. Efficient schemes were presented by Bonnetier, Bretin and Chambolle [6], where the convolution kernels are explicit and well-behaved in Fourier space but not necessarily in real space. The recent work [9] of Elsey and Esedoğlu is inspired by the variational viewpoint [14] and shows that not all anisotropies can be obtained when structural features such as positivity of the kernel are needed. However, variants of the scheme developed by Esedoğlu and Jacobs [13] share the same stability conditions even for more general kernels. The rigorous asymptotic analysis of thresholding schemes started with the independent convergence proofs of Evans [15] and Barles and Georgelin [5] in the isotropic two-phase case. Since the scheme preserves the geometric comparison principle of mean curvature flow, they were able to prove convergence towards the viscosity solution of mean curvature flow. Recently, Swartz and Yip [30] proved convergence for a smooth evolution by establishing consistency and stability of the scheme, very much in the flavor of classical numerical analysis. They prove explicit bounds on the curvature and injectivity radius of the approximations and get a good understanding of the transition layer. However, also their result does not generalize to the multi-phase case immediately. In our previous work [21] we established the convergence of thresholding to a distributional formulation of multi-phase mean-curvature flow based on the assumption of convergence of the energies. In [23], Swartz and the first author applied these techniques to the case of volume-preserving mean-curvature flow and other variants.

Since the works [5, 15] are based on the comparison principle, the proofs do not apply in the multi-phase case. Our guiding principle in this work is instead the gradient-flow structure of (multi-phase) mean curvature flow. In general, a gradient-flow structure is given by an energy functional and a dissipation mechanism, given by the geometry of the space of configurations through a Riemannian metric. A simple computation reveals this structure for mean curvature flow. If the hypersurface $\Sigma = \Sigma(t)$ evolves smoothly by its mean curvature (here and throughout we use the time scale such that $2V = H$) the change of area is given by

$$2\frac{d}{dt}|\Sigma| = -\int_\Sigma 2V\,H = -\int_\Sigma H^2, \tag{1}$$

where V denotes the normal velocity and H denotes the mean curvature of Σ. In view of (1), when fixing the energy to be the surface area, the metric tensor is given by the L^2-metric $\int_\Sigma V^2$ on the space of normal vector fields. However, some care needs to be taken when dealing with this metric as for example the geodesic distance vanishes identically [27]. The implicit time discretization of Almgren, Taylor and Wang [2], and Luckhaus and Sturzenhecker [24] makes use of the gradient-flow

structure. In fact, it inspired De Giorgi to define a similar implicit time discretization for abstract gradient flows which he named "minimizing movements". His abstract scheme consists of a family of minimization problems which mimic the principle of a gradient flow moving in direction of the steepest descent in an energy landscape. The configuration Σ^n at time step n is obtained from its predecessor Σ^{n-1} by minimizing $E(\Sigma) + \frac{1}{2h} \operatorname{dist}^2(\Sigma, \Sigma^{n-1})$, where dist denotes the geodesic distance induced by the Riemannian structure and $h > 0$ denotes the time-step size. In the Euclidean case, the scheme boils down to the implicit Euler scheme. It has been used for applications in partial differential equations and for instance allowed Jordan, Kinderlehrer and the second author [19] to interpret diffusion equations as gradient flows for the entropy w.r.t. the Wasserstein distance. In view of the degeneracy in the case of mean curvature flow it is evident that the scheme in [2, 24] uses a proxy for the geodesic distance. Their replacement for the distance of two boundaries $\Sigma = \partial\Omega$ and $\tilde{\Sigma} = \partial\tilde{\Omega}$ is the (non-symmetric) quantity $2 \int_{\Omega \Delta \tilde{\Omega}} d_{\tilde{\Omega}} \, dx$, where $d_{\tilde{\Omega}}$ denotes the (unsigned) distance to $\partial\tilde{\Omega}$. Chambolle [8] showed that the scheme [2, 24] which seems academic at a first glance can be implemented rather efficiently.

In retrospect, also Brakke's pioneering work [7] can be seen as a way of interpreting mean curvature flow as a gradient flow. His definition is similar to the one of an abstract gradient flow and characterizes solutions by the optimal dissipation of energy in the spirit of (1). Brakke's solutions are varifolds, a concept weak enough to obtain compactness under natural conditions and strong enough to give sense to either side of (1). In contrast to the abstract framework, Brakke measures the dissipation of energy only in terms of the gradient of the energy, here the mean curvature. Therefore he has to monitor *localized* versions of (1) and – as for an abstract gradient flow – only asks for an inequality instead of an equality. We refer to Definition 2.1 for a precise definition in our context of sets of finite perimeter. Since his definition does not involve the metric term, one loses control over the time derivative and thus weak solutions may be discontinuous in time and in particular mass can disappear instantly. Ilmanen [17] utilized a phase-field version of Huisken's monotonicity formula [16] to prove the convergence of solutions to the scalar Allen-Cahn Equation to Brakke's mean curvature flow. Extending his proof to the multi-phase case is a challenging open problem. Only recently, Simon and the first author [22] proved a conditional convergence result for the vector-valued Allen-Cahn Equation very much in the spirit of [24, 21]. However, an unconditional result is not yet available. Even the construction of non-trivial global solutions to multi-phase mean-curvature flow has only been done recently by Tonegawa and Kim [20].

In the present work we establish the convergence of the thresholding scheme to Brakke's motion by mean curvature. As our previous result [21], also this one is only a conditional convergence result in the sense that we assume the time-integrated energies to converge to those of the limit. Our proof is based on the observation that thresholding does not only have a *global* minimizing movements interpretation, but indeed solves a *family* of *localized* minimization problems. In Section 2 we state our main results, in particular Theorem 2.2. We use De Giorgi's variational interpolation for these localized minimization problems to derive an *exact* energy-dissipation relation and pass to the limit in the inequality with help of our strengthened convergence. Section 4 provides the tools for these results. We first recall the known results from the abstract framework of gradient flows in metric spaces (cf. Chapter 3 in [4]). Then we pass to the limit $h \to 0$ in these terms with help of our strengthened convergence.

The starting point for our analysis of thresholding schemes is the minimizing movements interpretation of Esedoğlu and the second author [14]. Let us explain this interpretation with help of

the example of the two-phase scheme. The combination $\chi^n = \mathbf{1}_{\{G_h * \chi^{n-1} > \frac{1}{2}\}}$ of convolution and thresholding is equivalent to minimizing $E_h(\chi) + \frac{1}{2h}\mathrm{d}_h^2(\chi, \chi^{n-1})$, where E_h is an approximation of the perimeter functional and d_h is a metric. The latter serves as a proxy for the induced distance, just like $2\int_{\Omega \Delta \Omega^{n-1}} d_{\Omega^{n-1}} dx$ in the minimizing movements scheme of Almgren, Taylor and Wang [2], and Luckhaus and Sturzenhecker [24]. The Γ-convergence of similar functionals has been developed some time ago by Alberti and Bellettini [1] and more recently by Ambrosio, De Philippis and Martinazzi [3], and was proven for the functionals E_h by Miranda, Pallara, Paronetto and Preunkert [28]. Esedoḡlu and the second author found an independent, much simpler proof in the case of the energies E_h, which extends to the multi-phase case.

In this first version of the paper we restrict ourselves to the two-phase case. Let us recall the thresholding scheme and the basic notation in this setting.

Algorithm 1.1. *Given the phase Ω^{n-1} at time $t = (n-1)h$, obtain the evolved phase Ω^n at time $t = nh$ by the following two operations:*

1. *Convolution step: $\phi := G_h * \mathbf{1}_{\Omega^{n-1}}$.*

2. *Thresholding step: $\Omega^n := \left\{ \phi > \frac{1}{2} \right\}$.*

Here and throughout the paper

$$G_h(z) := \frac{1}{(2\pi h)^{d/2}} \exp\left(-\frac{|z|^2}{2h} \right)$$

denotes a Gaussian of variance h. For convenience we will work with periodic boundary conditions, i.e., on the flat torus $[0, \Lambda)^d$. We write $\int dx$ short for $\int_{[0,\Lambda)^d} dx$ and $\int dz$ short for $\int_{\mathbb{R}^d} dz$. Furthermore, $\chi^n := \mathbf{1}_{\Omega^n}$ denotes the characteristic function of the phase Ω^n at time step n and we denote its piecewise constant interpolation by

$$\chi^h(t) := \chi^n = \mathbf{1}_{\Omega^n} \quad \text{for } t \in [nh, (n+1)h).$$

However, we will mostly use a nonlinear interpolation which will be introduced later. Selim Esedoḡlu and the second author [14] showed that thresholding preserves the gradient-flow structure of (multiphase) mean curvature flow in the sense that it can be viewed as a minimizing movements scheme

$$\chi^n = \arg\min_u \left\{ E_h(u) + \frac{1}{2h}\mathrm{d}_h^2(u, \chi^{n-1}) \right\}, \tag{2}$$

where the dissipation functional

$$\frac{1}{2h}\mathrm{d}_h^2(u, \chi) := \frac{1}{\sqrt{h}} \int \left[G_{h/2} * (u - \chi) \right]^2 dx \tag{3}$$

is the square of a metric and the energy is

$$E_h(u) := \frac{1}{\sqrt{h}} \int (1 - u)\, G_h * u\, dx, \tag{4}$$

an approximation of the perimeter functional. Indeed, these functionals Γ-converge to

$$E(\chi) := c_0 \int |\nabla \chi|, \quad \text{for } \chi \colon [0, \Lambda)^d \to \{0, 1\},$$

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where $c_0 = \frac{1}{\sqrt{2\pi}}$. This Γ-convergence is a consequence of the pointwise convergence of these functionals and the monotonicity property

$$E_{N^2 h}(u) \leq E_h(u) \quad \text{for all } u\colon [0, \Lambda)^d \to [0, 1], h > 0 \text{ and } N \in \mathbb{N}, \tag{5}$$

see [14]. We write $A \lesssim B$ to express that $A \leq CB$ for a generic constant $C < \infty$ that only depends on the dimension d and on the size Λ of the domain. By $A = O(B)$ we mean $|A| \lesssim B$ while $A = o(B)$ as $h \to 0$ means $\frac{A}{B} \to 0$ as $h \to 0$.

2 Brakke's inequality

The main statement of this work is Theorem 2.2 below. Assuming there was no drop of energy as $h \to 0$, i.e.,

$$\int\limits_0^T E_h(\chi^h)\, dt \to \int\limits_0^T E(\chi)\, dt, \tag{6}$$

it states that the limit of the approximate solutions satisfies a BV-version of Brakke's inequality [7].

Brakke's inequality is a weak formulation of motion by mean curvature $V = \frac{H}{2}$ and is motivated by the following characterization of the normal velocity. Given a smoothly evolving hypersurface $\partial\Omega(t) = \Sigma(t)$ with normal velocity V we have

$$\frac{d}{dt}\int\limits_\Sigma \zeta \leq \int\limits_\Sigma \left(-\zeta H\, V - V\, \nabla\zeta \cdot \nu + \partial_t \zeta\right) \tag{7}$$

for any smooth test function $\zeta \geq 0$. Here ν denotes inner normal of $\partial\Omega$ and we take the convention $V > 0$ for an expanding Ω and $H < 0$ for a convex Ω. The converse is also true: Given a function $V\colon \Sigma \to \mathbb{R}$ such that (7) holds for any such test function $\zeta \geq 0$ then V is the normal velocity of Σ. In the pioneering work [7], Brakke uses this inequality as a definition for the equation $V = \frac{H}{2}$ to extend the concept of motion by mean curvature to general varifolds. We recall his definition in our more restrictive setting of finite perimeter sets.

Definition 2.1. We say that $\chi\colon (0, T) \times [0, \Lambda)^d \to \{0, 1\}$ moves by mean curvature if there exists a $|\nabla\chi|\, dt$-measurable function $H\colon (0, T) \times [0, \Lambda)^d \to \mathbb{R}$ with

$$\int\limits_0^T \int H^2\, |\nabla\chi|\, dt < \infty,$$

which is the mean curvature in the sense that for all test vector fields $\xi \in C_0^\infty((0, T) \times [0, \Lambda)^d, \mathbb{R}^d)$

$$\int\limits_0^T \int (\nabla \cdot \xi - \nu \cdot \nabla\xi\, \nu)\, |\nabla\chi|\, dt = \int\limits_0^T \int H\, \xi \cdot \nu\, |\nabla\chi|\, dt, \tag{8}$$

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such that for any test function $\zeta \in C_0^\infty((0,T) \times [0,\Lambda)^d)$ with $\zeta \geq 0$ we have

$$\int_0^T \int \left(2\partial_t \zeta - \zeta H^2 - H\nu \cdot \nabla\zeta \right) |\nabla\chi| \, dt \geq 0. \tag{9}$$

Theorem 2.2 (Brakke's inequality)**.** *Given initial data $\chi^0 \colon [0,\Lambda)^d \to \{0,1\}$ with $E(\chi^0) < \infty$ and a finite time horizon $T < \infty$, for any sequence there exists a subsequence $h \downarrow 0$ such that the approximate solutions given by Algorithm 1.1 converge to a limit $\chi \colon (0,T) \times [0,\Lambda)^d \to \{0,1\}$ in L^1 and a.e. in space-time. Given the convergence assumption (6), χ moves by mean curvature in the sense of Definition 2.1.*

Remark 2.3. Given initial conditions χ^0 with $E(\chi^0) < \infty$ the compactness in [21] yields a subsequence such that $\chi^h \to \chi$ a.e. for a function χ with $\sup_t E(\chi(t)) \leq E(\chi^0)$.

This statement is similar to our result in [21]. There we proved the convergence of thresholding towards a distributional formulation of (multi-phase) mean-curvature flow. Under the same assumption (6) we constructed a $|\nabla\chi| \, dt$-measurable function $V \colon (0,T) \times [0,\Lambda)^d \to \mathbb{R}$ with

$$\int_0^T \int V^2 \, |\nabla\chi| \, dt < \infty,$$

which is the normal velocity in the sense that

$$\int_0^T \int \partial_t \zeta \, \chi \, dx \, dt = -\int_0^T \int \zeta V \, |\nabla\chi| \, dt$$

for all $\zeta \in C_0^\infty((0,T) \times [0,\Lambda)^d)$, such that $V = \frac{H}{2}$ in the following distributional sense:

$$\int_0^T \int \left(\nabla \cdot \xi - \nu \cdot \nabla\xi \, \nu - 2\, \xi \cdot \nu \, V \right) |\nabla\chi| \, dt = 0 \tag{10}$$

for all $\xi \in C_0^\infty((0,T) \times [0,\Lambda)^d, \mathbb{R}^d)$.

The connection of (10) to the strong equation $V = \frac{H}{2}$ comes from the integration by parts rule for smooth hypersurfaces:

$$\int_\Sigma \left(\nabla \cdot \xi - \nu \cdot \nabla\xi \, \nu \right) = \int_\Sigma H \, \xi \cdot \nu.$$

Without any regularity assumption, none of the two formulations is stronger in the sense that it implies the other. Nevertheless (10) requires more regularity as it is formulated for sets of finite perimeter, whereas Brakke's inequality naturally extends to general varifolds.

3   De Giorgi's variational interpolation

It is a well-appreciated fact that a classical gradient flow $\dot{u}(t) = -\nabla E(u(t))$ of a smooth energy functional E on a Hilbert space can be characterized by the optimal rate of dissipation of the energy

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E along the solution u:

$$\frac{d}{dt} E(u(t)) \leq -\frac{1}{2}|\dot{u}(t)|^2 - \frac{1}{2}|\nabla E(u(t))|^2. \tag{11}$$

This is the guiding principle in generalizing gradient flows to metric spaces where one replaces $|\dot{u}|$ by the metric derivative and $|\nabla E(u)|$ by some upper gradient, e.g. the *local slope* $|\partial E(u)|$, see (17) for a definition in our context.

Mean curvature flow can be viewed as a gradient flow in the sense that for a smooth evolution $\Sigma = \Sigma(t)$ the energy, which in this case is the surface area $|\Sigma(t)|$, satisfies the inequality

$$2\frac{d}{dt}|\Sigma| = \int_{\Sigma} H\, 2V \leq -\frac{1}{2}\int_{\Sigma} H^2 - \frac{1}{2}\int_{\Sigma}(2V)^2.$$

While in the abstract framework, the dissipation of the energy is measured w.r.t. both terms $|\dot{u}|^2 \hat{=} \int_{\Sigma}(2V)^2$ and $|\partial E(u)|^2 \hat{=} \int_{\Sigma} H^2$, Brakke measures the rate only in terms of the local slope $\int_{\Sigma} H^2$ but asks for the *localized* version (9).

The main result of this section and the basis of this work is the approximate version of Brakke's inequality, Lemma 3.1 below. In view of the minimizing movements interpretation (2) it should be feasible to obtain at least the *global* inequality

$$2\frac{d}{dt}|\Sigma| \leq -\int_{\Sigma} H^2$$

but the localized inequality (9) would be still out of reach. The lemma states that thresholding does not only solve the *global* minimization problem (2) but a whole *family* of *local* minimization problems, which will allow us to establish the family of localized inequalities (9).

Lemma 3.1 (Local minimization)**.** *Let χ^n be obtained from χ^{n-1} by one iteration of Algorithm 1.1 and $\zeta \geq 0$ an arbitrary test function. Then*

$$\chi^n = \arg\min_{u} \left\{ E_h(u, \chi^{n-1}; \zeta) + \frac{1}{2h} d_h^2(u, \chi^{n-1}; \zeta) \right\}, \tag{12}$$

where the minimum runs over all $u\colon [0, \Lambda)^d \to [0, 1]$. By $d_h(u, \chi; \zeta)$ we denote the localization of the metric $d_h(u, \chi)$ given by

$$\frac{1}{2h} d_h^2(u, \chi; \zeta) := \frac{1}{\sqrt{h}} \int \zeta \left[G_{h/2} * (u - \chi) \right]^2 dx, \tag{13}$$

which is again a (semi-)metric on the space of all such u's as above and in particular satisfies a triangle inequality. By $E_h(u, \chi; \zeta)$ we denote the localized (approximate) energy incorporating the localization error:

$$E_h(u, \chi; \zeta) := \frac{1}{\sqrt{h}} \int \zeta (1 - u)\, G_h * u \, dx + \frac{1}{\sqrt{h}} \int (u - \chi) [\zeta, G_h*] (1 - \chi)\, dx \tag{14}$$

$$+ \frac{1}{\sqrt{h}} \int (u - \chi) [\zeta, G_{h/2}*]\, G_{h/2} * (u - \chi)\, dx.$$

Here and throughout the paper

$$[\zeta, G_h *]\, u := \zeta\, G_h * u - G_h * (\zeta\, u) \approx -\nabla\zeta \cdot h\nabla G_h * u$$

denotes the commutator of the multiplication with the function ζ and the convolution with the kernel G_h.

Let us comment on the structure of the localized energy E_h. The first integral is an approximation of the localized surface energy $c_0 \int_\Sigma \zeta$. Expanding ζ, as $h \to 0$ the leading-order term of the second integral in the definition of $E_h(\chi^n, \chi^{n-1}; \zeta)$ is

$$h \int \frac{\chi^n - \chi^{n-1}}{h} \nabla\zeta \cdot \sqrt{h}\nabla G_h * \chi^{n-1}\, dx,$$

which at least formally (and after summation over the time steps) converges to $c_0 \int_0^T \int_\Sigma V\, \nabla\zeta \cdot \nu$ and hence we expect to recover the transport term $\frac{c_0}{2} \int_0^T \int_\Sigma H\, \nabla\zeta \cdot \nu$ in Brakke's inequality (9). We will see later that the last integral in the definition of E_h, the commutator in the metric term, is negligible in the limit $h \to 0$. By definition of E_h we have

$$E_h(u, u; \zeta) = \frac{1}{\sqrt{h}} \int \zeta\, (1 - u)\, G_h * u\, dx \quad \text{and} \quad E_h(u, \chi; 1) = E_h(u), \quad \text{cf.(4)},$$

so that in particular we recover the minimizing movements interpretation (2) in the case $\zeta \equiv 1$.

Thanks to the above *local* minimization property of the thresholding scheme we can apply the abstract framework of De Giorgi, cf. Chapters 1–3 in [4], to these localized energies. As for any minimizing movements scheme, the comparison of χ^n to the previous time step χ^{n-1} in the minimization problem (12) yields an energy-dissipation inequality which works well as an a priori estimate, but which is however not sharp. To obtain a *sharp* inequality we follow the ideas of De Giorgi. We introduce his *variational interpolation* u^h of χ^n and χ^{n-1}: For $t \in (0, h]$ and $n \in \mathbb{N}$ we let

$$u^h((n-1)h + t) := \arg\min_u \left\{ E_h(u, \chi^{n-1}; \zeta) + \frac{1}{2t} d_h^2(u, \chi^{n-1}; \zeta) \right\}. \tag{15}$$

Comparing $u^h(t)$ with $u^h(t + \delta t)$ in this minimization problem and taking the limit $\delta t \to 0$ while keeping h fixed, one obtains the sharp energy-dissipation inequality along this interpolation, the following approximate version of Brakke's inequality (9).

Corollary 3.2 (Approximate Brakke inequality). *For any test function $\zeta \geq 0$, a time-step size $h > 0$ and $T = Nh$ we have*

$$\frac{h}{2} \sum_{n=1}^N \left| \partial E_h(\,\cdot\,, \chi^{n-1}; \zeta) \right|^2 (\chi^n) + \frac{1}{2} \int_0^T \left| \partial E_h(\,\cdot\,, \chi^h(t); \zeta) \right|^2 (u^h(t))\, dt$$

$$+ \sum_{n=1}^N \left(E_h(\chi^n, \chi^{n-1}; \zeta) - E_h(\chi^n, \chi^n; \zeta) \right) \leq E_h(\chi^0, \chi^0; \zeta) - E_h(\chi^N, \chi^N; \zeta), \tag{16}$$

where $\left| \partial E_h(\,\cdot\,, \chi; \zeta) \right|(u)$ is the local slope of $E_h(\,\cdot\,, \chi; \zeta)$ at u defined by

$$\left| \partial E_h(\,\cdot\,, \chi; \zeta) \right|(u) := \limsup_{v \to u} \frac{(E_h(u, \chi; \zeta) - E_h(v, \chi; \zeta))_+}{d_h(u, v; \zeta)}. \tag{17}$$

The convergence $v \to u$ is in the sense of the metric d_h.

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Our goal is to derive Brakke's inequality (9) from its approximate version (16), i.e., we want to relate the limits of the expressions in (16) with the terms appearing in (9), cf. Propositions 4.5 and 4.8.

4 Some lemmas

Because of the localization, our energy (14) depends on the configuration at the previous time step. However, we can apply the abstract framework (cf. Chapter 3 of [4]) to this case if we only follow one time step. Both h and ζ are fixed parameters when applying these results.

Given χ we define the *Moreau-Yosida approximation* $E_{h,t}$ of E_h by

$$E_{h,t}(\chi;\zeta) := \min_u \left\{ E_h(u,\chi;\zeta) + \frac{1}{2t} \mathrm{d}_h^2(u,\chi;\zeta) \right\} \tag{18}$$

and furthermore we recall, cf. (15), the (not necessarily unique) *variational interpolation* $u^h(t)$ of χ and $\chi^1 := u^h(h)$ by

$$u^h(t) = \arg\min_u \left\{ E_h(u,\chi;\zeta) + \frac{1}{2t} \mathrm{d}_h^2(u,\chi;\zeta) \right\}.$$

As t decreases we have a stronger penalization. Thus we expect $u^h(t)$ to be "closer" to $\chi = u^h(0)$ than $\chi^1 = u^h(h)$ which justifies the name "interpolation". We will make this statement more rigorous later. Note that $E_h(u,\chi;\zeta)$ and $\mathrm{d}(u,\chi;\zeta)$ are, because of the smoothing property of the kernel G_h, weakly continuous in u and χ.

The following theorem monitors the evolution of the (approximate) energy along the interpolation $u^h(t)$ in terms of the distances at different time instances measured by the metric d_h, and gives a lower bound in terms of the local slope $|\partial E_h|$ of E_h, cf. (17).

Theorem 4.1 (Theorem 3.1.4 and Lemma 3.1.3 in [4]). *For every $\chi\colon [0,\Lambda)^d \to \{0,1\}$ the map $t \mapsto E_{h,t}(\chi;\zeta)$ is locally Lipschitz in $(0,h]$ and continuous in $[0,h]$ with*

$$\frac{d}{dt} E_{h,t}(\chi;\zeta) = -\frac{\mathrm{d}_h^2(u^h(t),\chi;\zeta)}{2t^2} \tag{19}$$

and furthermore we have

$$|\partial E_h(\,\cdot\,,\chi;\zeta)|\,(u^h(t)) \le \frac{\mathrm{d}_h(u^h(t),\chi;\zeta)}{t}. \tag{20}$$

In particular

$$\frac{t}{2} \left|\partial E_h(\,\cdot\,,\chi;\zeta)\right|^2 (u^h(t)) + \frac{1}{2} \int_0^t \left|\partial E_h(\,\cdot\,,\chi;\zeta)\right|^2 (u^h(s))\, ds$$

$$\le \frac{1}{2t} \mathrm{d}_h^2(u^h(t),\chi;\zeta) + \int_0^t \frac{\mathrm{d}_h^2(u^h(s),\chi;\zeta)}{2s^2}\, ds = E_h(\chi,\chi;\zeta) - E_h(u^h(t),\chi;\zeta). \tag{21}$$

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While the above statements are a mere application of the abstract theory in [4], we will now use the particular character of thresholding, i.e., the structure of the energy (14) and the metric term (13) in order to pass to the limit in the approximate Brakke inequality (16).

We start with the basic a priori estimate for the piecewise constant interpolation χ^h.

Corollary 4.2 (Energy-dissipation estimate). *Given initial conditions $\chi^0 \colon [0,\Lambda)^d \to \{0,1\}$ with finite energy $E_0 := E(\chi^0) < \infty$, a time-step size $h > 0$ and a finite time horizon $T = Nh$ we have*

$$\sup_n E_h(\chi^n) + h \sum_{n=1}^{N} \frac{d_h^2(\chi^n, \chi^{n-1})}{2h^2} \leq E_0. \tag{22}$$

We recall the following proposition from [21] which will allow us to pass to the limit in the approximate Brakke inequality for the scheme.

Proposition 4.3 (Lemma 2.8 and Proposition 3.5 in [21]). *Given $u^h \to \chi$ and $E_h(u^h) \to E(\chi)$, a test function $\zeta \in C^\infty([0,\Lambda)^d)$ and a test matrix field $A \in C^\infty([0,\Lambda)^d, \mathbb{R}^{d \times d})$ we have*

$$\lim_{h \to 0} \frac{1}{\sqrt{h}} \int \zeta (1 - u^h) G_h * u^h \, dx = c_0 \int \zeta \, |\nabla \chi| \quad and \tag{23}$$

$$\lim_{h \to 0} \frac{1}{\sqrt{h}} \int A \colon (1 - u^h) h \nabla^2 G_h * u^h \, dx = c_0 \int A \colon \nu \otimes \nu \, |\nabla \chi|. \tag{24}$$

In [21] we used the above proposition to pass to the limit in the first variation of the energy

$$\delta E_h(u, \xi) := \frac{d}{ds}\Big|_{s=0} E_h(u_s),$$

where the inner variations u_s of u along a vector field ξ are given by the transport equation

$$\partial_s u_s + \xi \cdot \nabla u_s = 0 \quad u_s|_{s=0} = u. \tag{25}$$

Proposition 4.4 (Proposition 3.2, Remark 3.3 and Lemma 3.4 in [21]). *Given $u \colon [0,\Lambda)^d \to [0,1]$ we have*

$$\delta E_h(u, \xi) = \frac{1}{\sqrt{h}} \int \nabla \xi \colon (1 - u) \left(G_h \, Id - h \nabla^2 G_h \right) * u \, dx + O(\sqrt{h} \|\nabla^2 \xi\|_\infty E_h(u)). \tag{26}$$

In particular if $u^h \to \chi \in \{0,1\}$ and $E_h(u^h) \to E(\chi) < \infty$ we have

$$\delta E_h(u^h, \xi) \to \delta E(\chi, \xi) = c_0 \int \nabla \xi \colon (Id - \nu \otimes \nu) \, |\nabla \chi|. \tag{27}$$

Although the proof is contained in [21], we will repeat the short argument for the proposition in this two-phase context based on (23) and (24) for the convenience of the reader in the following section.

For a constant test function, the right-hand side of (21) yields a telescoping sum, i.e., the last left-hand side term $\sum_{n=1}^{N} \left(E_h(\chi^n, \chi^{n-1}; \zeta) - E_h(\chi^n, \chi^n; \zeta) \right)$ in (16) disappears. However, for a non-constant test function we have to pass to the limit in this extra term. In the following proposition we prove the convergence towards the transport term $\frac{c_0}{2} \int_\Sigma H \nu \cdot \nabla \zeta$ in Brakke's inequality under the convergence assumption (6).

Proposition 4.5. *Given the convergence assumption* (6) *and* $T = Nh$ *we have*

$$\lim_{h \to 0} \sum_{n=1}^{N} \left(E_h(\chi^n, \chi^{n-1}; \zeta) - E_h(\chi^n, \chi^n; \zeta) \right) = \frac{c_0}{2} \int_0^T \int \nabla^2 \zeta \colon (Id - \nu \otimes \nu) |\nabla \chi| \, dt.$$

The following a priori estimate for the variational interpolation u^h defined in (15) follows now very easily.

Corollary 4.6 (A priori estimate). *Given initial conditions* $\chi^0 \colon [0, \Lambda)^d \to \{0, 1\}$ *with finite energy* $E_0 := E(\chi^0) < \infty$, *a time-step size* $h > 0$ *and a finite time horizon* $T = Nh$ *if the test function* ζ *is strictly positive, then for the interpolation* (15) *we have*

$$\sup_t E_h(u^h(t)) + \int_0^T \frac{\mathrm{d}_h^2(u^h(t), \chi^h(t))}{2h^2} dt \lesssim \frac{\|\zeta\|_{W^{2,\infty}}}{\inf \zeta} (1 + T) E_0 + o(1) \tag{28}$$

as $h \to 0$.

The following statement is a post-processed version of our assumption (6).

Lemma 4.7 (Lemma 2.8 in [21]). *Given the convergence assumption* (6), *for a subsequence, we also have the pointwise property*

$$E_h(\chi^h) \to E(\chi) \quad a.e. \ in \ (0, T) \tag{29}$$

and furthermore for the variational interpolation u^h *given by* (15)

$$E_h(u^h) \to E(\chi) \quad a.e. \ in \ (0, T) \tag{30}$$

and in particular the integrated version

$$\int_0^T E_h(u^h) \, dt \to \int_0^T E(\chi) \, dt. \tag{31}$$

Without the localization, i.e., if $\zeta \equiv 1$, we can show

$$\frac{c_0}{2} \int H^2 |\nabla \chi| \le \liminf_{h \to 0} |\partial E_h|^2 (u^h) \quad \text{whenever} \quad u^h \to \chi \text{ in } L^1 \text{ and } E_h(u^h) \to E(\chi).$$

In the following proposition we prove a similar estimate for the local slope $\left| \partial E_h(\,\cdot\,, \chi^h; \zeta) \right|^2 (u^h)$ after integration in time if additionally we have the following *quantitative* proximity of $u^h(t)$ to $\chi^h(t)$ in L^2 after mollification:

$$\sqrt{h} \int_0^T \int \left| G_{h/2} * \left(\frac{u^h - \chi^h}{h} \right) \right|^2 dx \, dt \quad \text{stays bounded as} \quad h \to 0.$$

In our case where $u^h(t)$ is the variational interpolation (15) or the approximate solution $\chi^h(t + h)$ itself, this rate is a direct consequence of the energy-dissipation estimate (22) or the a priori estimate (28), respectively.

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Proposition 4.8. *Let $\zeta > 0$ be smooth, $\chi^h(t)$ the approximate solution obtained by Algorithm 1.1 and let $u^h(t)$ be either the variational interpolation (15) or the approximate solution $\chi^h(t+h)$ at time $t + h$. Given the convergence assumption (6), there exists a measurable function $H \in L^2(|\nabla\chi| \, dt)$, which is the mean curvature in the sense of (8), such that*

$$\frac{c_0}{2} \int_0^T \int \zeta \, H^2 \, |\nabla\chi| \, dt \leq \liminf_{h\to 0} \int_0^T \left|\partial E_h(\,\cdot\,, \chi^h; \zeta)\right|^2 (u^h) \, dt. \tag{32}$$

5   Proofs

We first give the proofs of the main results, Theorem 2.2, Lemma 3.1 and Corollary 3.2 with help of the auxiliary statements in Section 4.

Proof of Theorem 2.2. Step 1: Time-freezing for ζ. We claim that it is enough to prove

$$\int_0^{\tilde{T}} \int \left(\zeta \frac{1}{2}H^2 + \frac{1}{2}H\nu \cdot \nabla\zeta\right) |\nabla\chi| \, dt \leq \int \zeta \, |\nabla\chi(0)| - \int \zeta |\nabla\chi(\tilde{T})| \tag{33}$$

for any time-independent, strictly positive test function $\zeta = \zeta(x) > 0$ and a.e. $\tilde{T}$.

This is a standard approximation argument: In order to reduce (9) to (33) we fix a time-dependent test function $\zeta = \zeta(t, x) \geq 0$ and two time instances $0 \leq s < t$. It is no restriction to assume $s = 0$. Writing $t =: \tilde{T}$ for the time horizon we take a regular partition $0 = T_0 < \cdots < T_M = \tilde{T}$ of the interval $(0, \tilde{T})$ of fineness $\tau = \tilde{T}/M$. We write ζ_M for the piecewise constant interpolation of ζ plus a small perturbation $\frac{1}{M}$ so that $\zeta_M \geq \frac{1}{M} > 0$:

$$\zeta_M(t) := \zeta(T_{m-1}) + \frac{1}{M} \quad \text{if } t \in [T_{m-1}, T_m).$$

Writing $\partial^{-\tau}\zeta_M(t) := \frac{1}{\tau}\left(\zeta_M(t) - \zeta_M(t - \tau)\right)$ for the discrete (backwards) time derivative we have

$$\zeta_M \to \zeta, \quad \nabla\zeta_M \to \nabla\zeta \quad \text{and} \quad \partial^{-\tau}\zeta_M \to \partial_t\zeta \quad \text{uniformly as } M \to \infty. \tag{34}$$

Using (33) for $\zeta_M \geq \frac{1}{M} > 0$ on each interval $[T_{m-1}, T_m)$ and summing over m we obtain (9).

Step 2: Proof of (33). Given a test function $\zeta = \zeta(x) > 0$ and $\tilde{T} > 0$, we want to prove (33). We may assume that $\tilde{T} = Nh$ is a multiple of the time step size h. Furthermore by (29) we may assume that $E_h(\chi^h(\tilde{T})) \to E(\chi(\tilde{T}))$. We pass to the limit in the approximate Brakke inequality (16) to prove Brakke's inequality (33) for this time-independent test function.

By (6) and (31) in Lemma 4.7 we may apply Proposition 4.8 to obtain

$$\frac{c_0}{4} \int_0^{\tilde{T}} \int \zeta \, H^2 \, |\nabla\chi| \, dt \leq \liminf_{h\to 0} \frac{h}{2} \sum_{n=1}^N \left|\partial E_h(\,\cdot\,, \chi^{n-1}; \zeta)\right|^2 (\chi^n),$$

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as well as

$$\frac{c_0}{4}\int_0^{\tilde{T}}\int \zeta\, H^2\, |\nabla\chi|\, dt \le \liminf_{h\to 0}\frac{1}{2}\int_0^{\tilde{T}} \left|\partial E_h(\,\cdot\,,\chi^h(t);\zeta)\right|^2 (u^h(t))\, dt.$$

In addition we may apply Proposition 4.5 for the transport term and after division by the common prefactor c_0 we obtain (33). $\qquad\square$

Proof of Lemma 3.1. Given initial conditions $\chi\in\{0,1\}$ and a time-step size $h>0$, one iteration of the thresholding scheme yields $\chi^1 = \mathbf{1}_{\{G_h*\chi>\frac{1}{2}\}}$. Then χ^1 clearly minimizes

$$(1-u)\, G_h*\chi + u\, G_h*(1-\chi)$$

among all $u\in[0,1]$ pointwise a.e. This expression is equal to

$$(1-u)\, G_h*u + (u-\chi)\, G_h*(u-\chi) + \left[-(1-\chi)\, G_h*(u-\chi)+u\, G_h*(1-\chi)\right].$$

The term in the parenthesis can be rewritten as

$$(u-\chi)\, G_h*(1-\chi) - (1-\chi)\, G_h*(u-\chi) + \chi\, G_h*(1-\chi),$$

where the last summand is independent of u and thus irrelevant for the minimization. Multiplying with $\zeta\ge 0$ and integrating shows that χ^1 minimizes

$$\int \zeta\left[(1-u)\, G_h*u + (u-\chi)\, G_h*(u-\chi) + (u-\chi)\, G_h*(1-\chi) - (1-\chi)\, G_h*(u-\chi)\right] dx + \text{const.}$$

Dividing by $\sqrt{h}$, recalling the definitions (13) and (14) of the localized distance and energy, and using the semi-group and symmetry properties of the kernel yield (12). $\qquad\square$

Proof of Corollary 3.2. We apply Theorem 4.1 with $\chi=\chi^{n-1}$ and $t=h$, and sum over $n=1,\ldots,N$. $\qquad\square$

Now we prove the auxiliary statements of Section 4 which we used for the proof of our main result.

Proof of Corollary 4.2. The statement simply follows from testing the global minimization problem (2) for χ^n with its predecessor χ^{n-1}. $\qquad\square$

Proof of Proposition 4.4. The first variation of E_h at u along the vector field ξ is given by

$$\begin{aligned}
\delta E_h(u,\xi) ={}& \frac{1}{\sqrt{h}}\int -\xi\cdot\nabla(1-u)\, G_h*u - (1-u)\, G_h*(\xi\cdot\nabla u)\, dx\\
={}& \frac{1}{\sqrt{h}}\int \xi\cdot((1-u)\,\nabla G_h*u) - (1-u)\,\nabla G_h*(\xi\, u)\, dx\\
& + \frac{1}{\sqrt{h}}\int (\nabla\cdot\xi)(1-u)\, G_h*u + (1-u)\, G_h*((\nabla\cdot\xi)\, u)\, dx.
\end{aligned}$$

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This can be compactly rewritten as

$$\delta E_h(u,\xi) = \frac{1}{\sqrt{h}}\int 2\,(\nabla\cdot\xi)\,(1-u)\,G_h * u + (1-u)\,[\xi\cdot,\nabla G_h *]\,u - (1-u)\,[\nabla\cdot\xi, G_h *]\,u\,dx.$$

We expand the first commutator

$$[\xi\cdot,\nabla G_h *]\,u = \int (\xi(x)-\xi(x-z))\cdot\nabla G_h(z)\,u(x-z)\,dz$$

$$= \nabla\xi\colon \int -\frac{z}{\sqrt{h}}\otimes\frac{z}{\sqrt{h}}G_h(z)\,u(x-z)\,dz + O\left(\|\nabla^2\xi\|_\infty\sqrt{h}\,k_h * u\right),$$

where the kernel k_h is given by the mask $k(z)=|z|^3 G(z)$ and can be controlled by a gaussian with slightly larger variance $k(z)\lesssim G(z/2)$. The second commutator can be estimated by

$$|[\nabla\cdot\xi, G_h *]\,u| \lesssim \|\nabla^2\xi\|_\infty\sqrt{h}\,\tilde{k} * u,$$

where $\tilde{k}_h$ is given by the mask $\tilde{k}(z)=|z|G(z)\lesssim G(z/2)$. By the identity $G(z)\,(Id - z\otimes z) = -\nabla^2 G(z)$ we indeed obtain (26) with an error of order $\|\nabla^2\xi\|_\infty\sqrt{h}\,E_{4h}(u)$, which by the monotonicity (5) of E_h yields the claim. $\qquad\square$

Proof of Proposition 4.5. We first note that by definition

$$E_h(\chi^n,\chi^{n-1};\zeta) - E_h(\chi^n,\chi^n;\zeta) = \frac{1}{\sqrt{h}}\int (\chi^n - \chi^{n-1})\,[\zeta, G_h *]\,(1-\chi^{n-1})\,dx$$

$$+ \frac{1}{\sqrt{h}}\int (\chi^n - \chi^{n-1})\,[\zeta, G_{h/2} *]\,G_{h/2} * (\chi^n - \chi^{n-1})\,dx.$$

By the antisymmetry of the commutator, we may replace $(1-\chi^{n-1})$ by $(1-\chi^n)$ on the right-hand side:

$$\frac{1}{\sqrt{h}}\int (\chi^n - \chi^{n-1})\,[\zeta, G_h *]\,(1-\chi^n) + (\chi^n - \chi^{n-1})\,[\zeta, G_{h/2} *]\,G_{h/2} * (\chi^n - \chi^{n-1})\,dx.$$

Now we prove the proposition in two steps. First, we show that the first term converges to the right-hand side of the claim:

$$\lim_{h\to 0}\int_0^T \int \partial_t^{-h}\chi^h\,\frac{1}{\sqrt{h}}\,[\zeta, G_h *]\,(1-\chi^h)\,dx = c_0\int_0^T\int \nabla^2\zeta\colon (Id - \nu\otimes\nu)\,|\nabla\chi|\,dt, \qquad (35)$$

where $\partial_t^{-h}\chi^h = \frac{\chi^h - \chi^h(\cdot-h)}{h}$ denotes the discrete backwards time-derivative of χ^h. Then we prove that the second term is negligible:

$$\lim_{h\to 0}\int_0^T \sqrt{h}\int \partial_t^{-h}\chi^h\,[\zeta, G_{h/2} *]\,G_{h/2} * \partial_t^{-h}\chi^h\,dx\,dt = 0. \qquad (36)$$

Step 1: Argument for (35). Expanding the commutator to second order

$$\frac{1}{\sqrt{h}}\,[\zeta, G_h *]\,v = -\sqrt{h}\nabla G_h * (\nabla\zeta\,v) + \frac{\sqrt{h}}{2}\,(G_h\,Id + h\nabla^2 G_h) * (\nabla^2\zeta\,v) + O\left(\|\nabla^3\zeta\|_\infty h\,k_h * |v|\right), \qquad (37)$$

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where the kernel k_h is given by the mask $k(z) = |z|^3 G(z)$, we obtain for the first-order term

$$h \sum_{n=1}^{N} \int \frac{\chi^n - \chi^{n-1}}{h} \sqrt{h} \nabla G_h * (-\nabla\zeta (1 - \chi^n)) \, dx$$

$$= h \sum_{n=1}^{N} \frac{1}{\sqrt{h}} \int (\chi^n - \chi^{n-1}) G_h * (\nabla\zeta \cdot \nabla\chi^n - \Delta\zeta (1 - \chi^n)) \, dx.$$

Now we recognize the first variation of the dissipation functional on the right-hand side:

$$\delta \left(\frac{1}{2h} d_h^2(\cdot, \chi^{n-1}) \right) (\chi^n, \xi) = \frac{2}{\sqrt{h}} \int (\chi^n - \chi^{n-1}) G_h * (-\xi \cdot \nabla\chi^n) \, dx.$$

Using the semi-group and symmetry properties of the kernel, the extra term involving the Laplacian of the test function can be estimated by Jensen's inequality and the energy-dissipation estimate (22):

$$\left| h \sum_{n=1}^{N} \frac{1}{\sqrt{h}} \int (\chi^n - \chi^{n-1}) G_h * (\Delta\zeta (1 - \chi^n)) \, dx \right|$$

$$\lesssim \|\Delta\zeta\|_\infty T^{1/2} \left(h \sum_{n=1}^{N} \frac{1}{\sqrt{h}} \int (G_{h/2} * (\chi^n - \chi^{n-1}))^2 \, dx \right)^{1/2} \leq \|\Delta\zeta\|_\infty T^{1/2} E_0^{1/2} h^{1/4}.$$

Formally, the leading-order term, i.e., the first variation of the dissipation functional, converges to $c_0 \int_\Sigma V \nabla\zeta \cdot \nu$ but we want to obtain the term $\frac{c_0}{2} \int_\Sigma H \nabla\zeta \cdot \nu$ instead. Therefore we use the minimizing movements interpretation (2) in form of the Euler-Lagrange equation

$$\delta E_h(\chi^n, \xi) + \delta \left(\frac{1}{2h} d_h^2(\cdot, \chi^{n-1}) \right) (\chi^n, \xi) = 0 \quad \text{for all } \xi \in C^\infty([0, \Lambda)^d, \mathbb{R}^d).$$

We thus have

$$h \sum_{n=1}^{N} \int \frac{\chi^n - \chi^{n-1}}{h} \sqrt{h} \nabla G_h * (\nabla\zeta \, \chi^n) \, dx = \frac{h}{2} \sum_{n=1}^{N} \delta E_h(\chi^n, \nabla\zeta) + o(1).$$

By the convergence of the energies (6) and Proposition 4.3 we may pass to the limit $h \to 0$ and obtain

$$\frac{1}{2} \int_0^T \delta E(\chi, \nabla\zeta) \, dt = \frac{c_0}{2} \int_0^T \int \nabla^2\zeta : (Id - \nu \otimes \nu) |\nabla\chi| \, dt.$$

Now we conclude the argument for (35) by showing that the contributions of the second- and third-order terms in the expansion (37) are negligible in the limit $h \to 0$. The contribution of the second-order term is estimated as follows

$$\int_0^T \int \partial_t^{-h} \chi^h \frac{\sqrt{h}}{2} (G_h \, Id + h\nabla^2 G_h) * (\nabla^2\zeta (1 - \chi^h)) \, dx \, dt$$

$$\leq \left(\int_0^T \sqrt{h} \int \left| (G_h \, Id + h\nabla^2 G_h) * \partial_t^{-h} \chi^h \right|^2 dx \, dt \right)^{\frac{1}{2}} \left(\int_0^T \sqrt{h} \int \left| \nabla^2\zeta (1 - \chi^h) \right|^2 dx \, dt \right)^{\frac{1}{2}}.$$

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The second right-hand side integral is bounded by $T\Lambda^d\|\nabla^2\zeta\|_\infty^2\sqrt{h} = o(1)$ while the first right-hand side integral can be estimated by

$$\sqrt{h}\int\left|\left(G_h\,Id + h\nabla^2 G_h\right)*\partial_t^{-h}\chi^h\right|^2 dx \lesssim \sqrt{h}\int\left(G_h*\partial_t^{-h}\chi^h\right)^2 + \left|h\nabla^2 G_h*\partial_t^{-h}\chi^h\right|^2 dx,$$

which by the semi-group property of the kernel $\nabla^2 G_h = \nabla^2 G_{h/2}*G_{h/2}$ is bounded by a constant times $\sqrt{h}\int\left(G_{h/2}*\partial_t^{-h}\chi^h\right)^2 dx$. Therefore the time integral stays bounded by the energy-dissipation estimate (22).

The contribution of the third-order term is controlled by

$$\int\limits_0^T\int h\left|\partial_t^{-h}\chi^h\right| dx\, dt = \int\limits_0^T\int \left|\chi^h(t) - \chi^h(t-h)\right| dx\, dt.$$

The following basic estimate, which is valid for any pair of characteristic functions,

$$|\chi - \tilde\chi| = |\chi - \tilde\chi|^2 \lesssim \left|G_{h/2}*(\chi - \tilde\chi)\right|^2 + \left|G_{h/2}*\chi - \chi\right|^2 + \left|G_{h/2}*\tilde\chi - \tilde\chi\right|^2,$$

and the fact that by the normalization $\int G_{h/2}(z)\,dz = 1$ and the pointwise estimate $G_{h/2}(z) \lesssim G_h(z)$ we have

$$\frac{1}{\sqrt{h}}\int\left|G_{h/2}*\chi - \chi\right| dx \leq \frac{1}{\sqrt{h}}\int G_{h/2}(z)\int|\chi(x) - \chi(x-z)|\, dx\, dz \lesssim E_h(\chi)$$

yield the estimate

$$\int\limits_0^T\int\left|\chi^h(t) - \chi^h(t-h)\right| dx\, dt \lesssim (1+T)\,E_0\sqrt{h} \to 0.$$

This concludes the proof of (35).

Step 2: Argument for (36). We expand the commutator to first order

$$\left[\zeta, G_{h/2}*\right]v = -\frac{h}{2}\nabla G_{h/2}*(\nabla\zeta\, v) + O\left(\|\nabla^2\zeta\|_\infty h\, k_h*|v|\right), \tag{38}$$

where the kernel k_h is given by the mask $k(z) = |z|^2 G_{1/2}(z)$, and first consider the contribution of the first-order term to (36), namely

$$-\frac{h}{2}\int\limits_0^T\sqrt{h}\int\partial_t^{-h}\chi^h\nabla G_{h/2}*\left(\nabla\zeta\, G_{h/2}*\partial_t^{-h}\chi^h\right) dx\, dt.$$

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Using the antisymmetry of ∇G, the chain rule and integration by parts this is equal to

$$\frac{h}{2}\int_0^T \sqrt{h}\int \nabla\left(G_{h/2}*\partial_t^{-h}\chi^h\right)\cdot\nabla\zeta\left(G_{h/2}*\partial_t^{-h}\chi^h\right)dx\,dt$$

$$=\frac{h}{2}\int_0^T \sqrt{h}\int \nabla\zeta\cdot\nabla\left(\frac{1}{2}\left(G_{h/2}*\partial_t^{-h}\chi^h\right)^2\right)dx\,dt$$

$$=-\frac{h}{4}\int_0^T \sqrt{h}\int \Delta\zeta\left(G_{h/2}*\partial_t^{-h}\chi^h\right)^2 dx\,dt.$$

By the energy-dissipation estimate (22) this term is $O(h)$ as $h\to 0$.

The second-order term coming from the expansion (38) is controlled by

$$\int_0^T \sqrt{h}\int \left|\partial_t^{-h}\chi^h\right| hk_h*\left|G_{h/2}*\partial_t^{-h}\chi^h\right| dx\,dt \lesssim \int_0^T \sqrt{h}\int \left|G_{h/2}*\partial_t^{-h}\chi^h\right| dx\,dt,$$

where the kernel k_h is given by the mask $k(z)=|z|^2 G_{1/2}(z)$. Therefore, this term vanishes as $h\to 0$ by Jensen's inequality and the energy-dissipation estimate (22). $\qquad\square$

Proof of Corollary 4.6. In contrast to the piecewise constant interpolation χ^h, the variational interpolation u^h is not given in an explicit form but only by the minimization problem (15). In particular, since in general u^h may depend on the test function ζ, we are tied to the local minimization problem (15). By (21) we have in particular

$$E_h(u^h(T),u^h(T);\zeta)+\int_0^T \frac{d_h^2(u^h,\chi^h;\zeta)}{2h^2}dt \le E_h(\chi^0,\chi^0;\zeta)-\sum_{n=1}^N\left(E_h(\chi^n,\chi^{n-1};\zeta)-E_h(\chi^n,\chi^n;\zeta)\right)$$

for any $T\in[Nh,(N+1)h)$, where $N\in\mathbb{N}$. The left-hand side is bounded from below by

$$\inf\zeta\left(E_h(u^h(T))+\int_0^T \frac{d_h^2(u^h,\chi^h)}{2h^2}dt\right)$$

while the right-hand side can be controlled by Proposition 4.5. $\qquad\square$

Proof of Lemma 4.7. The convergence assumption (6) and $\liminf$-inequality of the Γ-convergence imply the convergence of $E_h(\chi^h)\to E(\chi)$ in $L^1(0,T)$. In order to understand the behavior of the energies of the variational interpolations we compare them to the energies of the piecewise constant interpolation:

$$|E_h(u^h)-E_h(\chi^h)| \le \frac{2}{\sqrt{h}}\int \left|G_{h/2}*(u^h-\chi^h)\right| dx$$

and by Jensen we obtain

$$\int_0^T |E_h(u^h)-E_h(\chi^h)|\,dt \lesssim T^{1/2}\frac{1}{\sqrt{h}}\int_0^T\int \left(G_{h/2}*(u^h-\chi^h)\right)^2 dx\,dt,$$

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which by (28) is estimated by $T^{1/2} E_0^{1/2} h^{1/4} \to 0$. That means the approximate energies converge to the same limit in $L^1(0, T)$ and therefore we obtain the L^1-convergence (31) and – after the possible passage to a further subsequence – the pointwise convergences (29) and (30). $\qquad \square$

Proof of Proposition 4.8. We let the variations u_s defined in (25) play the role of v in the definition of the local slope (17) so that we obtain the inequality

$$|\partial E_h(\cdot, \chi^h; \zeta)|(u^h) \geq \limsup_{s \to 0} \frac{\left(E_h(u^h, \chi^h; \zeta) - E_h(u_s^h, \chi^h; \zeta)\right)_+}{d_h(u_s^h, u^h; \zeta)}.$$

As $s \to 0$ we expand the numerator in the following way

$$E_h(u_s^h, \chi^h; \zeta) = E_h(u^h, \chi^h; \zeta) + s \frac{d}{ds}\Big|_{s=0} E_h(u_s^h, \chi^h; \zeta) + o(s).$$

For the denominator we have

$$\frac{1}{2h} d_h^2(u_s^h, u^h; \zeta) = \frac{s^2}{\sqrt{h}} \int \zeta \left(G_{h/2} * (\xi \cdot \nabla u^h)\right)^2 dx + o(s^2)$$

as $s \to 0$. Taking the limit $s \to 0$ we obtain

$$|\partial E_h(\cdot, \chi^h; \zeta)|(u^h) \geq \frac{\frac{d}{ds}\big|_{s=0} E_h(u_s^h, \chi^h; \zeta)}{\sqrt{2\sqrt{h} \int \zeta \left(G_{h/2} * (\xi \cdot \nabla u^h)\right)^2 dx}} \quad \text{for all } \xi. \tag{39}$$

Now we expand ζ and ξ to analyze the leading order terms as $h \to 0$. Using (25) we can compute the first variation of the localized energy $E_h(u, \chi; \zeta)$:

$$\frac{d}{ds}\Big|_{s=0} E_h(u_s, \chi; \zeta)$$
$$= \frac{1}{\sqrt{h}} \int - \zeta \, \xi \cdot \nabla (1 - u) \, G_h * u - \zeta (1 - u) \, G_h * (\xi \cdot \nabla u)$$
$$\quad - \xi \cdot \nabla u \, [\zeta, G_h*] \, (1 - u) - \xi \cdot \nabla u \, [\zeta, G_h*] \, (u - \chi)$$
$$\quad - \xi \cdot \nabla u \, [\zeta, G_{h/2}*] G_{h/2} * (u - \chi) + \xi \cdot \nabla u \, G_{h/2} * [\zeta, G_{h/2}*] (u - \chi) \, dx.$$

The fourth term in the sum comes from replacing $(1 - \chi)$ by $(1 - u)$ in the third term, while for the last term we used the antisymmetry $\int u \, [\zeta, G_{h/2}*] \, v \, dx = - \int v \, [\zeta, G_{h/2}*] \, u \, dx$. Note that due to the symmetry of G there is a cancellation between the second and third term in this sum:

$$\int - \zeta (1 - u) \, G_h * (\xi \cdot \nabla u) - \xi \cdot \nabla u \, [\zeta, G_h*] \, (1 - u) \, dx = \int - \zeta \, \xi \cdot \nabla u \, G_h * (1 - u) \, dx$$
$$= \int - (1 - u) \, G_h * (\zeta \, \xi \cdot \nabla u) \, dx.$$

A direct computation based on the semi-group property $G_h = G_{h/2} * G_{h/2}$ yields

$$- [\zeta, G_h*] \, v - [\zeta, G_{h/2}*] \, G_{h/2} * v + G_{h/2} * [\zeta, G_{h/2}*] \, v = -2 \, [\zeta, G_{h/2}*] \, G_{h/2} * v \tag{40}$$

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so that the last three terms in the first variation of E_h above can be combined using once more the antisymmetry of the commutator, and we get

$$\frac{d}{ds}\Big|_{s=0} E_h(u_s, \chi; \zeta) = \frac{1}{\sqrt{h}} \int -\zeta\, \xi \cdot \nabla \left(1 - u\right) G_h * u - \left(1 - u\right) G_h * \left(\zeta\, \xi \cdot \nabla u\right) dx$$
$$+ \frac{2}{\sqrt{h}} \int G_{h/2} * (u - \chi) \left[\zeta, G_{h/2}*\right] (\xi \cdot \nabla u)\, dx. \tag{41}$$

Note that the first right-hand side integral is exactly $\delta E_h(u, \zeta\, \xi)$, the first variation of the energy along the "localized" vector field $\zeta\, \xi$. Now we plug $u = u^h$ into the above formula. Since $u^h \to \chi$ in L^1 and $E_h(u^h) \to E(\chi)$ for a.e. t, using Proposition 4.4 with $\zeta\, \xi$ playing the role of ξ, for a.e. t, along the sequence u^h the first right-hand side integral of (41) converges to

$$\delta E(\chi, \zeta\, \xi) = c_0 \int \nabla \left(\zeta\, \xi\right) : \left(Id - \nu \otimes \nu\right) |\nabla \chi|.$$

Now we give the argument that the second integral in (41) is negligible:

$$\frac{2}{\sqrt{h}} \int G_{h/2} * (u - \chi) \left[\zeta, G_{h/2}*\right] (\xi \cdot \nabla u)\, dx \to 0 \quad \text{in } L^1(0, T). \tag{42}$$

In view of the boundedness of

$$\sqrt{h} \int_0^T \int \left| G_{h/2} * \left(\frac{u^h - \chi^h}{h}\right) \right|^2 dx\, dt,$$

which is a direct consequence of the a priori estimate (28), by Cauchy-Schwarz it is enough to prove

$$\sqrt{h} \int_0^T \int \left(\left[\zeta, G_{h/2}*\right] \left(\xi \cdot \nabla u^h\right)\right)^2 dx\, dt \to 0. \tag{43}$$

Rewriting the commutator

$$\left[\zeta, G_{h/2}*\right] (\xi \cdot \nabla u) = \int G_{h/2}(z) \left(\zeta(x) - \zeta(x - z)\right) \xi(x - z) \cdot \nabla u^h(x - z)\, dz,$$

using $\xi \cdot \nabla u^h = \nabla(\xi \cdot u^h) - (\nabla \cdot \xi)\, u^h$ with ξ replaced by $\left(\zeta(\cdot) - \zeta(\cdot - z)\right) \xi(\cdot - z)$ and using the Lipschitz estimate $|\zeta(x) - \zeta(x - z)| \leq \|\nabla \zeta\|_\infty |z|$ we obtain the pointwise estimate

$$\left|\left[\zeta, G_{h/2}*\right] (\xi \cdot \nabla u)\right| = \left| \int \nabla G_{h/2}(z) \cdot \xi(x - z) \left(\zeta(x) - \zeta(x - z)\right) u^h(x - z)\, dz \right|$$
$$+ \left| \int G_{h/2}(z) \nabla \cdot \left[\left(\zeta(x) - \zeta(x - z)\right)\xi(x - z)\right] u^h(x - z)\, dz \right|$$
$$\leq \left(\|\nabla \zeta\|_\infty \|\xi\|_\infty + \|\zeta\, \xi\|_{W^{1,\infty}}\right) \lesssim \|\zeta\|_{W^{1,\infty}} \|\xi\|_{W^{1,\infty}}$$

and hence (43) holds with the rate $O(\|\zeta\|_{W^{1,\infty}}^2 \|\xi\|_{W^{1,\infty}}^2 \sqrt{h})$. Therefore we have proven the following convergence of the first variation of the localized energy (14):

$$\lim_{h \to 0} \int_0^T \frac{d}{ds}\Big|_{s=0} E_h(u_s^h, \chi^h; \zeta)\, dt = \lim_{h \to 0} \int_0^T \delta E_h(u^h, \zeta\, \xi)\, dt = c_0 \int_0^T \int \nabla \left(\zeta\, \xi\right) : \left(Id - \nu \otimes \nu\right) |\nabla \chi|\, dt.$$
$$\tag{44}$$

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With the same methods we can handle the term in the expansion of the metric term $\mathrm{d}_h(u_s^h, \chi^h)$:
We claim that

$$\lim_{h\to 0} 2\sqrt{h} \int \zeta \left(G_{h/2} * \left(\xi \cdot \nabla u^h\right)\right)^2 dx = \lim_{h\to 0} \frac{2}{\sqrt{h}} \int \zeta \left(\xi \otimes \xi\right) : \left(1 - u^h\right) \left(h\nabla^2 G_h\right) * u^h \, dx$$

$$= 2c_0 \int \zeta \left(\xi \cdot \nu\right)^2 |\nabla \chi| \quad \text{for a.e. } t. \tag{45}$$

To this end we plug $\xi \cdot \nabla u = \nabla \cdot (\xi u) - (\nabla \cdot \xi) u$ into the quadratic term on left-hand side and expand the square. First we note that only the term

$$2\sqrt{h} \int \zeta \left(G_{h/2} * \left(\nabla \cdot (\xi u^h)\right)\right)^2 dx = 2\sqrt{h} \int \zeta \left(\nabla G_{h/2} * \left(\xi u^h\right)\right)^2 dx \tag{46}$$

survives in the limit $h \to 0$. Indeed, we have

$$2\sqrt{h} \int \zeta \left(G_{h/2} * \left((\nabla \cdot \xi) u^h\right)\right)^2 dx \lesssim \|\nabla \xi\|_\infty^2 \sqrt{h} \int |\zeta| \, dx$$

and the mixed term can be estimated by Young's inequality and the boundedness of the leading-order term which we will show now. Using the antisymmetry of ∇G and in particular $\int \nabla G(z) \, dz = 0$ we may add a lower-order term to (46):

$$2\sqrt{h} \int \zeta \left(\nabla G_{h/2} * \left(\xi u^h\right)\right)^2 dx = 2\sqrt{h} \int u^h \, \xi \cdot \nabla G_{h/2} * \left(\zeta \nabla G_{h/2} * \left(-\xi u^h\right)\right) dx$$

$$= 2\sqrt{h} \int u^h \, \xi \cdot \nabla G_{h/2} * \left(\zeta \nabla G_{h/2} * \left(\xi (1 - u^h)\right)\right) dx + o(1).$$

The term involving $\nabla G_{h/2} * \left(\zeta \nabla G_{h/2} * \xi\right)$ is indeed of lower order since both gradients may be put on the test functions ζ and ξ. Now we want to commute the multiplication with ξ and the outer convolution and afterwards the multiplication with $\zeta \xi$ and the inner convolution. For this we use the L^∞-commutator estimates

$$\|[\xi\cdot, \nabla G_h*] \, u\|_\infty \lesssim \|\nabla \xi\|_\infty \quad \text{and} \quad \|[\zeta \, \xi\cdot, \nabla G_h*]\| \lesssim \|\zeta\|_{W^{1,\infty}} \|\xi\|_{W^{1,\infty}},$$

the L^1-estimate

$$\int \left|\nabla G_{h/2} * \left(\xi u^h\right)\right| dx \lesssim \|\nabla \xi\|_\infty + \|\xi\|_\infty \int \left|\nabla G_{h/2} * u^h\right| dx$$

and the a priori estimate (28) for the last term:

$$\int \left|\nabla G_{h/2} * u^h\right| dx \lesssim E_h(u^h) \lesssim \frac{\|\zeta\|_{W^{2,\infty}}}{\inf \zeta}(1 + T)E_0 + o(1).$$

Therefore, the leading-order term becomes

$$\frac{2}{\sqrt{h}} \int \zeta \left(\xi \otimes \xi\right) : \left(1 - u^h\right) \left(h\nabla^2 G_h\right) * u^h \, dx + o(1).$$

Then (45) follows from the convergence of the energies (cf. Lemma 4.7) and Proposition 4.3.

Using (44) for the numerator and (45) for the denominator of the right-hand side of (39) we obtain by Fatou's Lemma in t

$$\liminf_{h\to 0} \int_0^T |\partial E_h(\,\cdot\,,\chi^h;\zeta)|^2(u^h)\,dt \geq \frac{c_0}{2} \int_0^T \left(\sup_\xi \frac{\int \nabla\left(\zeta\,\xi\right) : \left(Id - \nu \otimes \nu\right)|\nabla\chi|}{\sqrt{\int \zeta\,|\xi|^2\,|\nabla\chi|}} \right)^2 dt,$$

which establishes the existence of $H \in L^2(|\nabla\chi|\,dt)$ and the estimate stated in the proposition. $\qquad\square$

Bibliography

[1] Giovanni Alberti and Giovanni Bellettini. "A non-local anisotropic model for phase transitions: asymptotic behaviour of rescaled energies". In: *European Journal of Applied Mathematics* 9.03 (1998), pp. 261–284.

[2] Fred Almgren, Jean E. Taylor, and Lihe Wang. "Curvature-driven flows: a variational approach". In: *SIAM Journal on Control and Optimization* 31.2 (1993), pp. 387–438.

[3] Luigi Ambrosio, Guido De Philippis, and Luca Martinazzi. "Gamma-convergence of nonlocal perimeter functionals". In: *Manuscripta Mathematica* 134.3 (2011), pp. 377–403.

[4] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Gradient flows in metric spaces and in the space of probability measures.* Birkhäuser, 2008.

[5] Guy Barles, H. Mete Soner, and Panagiotis E. Souganidis. "Front propagation and phase field theory". In: *SIAM Journal on Control and Optimization* 31.2 (1993), pp. 439–469.

[6] Eric Bonnetier, Elie Bretin, and Antonin Chambolle. "Consistency result for a non monotone scheme for anisotropic mean curvature flow". In: *Interfaces and Free Boundaries* 14.1 (2012), pp. 1–35.

[7] Kenneth A. Brakke. *The motion of a surface by its mean curvature.* Vol. 20. Princeton University Press Princeton, 1978.

[8] Antonin Chambolle and Matteo Novaga. "Approximation of the anisotropic mean curvature flow". In: *Mathematical Models and Methods in Applied Sciences* 17.06 (2007), pp. 833–844.

[9] Matt Elsey and Selim Esedoğlu. *Threshold Dynamics for Anisotropic Surface Energies.* Tech. rep. UM, 2016.

[10] Matt Elsey, Selim Esedoğlu, and Peter Smereka. "Diffusion generated motion for grain growth in two and three dimensions". In: *Journal of Computational Physics* 228.21 (2009), pp. 8015–8033.

[11] Matt Elsey, Selim Esedoğlu, and Peter Smereka. "Large-scale simulation of normal grain growth via diffusion-generated motion". In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science* 467.2126 (2011), pp. 381–401.

[12] Matt Elsey, Selim Esedoğlu, and Peter Smereka. "Large-scale simulations and parameter study for a simple recrystallization model". In: *Philosophical Magazine* 91.11 (2011), pp. 1607–1642.

[13] Selim Esedoğlu and Matt Jacobs. *Convolution kernels, and stability of threshold dynamics methods.* Tech. rep. UM, 2016., 2016.

[14] Selim Esedoğlu and Felix Otto. "Threshold dynamics for networks with arbitrary surface tensions". In: *Communications on Pure and Applied Mathematics* 68.5 (2015), pp. 808–864.

[15] Lawrence C. Evans, H. Mete Soner, and Panagiotis E. Souganidis. "Phase transitions and generalized motion by mean curvature". In: *Communications on Pure and Applied Mathematics* 45.9 (1992), pp. 1097–1123.

[16] Gerhard Huisken. "Asymptotic-behavior for singularities of the mean-curvature flow". In: *Journal of Differential Geometry* 31.1 (1990), pp. 285–299.

[17] Tom Ilmanen. "Convergence of the Allen-Cahn equation to Brakkes motion by mean curvature". In: *Journal of Differential Geometry* 38.2 (1993), pp. 417–461.

[18] Hitoshi Ishii, Gabriel E. Pires, and Panagiotis E. Souganidis. "Threshold dynamics type approximation schemes for propagating fronts". In: *Journal of the Mathematical Society of Japan* 51.2 (1999), pp. 267–308.

[19] Richard Jordan, David Kinderlehrer, and Felix Otto. "The variational formulation of the Fokker–Planck equation". In: *SIAM Journal on Mathematical Analysis* 29.1 (1998), pp. 1–17.

[20] Lami Kim and Yoshihiro Tonegawa. "On the mean curvature flow of grain boundaries". In: *arXiv preprint arXiv:1511.02572* (2015).

[21] Tim Laux and Felix Otto. "Convergence of the thresholding scheme for multi-phase mean-curvature flow". In: *Calculus of Variations and Partial Differential Equations* 55.5 (2016), pp. 1–74.

[22] Tim Laux and Thilo Simon. "Convergence of the Allen-Cahn Equation to multi-phase mean-curvature flow". In: *arXiv preprint arXiv:1606.07318* (2016).

[23] Tim Laux and Drew Swartz. "Convergence of thresholding schemes incorporating bulk effects". In: *arXiv preprint arXiv:1601.02467* (2016).

[24] Stephan Luckhaus and Thomas Sturzenhecker. "Implicit time discretization for the mean curvature flow equation". In: *Calculus of variations and partial differential equations* 3.2 (1995), pp. 253–271.

[25] Barry Merriman, James K. Bence, and Stanley J. Osher. *Diffusion generated motion by mean curvature.* CAM Report 92-18. Department of Mathematics, University of California, Los Angeles. 1992.

[26] Barry Merriman, James K. Bence, and Stanley J. Osher. "Motion of multiple junctions: A level set approach". In: *Journal of Computational Physics* 112.2 (1994), pp. 334–363.

[27] Peter W. Michor and David Mumford. "Riemannian geometries on spaces of plane curves". In: *Journal of the European Mathematical Society* 8.1 (2006), pp. 1–48.

[28] Michele Miranda, Diego Pallara, Fabio Paronetto, and Marc Preunkert. "Short-time heat flow and functions of bounded variation in R^N". In: *Annales-Faculté des Sciences Toulouse Mathématiques.* Vol. 16. 1. Université Paul Sabatier. 2007, pp. 125–145.

[29] William T. Read and William B. Shockley. "Dislocation models of crystal grain boundaries". In: *Physical Review* 78.3 (1950), p. 275.

[30] Drew Swartz and Nung Kwan Yip. "Convergence of diffusion generated motion to motion by mean curvature". In: *arXiv preprint arXiv:1703.06519* (2017).