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by

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We study the unextendible maximally entangled bases (UMEB) in $\mathbb{C}^d \bigotimes \mathbb{C}^d$ and connect the problem to the partial Hadamard matrices. We show that for a given special UMEB in $\mathbb{C}^d \bigotimes \mathbb{C}^d$, there is a partial Hadamard matrix which can not be extended to a complete Hadamard matrix in $\mathbb{C}^d$. As a corollary, any $(d-1) \times d$ partial Hadamard matrix can be extended to a complete Hadamard matrix, which answers a conjecture about $d = 5$. We obtain that for any d there is a UMEB except for $d = p$ or $2p$, where $p \equiv 3 \mod 4$ and p is a prime. The existence of different kinds of constructions of UMEBs in $\mathbb{C}^{nd} \bigotimes \mathbb{C}^{nd}$ for any $n \in \mathbb{N}$ and $d = 3 \times 5 \times 7$ is also discussed.

I. INTRODUCTION

It is well known that quantum states are divided into two classes: separable states and entangled ones. The pure product states are a special set of separable states. While the maximally entangled states are another set of states that plays important roles in many information processing [1, 2, 3, 4]. One of the significant properties of quantum systems is the quantum nonlocality. An unextendible product basis (UPB) in bipartite quantum system $\mathbb{C}^m \bigotimes \mathbb{C}^n$ is a basis consisting a set of less than mn orthogonal product states such that no further product states are orthogonal to every state in that set [5, 6]. It is proven that the UPBs may display nonlocality without entanglement [6, 7]. Similar to UPBs, Bravyi and Smolin first proposed the unextendible maximally entangled basis (UMEB) in 2009, which consists a set of less than d^2 orthonormal maximally entangled states in $\mathbb{C}^d \bigotimes \mathbb{C}^d$ such that no more maximally entangled states are orthogonal to all the states in the set. The UMEBs are helpful in constructing quantum states with special properties of the entanglement of assistance (EOF) and can be used to find quantum channels that are unital but not convex mixtures of unitary operations [8].

It has been proved that there do not exist UMEBs for $d = 2$, and a 6-member UMEB for $d = 3$ and a 12-member UMEB for $d = 4$ have been constructed [8]. UMEBs have been investigated extensively since then. Many UMEBs have been constructed in $\mathbb{C}^d \bigotimes \mathbb{C}^{d'} (d \neq d')$ [9, 10]. In Ref. [11], the authors studied the UMEB in $\mathbb{C}^d \bigotimes \mathbb{C}^d$, and shew that if there is an UMEB in $\mathbb{C}^d \bigotimes \mathbb{C}^d$, then there is also an UMEB in $\mathbb{C}^{qd} \bigotimes \mathbb{C}^{qd}$ for any $q \in \mathbb{N}$. However, for UMEBs in $\mathbb{C}^d \bigotimes \mathbb{C}^d$, some results are obtained only for the cases of $d = 3, 4, 3n, 4n$. It is of significance to consider the UMEBs in higher-dimensional system $\mathbb{C}^d \bigotimes \mathbb{C}^d$ for general d.

The construction of Hadamard matrices is also an interesting topic. In this paper, we mainly concern about the complex Hadamard matrices, see Refs. [12, 13, 14]. A partial Hadamard matrix is a matrix $H \in M_{m \times n}(\mathbb{T})$ with entries in the circle $\mathbb{T}$, whose rows are pairwise orthogonal. Given a partial Hadamard matrix $H \in M_{m \times n}(\mathbb{T})$ ($n > m$), one interesting problem is to justify whether this matrix can be extended to an $n \times n$ complex Hadamard matrix. For the real case, there are already many results [15, 16]. But for the general complex case, fewer result is known about this problem [17].

In this paper, we show a relation between UMEBs and partial Hadamard matrices. In particular, we show that if there is a special UMEB in $\mathbb{C}^d \bigotimes \mathbb{C}^d$, then we can find a corresponding partial Hadamard matrix which can not be extended to a complete Hadamard matrix, and vice versa. Then by using the extendibility of any $d^2 - 1$ orthogonal maximally entangled states, we give an answer to the conjecture in [17]. The relation between UMEBs and partial Hadamard matrices also gives us a method construct UMEB. As a example, we first construct a 23-member UMEB in $\mathbb{C}^5 \bigotimes \mathbb{C}^5$. Then we generalize the example to higher dimension $\mathbb{C}^{4n+1} \bigotimes \mathbb{C}^{4n+1}$. We show that for any $n \in \mathbb{N}$, there exists a UMEB in $\mathbb{C}^{4n+1} \bigotimes \mathbb{C}^{4n+1}$. Then from the results in [11] we obtain that there is an UMEB in $\mathbb{C}^{q(4n+1)} \bigotimes \mathbb{C}^{q(4n+1)}$ for any $q \in \mathbb{N}$. Therefore, we solve the problem for most of the cases except for $d = p$ or $2p$, where $p \equiv 3 \mod 4$ and p is a prime. In addition, we also give a UMEB in $\mathbb{C}^7 \bigotimes \mathbb{C}^7$, as the most easy case for the remained unsolved cases. Then by using the UMEBs constructed from $d = 3, 5, 7$, we show that there are different kinds of UMEBs in $\mathbb{C}^{(3 \times 5 \times 7)n} \bigotimes \mathbb{C}^{(3 \times 5 \times 7)n}$ for any $n \in \mathbb{N}$.

II. UMEBS IN $\mathbb{C}^d \bigotimes \mathbb{C}^d$ AND PARTIAL HADAMARD MATRIX

Definition 1. A set of states $\{|\phi_a\rangle \in \mathbb{C}^d \bigotimes \mathbb{C}^d : a = 1, 2, \cdots, n, n < d^2\}$ is called an n-number UMEB if and only if (i) $|\phi_a\rangle$, $a = 1, 2, \cdots, n$, are maximally en-

tangled; (ii) $\langle\phi_a|\phi_b\rangle = \delta_{ab}$; (iii) if $\langle\phi_a|\psi\rangle = 0$ for all $a = 1, 2, \cdots, n$, then $|\psi\rangle$ cannot be maximally entangled.

Here under computational basis a maximally entangled state $|\phi_a\rangle$ can be expressed as

$$|\phi_a\rangle = (I \otimes U_a)\frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i\rangle \otimes |i\rangle, \tag{1}$$

where I is the $d \times d$ identity matrix, U_a is any unitary matrix. According to (1), a set of unitary matrices $\{U_a \in M_d(\mathbb{C})|a = 1, ..., n\}$ gives an n-number UMEB in $\mathbb{C}^d \bigotimes \mathbb{C}^d$ if and only if
(i) $n < d^2$;
(ii) $Tr(U_a^\dagger U_b) = d\,\delta_{ab}, \quad \forall a, b = 1, \cdots, n$;
(iii) For any $U \in M_d(\mathbb{C})$, if $Tr(U_a^\dagger U) = 0, \ \forall\, a = 1, \cdots, n$, then U cannot be unitary.

Definition 2. Partial Hadamard matrices: A partial Hadamard matrix is a rectangular matrix with entries in the circle $\mathbb{T}$, $H \in M_{m \times n}(\mathbb{T})$, whose rows are pairwise orthogonal.

In studying the UMEB in $\mathbb{C}^d \bigotimes \mathbb{C}^d$ we need the following result from [11].

Lemma. If there is an N-number UMEB in $\mathbb{C}^d \bigotimes \mathbb{C}^d$, then for any $q \in \mathbb{N}$, there is a $\widetilde{N}$-number, $\widetilde{N} = (qd)^2 - (d^2 - N)$, UMEB in $\mathbb{C}^{qd} \bigotimes \mathbb{C}^{qd}$.

In this paper, we mainly study the UMEBs in $\mathbb{C}^d \bigotimes \mathbb{C}^d$ containing the following set of unitary matrices, we call them a special form of UMEB if exists,

$$S_0 = \{X^m Z^n \mid m = 1, 2, ..., d-1, n = 0, 1, ..., d-1\},$$

where $X = \sum_{j=0}^{d-1}|j+1\rangle\langle j|$, $Z = \sum_{j=0}^{d-1}\omega_d^j|j\rangle\langle j|$, $\omega_d = e^{\frac{2\pi i}{d}}$.

Suppose $A = (a_{st})_{k \times d}$ is a $k \times d$ partial Hadamard matrix, $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the rows of A, and $|a_{st}| = 1$, $\alpha_t^\dagger \alpha_s = d\,\delta_{st}$. Given a partial Hadamard matrix $A = (\alpha_1, \alpha_2, ..., \alpha_k)^t$, where t denotes transposition, we construct a set of unitary matrices, denoted by

$$S(A) = \{diag(\alpha_s) \mid s = 1, 2, \ldots, k\},$$

where $diag(\alpha_s) = \sum_{t=1}^{d}a_{st}|t\rangle\langle t|$. Then the elements in $S(A)$ are unitary and orthogonal with each other under inner product $(A, B) = Tr(AB^\dagger)$.

Proposition 1. Given a $k \times d$ partial Hadamard matrix A, then $S_0 \cup S(A)$ can not be extended to maximally entangled basis (MEB) if and only if A can not be extended to a complete Hadamard matrix.

Proof. $\Rightarrow$: Suppose A can be extended to a complete Hadamard matrix. That is, there are $d - k$ mutually orthogonal vectors $\nu_1, \nu_2, \ldots, \nu_{d-k}$ with modules 1 for each

entry which are orthogonal to all the rows of A. Then $U_j = diag(\nu_j)$ are unitary matrices which are orthogonal to each other, and lie in the orthogonal complement of $S_0 \cup S(A)$. Then $S_0 \cup S(A) \cup \{U_j \mid j = 1, 2, \ldots, d-k\}$ is a MEB. This is contradicted to that $S_0 \cup S(A)$ can not be extended to MEB.

$\Leftarrow$: If $S_0 \cup S(A)$ can be extend to MEB, then there are $d - k$ orthogonal matrices $U_1, U_2, \ldots, U_{d-k}$ which lie in $(S_0 \cup S(A))^\perp$. However, $S_0^\perp$ are the set of diagonal matrices. Hence, $(S_0 \cup S(A))^\perp \subseteq S_0^\perp$ is a subset of diagonal matrices. Suppose $U_j = diag(\nu_j)$ for some vector ν_j in $\mathbb{C}^d$ for $j = 1, 2, \ldots, d-k$. Then the unitarity of the matrix U_j gives that the entries of ν_j are module 1. The orthogonality of $S(A) \cup \{U_1, U_2, \ldots, U_{d-k}\}$ gives that $(A, \nu_1, \nu_2, ..., \nu_{d-k})^t$ is a Hadamard matrix. ∎

Now we give an answer to the conjecture in [5]: any 4×5 partial Hadamard matrix can be complemented to a complete Hadamard matrix.

Corollary 1. If A is a $(d-1) \times d$ partial Hadamard matrix, $d \geq 2$ is an integer, then A can be complemented to a complete Hadamard matrix.

Proof. Suppose $A = (\alpha_1, \alpha_2, ..., \alpha_{d-1})^t$, then we have

$$\dim_{\mathbb{C}}(span_{\mathbb{C}}\{\alpha_1, \alpha_2, \ldots, \alpha_{d-1}\}) = d - 1,$$
$$\dim_{\mathbb{C}}(span_{\mathbb{C}}\{\alpha_1, \alpha_2, \ldots, \alpha_{d-1}\})^\perp = 1.$$

Choosing

$$\nu_d = (\nu_{d1}, \nu_{d2}, \ldots, \nu_{dd}) \in span\{\alpha_1, \alpha_2, \ldots, \alpha_{d-1}\}^\perp$$

such that $\|\nu_d\| = 1$, we have that $U = \left(\alpha_1/\sqrt{d}, \alpha_2/\sqrt{d}, ..., \alpha_{d-1}/\sqrt{d}, \nu\right)^t$ is a matrix with orthonormal rows. Namely, U is a unitary matrix. Then all the columns of U are also orthonormal. Hence, $\nu_{dk} = 1/\sqrt{d}$ for $k = 1, 2, \ldots, d$. And $\left(\alpha_1, \alpha_2, ..., \alpha_{d-1}, \sqrt{d}\nu\right)^t$ is a Hadamard matrix. ∎

Corollary 1 can be also understood in the following way. Since A is a $(d - 1) \times d$ matrix, one has that $S_0 \cup S(A)$ is a set of maximally entangled states with $d^2 - 1$ states. By [8] it can be extended to MEB. Hence from Proposition 1, A can be complemented to be a Hadamard matrix.

Remark 1: Proposition 1 gives a way to construct UMEBs. Suppose there is a partial Hadamard matrix A whose orthogonal complement contains no vectors with each entry module 1. Then $S_0 \cup S(A)$ is a UMEB.

Example 1. In $\mathbb{C}^5 \bigotimes \mathbb{C}^5$, there exists a UMEB with 23 elements.

Let

$$A = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & \omega & \omega^2 \end{pmatrix},$$

where $\omega = e^{\frac{2\pi i}{3}}$. If we denote

$$\nu_1 = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 0, 0),$$

$$\nu_2 = (\frac{1}{\sqrt{10}}, 0, \frac{1}{\sqrt{10}}, \frac{2\omega^2}{\sqrt{10}}, \frac{2\omega}{\sqrt{10}}),$$

$$\nu_3 = (0, \sqrt{\frac{3}{5}}, 0, \frac{\omega-1}{\sqrt{15}}, \frac{\omega^2-1}{\sqrt{15}}),$$

then $\text{span}_{\mathbb{C}}\{\alpha_1, \alpha_2\}^{\perp} = \{\nu_1, \nu_2, \nu_3\}$. Let $\alpha = k_1\nu_1 + k_2\nu_2 + k_3\nu_3$ be a vector with each entries module 1, that is

$$\begin{cases} |\frac{k_1}{\sqrt{2}} + \frac{k_2}{\sqrt{10}}| = 1, \\ |-\frac{k_1}{\sqrt{2}} + \frac{k_2}{\sqrt{10}}| = 1, \\ |\sqrt{\frac{3}{5}}k_3| = 1, \\ |k_1|^2 + |k_2|^2 + |k_3|^2 = 1. \end{cases}$$

From these equations we have $|k_1| = |k_2| = |k_3| = \sqrt{\frac{5}{3}}$.

Moreover, $k_2 = \pm ik_1, k_3 = \pm ik_1$.

If we set $\alpha_3 = \sqrt{\frac{5}{3}}\nu_1 + i\sqrt{\frac{5}{3}}\nu_2 + i\sqrt{\frac{5}{3}}\nu_3$, then $\begin{pmatrix} A \\ \alpha_3 \end{pmatrix}$ is also a partial Hadamard matrix. However, any vector lies in $\text{span}_{\mathbb{C}}\{\alpha_1, \alpha_2, \alpha_3\}^{\perp} \subseteq \text{span}_{\mathbb{C}}\{\alpha_1, \alpha_2\}^{\perp}$. Hence, if $\nu \in \text{span}_{\mathbb{C}}\{\alpha_1, \alpha_2, \alpha_3\}^{\perp}$ with each entry module 1, then ν has the form $\nu = k_1\nu_1 \pm ik_1\nu_2 \pm ik_1\nu_3$. However, ν can not be orthogonal to α_3, Hence, we have a UMEB in $\mathbb{C}^5 \otimes \mathbb{C}^5$. ∎

Proposition 2. In $\mathbb{C}^{4n+1}$ there exists a partial Hadamard matrix which can not be complemented to a complete Hadamard matrix.

Proof: Set

$$A = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{2n} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{2n-1} & 1 & \sigma & \sigma^2 & \cdots & \sigma^{2n} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(2n-1)} & 1 & \sigma^2 & \sigma^4 & \cdots & \sigma^{2(2n)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{2n-1} & \omega^{2(2n-1)} & \cdots & \omega^{(2n-1)(2n-1)} & 1 & \sigma^{2n-1} & \sigma^{2(2n-1)} & \cdots & \sigma^{2n(2n-1)} \end{pmatrix},$$

where $\omega = e^{\frac{2\pi i}{2n}}$, $\sigma = e^{\frac{2\pi i}{2n+1}}$. Firstly, we compute the orthogonal complements of the subspace V spanned by the rows of A. Obviously,

$$\beta_1 = (\frac{1}{\sqrt{2}}, \overbrace{0, 0, \cdots, 0}^{2n-1}, -\frac{1}{\sqrt{2}}, \overbrace{0, 0, \cdots, 0}^{2n}),$$

$$\beta_2 = (\frac{1}{\sqrt{8n+2}}, \overbrace{0, 0, \cdots, 0}^{2n-1}, \frac{1}{\sqrt{8n+2}}, \frac{2\sigma^{2n}}{\sqrt{8n+2}},$$
$$\frac{2\sigma^{2n-1}}{\sqrt{8n+2}}, \cdots, \frac{2\sigma}{\sqrt{8n+2}}),$$

are orthogonal to $\alpha_1, \alpha_2, \ldots, \alpha_{2n}$ and $\beta_1 \perp \beta_2$. Set

$$\gamma_1 = (\overbrace{0, 1, 0, 0, \cdots, 0, 0}^{2n}, \overbrace{0, 0, \cdots, 0}^{2n+1}),$$
$$\gamma_2 = (0, 0, 1, 0, \cdots, 0, 0, 0, 0, \cdots, 0),$$
$$\vdots$$
$$\gamma_{2n-1} = (0, 0, 0, 0, \cdots, 1, 0, 0, 0, \cdots, 0).$$

By Schmidt orthogonalization we have

$$\beta_3 = (0, \overbrace{\sqrt{\frac{2n+1}{4n+1}}, 0, 0, \ldots, 0, 0, 0}^{2n+1}, \beta_{31}, \beta_{32}, \cdots, \beta_{3,2n}),$$

$$\beta_4 = (0, 0, \sqrt{\frac{2n+1}{4n+1}}, 0, \ldots, 0, 0, 0, \beta_{41}, \beta_{42}, \cdots, \beta_{4,2n}),$$

$$\vdots$$

$$\beta_{2n+1} = (0, 0, 0, 0, \ldots, 0, \sqrt{\frac{2n+1}{4n+1}}, 0, \beta_{2n+1,1}, \cdots, \beta_{2n+1,2n}).$$

Then we have $\text{span}_{\mathbb{C}}\{\alpha_1, \alpha_2, \ldots, \alpha_{2n}\}^{\perp} = \text{span}_{\mathbb{C}}\{\beta_1, \beta_2, \ldots, \beta_{2n+1}\}$. Suppose $\nu = k_1\beta_1 + k_2\beta_2 + \cdots + k_{2n+1}\beta_{2n+1}$ is a vector with entries module 1. We

must have

$$\begin{cases} |\frac{1}{\sqrt{2}}k_1 + \frac{1}{\sqrt{8n+2}}k_2| = 1, \\ |\frac{1}{\sqrt{2}}k_1 - \frac{1}{\sqrt{8n+2}}k_2| = 1, \\ |\sqrt{\frac{2n+1}{4n+1}}k_3| = 1, \\ |\sqrt{\frac{2n+1}{4n+1}}k_3| = 1, \\ \quad \vdots, \\ |\sqrt{\frac{2n+1}{4n+1}}k_{2n+1}| = 1, \\ |k_1|^2 + |k_2|^2 + \cdots + |k_{2n+1}|^2 = |\nu|^2 = 4n+1. \end{cases}$$

Solving the above equations we obtain

$$|k_1| = |k_2| = \cdots = |k_{2n+1}| = \sqrt{\frac{4n+1}{2n+1}}$$

and $k_1 = \pm ik_2$. If A can be extended to a complete Hadamard matrix by adding $2n + 1$ the rows $\nu_1, \nu_2, \ldots, \nu_{2n+1}$, we have

$$\begin{aligned} \nu_1 &= k_{11}\beta_1 + k_{12}\beta_2 + \cdots + k_{1,2n+1}\beta_{2n+1}, \\ \nu_2 &= k_{21}\beta_1 + k_{22}\beta_2 + \cdots + k_{2,2n+1}\beta_{2n+1}, \\ &\vdots \\ \nu_{2n+1} &= k_{2n+1,1}\beta_1 + k_{2n+1,2}\beta_2 + \cdots + k_{2n+1,2n+1}\beta_{2n+1}. \end{aligned}$$

The above analysis gives that $|k_{st}| = \sqrt{\frac{4n+1}{2n+1}}$. Clearly, the orthogonality of $\nu_1, \nu_2, \ldots, \nu_{2n+1}$ implies that the vectors $(k_{11}, k_{12}, \ldots, k_{1,2n+1})$, $(k_{21}, k_{22}, \ldots, k_{2,2n+1})$, ..., $2n + 1, 1, k_{2n+1,2}, \ldots, k_{2n+1,2n+1})$ are orthogonal each other. Hence, if we denote $K = (k_{st})_{(2n+1)\times(2n+1)}$, then $\sqrt{\frac{2n+1}{4n+1}}K$ is a matrix with entries module 1 and the rows being mutually orthogonal. Hence, $H = \sqrt{\frac{2n+1}{4n+1}}K$ is a Hadamard matrix.

However, $k_{j2} = \pm ik_{j1}$ for $j = 1, 2, \cdots, 2n + 1$. If we replace each row $(k_{j1}, k_{j2}, \ldots, k_{j,2n+1})$ by $\frac{1}{k_{j1}}(k_{j1}, k_{j2}, \ldots, k_{j,2n+1})$, then the new matrix $\widetilde{H}$ is also a Hadamard matrix with the entries of the first column all being 1, and the entries of the second column being either i or $-i$. The Hadamard matrix $\widetilde{H}$ also gives that the columns of $\widetilde{H}$ are mutually orthogonal. However $(1, 1, \ldots, 1)^T$ can not be orthogonal to the second column. Therefore A can not be extended to a complete Hadamard matrix. ∎

From Proposition 2 we have

Corollary 2. There exists a UMEB in $\mathbb{C}^{4n+1} \otimes \mathbb{C}^{4n+1}$ for any integer n.

Corollary 3. There exists a UMEB in $\mathbb{C}^d \otimes \mathbb{C}^d$, whenever $d \neq p$ or $2p$ ($p \equiv 3 \bmod 4$ and p is a prime).

Proof: Let $d = p_1^{r_1}p_2^{r_2}\ldots p_k^{r_k}$, where p_i are primes, $p_1 < p_2 < \ldots < p_k$, and $r_i \in \mathbb{N}$ for $i = 1, 2, \ldots, k$. If $p_1 = 2, r_1 \geq 2$, then we have a UMEB when d is multiple of 4. If $p_j = 4n + 1(n \in \mathbb{N})$, from the corollary 2 we have an UMEB. Hence we can suppose that all the primes are of the form $p_j = 4n + 3$ except for the case that the first one is 2. Now suppose there are two primes $p_j = 4n + 3$ and $p_s = 4m + 3$ ($m \in \mathbb{N}$). Then we have $4t + 1|p_jp_s$ for some integer t, and we can also get a UMEB. ∎

Remark 2. From the Corollary 3, we have solved the problem for all d, except for $d = p$ or $2p$, where $p = 3 \bmod 4$ and p is a prime. The most simple unsolved case is $d = 7 = 4 + 3$. If one can construct a UMEB in $\mathbb{C}^7 \otimes \mathbb{C}^7$, then from Lemma one can get a UMEB in $\mathbb{C}^{7n} \otimes \mathbb{C}^{7n}(n \in \mathbb{N})$.

Example 2. In $\mathbb{C}^7 \otimes \mathbb{C}^7$, there exists an UMEB with 45 elements.

Let $A = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & 1 & i & -1 & -i \\ 1 & \omega^2 & \omega & 1 & -1 & 1 & -1 \end{pmatrix}$,

where $\omega = e^{\frac{2\pi i}{3}}$. Obviously,

$$\begin{aligned} \beta_1 &= (\frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0, 0), \\ \beta_2 &= (\frac{1}{\sqrt{14}}, 0, 0, \frac{1}{\sqrt{14}}, \frac{-2i}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{2i}{\sqrt{14}}), \end{aligned}$$

are orthogonal to $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1 \perp \beta_2$. Set $\gamma_1 = (0, 0, 0, 0, 0, 1, 0)$. By Schmidt orthogonalization we have

$$\beta_3 = (0, \frac{2\omega}{\sqrt{14}}, \frac{2\omega^2}{\sqrt{14}}, 0, \frac{-i}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{i}{\sqrt{14}}).$$

Let β_4 be a normalized vector and orthogonal to $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$. Then we obtain

$$\beta_4 = (0, \frac{2}{\sqrt{14}}, \frac{-2\omega^2}{\sqrt{14}}, 0, \frac{\omega-1}{\sqrt{14}}, 0, \frac{\omega-1}{\sqrt{14}}).$$

Therefore $\mathrm{span}_{\mathbb{C}}\{\alpha_1, \alpha_2, \alpha_3\}^{\perp} = \mathrm{span}_{\mathbb{C}}\{\beta_1, \beta_2, \beta_3, \beta_4\}$. Let $\alpha = k_1\beta_1 + k_2\beta_2 + k_3\beta_3 + k_4\beta_4$ be a vector with each entries module 1, that is,

$$\begin{cases} |\frac{k_1}{\sqrt{2}} + \frac{k_2}{\sqrt{14}}| = 1, \\ |-\frac{k_1}{\sqrt{2}} + \frac{k_2}{\sqrt{14}}| = 1, \\ |\frac{2\omega k_3}{\sqrt{14}} + \frac{2k_4}{\sqrt{14}}| = 1, \\ |\frac{2\omega^2 k_3}{\sqrt{14}} - \frac{2\omega k_4}{\sqrt{14}}| = 1, \\ |\frac{-2ik_2}{\sqrt{14}} - \frac{ik_3}{\sqrt{14}} + \frac{(\omega-1)k_4}{\sqrt{14}}| = 1, \\ |\frac{2ik_2}{\sqrt{14}} + \frac{ik_3}{\sqrt{14}} + \frac{(\omega-1)k_4}{\sqrt{14}}| = 1, \\ |\frac{-2k_2}{\sqrt{14}} + \frac{2k_3}{\sqrt{14}}| = 1, \\ |k_1|^2 + |k_2|^2 + |k_3|^2 + |k_4|^2 = 7. \end{cases}$$

From the above equations we have

$$
\begin{cases}
\frac{|k_1|^2}{2} + \frac{|k_2|^2}{14} = 1, \\
k_1 \perp k_2, \\
\frac{4|k_3|^2}{14} + \frac{4|k_4|^2}{14} = 1, \\
\omega k_3 \perp k_4 \Rightarrow k_3 \perp \omega^2 k_4, \\
\frac{|(\omega-1)k_4|^2}{14} + \frac{|2ik_2+ik_3|^2}{14} = 1, \\
(\omega - 1)k_4 \perp (2ik_2 + ik_3) \Rightarrow -\sqrt{3}i\omega^2 k_4 \perp (2ik_2 + ik_3), \\
|k_1|^2 + |k_2|^2 + |k_3|^2 + |k_4|^2 = 7.
\end{cases}
$$

The above equations give rise to $|k_1| = |k_2| = \sqrt{\frac{7}{4}}$, $|k_3|^2 + |k_4|^2 = \frac{7}{2}$. Since $\omega^2 k_4 \perp k_3$, $\omega^2 k_4 \perp (2k_2 + k_3)$, we get $\omega^2 k_4 \perp 2k_2$, and k_2 and k_3 are collinear. Hence we can suppose $k_2 = rk_3$, where r is a real number satisfying

$$
\begin{cases}
r^2 - 2r - \frac{1}{2} = 0, \\
r^2 - 2r - 1 = 0.
\end{cases}
$$

As r has no solutions to the above equations, no k_1, k_2, k_3, k_4 satisfy the required conditions. That is, there is no vector in $\mathrm{span}_{\mathbb{C}}\{\alpha_1, \alpha_2, \alpha_3\}^{\perp}$ with each entry module 1. Hence, by Proposition 1 we have a UMEB in $\mathbb{C}^7 \otimes \mathbb{C}^7$. ∎

Actually, for $d = 3$ the UMEB contains 6 MEBs and needs other 3 states to form a full base. Similarly, for $d = 5$ and $d = 7$, 2 and 4 more states are needed to form a full base, respectively. Then there are three ways to obtain the UMEBs for $d = 3 \times 5 \times 7$ by the method of lemma, respectively from the constructions of $d = 3, 5, 7$. The one obtained from $d = 3$ needs $3 \times 35 = 105$ more states. The one obtained from $d = 5$ needs $2 \times 21 = 42$ more states, while the one obtained from $d = 7$ needs $4 \times 15 = 60$ more states. Hence these three UMEBs are different from each other. For the case $d = 3 \times 5 \times 7$, there are at least three UMEBs. Moreover, the approach can be generalized to the case of $d = 3 \times 5 \times 7 \times n$ for any integer n.

III. CONCLUSION AND DISCUSSION

We have studied the UMEBs in $\mathbb{C}^d \bigotimes \mathbb{C}^d$ and connected the problem to the partial Hadamard matrix. It has been shown that the existence of a special UMEB in $\mathbb{C}^d \bigotimes \mathbb{C}^d$ is equivalent to the existence of an uncompletable partial Hadamard matrix. In particular, as a corollary, we have shown that any $(d - 1) \times d$ partial Hadamard matrices can be always extended to a complete Hadamard matrix, which gives an answer to the conjecture in [17]. Actually, the Proposition 1 also gives us a method to construct UMEB by using an uncompletable partial Hadamard matrix. We have proven that there exists an uncompletable partial Hadamard matrix for $d = 4n+1$, which implies the existence of a UMEB in $\mathbb{C}^{4n+1} \bigotimes \mathbb{C}^{4n+1}$. At last, combining the Lemma and the proposition 2, we have obtained that for any d there is a UMEB except for $d = p$ or $2p$, where $p \equiv 3 \mod 4$ and p is a prime. In addition, we have also presented a UMEB by the partial Hadamard approach for $d = 7$. We have concluded that there are at least three different sets of UMEBs in $\mathbb{C}^d \bigotimes \mathbb{C}^d$ when d is multiple of $3 \times 5 \times 7$. Our results may highlight the further researches on the constructions of UMEBs and the partial Hadamard matrices. Moreover, our results may be also helpful in studying the constructions of non-UMEBs such as the ones with the maximality of states given by Schmidt numbers [18, 19].

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