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by

Xue-Na Zhu, Xianqing Li-Jost, and Shao-Ming Fei

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Xue-Na Zhu[1], Xianqing Li-Jost[2,3], and Shao-Ming Fei[3,4*]
[1]*School of Mathematics and Statistics Science, Ludong University, Yantai 264025, China*
[2]*School of Mathematics and Statistics, Hainan Normal University, Haikou, 571158, China*
[3]*Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany*
[4]*School of Mathematical Sciences, Capital Normal University Beijing 100048, China*

Quantum correlations characterized by quantum entanglement and quantum discord play important roles in many quantum information processing. We study the relations among the entanglement of formation, concurrence, tangle, linear entropy based classical correlation and von Neumann entropy based classical correlation. We present analytical formulae of both linear entropy based and von Neumann entropy based classical correlations for arbitrary $d \times 2$ quantum states. From the von Neumann entropy based classical correlation, we derive an explicit formula of quantum discord for arbitrary rank-2 two-qubit quantum states.

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I. INTRODUCTION

Correlations between the subsystems of a bipartite system play significant roles in many information processing tasks and physical processes. The quantum entanglement [1] is an important kind of quantum correlation which plays significant roles in many quantum tasks such as quantum teleportation, dense coding, swapping, error correction and remote state preparation. A bipartite state is called separable if it has zero entanglement between subsystems A and B: the probabilities of the measurement outcomes from measuring the subsystem A are independent of the probabilities of the measurement outcomes from measuring the subsystem B. Nevertheless, a separable state may still have quantum correlation – quantum discord, if it is impossible to learn all the mutual information by measuring one of the subsystems. Quantum discord is the minimum amount of correlation, as measured by mutual information, that is necessarily lost in a local measurement of bipartite quantum states. It has been shown that the quantum discord is required for some information processing like assisted optimal state discrimination [2, 3].

Let ρ_{AB} denote the density operator of a bipartite system $H_A \otimes H_B$. The quantum mutual information is defined by

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \tag{1}$$

where $\rho_{A(B)} = Tr_{B(A)}(\rho_{AB})$ are reduced density matrices, $S(\rho) = -Tr(\rho \log \rho)$ is the Von Neumann entropy. Quantum mutual information is the information-theoretic measure of the total correlation in bipartite quantum states. In terms of measurement-based conditional density operators, the classical correlation of bipartite states ρ_{AB} is defined by [4],

$$I^{\leftarrow}(\rho_{AB}) = \max_{\{P_i\}}[S(\rho_A) - \sum_i p_i S(\rho_A^i)], \tag{2}$$

where the maximum is taken over all positive operator-valued measure (POVM) P_i performed on subsystem B, satisfying $\sum_i P_i^\dagger P_i = I$ with probability of i as an outcome, $p_i = Tr[(I_A \otimes P_i)\rho_{AB}(I_A \otimes P_i^\dagger)]$, $\rho_A^i = Tr_B[(I_A \otimes P_i)\rho_{AB}(I_A \otimes P_i^\dagger)]/p_i$ is the conditional states of system A associated with outcome i, I_A and I are the corresponding identity operators.

The quantum discord is defined as the difference between the total correlation and the classical correlation [4, 5]:

$$Q^{\leftarrow}(\rho_{AB}) = I(\rho_{AB}) - I^{\leftarrow}(\rho_{AB}). \tag{3}$$

Generally it is a challenging problem to compute the quantum correlation $Q^{\leftarrow}(\rho_{AB})$ due to difficulty in computing the classical correlation $I^{\leftarrow}(\rho_{AB})$. Analytically formulae of $\mathcal{Q}(\rho)$ can be obtained only for some special quantum states like Bell-diagonal states [6], X-type states [7, 8] with respect to projective measurements, as well as some special two-qubit states [9]. In stead of analytical formulae, some estimation on the lower and upper bounds of quantum discord are also obtained [10, 11]. A lower bound of quantum discord for the 2-qutrit systems is obtained in [12]. In [13] a hierarchy of computationally efficient lower bounds to the standard quantum discord has been presented.

In this paper, by studying the classical correlations of $d \otimes 2$ quantum states, we present the analytical formula of quantum discord for any two-qubit states with rank-2.

II. ANALYTICAL FORMULA OF QUANTUM DISCORD FOR RANK-2 TWO-QUBIT STATES

To derive an analytical formula of quantum discord for rank-2 two-qubit states under von Neumann entropy, we

*feishm@cnu.edu.cn

first study the classical correlation under linear entropy. The linear entropy $S_2(\rho)$ of a quantum state ρ is given by $S_2(\rho)=2[1-Tr(\rho^2)]$. The linear entropy version of the classical correlation (2) of a bipartite state ρ_{AB} is given by $I_2^{\leftarrow}(\rho_{AB})=\max[S_2(\rho_A)-\sum_i p_i S_2(\rho_A^i)]$.

Any $d\otimes 2$ bipartite quantum state ρ_{AB} may be written as

$$\rho_{AB}=\Lambda_\rho\otimes I_B(|r_{B'B}\rangle\langle r_{B'B}|), \tag{4}$$

where $|r_{B'B}\rangle$ is the symmetric two qubit purification of the reduced density operator ρ_B on an auxiliary qubit system B' and Λ_ρ is a qubit channel from B' to A.

A qudit states can be written as the Bloch expression $\rho=\frac{I_d+\vec{r}\gamma}{d}$, where I_d denotes the $d\times d$ identity matrix, $\vec{r}$ is a d^2-1 dimensional real vector, $\gamma=(\lambda_1,\lambda_2,...,\lambda_{d^2-1})^T$ is the vector of the generators of $SU(d)$ and T stands for transpose. The linear entropy written in terms of the Bloch vector $\vec{r}$ of a qudit state, is given by $S_2(\frac{I_d+\vec{r}\gamma}{d})=\frac{2d^2-2d-4|\vec{r}|^2}{d^2}$. The action of a qubit channel Λ on a single-qubit state $\rho=\frac{I_2+\vec{r_B}\sigma}{2}$, where $\vec{r_B}$ is the Bloch vector and σ is the vector of Pauli operators, has the following form,

$$\Lambda(\rho)=\frac{I_d+(L\vec{r_B}+l)\gamma}{d}, \tag{5}$$

where L is a $(d^2-1)\times 3$ real matrix and l is a three-dimensional vector.

We first give a Lemma about the linear entropy version of the classical correlation.

[Lemma] For arbitrary $d\otimes 2$ quantum states,

$$I_2^{\leftarrow}(\rho_{AB})=\frac{4}{d^2}\lambda_{\max}(L^TL)S_2(\rho_B), \tag{6}$$

where $\lambda_{\max}(L^TL)$ stands for the largest eigenvalues of the matrix L^TL.

[Proof] Any $d\otimes 2$ bipartite quantum state ρ_{AB} can be written as

$$\rho_{AB}=\Lambda_\rho\otimes I_B(|r_{B'B}\rangle\langle r_{B'B}|),$$

Let $\rho_B=\sum\lambda_i|\phi_i\rangle\langle\phi_i|$ be the spectral decomposition of ρ_B. Then $|V_{B'B}\rangle=\sum\sqrt{\lambda_i}|\phi_i\rangle|\phi_i\rangle$. One has [16],

$$I_2^{\leftarrow}(\rho_{AB})=\max_{\{p_i,\psi_i\}}\left(S_2[\Lambda(\rho_B)]-\sum_i p_i S_2[\Lambda(|\psi_i\rangle\langle\psi_i|)]\right),$$

where the maximization goes over all possible pure state decompositions of ρ_B. Taking into account (5), we have

$$\begin{aligned}S_2[\Lambda(\rho_B)] &= S_2[\Lambda(\frac{I_2+\vec{r}_B\sigma}{2})]\\ &= \frac{2d^2-2d-4(L\vec{r}_B+l)^T(L\vec{r}_B+l)}{d^2}.\end{aligned}$$

In the Pauli basis, the possible pure state decompositions of ρ_B are represented by all possible sets of probability $\{p_j\}$ and $\vec{r}_j$ such that $\rho_B=\sum_j p_j\frac{I_2+\vec{r}_j\sigma}{2}$. Set $\vec{r}_j=\vec{r}_B+\vec{x}_j$. One can easily check that the calculation of $I_2^{\leftarrow}(\rho_{AB})$ reduces to determine $p_j,\vec{x}_j$, subject to the conditions $\sum_j p_j\vec{x}_j=0$ and $|\vec{r}_B+\vec{x}_j|=1$, in the following maximization,

$$\frac{4}{d^2}\max_{\{p_j,\vec{x}_j\}}\sum_j p_j\vec{x}_j^TL^TL\vec{x}_j.$$

By using the method used in calculating the linear Holevo capacity for qubit channels [16], we have (6). $\square$

Remark By proving the Lemma, we have corrected a error in [11], where the factor $4/d^2$ in (6) was missed.

To get the analytical formula of classical correlation $I^{\leftarrow}(\rho_{AB})$ under von Neumann entropy from $I_2^{\leftarrow}(\rho_{AB})$ under linear entropy for any bipartite states ρ_{AB}, we consider the relations among entanglement of formation, concurrence, tangle, $I^{\leftarrow}(\rho_{AB})$ and $I_2^{\leftarrow}(\rho_{AB})$. The tangle $\tau(\rho_{AB})$ is defined by

$$\tau(\rho_{AB})=\inf_{\{p_i,|\psi\rangle_i\}}\sum p_i S_2(\rho_B^i), \tag{7}$$

where the infimum runs over all pure-state decompositions $\{p_i,|\psi\rangle_i\}$ of ρ_{AB} and $\rho_B^i=Tr_A(|\psi\rangle_i\langle\psi|)$. Due to the convexity, one has $C^2(\rho_{AB})\le\tau(\rho_{AB})$ for two-qubit states. Generally, $\tau(\rho_{AB})$ is not equal to the square of the concurrence [14].

The entanglement of formation $E(|\psi\rangle_{AB})$ [19–21] and the concurrence $C(|\psi\rangle_{AB})$ [22–24] of a pure state $|\psi\rangle_{AB}$ are defined by $E(|\psi\rangle_{AB})=S(\rho_A)$ and $C(|\psi\rangle_{AB})=\sqrt{2[1-Tr(\rho_A^2)]}$, respectively. They are extended to mixed states ρ_{AB} by convex-roof construction, $E(\rho_{AB})=\inf_{\{p_i,|\psi_i\rangle\}}\sum_i p_iE(|\psi_i\rangle)$, $C(\rho_{AB})=\inf_{\{p_i,|\psi_i\rangle\}}\sum_i p_iC(|\psi_i\rangle)$, with the infimum taking over all possible pure state decompositions of ρ_{AB}.

For the two-qubit quantum statesρ_{AB}, the entanglement of formation $E_f(\rho_{AB})$ and concurrence $C(\rho_{AB})$ have the following relation [14]:

$$E_f(\rho_{AB})=h(\frac{1+\sqrt{1-C^2(\rho_{AB})}}{2}) \tag{8}$$

where $h(x)=-x\log_2(x)-(1-x)\log_2(1-x)$.

For a tripartite pure state $|\psi\rangle_{ABC}$, one has the following relations [15],

$$E_f(\rho_{AC})+I^{\leftarrow}(\rho_{AB})=S(\rho_A). \tag{9}$$

In the following we denote $f(x)=h(\frac{1+\sqrt{1-x}}{2})$ for simplicity.

[Theorem] For rank-2 two-qubit quantum states ρ_{AB}, the quantum discord is given by

$$Q^{\leftarrow}(\rho_{AB})=S(\rho_B)-S(\rho_{AB})+f(S_2(\rho_A)-I_2^{\leftarrow}(\rho_{AB})). \tag{10}$$

[Proof]: For two-qubit quantum states ρ_{AB} with rank-2, they have spectral decompositions, $\rho_{AB}=\lambda_1|\psi\rangle_1\langle\psi|+\lambda_2|\psi\rangle_2\langle\psi|$, where λ_i and $|\psi\rangle_i$, $i=1,2$, $\lambda_1+\lambda_2=1$, are respectively the eigenvalues and eigenvectors. Then the purified tripartite qubit state can be written as $|\psi\rangle_{ABC} = \sqrt{\lambda_1}|\psi\rangle_1|0\rangle + \sqrt{\lambda_2}|\psi\rangle_2|1\rangle$, satisfying $\rho_{AB} = Tr_C(|\psi\rangle_{ABC}\langle\psi|)$. We have the following monogamy relation holds [16],

$$\tau(\rho_{AC}) + I_2^{\leftarrow}(\rho_{AB}) = S_2(\rho_A). \tag{11}$$

As ρ_{AC} is a two-qubit state, one has $\tau(\rho_{AC}) = C^2(\rho_{AC})$ [18]. Moreover, $S(\rho_A) = E_f(|\psi\rangle_{A|BC}) = f(C^2(|\psi\rangle_{A|BC})) = f(S_2(\rho_A))$,

$$\begin{aligned} E_f(\rho_{AC}) &= f(C^2(\rho_{AC})) \\ &= f\left(S_2(\rho_A) - I_2^{\leftarrow}(\rho_{AB})\right). \end{aligned}$$

where the fist and second equations are due to (8) and (11). From (9), we have

$$I^{\leftarrow}(\rho_{AB}) = S(\rho_A) - f\big(S_2(\rho_A) - I_2^{\leftarrow}(\rho_{AB})\big). \tag{12}$$

According to (3), we have the quantum discord for any rank-2 two-qubit states. □

Theorem 2 provides an analytical formula (10) of quantum discord in terms of the original on Von Neumann entropy for arbitrary rank-2 two-qubit quantum states. Besides, the Von Neumann entropy based classical correlation (12) is also analytically presented. It should be emphasized that, the analytical formula of quantum discord (10) is only for rank-2 two-qubit quantum states, but the formula for classical correlation (12) is valid for any $d \otimes 2$ bipartite states with any ranks. In the following, we give some detailed examples for quantum discords and also classical correlations.

III. EXAMPLES

Let us first consider the rank-2 of two-qubit Bell-diagonal states,

$$\rho = \frac{1}{4}\left(I + \sum_{j=1}^{3} c_i\sigma_j \otimes \sigma_j\right).$$

By Theorem 2, we have $S(\rho_A) = 1$ and $S_2(\rho_A) = 1$ and $I_2^{\leftarrow}(\rho) = c^2$. Then

$$I^{\leftarrow}(\rho) = 1 - f(1-c^2) = \frac{1-c}{2}\log_2(1-c) + \frac{1+c}{2}\log_2(1+c),$$

which coincides with the result in Ref. [9].

Example 1: Now consider the following two-qubit states,

$$\begin{aligned} \rho_1 &= \frac{2-x}{6}|00\rangle\langle 00| + \frac{1+x}{6}|01\rangle\langle 01| + \frac{1}{6}|01\rangle\langle 10| \quad (13) \\ &+ \frac{1}{6}|10\rangle\langle 01| + \frac{1+x}{6}|10\rangle\langle 10| + \frac{2-x}{6}|11\rangle\langle 11|, \end{aligned}$$

where $x \in [0, 2]$. By computation we have $S_2(\rho_B) = 1$, and the qubit channel Λ is given by $\Lambda(|0\rangle\langle 0|) = \frac{2-x}{3}|0\rangle\langle 0| + \frac{1+x}{3}|1\rangle\langle 1|$, $\Lambda(|0\rangle\langle 1|) = \frac{1}{3}|1\rangle\langle 0|$, $\Lambda(|1\rangle\langle 0|) = \frac{1}{3}|0\rangle\langle 1|$ and $\Lambda(|1\rangle\langle 1|) = \frac{1+x}{3}|0\rangle\langle 0| + \frac{2-x}{3}|1\rangle\langle 1|$. Therefore we obtain

$$L = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1-2x}{3} \end{pmatrix} \tag{14}$$

and $I_2^{\leftarrow}(\rho_1) = \max_{\{x\in[0.2]\}}\{\frac{1}{9}, \frac{(1-2x)^2}{9}\}$, see Fig.1.

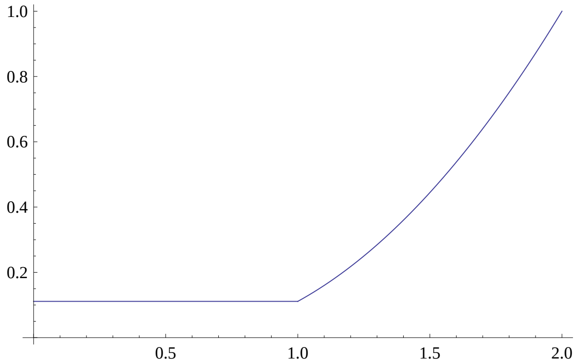

Fig. 1: The classical correlation $I_2^{\leftarrow}(\rho_1)$ with $x \in [0, 2]$.

The rank of ρ_1 is two when $x = 2$. In this case, we have $S(\rho_B) = S_2(\rho_A) = 1$ and $S(\rho_{AB}) = \log_2 3 - \frac{2}{3}$. Hence $Q^{\leftarrow}(\rho_{AB}) = \frac{5}{3} - \log_2 3$.

Example 2: We calculate now the discord of the Horodecki state [**?**],

$$\rho^H(p) = p|\varphi^+\rangle\langle\varphi^+| + (1-p)|00\rangle\langle 00|,$$

where $|\varphi\rangle^+ = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. The qubit channel Λ can be explicitly calculated: $\Lambda(|0\rangle\langle 0|) = \frac{2(1-p)}{2-p}|0\rangle\langle 0| + \frac{p}{2-p}|1\rangle\langle 1|$, $\Lambda(|1\rangle\langle 0|) = \sqrt{\frac{p}{2-p}}|1\rangle\langle 0|$, $\Lambda(|0\rangle\langle 1|) = \sqrt{\frac{p}{2-p}}|0\rangle\langle 1|$ and $\Lambda(|1\rangle\langle 1|) = |0\rangle\langle 0|$. By applying Theorem 1, we get the matrix

$$L = \begin{pmatrix} \sqrt{\frac{p}{2-p}} & 0 & 0 \\ 0 & -\sqrt{\frac{p}{2-p}} & 0 \\ 0 & 0 & -\frac{p}{2-p} \end{pmatrix}.$$

It is straightforward to verify that $S_2(\rho^H(p)_B) = S_2(\rho^H(p)_A) = p(2-p)$ and $S(\rho^H(p)) = h(p)$. Thus, the discord of $\rho^H(p)$ is given by

$$Q^{\leftarrow}(\rho^H(p)) = h(\frac{p}{2}) - h(p) + f(2p(1-p)),$$

see Fig.3.

Now we consider some more general rank-2 states,

$$\rho_2 = x|\varphi\rangle\langle\varphi| + (1-x)|\phi\rangle\langle\phi|,$$

where $|\varphi\rangle = Sin\theta\,|00\rangle + Cos\theta\,|11\rangle$, $|\phi\rangle = Sin\eta\,|01\rangle + Cos\eta\,|10\rangle$, $x \in [0, 1]$ and $\theta, \eta \in [0, 2\pi]$. Direct computation shows $L = diag\{L_1, L_2, L_3\}$, where

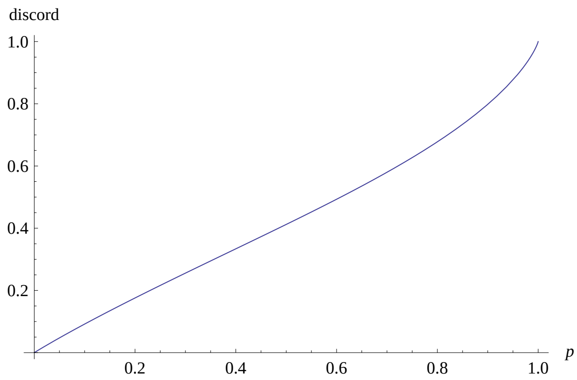

Fig. 2: The discord of the Horodecki state $\rho^H(p)$.

$$
\begin{aligned}
L_1 &= \frac{xSin\theta Cos\theta + (1-x)Sin\eta Cos\eta}{\sqrt{[xCos^2\theta + (1-x)Sin^2\eta]\,[xSin^2\theta + (1-x)Cos^2\eta]}},\\
L_2 &= \frac{xSin\theta Cos\theta - (1-x)Sin\eta Cos\eta}{\sqrt{[xCos^2\theta + (1-x)Sin^2\eta]\,[xSin^2\theta + (1-x)Cos^2\eta]}},\\
L_3 &= \frac{x^2Sin^2\theta Cos^2\theta - (1-x)^2Sin^2\eta Cos^2\eta}{[xCos^2\theta + (1-x)Sin^2\eta][xSin^2\theta + (1-x)Cos^2\eta]},\\
L_4 &= 4x(1-x) + x^2Sin^22\theta + (1-x)^2Sin^22\eta - 4x(1-x)Cos^2(\theta-\eta) - 2x(1-x)Sin2\theta Sin2\eta,\\
L_5 &= 4\left[xSin^2\theta + (1-x)Cos^2\eta\right]\left[xCos^2\theta + (1-x)Sin^2\eta\right],
\end{aligned}
$$

$S(\rho_B) = h\left(xSin^2\theta + (1-x)Cos^2\eta\right)$, $S(\rho_2) = h(x)$, and $S_2(\rho_A) = L_4$. Therefore we obtain

$$
Q^{\leftarrow}(\rho_{AB}) = h\left(xSin^2\theta + (1-x)Cos^2\eta\right) - h(x) + f(L_4 - \max_{\{i=1,2,3\}}\{L_i^2\}L_5).
$$

The Horodecki state $\rho^H(p)$ is a special case of ρ_2 at $\theta = \frac{\pi}{2}$, $\eta = \frac{\pi}{4}$ and $x = 1-p$.

IV. CONCLUSION AND REMARKS

By analyzing the relations among the entanglement of formation, concurrence, tangle, linear entropy classical correlation and von Neumann entropy classical correlation, we have derived the analytical formulae of classical correlations under both linear and von Neumann entropic ones for arbitrary $d \otimes 2$ states, From the von Neumann entropy based classical correlation, we have presented explicit formula of quantum discord for arbitrary rank-2 two-qubit quantum states. If one can further get the relation between $\tau(\rho_{AB})$ and $E(\rho_{AB})$ for rank-2 $d \otimes 2$ systems, it would be possible to compute the quantum discord for rank-2 $d \otimes 2$ states. And if one is able to get the relation between $\tau(\rho_{AB})$ and $E(\rho_{AB})$ for $4 \otimes 2$ systems, maybe one can compute the discord for any two-qubit states. However, for the rank-2 mixed states ρ_{AB}, the corresponding entanglement of formation satisfies the inequality $E(\rho_{AB}) \leq f(\tau)$ [18]. The tangle $\tau(\rho_{AB})$ is not, in general, equal to the square of the square of concurrence $C^2(\rho_{AB})$. It is of difficulty to calculate the discord of any rank-2 $d \otimes 2$ quantum states and any two-qubit states.

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