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ASYMPTOTIC ANALYSIS AND QUALITATIVE BEHAVIOR AT THE FREE BOUNDARY FOR SACKS-UHLENBECK α-HARMONIC MAPS

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Abstract. We investigate the possible blow-up behavior of sequences of Sacks-Uhlenbeck α-harmonic maps from a compact Riemann surface with boundary to a compact Riemannian manifold N with a free boundary on a closed submanifold $K \subset N$. We discover and explore a new phenomenon, that the connection between bubbles, instead of being a geodesic joining them, can be a more general curve that involves the geometry of both N and K. In technical terms, by comparing the blow-up radius with the distance between the blow-up position and the boundary, we define a new quantity, based on which we show a generalized energy identity for the blow-up sequence and give new length formulas for the necks in the case that there is only one bubble occurring at a boundary blow-up point.

1. introduction

The blow-up behavior of sequences of two-dimensional harmonic maps and various variants, in particular the α-harmonic maps introduced by Sacks-Uhlenbeck [21], has been studied extensively, as one of the model problems of geometric analysis. Very precise results are known, like the energy identity and the no-neck theorem. More precisely, bubbles are minimal spheres, connected by necks which are geodesics in the target manifold. One might therefore think that all is known. Here, however, we show that there still is a surprise in store. More precisely, we investigate the blow-up behavior of α-harmonic maps at a free boundary and show that the limits of the necks between bubbles can be of a more general nature than just described, involving also the geometry of the free boundary.

Let us now describe first the setting and then our results in more precise terms. Let (M, g) be a compact Riemannian manifold with smooth boundary ∂M and (N, h) be a compact Riemannian manifold of dimension n. For a map $u \in C^2(M, N)$, the energy density of u is defined by

$$e(u) = |\nabla_g u|^2 = \text{Trace}_g u^* h,$$

where u^*h is the pull-back of the metric tensor h. The energy of the map u is defined as

$$E(u) = \int_M e(u) dvol_g.$$

Let $K \subset N$ be a $k-$dimensional closed submanifold where $1 \le k \le n-1$. Define

$$C(K) = \left\{ u \in C^2(M, N);\ u(\partial M) \subset K \right\}.$$

A critical point of the energy E over $C(K)$ is a harmonic map with free boundary $u(\partial M)$ on K. The problem of the existence, uniqueness and regularity of such harmonic maps with a free boundary was first systematically investigated in [9].

By Nash's embedding theorem, (N, h) can be isometrically embedded into some Euclidean space $\mathbb{R}^N$. Then we can get the Euler-Lagrange equation

$$\Delta_g u = A(u)(\nabla_g u, \nabla_g u),$$

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where A is the second fundamental form of $N \subset \mathbb{R}^N$ and Δ_g is the Laplace-Beltrami operator on M which is defined by

$$\Delta_g := \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\beta}(\sqrt{g}g^{\alpha\beta}\frac{\partial}{\partial x^\alpha}).$$

Moreover, for $1 \leq k \leq n-1$, u has free boundary $u(\partial M)$ on K, that is

$$u(x) \in K, \quad du(x)(\overrightarrow{n}) \perp T_{u(x)}K, \quad a.e.\ x \in \partial M, \tag{1.1}$$

where $\overrightarrow{n}$ is the outward unite normal vector on ∂M and $\perp$ means orthogonal.

Due to the nonlinearity in the Euler-Lagrange equations, which are quadratic in the gradient of the solutions, the existence problem is not easy. For instance, the associated heat flow may blow up at finite time, see [2] for an example. However, when the domain is of dimension two, the case of interest in this paper, Struwe [24] showed that there is a global weak solution to this flow, which is unique and regular with the exception of at most finitely many singularities and hence some existence results follow by letting the time goes to infinity. Struwe's result was extended to the free boundary case in [20], which leads to some existence results of harmonic maps with free boundary. For harmonic map flows with Dirichlet boundary, we refer to [10, 1, 13].

Classical variational methods likewise encounter problems, since the energy functional E does not satisfy the Palais-Smale condition. To overcome these problems and to prove the existence of harmonic maps from a closed Riemann surface, Sacks-Uhlenbeck introduced the following perturbed energy functional in their seminal work [21]

$$E_\alpha(u) = \int_M \left(1 + |\nabla_g u|^2\right)^\alpha dM, \quad \alpha > 1,$$

which is called the α-energy functional. This functional can be seen as a perturbation of the energy functional E that, in contrast to E itself, satisfies the Palais-Smale condition. Critical points of E_α, which are called α-harmonic maps, satisfy the following equation

$$\Delta_g u_\alpha + (\alpha - 1)\frac{\nabla_g |\nabla_g u_\alpha|^2 \nabla_g u_\alpha}{1 + |\nabla_g u_\alpha|^2} + A(u_\alpha)(du_\alpha, du_\alpha) = 0. \tag{1.2}$$

Furthermore, it was proved that for a sequence of α-harmonic maps with uniformly bounded α-energy, either there exists a nontrivial harmonic sphere or u_α sub-converges smoothly to a limit harmonic map. The finer qualitative bubbling behaviour and asymptotic analysis at the blow-up points was explored in various situations. For the energy identity of a minimax sequence of α-harmonic maps, see [11, 14]. For a sequence of minimizing α-harmonic maps, it was proved in [3] that the necks converges to some limit geodesics of finite length which implies that the energy identity holds true. See [7] for the analysis of energy minimizing sequences in homotopy classes. For the energy identity of a sequence of α-harmonic maps with an additional entropy-type condition, see [14]. The energy identity for a sequence of α-harmonic maps into certain special target manifolds was proved in [16, 15]. For a general blow-up sequence of α-harmonic maps, however, the energy identity is not necessary true, see the counter example constructed in [18]. Nevertheless, in [17] the authors investigated the general situation and established a generalized energy identity and studied the asymptotic behaviors of the necks, the limit of which are geodesics in the target manifold but not necessary of finite lengths. Sacks-Uhlenbeck's scheme was applied to the free boundary case in [23, 8] to get the existence of minimal disks with free boundary, i.e. harmonic maps from a disk with free boundary conditions, and with controlled index. However, the finer qualitative behavior at the free boundary blow-up points remained left open for a long time.

In fact, as we shall see in this paper, a new phenomenon can occur at the free boundary. For interior blow-ups, the bubbles are connected by geodesics. At a free boundary, however, the connection between bubbles, instead of being a geodesic joining them, can be a more general curve that involves the geometry of both the target manifold N and the submanifold $K \subset N$ supporting the free boundary. Thus, in this paper, we shall complete the boundary blow-up theory and analyze this phenomenon in detail.

Thus, at the free boundary, In comparison with the interior case there are new phenomena and new difficulties compared to the interior case. Therefore, we need to make new observations and develop new methods to handle the more complicated situation. We start with the fact that the equation for α-harmonic maps has an equivalent form (1.5) which sahll allow us to extend the Pohozaev type arguments, see Lemma 2.6 and Lemma 2.8. This is crucial when estimating the energy concentration in the neck domains occurring in the free boundary case. Secondly, in addition to considering the relations between the blow-up radius and the parameter α as is done in the interior case, here in the case of a boundary blow-up, by comparing the blow-up radius with the distance between the blow-up position and the boundary ∂M, we shall define a new quantity, which determines the limit behavior of the necks and leads to new length formulas for the necks. Moreover, in contrast to the interior bubbling case where the limits of the necks are geodesics in N, here in the free boundary bubbling case, we will see that the limits of the necks are either geodesics in N or geodesics-like curves in K, see Theorem 1.2. Thirdly, when applying the Pohozaev identities, we need to deal with some terms which are technically not easy to handle. For instance, besides the "bad" term

$$\frac{\alpha-1}{t}\int_{D_t}(1+|\nabla u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2 dM,$$

which already appeared in the interior case in [17], see also Corollary 2.7, here in the free boundary case, there is a new "bad" term

$$-\frac{2(\alpha-1)}{t}\int_{D_t^+(x'_\alpha)}\frac{\nabla|\nabla_g u_\alpha|^2\nabla u_\alpha}{\sigma_\alpha+|\nabla_g u_\alpha|^2}|x-x'_\alpha|\frac{\partial u_\alpha}{\partial|x-x'_\alpha|}dx,$$

see Lemma 2.11. Here x_α is the blow-up position and $x'_\alpha \in \partial M$ is the projection of x_α on ∂M, i.e. $|x_\alpha - x'_\alpha| = dist(x_\alpha, \partial M)$. One of the main challenges and achievements in the present work is to handle this new term, see Section 4.

Since in general multiple bubbles can split off at a blow-up point, similarly to the interior case in [17], we shall consider the following more general α-energy functionals

$$E_{\alpha,\sigma_\alpha}(u)=\int_M(\sigma_\alpha+|\nabla_{g_\alpha}u|^2)^\alpha dM,\ \alpha>1,$$

where $0<\sigma_\alpha\le 1$ satisfies $0<\beta_0\le \liminf_{\alpha\searrow 1}\sigma_\alpha^{\alpha-1}\le 1$. Critical points of E_{α,σ_α} are also called *α-harmonic maps*, which satisfy the following Euler-Lagrange equation

$$\Delta_{g_\alpha}u_\alpha+(\alpha-1)\frac{\nabla_{g_\alpha}|\nabla_{g_\alpha}u_\alpha|^2\nabla_{g_\alpha}u_\alpha}{\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2}+A(u_\alpha)(du_\alpha,du_\alpha)=0. \tag{1.3}$$

It is easy to see that the equation (1.3) has the following two equivalent forms:

$$\Delta u_\alpha+(\alpha-1)\frac{\nabla|\nabla_{g_\alpha}u_\alpha|^2\nabla u_\alpha}{\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2}+A(u_\alpha)(du_\alpha,du_\alpha)=0, \tag{1.4}$$

$$div\{(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}\nabla u_\alpha\}+(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}A(u_\alpha)(du_\alpha,du_\alpha)=0. \tag{1.5}$$

To present our results, we shall first give a general description of the blow-up procedure and the bubbling phenomena. We refer to [21, 23, 8, 17] for details on some well known results. More detailed new results in the free boundary case are developed in Section 3 and Section 4.

Given a sequence of α-harmonic maps $\{u_\alpha\}: M \to N$ with free boundary $u_\alpha(\partial M)$ on K and with uniformly bounded α-energy

$$E_{\alpha,\sigma_\alpha}(u_\alpha)\le\Lambda_1,$$

following Sacks-Uhlenbeck's scheme, by passing to a subsequence, we know that u_α converges to some limit harmonic map $u_0 : M \to N$ smoothly away from at most finitely many points $\{x_i\}_{i=1}^I$ as $\alpha \searrow 1$ [1]. For a fixed point x_i, $1 \le i \le I$, we may assume there are k_i bubbles occurring at this point, i.e. there are a sequence of points $\{x_\alpha^{ij}\}$, $j = 1, ..., k_i$, and a sequence of positive numbers $\{\lambda_\alpha^{ij}\}$ with $x_\alpha^{ij} \to x_i$, $\lambda_\alpha^{ij} \to 0$ as $\alpha \searrow 1$ and one of the following two alternatives holds true: if $1 \le j_1, j_2 \le k_i$ and $j_1 \ne j_2$,

(A1) for any fixed $R > 0$, $D^M_{R\lambda_\alpha^{ij_1}}(x_\alpha^{ij_1}) \cap D^M_{R\lambda_\alpha^{ij_2}}(x_\alpha^{ij_2}) = \emptyset$, whenever α is sufficiently close to 1.

(A2) $\frac{\lambda_\alpha^{ij_1}}{\lambda_\alpha^{ij_2}} + \frac{\lambda_\alpha^{ij_2}}{\lambda_\alpha^{ij_1}} \to \infty$, as $\alpha \searrow 1$.

Moreover, the rescaled maps

$$w_\alpha^{ij} := u_\alpha(x_\alpha^{ij} + \lambda_\alpha^{ij} x)$$

converge in $C^k_{loc}(\mathbb{R}^2 \setminus \{p_1^{ij}, ..., p_{s_j}^{ij}\})$ to a nontrivial harmonic map w^{ij} or converge in $C^k_{loc}(\mathbb{R}^2_a \setminus \{p_1^{ij}, ..., p_{s_j}^{ij}\})$ to a nontrivial harmonic map w^{ij} with free boundary $w^{ij}(\partial\mathbb{R}^2_a)$ on K, where $\mathbb{R}^2_a := \{y = (y_1, y_2) \in \mathbb{R}^2 \mid y_2 \ge -a\}$. Usually, we call the above harmonic map w^{ij} (or the corresponding scaling factors $(x_\alpha^{ij}, \lambda_\alpha^{ij})$) a bubble.

To study the asymptotic behavior of the necks, we consider the following two types of quantities:

$$\mu_{ij} = \liminf_{\alpha \searrow 1} (\lambda_\alpha^{ij})^{2-2\alpha} \tag{1.6}$$

and

$$\nu_{ij} = \liminf_{\alpha \searrow 1} (\lambda_\alpha^{ij})^{-\sqrt{\alpha-1}}. \tag{1.7}$$

It is easy to see that $\nu_{ij} \in [0, \infty]$. Also, we can see that there exists a positive constant $\mu_{max} \ge 1$ such that $\mu_{ij} \in [1, \mu_{max}]$. In fact, for simplicity of notations, we may assume there is only one blow-up point which is a boundary point denoted by $x \in M$, and there are k_1 bubbles occurring at this point, i.e., there are a sequence of points $\{x_\alpha^j\}$ and a sequence of positive numbers $\{\lambda_\alpha^j\}$, $1 \le j \le k_1$ satisfying (A1) or (A2). Without loss of generality we assume λ_α^1 is the smallest one, i.e. $\frac{\lambda_\alpha^1}{\lambda_\alpha^j} \le C < \infty$ for all $j = 2, ..., k_1$, as $\alpha \searrow 1$. We just need to show that

$$\mu_1 = \liminf_{\alpha \searrow 1} (\lambda_\alpha^1)^{2-2\alpha} \le \mu_{max}.$$

By classical blow-up theory, we have:

(1) if $\limsup_{\alpha \searrow 1} \frac{dist(x_\alpha^1, \partial M)}{\lambda_\alpha^1} = \infty$, then $w^1 : \mathbb{R}^2 \to N$ is a nontrivial smooth harmonic map and

$$w_\alpha(x) = u_\alpha(x_\alpha^1 + \lambda_\alpha^1 x) \to w^1 \ in\ C^k_{loc}(\mathbb{R}^2).$$

Therefore, we have

$$\Lambda_1 \ge \lim_{R\to\infty} \lim_{\alpha \searrow 1} \int_{D_{\lambda_\alpha^1 R}(x_\alpha^1)} |\nabla u_\alpha|^{2\alpha} dx = \lim_{R\to\infty} \lim_{\alpha \searrow 1} (\lambda_\alpha^1)^{2-2\alpha} \int_{D_R(0)} |\nabla w_\alpha|^2 dx = \mu_1 E(w^1).$$

(2) if $\limsup_{\alpha \searrow 1} \frac{dist(x_\alpha^1, \partial M)}{\lambda_\alpha^1} = a < \infty$, then $w^1 : \mathbb{R}^2_a \to N$ is a nontrivial smooth harmonic map with free boundary $w^1(\partial\mathbb{R}^2_a)$ on K and

$$\lim_{\alpha \searrow 1} \|w_\alpha(x)\|_{W^{1,2}(D_R(0)\cap B_\alpha)} = \|w^1(x)\|_{W^{1,2}(D_R(0)\cap \mathbb{R}^2_a)},$$

[1] Here and in the sequel, for simplicity of notations, when talking about a sequence of u_α for $\alpha \searrow 1$, we mean the sequence of u_{α_k} for a given sequence of $\alpha_k \searrow 1$.

where $B_\alpha := \left\{x \in \mathbb{R}^2 \mid x^1_\alpha + \lambda^1_\alpha x \in M\right\}$. Therefore, we have

$$\Lambda_1 \geq \lim_{R\to\infty}\lim_{\alpha\searrow 1}\int_{D_{\lambda^1_\alpha R}(x_\alpha)} |\nabla u_\alpha|^{2\alpha} dx = \lim_{R\to\infty}\lim_{\alpha\searrow 1}(\lambda^1_\alpha)^{2-2\alpha}\int_{D_R(0)\cap B_\alpha} |\nabla w_\alpha|^2 dx = \mu_1 E(w^1).$$

See the proof of Theorem 3.1 at the beginning of Section 3 for more details. Thus, we have

$$\mu_1 \leq \frac{\Lambda_1}{E(w^1)}$$

for both cases. Here $E(w^1) \geq \vartheta > 0$ with

$$\vartheta := \inf\{\, E(u) \mid u \text{ is a nontrivial harmonic sphere into } N \text{ or } u \text{ is a nontrivial harmonic disk into } N \text{ with free boundary on } K\,\}.$$

We first state a result about a generalized energy identity.

Theorem 1.1. *Let M be a smooth compact Riemann surface with smooth boundary ∂M, N be a n-dimensional smooth closed Riemannian manifold and $K \subset N$ be a smooth submanifold with dimension $1 \leq k \leq n-1$. Let $u_{\alpha_k} \in C^\infty(M,N)$ ($\alpha_k \searrow 1$) be a sequence of α_k-harmonic maps with free boundary $u_{\alpha_k}(\partial M)$ on K and with uniformly bounded α-energy, i.e. $E_{\alpha_k,\sigma_{\alpha_k}}(u_{\alpha_k}) \leq \Lambda$. We define the blow-up set*

$$\mathbf{S} := \left\{x \in M \mid \liminf_{k\to\infty}\int_{D^M_r(x)} |\nabla u_{\alpha_k}|^2 dM \geq \overline{\epsilon},\ \forall r > 0\right\},$$

where $D^M_r(x) = \left\{y \in M \mid dist^M(x,y) \leq r\right\}$ denotes the geodesic ball in M and $\overline{\epsilon} > 0$ is the constant defined in (3.2)*. Then $\mathbf{S}$ is a finite set $\{p_1, ..., p_I\}$. Passing to a subsequence, there exists $u_0 : M \to N$ which is a smooth harmonic map with free boundary $u_0(\partial M)$ on K, and there are finitely many bubbles: a finite set of harmonic spheres $w^l_i : S^2 \to N$, $l = 1, ...l_i$ and a finite set of harmonic disks $\overline{w}^k_i : D_1(0) \to N$, $k = 1, ...k_i$ with free boundaries $\overline{w}^k_i(\partial D_1(0))$ on K, where $l_i + k_i \geq 1$, $i = 1, ..., I$, such that $u_{\alpha_k} \to u_0$ weakly in $W^{1,2}(M,N)$ and strongly in $C^\infty_{loc}(M \setminus \mathbf{S}, N)$. Moreover, the following generalized identity holds:*

$$\lim_{k\to\infty} E_{\alpha_k}(u_{\alpha_k}) = E(u_0) + |M| + \sum_{i=1}^{I}\sum_{l=1}^{l_i}\mu^2_{il}E(w^l_i) + \sum_{i=1}^{I}\sum_{k=1}^{k_i}\mu^2_{ik}E(\overline{w}^k_i). \tag{1.8}$$

In Theorem 1.1, when a bubble with bubbling data $(x_\alpha, \lambda_\alpha)$ is occuring at a blow-up point $x_0 \in \partial M$, where x_α is the blow-up position, λ_α is the blow-up radius and x_α converges to x_0, we need to consider two cases. Denote $d_\alpha = dist(x_\alpha, \partial M)$. The case of $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha} < \infty$ can be handled in an analogous way as in the interior case, while, the case of $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha} = \infty$ is much more complicated and there are new difficulties arising in this situation. To overcome these, we need to develop a new method. In such a case, we shall introduce a new quantity s_α, defined by

$$s_\alpha := \log_{\lambda_\alpha} d_\alpha, \quad if\ \lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha} = \infty,$$

Since $\lambda_\alpha \to 0$ and $d_\alpha \to 0$, it is easy to see that when $\alpha - 1$ is small enough, there holds

$$0 < s_\alpha < 1.$$

By passing to a subsequence, we assume the following limit exists:

$$\lim_{\alpha\searrow 1} s_\alpha = s_0 \in [0,1].$$

This quantity can be used to distinguish the "interior" and "boundary" neck domain. With the help of this new quantity, we are able to handle the more difficult case of $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$, see Section 3 for details. Moreover, it turns out that this quantity also plays a key role in characterizing the limiting behaviors of the necks. This is our second main achievement in the present work, which will be explored in the next step.

To proceed, let us recall a useful technique, namely, the geodesic reflection argument, which has played a key role in the study of free boundary problems for harmonic type systems [10, 9, 22, 4]. Denote by K_{δ_0} the δ_0-tubular neighborhood of K in N. Taking $\delta_0>0$ small enough, then for any $y\in K_{\delta_0}$, there exists a unique projection $y'\in K$. Set $\overline{y}=\exp_{y'}\left\{-\exp_{y'}^{-1}y\right\}$ and define an involution $\sigma: K_{\delta_0}\to K_{\delta_0}$, *i.e.* satisfying $\sigma^2=Id$, by the following

$$\sigma(y)=\overline{y}\quad for\quad y\in K_{\delta_0}. \tag{1.9}$$

Then it is easy to check that the linear operator $D\sigma: TN|_{K_{\delta_0}}\to TN|_{K_{\delta_0}}$ satisfies

$$D\sigma(V)=V,\ \forall V\in TK \text{ and } D\sigma(\xi)=-\xi,\ \forall\xi\in T^\perp K.$$

Now, we present our main result in this paper, which provides a complete geometric picture of all possible limiting behaviors of the necks occurring in the blow-up process at the free boundary. More precisely, we have

Theorem 1.2. *Under the same assumptions as in Theorem 1.1, assume* $\mathbf{S}=\{x_1\}$ *and there is only one bubble in* $D_r^M(x_1)\subset M$ *for some* $r>0$*, for the sequence* $\{u_{\alpha_k}\}$*, denoted by* w^1*, which is either a harmonic sphere or a harmonic disk with free boundary* $w(\partial D_1(0))$ *on* K*. Let*

$$\nu^1=\liminf_{\alpha\searrow 1}(\lambda_\alpha^1)^{-\sqrt{\alpha-1}}. \tag{1.10}$$

Denoting $d_\alpha^1:=dist(x_\alpha^1,\partial M)$*, by taking a subsequence, we assume the following limit exists:*

$$\lim_{\alpha\searrow 1}\frac{d_\alpha^1}{\lambda_\alpha^1}\in[0,\infty]$$

and define a quantity s_α *as follows:*

$$d_\alpha^1=(\lambda_\alpha^1)^{s_\alpha},\quad if\ \lim_{\alpha\searrow 1}\frac{d_\alpha^1}{\lambda_\alpha^1}=\infty,$$

where $0<s_\alpha<1$*. By passing to a subsequence, we assume the following limit exists:*

$$\lim_{\alpha\searrow 1}s_\alpha=s_0\in[0,1].$$

Then we have the following alternatives:

(1) *when* $\nu^1=1$*, the set* $u_0(D_r^M(x_1))\cup w^1(S^2)$ *or* $u_0(D_r^M(x_1))\cup w^1(D_1(0))$ *is a connected set in* N*;*

(2) *when* $\nu^1\in(1,\infty)$*, then*

(2-a) *if* $\lim_{\alpha\searrow 1}\frac{d_\alpha^1}{\lambda_\alpha^1}=\infty$ *and* $s_0=0$*, then* w^1 *is a harmonic sphere and the set* $u_0(D_r^M(x_1))$ *and* $w(S^2)$ *are connected by a geodesic* Γ^2 *in* N *with length*

$$L(\Gamma^2)=\sqrt{\frac{E(w^1)}{\pi}}\log\nu^1;$$

(2-b) *if* $\lim_{\alpha\searrow 1}\frac{d_\alpha^1}{\lambda_\alpha^1}=a<\infty$ *or* $\lim_{\alpha\searrow 1}\frac{d_\alpha^1}{\lambda_\alpha^1}=\infty$ *and* $s_0=1$*, then* w^1 *is a harmonic disk with free boundary* $w^1(\partial D_1(0))$ *on* K *or a harmonic sphere, respectively and the set* $u_0(D_r^M(x_1))$ *and* $w(S^2)$ *are connected by a geodesic-like curve* Γ^1 *in* K *with length*

$$L(\Gamma^1)=\sqrt{\frac{2E(w^1)}{\pi}}\log\nu^1;$$

Here, the geodesic-like curve Γ^1 *in* K *is defined by* ω^1 *satisfying the following equation*

$$\frac{d^2\omega^1}{ds^2}+A(\omega^1)(\frac{d\omega^1}{ds},\frac{d\omega^1}{ds})+\frac{1}{2}D^2\sigma(\omega^1)(\frac{d\omega^1}{ds},\frac{d\omega^1}{ds})=0,\tag{1.11}$$

where A *is the second fundamental form of* $N\subset\mathbb{R}^N$ *and* σ *is the involution in* (1.9).

(2-c) *if* $\lim_{\alpha\searrow 1}\frac{d_\alpha^1}{\lambda_\alpha^1}=\infty$ *and* $s_0\in(0,1)$*, then* w^1 *is a harmonic sphere and the set* $u_0(D_r^M(x_1))$ *and* $w(S^2)$ *are connected by a curve* $\Gamma=\Gamma^1\cup\Gamma^2$ *where* Γ^2 *is a geodesic in* N *with length*

$$L(\Gamma^2)=(1-s_0)\sqrt{\frac{E(w^1)}{\pi}}\log\nu^1,$$

and Γ^1 *is a geodesic-like curve in* K *as in* (1.11) *with length*

$$L(\Gamma^1)=s_0\sqrt{\frac{2E(w^1)}{\pi}}\log\nu^1.$$

Moreover, Γ *is a continuous curve,* Γ^1 *and* Γ^2 *are smooth curves which are connected at the point* $y\in K$ *defined by the following limit (taking subsequences if necessary)*

$$y=\lim_{\alpha\searrow 1}u_\alpha(\partial D_{d_\alpha^1}(x_\alpha^1)).$$

(3) *when* $\nu^1=\infty$*, the neck contains at least an infinite length curve which is either a geodesic in* N *or a geodesic-like curve in* K *as in* (1.11)*;*

In particular, if $K\subset N$ *is a totally geodesic submanifold, then the geodesic-like curves in (2) and (3) are geodesics in* N*.*

From the equation (1.11) of the geodesic-like curve in K, we can view ω^1 as a kind of average of geodesics in N and geodesics in K.

The rest of this paper is organized as follows. In Section 2, we shall extend the α-harmonic map across the boundary via the geodesic reflection and derive the equation of the new map. Using the new equation of the involuted map, we obtain the small energy regularity in the free boundary case. Thanks to the free boundary condition, several Pohozaev type identities are established in this section. In the end of this section, we recall some basic lemmas which will be used in this paper, such as a removable singularity theorem and a gap theorem. In Section 3, we prove our first main Theorem 1.1. The asymptotic behaviors of the necks are studied in Section 4. We give new length formulas of the necks in the "one bubble" case in this section.

Notation: $D_r(x_0)$ denotes the closed disk of radius r and center x_0 in $\mathbb{R}^2$. Denote

$$D_r^+(x_0):=\left\{x=(x^1,x^2)\in D_r(x_0)\mid x^2\geq 0\right\},D_r^-(x_0):=\left\{x=(x^1,x^2)\in D_r(x_0)\mid x^2\leq 0\right\},$$
$$\partial^+D_r(x_0):=\left\{x=(x^1,x^2)\in\partial D_r(x_0)\mid x^2\geq 0\right\},\partial^-D_r(x_0):=\left\{x=(x^1,x^2)\in\partial D_r(x_0)\mid x^2\leq 0\right\},$$
$$\partial^0D_r^+(x_0)=\partial^0D_r^-(x_0):=\partial D_r^+(x_0)\setminus\partial^+D_r(x_0).$$

Let $a \geq 0$ be a constant, denote

$$\mathbb{R}^2_a := \left\{(x^1, x^2) \mid x^2 \geq -a\right\} \ \ and \ \ \mathbb{R}^{2+}_a := \left\{(x^1, x^2) \mid x^2 > -a\right\}.$$

For convenience, we denote $D_r = D_r(0)$, $D = D_1(0)$ and $\mathbb{R}^2_+ = \mathbb{R}^2_a$ when $a = 0$.

In this paper, we use the notation Δ_g and ∇_g to denote the Laplace-Beltrami operator and the gradient on the Riemannian manifold (M, g) and use $\Delta := \partial_x^2 + \partial_y^2$ and ∇ to denote the usual Laplace operator and the gradient on the Euclidean space $\mathbb{R}^2$.

2. Some basic lemmas

In this section, we will derive some basic lemmas for α-harmonic maps at the free boundary, such as small energy localization, small energy regularity and several Pohozaev type identities. In the end of this section, we will recall the removable singularity theorem and the energy gap theorem for harmonic maps with free boundary.

By the Riemann mapping theorem, for any $p \in \partial M$, where M is a Riemann surface with smooth boundary, there exists a neighborhood of p such that the metric can be expressed as

$$g = e^{\varphi}((dx^1)^2 + (dx^2)^2),$$

where $(x^1, x^2) \in D_1^+(0)$ and φ is a smooth function satisfying $\varphi(p) = 0$. Therefore, to study the blow-up behavior at the boundary point, it suffices to consider the case of a domain being the unit upper half disk $D_1^+(0) \subset \mathbb{R}^2$ with the center 0 and with the metric $g = e^{\varphi}((dx^1)^2 + (dx^2)^2)$.

Scheven [22] showed that the image of a harmonic map with free boundary on K is contained in a small tubular neighborhood of K if the energy of this map is small. Here, we have the following analogue for α-harmonic maps:

Lemma 2.1. *There exist two positive constants ϵ_1 and $\alpha_1 > 1$ depending only on g, N, such that if u_α : $(D_1^+(0), g_\alpha) \to N$ is a α-harmonic map with free boundary $u_\alpha(\partial^0 D_1^+)$ on K and with*

$$E(u_\alpha) \leq \epsilon_1, \ 1 \leq \alpha \leq \alpha_1,$$

where $g_\alpha = e^{\varphi_\alpha}((dx^1)^2 + (dx^2)^2)$ and $\varphi_\alpha(0) = 0$, $\varphi_\alpha \to \varphi$ in $C^\infty(D_1^+(0))$ as $\alpha \searrow 1$, i.e., g_α converges smoothly to g, then there holds

$$dist(u_\alpha(x), K) \leq C(g, N)\|\nabla u_\alpha\|_{L^2(D_1^+(0))}$$

for all $x \in D_{1/2}^+(0)$.

Proof. With the help of the interior small energy regularity for α-harmonic maps (see [21] or Lemma 2.3), the proof is almost the same as in the case of harmonic map, see the proof of Lemma 3.1 in [22] (the key idea is the interior small energy regularity of harmonic maps and the Green's representation of Laplace operator). □

By Lemma 2.1, there exist two small constants $\epsilon_1 = \epsilon_1(g, N, \delta_0) > 0$ and $\alpha_1 = \alpha_1(g, N, \delta_0) > 0$ such that if $\|\nabla u_\alpha\|_{L^2(D_2^+)} \leq \epsilon_1$ and $1 < \alpha \leq \alpha_1$, then $u_\alpha(D^+) \subset K_{\delta_0}$. Then we can define an extension of u_α to $D_1(0)$ by

$$\widehat{u_\alpha}(x) = \begin{cases} u_\alpha(x), & if \quad x \in D^+; \\ \sigma(u_\alpha(\rho(x))), & if \quad x \in D^-, \end{cases} \tag{2.1}$$

where σ is the involution defined in (1.9) and $\rho(x) = (x^1, -x^2)$ for $x = (x^1, x^2) \in D_1(0)$.

Next, we derive the equation of the extended map $\widehat{u}_\alpha$.

Proposition 2.2. *$\widehat{u}_\alpha \in W^{2,\infty}(M,N)$ satisfies*

$$\triangle\widehat{u}_\alpha + \Upsilon_{\widehat{u}_\alpha}(\nabla\widehat{u}_\alpha, \nabla\widehat{u}_\alpha) + \widehat{F}(\nabla^2\widehat{u}_\alpha, \nabla\widehat{u}_\alpha) = 0 \quad in \quad D. \tag{2.2}$$

Here, $\Upsilon_{\widehat{u}_\alpha}(\cdot,\cdot)$ is a bounded bilinear form given by

$$\Upsilon_{\widehat{u}_\alpha}(\cdot,\cdot) = \begin{cases} A(\widehat{u}_\alpha)(\cdot,\cdot) \text{ in } D^+, \\ D^2(\sigma\circ\Pi_N)|_{\sigma(\widehat{u}_\alpha)}(D\sigma(\widehat{u}_\alpha)\cdot\,, D\sigma(\widehat{u}_\alpha)\cdot\,) \text{ in } D^-, \end{cases} \tag{2.3}$$

where $\Pi_N : N_{\delta_0'} \to N$ is the nearest projection map for some δ_0'−neighborhood of N in $\mathbb{R}^N$ and $\widehat{F}$ is given by

$$\widehat{F} = \begin{cases} (\alpha-1)\frac{\nabla|\nabla_{g_\alpha}u_\alpha|^2\nabla u_\alpha}{\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2} \text{ in } D^+, \\ (\alpha-1)\frac{\nabla\langle D\sigma(\widehat{u}_\alpha)\cdot\nabla\widehat{u}_\alpha, D\sigma(\widehat{u}_\alpha)\cdot\nabla\widehat{u}_\alpha\rangle_{g_\alpha(\rho(x))}\nabla\widehat{u}_\alpha}{\sigma_\alpha+\langle D\sigma(\widehat{u}_\alpha)\cdot\nabla\widehat{u}_\alpha, D\sigma(\widehat{u}_\alpha)\cdot\nabla\widehat{u}_\alpha\rangle_{g_\alpha(\rho(x))}} \text{ in } D^-. \end{cases} \tag{2.4}$$

Proof. The proof is similar to the case of approximate harmonic maps with uniformly bounded L^2-tension fields, see Proposition 3.3 in [12]. Firstly, by the property of $D\sigma$, it is easy to see $\widehat{u}_\alpha \in W^{2,\infty}(D)$, since u_α satisfies free boundary condition. Computing directly, we have

$$\begin{aligned} \nabla_{e_\beta}\widehat{u}_\alpha(x) &= D\sigma|_{u_\alpha(\rho(x))} \circ Du_\alpha|_{\rho(x)} \circ D\rho|_x(e_\beta) \\ &= D\sigma|_{u_\alpha(\rho(x))} \circ D\Pi_N|_{u_\alpha(\rho(x))} \circ Du_\alpha|_{\rho(x)} \circ D\rho|_x(e_\beta), \quad x \in D^-. \end{aligned}$$

By straightforward calculations, we obtain

$$\begin{aligned} \Delta\widehat{u}_\alpha(x) &= D^2(\sigma\circ\Pi_N)|_{\sigma(\widehat{u}_\alpha)}(\nabla(u_\alpha\circ\rho), \nabla(u_\alpha\circ\rho)) + D\sigma(\sigma(\widehat{u}_\alpha))\cdot(\Delta u_\alpha)^\top(\rho(x)) \\ &= D^2(\sigma\circ\Pi_N)|_{\sigma(\widehat{u}_\alpha)}(D\sigma(\widehat{u}_\alpha)\cdot\nabla\widehat{u}_\alpha(x), D\sigma(\widehat{u}_\alpha)\cdot\nabla\widehat{u}_\alpha(x)) + D\sigma(\sigma(\widehat{u}_\alpha))\cdot(\Delta u_\alpha)^\top(\rho(x)). \end{aligned}$$

Combining this with the fact that $\widehat{u}_\alpha$ satisfies equation (1.4) in D^+, the equation (2.2) follows immediately. □

Now, we give two small energy regularity lemmas for α-harmonic maps for both the interior case and the free boundary case. For the interior case, we have

Lemma 2.3. *([21]) For any $1 < p < \infty$, there exist two constants $\epsilon_0 > 0$ and $\alpha_0 > 1$ depending only on p, g and N, such that if $u_\alpha : (D_1(0), g_\alpha) \to N$ is an α-harmonic map with*

$$E(u_\alpha) \le \epsilon_0,\ 1 \le \alpha \le \alpha_0,$$

where $g_\alpha = e^{\rho_\alpha}((dx^1)^2 + (dx^2)^2)$ and $\rho_\alpha(0) = 0$, $\rho_\alpha \to \rho$ in $C^\infty(D_1(0))$ as $\alpha \searrow 1$, i.e., g_α converges smoothly to g, then there holds

$$\|\nabla u_\alpha\|_{W^{1,p}(D_{1/2}(0))} \le C(p,g,N)\|\nabla u_\alpha\|_{L^2(D_1(0))}.$$

for some $C(p,g,N) > 0$.

Near the boundary, we have

Lemma 2.4. *For any $1 < p < \infty$, there exist two constants $\epsilon_0 > 0$ and $\alpha_0 > 1$ depending only on p, g and N, such that if $u_\alpha : (D_1^+(0), g_\alpha) \to N$ is an α-harmonic map with free boundary $u_\alpha(\partial^0 D_1^+(0))$ on K and satisfying*

$$E(u_\alpha) \le \epsilon_0,\ 1 \le \alpha \le \alpha_0,$$

where $g_\alpha = e^{\varphi_\alpha}((dx^1)^2 + (dx^2)^2)$ *and* $\varphi_\alpha(0) = 0$, $\varphi_\alpha \to \varphi$ *in* $C^\infty(D_1^+(0))$ *as* $\alpha \searrow 1$, *i.e.,* g_α *converges smoothly to* g, *then there holds*

$$\|\nabla u_\alpha\|_{W^{1,p}(D^+_{1/2}(0))} \le C(p,g,N)\|\nabla u_\alpha\|_{L^2(D^+)}.$$

for some $C(p,g,N) > 0$.

Proof. By Lemma 2.1, we can extend u to $\widehat{u} \in W^{2,p}(D)$ which is defined in D and satisfies

$$|\Delta\widehat{u}_\alpha| \le C(g,N)|\nabla\widehat{u}_\alpha|^2 + C(g,N)(\alpha-1)|\nabla^2\widehat{u}_\alpha| \quad in \quad D. \tag{2.5}$$

We first consider the case $1 < p < 2$. Take a cut-off function $\eta \in C_0^\infty(D)$, such that $0 \le \eta \le 1$, $\eta|_{D_{3/4}} \equiv 1$ and $|\nabla\eta| \le C$. Then, we have

$$\Delta(\eta\widehat{u}_\alpha) = \eta\Delta\widehat{u}_\alpha + 2\nabla\eta\nabla\widehat{u}_\alpha + \widehat{u}_\alpha\Delta\eta \le C(g,N)|\nabla\widehat{u}_\alpha||\nabla(\eta\widehat{u}_\alpha)| + C(g,N)(|\nabla\widehat{u}_\alpha| + |\widehat{u}_\alpha| + (\alpha-1)|\nabla^2(\eta\widehat{u}_\alpha)|).$$

Without loss of generality, we may assume $\frac{1}{|D^+|}\int_{D^+}\widehat{u}_\alpha dx = \frac{1}{|D^+|}\int_{D^+} u_\alpha dx = 0$. By standard elliptic estimates, Sobolev's embedding and Poincaré's inequality, we have

$$\begin{aligned}
\|\eta\widehat{u}_\alpha\|_{W^{2,p}(D)} &\le C(p,g,N)\||\nabla\widehat{u}_\alpha||\nabla(\eta\widehat{u}_\alpha)|\|_{L^p(D)} + C(p,g,N)(\|\nabla\widehat{u}_\alpha\|_{L^p(D)} + \|\widehat{u}_\alpha\|_{L^p(D)} + (\alpha-1)\|\nabla^2(\eta\widehat{u}_\alpha)\|_{L^p(D)}) \\
&\le C(p,g,N)\|\nabla\widehat{u}_\alpha\|_{L^2(D)}\|\nabla(\eta\widehat{u}_\alpha)\|_{L^{\frac{2p}{2-p}}(D)} + C(p,g,N)(\|\nabla\widehat{u}_\alpha\|_{L^p(D)} + (\alpha-1)\|\nabla^2(\eta\widehat{u}_\alpha)\|_{L^p(D)}) \\
&\le C(p,g,N)\epsilon_0\|\eta\widehat{u}_\alpha\|_{W^{2,p}(D)} + C(p,g,N)(\|\nabla\widehat{u}_\alpha\|_{L^p(D)} + (\alpha-1)\|\nabla^2(\eta\widehat{u}_\alpha)\|_{L^p(D)}).
\end{aligned}$$

Taking $\epsilon_0 > 0$ and $\alpha_0 > 0$ sufficiently small, we have

$$\|u_\alpha\|_{W^{2,p}(D^+_{1/2})} \le \|\eta\widehat{u}_\alpha\|_{W^{2,p}(D)} \le C(p,g,N)\|\nabla u_\alpha\|_{L^p(D^+)},$$

where we also used the fact that $\|\nabla\widehat{u}_\alpha\|_{L^p(D)} \le C(N)\|\nabla u_\alpha\|_{L^p(D^+)}$.

So, we have proved the lemma in the case $1 < p < 2$. This estimate with $p = \frac{4}{3}$ gives an $L^4(D_{3/4})$ bound for $\nabla\widehat{u}_\alpha$. Then one can apply the $W^{2,2}$boundary estimate to the equation and get the conclusion of the lemma with $p = 2$, which implies $\nabla\widehat{u}_\alpha \in L^p(D_{5/8})$ for any $1 < p < \infty$. Then the conclusion of the lemma follows immediately from the standard $W^{2,p}$-estimate for the Laplace operator. □

As an application of the Lemma 2.4, we have

Lemma 2.5. *Let* $D_1^+(0) \subset \mathbb{R}^2$ *be the unit normal upper half disk. Let* $g_\alpha = e^{\varphi_\alpha(x)}((dx^1)^2 + (dx^2)^2)$ *and* $g = e^{\varphi(x)}((dx^1)^2 + (dx^2)^2)$ *be a family of metrics on* $D_1(0)$, *where* $\varphi_\alpha \in C^\infty(D_1)$, $\varphi_\alpha(0) = 0$ *and* $\varphi_\alpha \to \varphi$ *in* $C^\infty(D_1)$ *as* $\alpha \searrow 1$. *Let* $u_\alpha \in C^\infty(D_1^+(0), N)$ *be a sequence of* α*-harmonic maps with free boundaries* $u_\alpha(\partial^0 D^+)$ *on* K, *with uniformly bounded energy* $E_{\alpha,\sigma_\alpha}(u_\alpha) \le \Lambda_1$ *and with* $\lim_{\alpha\searrow 1}(\sigma_\alpha)^{\alpha-1} \ge \beta_0 > 0$, *then there exists a positive* β_1 *independent of* α, *such that*

$$\beta_0 \le \liminf_{\alpha\searrow 1}\|(\sigma_\alpha + |\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}\|_{C^0(D^+)} \le \limsup_{\alpha\searrow 1}\|(\sigma_\alpha + |\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}\|_{C^0(D^+)} \le \beta_1.$$

Proof. Without loss of generality, we assume 0 is the only energy concentration point for the sequence $\{u_\alpha\}$. Then by the blow-up process described in the introduction, we can get finite bubbles at this point, i.e. there exist a positive sequence $\lambda_\alpha^i \to 0$ and a sequence of points $x_\alpha^i \to 0$, $i = 1, ..., I$, as $\alpha \searrow 1$, which satisfy (A1) or (A2). Also, without loss of generality, we assume λ_α^1 is the smallest one, i.e. $\limsup_{\alpha\searrow 1}\frac{\lambda_\alpha^1}{\lambda_\alpha^j} \le C$ for $j = 2, ..., I$. By a standard blow-up argument, we know $w_\alpha(x) = u_\alpha(x_\alpha^1 + \lambda_\alpha^1 x)$ has no energy concentration points. By Lemma 2.4, we have

$$\|(\sigma_\alpha + |\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}\|_{C^0(D^+)} \le C(1 + (\lambda_\alpha^1)^{2-2\alpha}) \le C(1 + \mu_{max}),$$

when $\alpha - 1$ is small enough. Then the conclusion of the lemma follows immediately. □

Next, we derive several versions of Pohozaev type identities.

For the interior case, with the help of the divergence structure of equation (1.5), we establish a Pohozaev type identity.

Lemma 2.6. *Let (D, g_α) be a unit disk in $\mathbb{R}^2$ equipped with a metric $g_\alpha = e^{\varphi_\alpha}\left((dx^1)^2 + (dx^2)^2\right)$, where $\varphi_\alpha \in C^\infty(D)$. If u_α is a critical point of $E_{\alpha,\sigma_\alpha}(u; D)$, then for any $0 < t < 1$, there holds*

$$
\begin{aligned}
&(1-\frac{1}{2\alpha})\int_{\partial D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\frac{\partial u_\alpha}{\partial r}|^2-\frac{1}{2\alpha}\int_{\partial D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|x|^{-2}|\frac{\partial u_\alpha}{\partial\theta}|^2\\
&=(1-\frac{1}{\alpha})\frac{1}{t}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx+\frac{\sigma_\alpha}{2\alpha}\int_{\partial D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}e^{\varphi_\alpha}\\
&\quad+(1-\frac{1}{\alpha})\frac{1}{2t}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2r\frac{\partial\varphi_\alpha}{\partial r}dx\\
&\quad-\frac{\sigma_\alpha}{\alpha t}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}(1+\frac{r}{2}\frac{\partial\varphi_\alpha}{\partial r})e^{\varphi_\alpha}dx.
\end{aligned}\tag{2.6}
$$

Proof. Multiplying (1.5) by $r\frac{\partial u_\alpha}{\partial r}$ and integrating by parts, we have

$$
\begin{aligned}
0&=\int_{D_t}div\{(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}\nabla u_\alpha\}r\frac{\partial u_\alpha}{\partial r}dx\\
&=\int_{\partial D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}r|\frac{\partial u_\alpha}{\partial r}|^2-\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}\nabla u_\alpha\nabla(r\frac{\partial u_\alpha}{\partial r})dx\\
&=\int_{\partial D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}r|\frac{\partial u_\alpha}{\partial r}|^2-\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx\\
&\quad-\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}\frac{1}{2}r\frac{\partial}{\partial r}|\nabla u_\alpha|^2dx.
\end{aligned}\tag{2.7}
$$

Noting that

$$|\nabla u_\alpha|^2 = e^{\varphi_\alpha}|\nabla_{g_\alpha}u_\alpha|^2,$$

we get

$$
\begin{aligned}
&-\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}\frac{1}{2}r\frac{\partial}{\partial r}|\nabla u_\alpha|^2dx\\
&=-\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}\frac{1}{2}r\frac{\partial}{\partial r}|\nabla_{g_\alpha}u_\alpha|^2e^{\varphi_\alpha}dx-\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2\frac{1}{2}r\frac{\partial\varphi_\alpha}{\partial r}dx\\
&=-\frac{1}{2\alpha}\int_{D_t}r\frac{\partial}{\partial r}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha}e^{\varphi_\alpha}dx-\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2\frac{1}{2}r\frac{\partial\varphi_\alpha}{\partial r}dx.
\end{aligned}\tag{2.8}
$$

Integrating by parts yields that

$$
\begin{aligned}
&-\frac{1}{2\alpha}\int_{D_t} r\frac{\partial}{\partial r}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^\alpha e^{\varphi_\alpha}dx\\
&=-\frac{1}{2\alpha}\int_{\partial D_t} r(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^\alpha e^{\varphi_\alpha}+\frac{1}{2\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^\alpha div\{xe^{\varphi_\alpha}\}dx\\
&=-\frac{1}{2\alpha}\int_{\partial D_t} r(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2-\frac{\sigma_\alpha}{2\alpha}\int_{\partial D_t} r(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}e^{\varphi_\alpha}\\
&\quad+\frac{1}{\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^\alpha e^{\varphi_\alpha}dx+\frac{1}{2\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^\alpha r\frac{\partial\varphi_\alpha}{\partial r}e^{\varphi_\alpha}dx. \qquad (2.9)
\end{aligned}
$$

By (2.7)-(2.9), we obtain

$$
\begin{aligned}
&\int_{\partial D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}r|\frac{\partial u_\alpha}{\partial r}|^2-\frac{1}{2\alpha}\int_{\partial D_t} r(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2\\
&=\frac{\sigma_\alpha}{2\alpha}\int_{\partial D_t} r(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}e^{\varphi_\alpha}+\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx-\frac{1}{\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^\alpha e^{\varphi_\alpha}dx\\
&\quad+\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2\frac{1}{2}r\frac{\partial\varphi_\alpha}{\partial r}dx-\frac{1}{2\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^\alpha r\frac{\partial\varphi_\alpha}{\partial r}e^{\varphi_\alpha}dx\\
&=\frac{\sigma_\alpha}{2\alpha}\int_{\partial D_t} r(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}e^{\varphi_\alpha}+(1-\frac{1}{\alpha})\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx\\
&\quad-\frac{\sigma_\alpha}{\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}e^{\varphi_\alpha}dx+\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2\frac{1}{2}r\frac{\partial\varphi_\alpha}{\partial r}dx\\
&\quad-\frac{1}{2\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^\alpha r\frac{\partial\varphi_\alpha}{\partial r}e^{\varphi_\alpha}dx\\
&=\frac{\sigma_\alpha}{2\alpha}\int_{\partial D_t} r(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}e^{\varphi_\alpha}+(1-\frac{1}{\alpha})\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx\\
&\quad-\frac{\sigma_\alpha}{\alpha}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}(1+\frac{r}{2}\frac{\partial\varphi_\alpha}{\partial r})e^{\varphi_\alpha}dx+(1-\frac{1}{\alpha})\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2\frac{1}{2}r\frac{\partial\varphi_\alpha}{\partial r}dx
\end{aligned}
$$

which implies the conclusion of the lemma. □

As a direct corollary of the above lemma, we give a new proof of following Pohozaev type estimate (Lemma 2.3 in [17]), which was proved via a variational method.

Corollary 2.7. *Let (D, g_α) be a unit disk in $\mathbb{R}^2$ equipped with a metric $g_\alpha = e^{\varphi_\alpha}\left((dx^1)^2+(dx^2)^2\right)$, where $\varphi_\alpha(0) = 0$ and $\varphi_\alpha \to \varphi \in C^\infty(D)$ smoothly. If u_α is a critical point of $E_{\alpha,\sigma_\alpha}(u; D)$ where $0 < \beta_0 < \lim_{\alpha\searrow 1}\sigma_\alpha^{\alpha-1} \le 1$ and $E_{\alpha,\sigma_\alpha}(u_\alpha) \le \Lambda_1$, then for any $0 < t < 1$, we have*

$$
\begin{aligned}
&(1-\frac{1}{2\alpha})\int_{\partial D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\frac{\partial u_\alpha}{\partial r}|^2-\frac{1}{2\alpha}\int_{\partial D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|x|^{-2}|\frac{\partial u_\alpha}{\partial\theta}|^2\\
&=(1-\frac{1}{\alpha})\frac{1}{t}\int_{D_t}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx+O(t)+O(\alpha-1). \qquad (2.10)
\end{aligned}
$$

Thanks to the free boundary condition, we can also derive a Pohozaev type identity at a boundary point.

Lemma 2.8. *Let (D^+, g_α) be a upper half unit disk in $\mathbb{R}^2_+$ with metric $g_\alpha = e^{\varphi_\alpha}\left((dx^1)^2+(dx^2)^2\right)$, where $\varphi_\alpha \in C^\infty(D)$. If u_α is a critical point of $E_{\alpha,\sigma_\alpha}(u; D^+)$ with free boundary $u_\alpha(\partial^0 D^+)$ on K, then for any*

$0 < t < 1$, *there holds*

$$(1-\frac{1}{2\alpha})\int_{\partial^+ D_t^+}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\frac{\partial u_\alpha}{\partial r}|^2-\frac{1}{2\alpha}\int_{\partial^+ D_t^+}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|x|^{-2}|\frac{\partial u_\alpha}{\partial\theta}|^2$$
$$=(1-\frac{1}{\alpha})\frac{1}{t}\int_{D_t^+}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx+\frac{\sigma_\alpha}{2\alpha}\int_{\partial^+ D_t^+}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}e^{\varphi_\alpha}$$
$$+(1-\frac{1}{\alpha})\frac{1}{2t}\int_{D_t^+}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2 r\frac{\partial\varphi_\alpha}{\partial r}dx$$
$$-\frac{\sigma_\alpha}{\alpha t}\int_{D_t^+}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}(1+\frac{r}{2}\frac{\partial\varphi_\alpha}{\partial r})e^{\varphi_\alpha}dx. \tag{2.11}$$

Proof. The proof is similar to the interior case done in Lemma 2.6. Multiplying (1.5) by $r\frac{\partial u_\alpha}{\partial r}$ and integrating by parts, by (2.7) and (2.8), there holds

$$0=\int_{D_t^+}div\{(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}\nabla u_\alpha\}r\frac{\partial u_\alpha}{\partial r}dx$$
$$=\int_{\partial^+ D_t^+}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}r|\frac{\partial u_\alpha}{\partial r}|^2-\int_{D_t^+}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx$$
$$-\frac{1}{2\alpha}\int_{D_t^+}r\frac{\partial}{\partial r}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha}e^{\varphi_\alpha}dx-\int_{D_t^+}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2\frac{1}{2}r\frac{\partial\varphi_\alpha}{\partial r}dx, \tag{2.12}$$

where we used the free boundary condition that $\frac{\partial u_\alpha}{\partial\vec{n}}\perp\frac{\partial u_\alpha}{\partial r}$ on $\partial^0 D^+$. Here, $\vec{n}$ is the unit outward normal vector field of ∂D^+.

Integrating by parts yields that

$$-\frac{1}{2\alpha}\int_{D_t^+}r\frac{\partial}{\partial r}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha}e^{\varphi_\alpha}dx$$
$$=-\frac{1}{2\alpha}\int_{\partial D_t^+}\langle x,\vec{n}\rangle(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha}e^{\varphi_\alpha}+\frac{1}{2\alpha}\int_{D_t^+}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha}div\{xe^{\varphi_\alpha}\}dx$$
$$=-\frac{1}{2\alpha}\int_{\partial^+ D_t^+}\langle x,\vec{n}\rangle(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha}e^{\varphi_\alpha}+\frac{1}{2\alpha}\int_{D_t^+}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha}div\{xe^{\varphi_\alpha}\}dx, \tag{2.13}$$

where the last equality we used the fact that $\langle x,\vec{n}\rangle|_{\partial^0 D^+}=0$. The rest of the proof is the same as in Lemma 2.6 and we omit it. □

Consequently, we have the following Pohozaev type estimate for the free boundary case.

Corollary 2.9. *Let (D^+,g_α) be a upper half unit disk in $\mathbb{R}^2_+$ with metric $g_\alpha=e^{\varphi_\alpha}\left((dx^1)^2+(dx^2)^2\right)$, where $\varphi_\alpha(0)=0$ and $\varphi_\alpha\to\varphi\in C^\infty(D^+)$ smoothly. If $E_{\alpha,\sigma_\alpha}(u_\alpha)\leq\Lambda_1$ and u_α is a critical point of $E_{\alpha,\sigma_\alpha}(u;D^+)$ with free boundary $u_\alpha(\partial^0 D^+)$ on K, where $0<\beta_0<\lim_{\alpha\searrow1}\sigma_\alpha^{\alpha-1}\leq1$, then for any $0<t<1$, we have*

$$(1-\frac{1}{2\alpha})\int_{\partial^+ D_t^+}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\frac{\partial u_\alpha}{\partial r}|^2-\frac{1}{2\alpha}\int_{\partial^+ D_t^+}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|x|^{-2}|\frac{\partial u_\alpha}{\partial\theta}|^2$$
$$=(1-\frac{1}{\alpha})\frac{1}{t}\int_{D_t^+}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx+O(t)+O(\alpha-1). \tag{2.14}$$

Next, we recall a different version of Pohozaev type identities in the interior case and then extend it to the free boundary case.

Lemma 2.10 (Lemma 2.4 in [17]). *Let (D, g_α) be a unit disk in $\mathbb{R}^2$ with metric $g_\alpha = e^{\varphi_\alpha}\Big((dx^1)^2 + (dx^2)^2\Big)$, where $\varphi_\alpha \in C^\infty(D)$. If u_α is a critical point of $E_{\alpha,\sigma_\alpha}(u; D)$, then for any $0 < t < 1$, there holds*

$$\int_{\partial D_t} (|\frac{\partial u_\alpha}{\partial r}|^2 - r^{-2}|\frac{\partial u_\alpha}{\partial \theta}|^2) = -\frac{2(\alpha-1)}{t}\int_{D_t} \frac{\nabla|\nabla_{g_\alpha} u_\alpha|^2 \nabla u_\alpha}{\sigma_\alpha + |\nabla_{g_\alpha} u_\alpha|^2} r \frac{\partial u_\alpha}{\partial r} dx. \tag{2.15}$$

Lemma 2.11. *Let (D^+, g_α) be a upper half unit disk in $\mathbb{R}^2_+$ with metric $g_\alpha = e^{\varphi_\alpha}\Big((dx^1)^2 + (dx^2)^2\Big)$, where $\varphi_\alpha \in C^\infty(D)$. If u_α is a critical point of $E_{\alpha,\sigma_\alpha}(u; D^+)$ with free boundary $u_\alpha(\partial^0 D^+)$ on K, then for any $0 < t < 1$, there holds*

$$\int_{\partial^+ D_t^+} (|\frac{\partial u_\alpha}{\partial r}|^2 - r^{-2}|\frac{\partial u_\alpha}{\partial \theta}|^2) = -\frac{2(\alpha-1)}{t}\int_{D_t^+} \frac{\nabla|\nabla_{g_\alpha} u_\alpha|^2 \nabla u_\alpha}{\sigma_\alpha + |\nabla_{g_\alpha} u_\alpha|^2} r \frac{\partial u_\alpha}{\partial r} dx. \tag{2.16}$$

Proof. Multiplying the equation (1.4) by $r\frac{\partial u_\alpha}{\partial r}$ and integrating over D^+ yields

$$\int_{D_t^+} r\frac{\partial u_\alpha}{\partial r}\Delta u_\alpha dx = -(\alpha-1)\int_{D_t^+} \frac{\nabla|\nabla_{g_\alpha} u_\alpha|^2 \nabla u_\alpha}{\sigma_\alpha + |\nabla_{g_\alpha} u_\alpha|^2} r \frac{\partial u_\alpha}{\partial r} dx. \tag{2.17}$$

Integrating by parts and using the free boundary condition $\frac{\partial u_\alpha}{\partial \vec{n}} \perp \frac{\partial u_\alpha}{\partial r}$ on $\partial^0 D^+$, it is standard to get the following:

$$\int_{D_t^+} r\frac{\partial u_\alpha}{\partial r}\Delta u_\alpha dx = \int_{\partial^+ D_t^+} r(|\frac{\partial u_\alpha}{\partial r}|^2 - \frac{1}{2}|\nabla u_\alpha|^2). \tag{2.18}$$

See Lemma 4.3 in [12] for a detailed proof. Then the conclusion of the lemma follows immediately. □

In the end of this section, we recall two basic theorems which are useful in the bubbling analysis.

Theorem 2.12 (Removable singularity, see Theorem 1.10 in [8] or Theorem 3.6 in [12]). *Let $x_0 \in \partial D$ and $u \in C^\infty_{loc}(D \setminus \{x_0\}, N)$ be a harmonic map with finite energy and with free boundary condition*

$$u(x) \in K, \quad du(x)(\vec{n}) \perp T_{u(x)}K, \quad a.e.\ x \in \partial D, \tag{2.19}$$

then u can be extended to a smooth map in D.

Theorem 2.13 (Energy gap, see Lemma 4.2 in [12]). *There exists a positive constant $\epsilon_2 = \epsilon_2(M, N) > 0$ such that if u is a smooth harmonic map from M to N with free boundary on K and satisfying*

$$\int_M |\nabla u|^2 dvol \leq \epsilon_2,$$

then u is a constant map.

3. Energy identity

In this section, we shall prove our first main result Theorem 1.1.

We first consider the following simpler case of a single boundary blow-up point.

Theorem 3.1. *Let $D_1^+(0) \subset \mathbb{R}^2$ be the unit normal upper half disk, $g_\alpha = e^{\varphi_\alpha(x)}((dx^1)^2 + (dx^2)^2)$ and $g = e^{\varphi(x)}((dx^1)^2 + (dx^2)^2)$ be a family of metrics on $D_1(0)$, where $\varphi_\alpha \in C^\infty(D_1)$, $\varphi_\alpha(0) = 0$ and $\varphi_\alpha \to \varphi$ in $C^\infty(D_1)$ as $\alpha \searrow 1$. Let $u_\alpha \in C^\infty(D_1^+(0), N)$ be a sequence of α-harmonic maps with free boundaries $u_\alpha(\partial^0 D^+)$ on K and satisfying*

(a) $\sup_\alpha E_{\alpha,\sigma_\alpha}(u_\alpha) \le \Lambda_1$, *and* $0 < \beta_0 \le \lim_{\alpha\searrow 1}(\sigma_\alpha)^{\alpha-1} \le 1$,

(b) $u_\alpha \to u$ *strongly in* $C^\infty_{loc}(D^+ \setminus \{0\}, \mathbb{R}^N)$ *as* $\alpha \searrow 1$.

Then there exist a subsequence of u_α (still denoted by u_α) and a nonnegative integer L such that, for any $i = 1, ..., L$, there exist a point x^i_α, positive numbers λ^i_α and a nonconstant harmonic sphere w^i or a nonnegative constant a^i and a nonconstant harmonic disk w^i (which we view as a map from $\mathbb{R}^2_{a^i} \cup \{\infty\} \to N$) with free boundary $w^i(\partial\mathbb{R}^2_{a^i})$ on K such that:

(1) $x^i_\alpha \to 0,\ \lambda^i_\alpha \to 0$, *as* $\alpha \searrow 1$;

(2) $\frac{dist(x^i_\alpha,\partial^0 D^+)}{\lambda^i_\alpha} \to a^i$ *or* $\frac{dist(x^i_\alpha,\partial^0 D^+)}{\lambda^i_\alpha} \to \infty$ *(i.e. $a^i = \infty$), as* $\alpha \searrow 1$;

(3) $\lim_{\alpha\searrow 1} \left(\frac{\lambda^i_\alpha}{\lambda^j_\alpha} + \frac{\lambda^j_\alpha}{\lambda^i_\alpha} + \frac{|x^i_\alpha - x^j_\alpha|}{\lambda^i_\alpha + \lambda^j_\alpha}\right) = \infty$ *for any* $i \neq j$;

(4) w^i *is the weak limit of* $u_\alpha(x^i_\alpha + \lambda^i_\alpha x)$ *in* $W^{1,2}_{loc}(\mathbb{R}^2)$, *if* $\frac{dist(x^i_\alpha,\partial^0 D^+)}{\lambda^i_\alpha} \to \infty$ *or* w^i *is the weak limit of* $u_\alpha(x^i_\alpha + \lambda^i_\alpha x)$ *in* $W^{1,2}_{loc}(\mathbb{R}^{2+}_{a^i})$, *if* $\frac{dist(x^i_\alpha,\partial^0 D^+)}{\lambda^i_\alpha} \to a^i$, *where* $\mathbb{R}^{2+}_{a^i} := \{(y_1, y_2) \in \mathbb{R}^2 | y^2 > -a^i\}$.

(5) ***Energy identity:*** *we have*

$$\lim_{\delta\to 0}\lim_{\alpha\searrow 1} E_{\alpha,\sigma_\alpha}(u_\alpha, D^+_\delta(0)) = \sum_{i=1}^{L} \mu_i^2 E(w^i), \tag{3.1}$$

where $\mu_i = \lim_{\alpha\searrow 1}(\lambda^i_\alpha)^{2-2\alpha}$.

Proof of Theorem 3.1. By assumptions, without loss of generality, we assume that 0 is the only blow-up point of the sequence $\{u_\alpha\}$ in D^+, *i.e.*

$$\liminf_{\alpha\searrow 1} E(u_\alpha; D^+_r) \ge \frac{\overline{\epsilon}^2}{8} \text{ for all } r > 0 \tag{3.2}$$

where $\overline{\epsilon} = \min\{\epsilon_1, \epsilon_0\} > 0$ and $\epsilon_0 > 0,\ \epsilon_1 > 0$ are the constants in Lemma 2.1, Lemma 2.3 and Lemma 2.4. By standard blow-up argument, we can assume that there exist sequences $x_\alpha \to 0$ and $\lambda_\alpha \to 0$ such that

$$E(u_\alpha; D^+_{\lambda_\alpha}(x_\alpha)) = \sup_{\substack{x\in D^+, r\le\lambda_\alpha \\ D^+_r(x)\subset D^+}} E(u_\alpha; D^+_r(x)) = \frac{\overline{\epsilon}^2}{32}. \tag{3.3}$$

Denoting $d_\alpha = dist(x_\alpha, \partial^0 D^+)$, we have the following two cases:

Case 1: $\limsup_{\alpha\searrow 1} \frac{d_\alpha}{\lambda_\alpha} < \infty$.

Set

$$w_\alpha(x) := u_\alpha(x_\alpha + \lambda_\alpha x)$$

and

$$B_\alpha := \{x \in \mathbb{R}^2 | x_\alpha + \lambda_\alpha x \in D^+\}.$$

After taking a subsequence, we may assume $\lim_{\alpha\searrow 1} \frac{d_\alpha}{\lambda_\alpha} = a \ge 0$. Then

$$B_\alpha \to \mathbb{R}^2_a := \{(x^1, x^2) | x^2 \ge -a\}.$$

It is easy to see that $w_\alpha(x)$ lives in B_α which is a α-harmonic map with the free boundary condition

$$w_\alpha(x) \in K, \quad dw_\alpha(x)(\overrightarrow{n}) \perp T_{w_\alpha(x)}K, \text{ if } x_\alpha + \lambda_\alpha x \in \partial^0 D^+. \tag{3.4}$$

By Lemma 2.3 and Lemma 2.4, we have

$$\|w_\alpha\|_{W^{2,2}(D_{4R}(0)\cap B_\alpha)} \leq C(R,N) \tag{3.5}$$

for any $D_{4R}(0) \subset \mathbb{R}^2$. Noting that for any $x \in \{x^2 = -a\}$ on the boundary, $x_\alpha + \lambda_\alpha x \to 0$, by a similar discussion as in [12], there exist a subsequence of w_α (also denoted by w_α) and a nontrivial harmonic map $w^1 : \mathbb{R}^2_a \to N$ with free boundary $w^1(\partial\mathbb{R}^2_a)$ on K such that for any $R > 0$, there hold

$$\lim_{\alpha\searrow 1} \|w_\alpha(x) - w^1(x)\|_{W^{1,2}(D_R(0)\cap B_\alpha\cap\mathbb{R}^2_a)} = 0 \tag{3.6}$$

and

$$\lim_{\alpha\searrow 1} \|w_\alpha(x)\|_{W^{1,2}(D_R(0)\cap B_\alpha)} = \|w^1(x)\|_{W^{1,2}(D_R(0)\cap\mathbb{R}^2_a)}. \tag{3.7}$$

By the conformal invariance and removable singularity Theorem 2.12, $w^1(x)$ can be extended to a nontrivial harmonic disk with free boundary $w^1(\partial D)$ on K.

Case 2: $\limsup_{\alpha\searrow 1} \frac{d_\alpha}{\lambda_\alpha} = \infty$.

In this case, we can see that $w_\alpha(x)$ lives in B_α which tends to $\mathbb{R}^2$ as $\alpha \searrow 1$. Moreover, for any $x \in \mathbb{R}^2$, when α is sufficiently closed to 1, by (3.3), we have

$$E(w_\alpha; D_1(x)) \leq \frac{\bar{\epsilon}^2}{32}. \tag{3.8}$$

According to Lemma 2.3, there exist a subsequence of w_α (we still denote it by w_α) and a harmonic map $w^1(x) \in W^{1,2}(\mathbb{R}^2, N)$ such that

$$\lim_{\alpha\searrow 1} w_\alpha(x) = w^1(x) \text{ in } W^{1,2}_{loc}(\mathbb{R}^2).$$

Besides, we know $E(w^1; D_1(0)) = \frac{\bar{\epsilon}^2}{32}$. By the standard theory of harmonic maps, $w^1(x)$ can be extended to a nontrivial harmonic sphere. We call the above harmonic sphere $w^1(x)$ or harmonic disk $w^1(x)$ the first bubble.

By a standard induction argument as in the case of approximate harmonic maps with uniformly bounded L^2-tension fields [6], we only need to prove the theorem in the case where there is only one bubble. Under the "one bubble" assumption, we first make the following:

Claim: for any $\epsilon > 0$, there exist $\delta > 0$ and $R > 0$ such that

$$\int_{D^+_{8t}(x_\alpha)\setminus D^+_t(x_\alpha)} |\nabla u_\alpha|^2 dx \leq \epsilon^2 \text{ for any } t \in (\frac{1}{2}\lambda_\alpha R, 2\delta) \tag{3.9}$$

when $\alpha - 1$ is small enough.

The proof of this claim is now standard and it follows from a contradiction argument. In fact, if the claim is not true, one can then get a second bubble. For more details on such a contradiction argument, one can refer to [6, 17] for the interior case and [12] for the free boundary case.

To proceed, we need to establish a series of auxiliary lemmas and we leave the rest of the proof of Theorem 3.1 in the end of this section. □

Let $x'_\alpha \in \partial^0 D^+$ be the projection of x_α, *i.e.* $d_\alpha = dist(x_\alpha, \partial^0 D^+) = |x_\alpha - x'_\alpha|$.

Lemma 3.2. *Under the assumption of Theorem 3.1 and "one bubble" assumption, when $\frac{1}{R}$, δ and $\alpha - 1$ are sufficiently small, we have:*

(1) *If* $\lim_{\alpha\searrow 1} \frac{d_\alpha}{\lambda_\alpha} = a$, *then*

$$\int_{D_t^+(x'_\alpha)\setminus D_s^+(x'_\alpha)} |\nabla^2 u_\alpha||x - x'_\alpha||\nabla u_\alpha| dx \le C \int_{D_{2t}^+(x'_\alpha)\setminus D_{s/2}^+(x'_\alpha)} |\nabla u_\alpha|^2 dx, \ \text{for any } \lambda_\alpha R \le s < t \le \delta;$$

(2) *If* $\lim_{\alpha\searrow 1} \frac{d_\alpha}{\lambda_\alpha} = \infty$, *then*

$$\int_{D_t^+(x'_\alpha)\setminus D_s^+(x'_\alpha)} |\nabla^2 u_\alpha||x - x'_\alpha||\nabla u_\alpha| dx \le C \int_{D_{2t}^+(x'_\alpha)\setminus D_{s/2}^+(x'_\alpha)} |\nabla u_\alpha|^2 dx, \ \text{for any } 4d_\alpha \le s < t \le \delta,$$

$$\int_{D_t^+(x_\alpha)\setminus D_s^+(x_\alpha)} |\nabla^2 u_\alpha||x - x_\alpha||\nabla u_\alpha| dx \le C \int_{D_{2t}^+(x_\alpha)\setminus D_{s/2}^+(x_\alpha)} |\nabla u_\alpha|^2 dx, \ \text{for any } \lambda_\alpha R \le s < t \le \delta.$$

Proof. We first prove the case (1). Without loss of generality, we assume $t = 2^I s$ for some positive integer I. Under the "one bubble" assumption, (3.9) holds which implies

$$|x - x'_\alpha||\nabla u_\alpha(x)| \le C \int_{D_{2^{i+1}s}^+(x'_\alpha)\setminus D_{2^{i-2}s}^+(x'_\alpha)} |\nabla u_\alpha|^2 dx, \ \ 2^{i-1}s \le |x - x_\alpha| \le 2^i s, \ \ 1 \le i \le I,$$

and

$$\int_{D_{2^i s}^+(x'_\alpha)\setminus D_{2^{i-1}s}^+(x'_\alpha)} |\nabla^2 u_\alpha|^2 dx \le C \int_{D_{2^{i+1}s}^+(x'_\alpha)\setminus D_{2^{i-2}s}^+(x'_\alpha)} |\nabla u_\alpha|^2 dx, \ \ 1 \le i \le I$$

by Lemma 2.4.

Then we get

$$\begin{aligned}
\int_{D_t^+(x'_\alpha)\setminus D_s^+(x'_\alpha)} |\nabla^2 u_\alpha||x - x'_\alpha||\nabla u_\alpha| dx &= \sum_{i=1}^{I} \int_{D_{2^i s}^+(x'_\alpha)\setminus D_{2^{i-1}s}^+(x'_\alpha)} |\nabla^2 u_\alpha||x - x'_\alpha||\nabla u_\alpha| dx \\
&\le \sum_{i=1}^{I} \sup_{2^{i-1}s\le|x-x_\alpha|\le 2^i s} |x - x'_\alpha||\nabla u_\alpha| \int_{D_{2^i s}^+(x'_\alpha)\setminus D_{2^{i-1}s}^+(x'_\alpha)} |\nabla^2 u_\alpha| dx \\
&\le C \sum_{i=1}^{I} \int_{D_{2^{i+1}s}^+(x'_\alpha)\setminus D_{2^{i-2}s}^+(x'_\alpha)} |\nabla u_\alpha|^2 dx \\
&= C \int_{D_{2t}^+(x'_\alpha)\setminus D_{s/2}^+(x'_\alpha)} |\nabla u_\alpha|^2 dx.
\end{aligned}$$

With the help of Lemma 2.3, the proof of case (2) is similar. □

Lemma 3.3. *Under the assumption of Lemma 3.2, there hold:*

(1) *If* $\lim_{\alpha\searrow 1} \frac{d_\alpha}{\lambda_\alpha} = a < \infty$, *then*

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{\alpha\searrow 1} \int_{D_\delta^+(x'_\alpha)\setminus D_{R\lambda_\alpha}^+(x'_\alpha)} |x - x'_\alpha|^{-2} \left|\frac{\partial u_\alpha}{\partial \theta'}\right|^2 dx = 0;$$

(2) *If* $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$, *then*

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{\alpha\searrow 1}\left(\int_{D^+_\delta(x'_\alpha)\setminus D^+_{2d_\alpha}(x'_\alpha)}|x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2dx+\int_{D_{d_\alpha}(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)}|x-x_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta}|^2dx\right)=0.$$

Here, (r,θ) *and* (r',θ') *are the polar coordinates around the points* x_α *and* x'_α *respectively.*

Proof. We divide the proof into two steps.

Step 1: We prove the lemma for **Case 1**, *i.e.*, $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=a<\infty$.

Firstly, it is easy to see that for any $\lambda_\alpha R\le t\le\delta$, there holds

$$D^+_{2t}(x'_\alpha)\setminus D^+_t(x'_\alpha)\subset D^+_{4t}(x_\alpha)\setminus D^+_{t/2}(x_\alpha),$$

when $\frac{1}{\alpha-1}$ and R are big enough.

By assumption (3.9), we have

$$\int_{D^+_{2t}(x'_n)\setminus D^+_t(x'_n)}|\nabla u_n|^2dx\le\epsilon^2 \text{ for any } t\in(\lambda_\alpha R,\delta). \tag{3.10}$$

According to the small energy regularity results Lemma 2.3 and Lemma 2.4, we obtain

$$Osc_{D^+_{2t}(x'_\alpha)\setminus D^+_t(x'_\alpha)}u_\alpha\le C\|\nabla u_\alpha\|_{L^2(D^+_{4t}(x'_\alpha)\setminus D^+_{t/2}(x'_\alpha))} \tag{3.11}$$

for any $t\in(\lambda_\alpha R,\delta)$. Denote $\Omega:=D^+_\delta(x'_\alpha)\setminus D_{\lambda_\alpha R}(x'_\alpha)$, then $u_\alpha(\Omega)\subset K_{\delta_0}$ and we can extend the definition of u_α to the domain $\widehat{\Omega}:=D_\delta(x'_\alpha)\setminus D_{\lambda_\alpha R}(x'_\alpha)$ by defining $\widehat{u_\alpha}$ as (2.1). Then $\widehat{u_\alpha}\in W^{2,\infty}(\widehat{\Omega})$ and satisfies equation (2.2).

Define

$$\widehat{u_\alpha^*}(r'):=\frac{1}{2\pi r'}\int_{\partial D_{r'}(x'_\alpha)}\widehat{u_\alpha}.$$

Then by (3.11), we have

$$\begin{aligned}\|\widehat{u_\alpha}(x)-\widehat{u_\alpha^*}(x)\|_{L^\infty(\widehat{\Omega})}&\le\sup_{\lambda_\alpha R\le t\le\delta}\|\widehat{u_\alpha}(x)-\widehat{u_\alpha^*}(x)\|_{L^\infty(D_{2t}(x'_\alpha)\setminus D_t(x'_\alpha))}\\&\le C(1+\|D\sigma\|_{L^\infty})\sup_{\lambda_\alpha R\le t\le\delta}Osc_{D^+_{2t}(x'_\alpha)\setminus D^+_t(x'_\alpha)}u_\alpha\le C(N)\epsilon.\end{aligned}$$

On the one hand, by Jensen's inequality, we have

$$\begin{aligned}\int_{\widehat{\Omega}}\nabla\widehat{u_\alpha}\nabla(\widehat{u_\alpha}-\widehat{u_\alpha^*})dx&=\int_{\widehat{\Omega}}|\nabla\widehat{u_\alpha}|^2dx-\int_{\widehat{\Omega}}\frac{\partial\widehat{u_\alpha}}{\partial r'}\frac{\partial\widehat{u_\alpha^*}}{\partial r'}dx\\&\ge\int_{\widehat{\Omega}}|\nabla\widehat{u_\alpha}|^2dx-(\int_{\widehat{\Omega}}|\frac{\partial\widehat{u_\alpha}}{\partial r'}|^2dx)^{1/2}(\int_{\widehat{\Omega}}|\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial\widehat{u_\alpha}}{\partial r'}(r',\theta)d\theta'|^2dx)^{1/2}\\&\ge\int_{\widehat{\Omega}}|\nabla\widehat{u_\alpha}|^2dx-\int_{\widehat{\Omega}}|\frac{\partial\widehat{u_\alpha}}{\partial r'}|^2dx=\int_{\widehat{\Omega}}|x-x'_\alpha|^{-2}|\frac{\partial\widehat{u_\alpha}}{\partial\theta'}|^2dx.\end{aligned}$$

On the other hand, using equation (2.2), we get

$$\begin{aligned}-\int_{\widehat{\Omega}}\Delta\widehat{u_\alpha}(\widehat{u_\alpha}-\widehat{u_\alpha^*})dx&\le\int_{\widehat{\Omega}}|\Upsilon_{\widehat{u_\alpha}}(\nabla\widehat{u_\alpha},\nabla\widehat{u_\alpha})+\widehat{F}(\nabla^2\widehat{u_\alpha},\nabla\widehat{u_\alpha})||\widehat{u_\alpha}-\widehat{u_\alpha^*}|dx\\&\le C\epsilon\int_{\widehat{\Omega}}|\nabla\widehat{u_\alpha}|^2dx+C\int_{\widehat{\Omega}}|\widehat{F}(\nabla^2\widehat{u_\alpha},\nabla\widehat{u_\alpha})||\widehat{u_\alpha}-\widehat{u_\alpha^*}|dx.\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{\widehat{\Omega}} |x - x'_\alpha|^{-2} |\frac{\partial \widehat{u_\alpha}}{\partial \theta'}|^2 dx \\
\le& \int_{\partial(\widehat{\Omega})} \frac{\partial \widehat{u_n}}{\partial n} (\widehat{u_\alpha} - \widehat{u_\alpha^*}) + C\epsilon \int_{\widehat{\Omega}} |\nabla \widehat{u_\alpha}|^2 dx + C \int_{\widehat{\Omega}} |\widehat{F}(\nabla^2 \widehat{u_\alpha}, \nabla \widehat{u_\alpha})| |\widehat{u_\alpha} - \widehat{u_\alpha^*}| dx.
\end{aligned} \tag{3.12}$$

By the definition of $\widehat{u_n}$ (see (2.1)), we obtain

$$\begin{aligned}
\int_{\widehat{\Omega}} |x - x'_\alpha|^{-2} |\frac{\partial \widehat{u_\alpha}}{\partial \theta'}|^2 dx &= \int_{\Omega} |x - x'_\alpha|^{-2} |\frac{\partial \widehat{u_\alpha}}{\partial \theta'}|^2 dx + \int_{\widehat{\Omega} \setminus \Omega} |x - x'_\alpha|^{-2} |D\sigma \cdot \frac{\partial u_\alpha(\rho(x))}{\partial \theta'}|^2 dx \\
&= \int_{\Omega} |x - x'_\alpha|^{-2} |\frac{\partial u_\alpha}{\partial \theta'}|^2 dx + \int_{\Omega} |x - x'_\alpha|^{-2} |D\sigma \cdot \frac{\partial u_\alpha(x)}{\partial \theta'}|^2 dx.
\end{aligned}$$

Direct computations yields that

$$\begin{aligned}
|D\sigma \cdot \frac{\partial u_\alpha(x)}{\partial \theta'}|^2 &= \langle D\sigma \cdot \frac{\partial u_\alpha(x)}{\partial \theta'}, D\sigma \cdot \frac{\partial u_\alpha(x)}{\partial \theta'} \rangle = \langle D\sigma|_{\sigma(u_\alpha(x))} D\sigma|_{u_\alpha(x)} \cdot \frac{\partial u_\alpha(x)}{\partial \theta'}, \frac{\partial u_\alpha(x)}{\partial \theta'} \rangle \\
&= \langle (D\sigma|_{\sigma(u_\alpha(x))} D\sigma|_{u_\alpha(x)} - Id) \frac{\partial u_\alpha(x)}{\partial \theta'}, \frac{\partial u_\alpha(x)}{\partial \theta'} \rangle + |\frac{\partial u_\alpha(x)}{\partial \theta'}|^2.
\end{aligned}$$

Noting that $D\sigma|_{\sigma(y)} D\sigma|_y = Id$ when $y \in K$, by the continuity of eigenvalues of $D\sigma|_{\sigma(y)} D\sigma|_y$, we have that for any $\delta' > 0$, there exists a constant $\delta_1 = \delta_1(\delta') > 0$, such that for any $y \in K_{\delta_1}$ and $\xi \in T_y N$, there holds

$$\langle D\sigma|_{\sigma(y)} D\sigma|_y \xi, \xi \rangle \le (1 + \delta') |\xi|^2.$$

By (3.11), we have $\|dist(u_\alpha, K)\|_{L^\infty(\Omega)} \le C\epsilon$. Thus, for any $\delta' > 0$, $\xi \in T_{u_\alpha(x)} N$, there holds

$$|\langle (D\sigma|_{\sigma(u_\alpha(x))} D\sigma|_{u_\alpha(x)} - Id) \xi, \xi \rangle| \le \delta' |\xi|^2 \tag{3.13}$$

when $\epsilon > 0$ is small enough.

Thus, we have

$$\begin{aligned}
(2 - \delta') \int_{\Omega} |x - x'_\alpha|^{-2} |\frac{\partial u_\alpha}{\partial \theta'}|^2 dx &\le \int_{\partial(\widehat{\Omega})} \frac{\partial \widehat{u_\alpha}}{\partial \vec{n}} (\widehat{u_\alpha} - \widehat{u_\alpha^*}) \\
&\quad + C\epsilon \int_{\widehat{\Omega}} |\nabla \widehat{u_\alpha}|^2 dx + C \int_{\widehat{\Omega}} |\widehat{F}(\nabla^2 \widehat{u_\alpha}, \nabla \widehat{u_\alpha})| |\widehat{u_\alpha} - \widehat{u_\alpha^*}| dx.
\end{aligned} \tag{3.14}$$

As for the boundary term, by trace theory, we have

$$\begin{aligned}
\int_{\partial D_\delta(x'_\alpha)} \frac{\partial \widehat{u_\alpha}}{\partial \vec{n}} (\widehat{u_\alpha} - \widehat{u_\alpha^*}) &\le C\epsilon \int_{\partial^+ D_\delta(x'_\alpha)} |\nabla u_\alpha| \\
&\le C\epsilon \left(\|\nabla u_\alpha\|_{L^2(D^+_{2\delta}(x'_\alpha) \setminus D^+_{\frac{1}{2}\delta}(x'_\alpha))} + \delta \|\nabla^2 u_\alpha\|_{L^2(D^+_{2\delta}(x'_\alpha) \setminus D^+_{\frac{1}{2}\delta}(x'_\alpha))} \right) \\
&\le C\epsilon \|\nabla u_\alpha\|_{L^2(D^+_{4\delta}(x'_\alpha) \setminus D^+_{\frac{1}{8}\delta}(x'_\alpha))} \le C\epsilon,
\end{aligned}$$

where the last second inequality can be derived from Lemma 2.3 and Lemma 2.4.

Also, there holds

$$\int_{\partial D_{\lambda_\alpha R}(x'_\alpha)} \frac{\partial \widehat{u_n}}{\partial n} (\widehat{u_n} - \widehat{u_n^*}) \le C\epsilon.$$

Noting that

$$|\widehat{F}(\nabla^2 \widehat{u_\alpha}, \nabla \widehat{u_\alpha})| \le \begin{cases} C(\alpha - 1) |\nabla^2 u_\alpha(x)| & in\ D^+, \\ C(\alpha - 1) |\nabla^2 u_\alpha(\rho(x))| & in\ D^-, \end{cases} \tag{3.15}$$

by Lemma 3.2, there holds

$$\int_{\widehat{\Omega}} |\widehat{F}(\nabla^2 \widehat{u_\alpha}, \nabla \widehat{u_\alpha})||\widehat{u_\alpha} - \widehat{u_\alpha^*}|dx \le C(\alpha-1) \int_\Omega |\nabla^2 u_\alpha||x - x'_\alpha||\nabla u_\alpha| dx \le C(\alpha - 1). \tag{3.16}$$

Combining these together and taking $\epsilon > 0$ sufficiently small (then $\delta' > 0$ is small), it is easy to see that (3.14) yields

$$\int_\Omega |x - x'_\alpha|^{-2} |\frac{\partial u_\alpha}{\partial \theta'}|^2 dx \le C(\alpha - 1 + \epsilon). \tag{3.17}$$

We finished the proof of this lemma in the **Case 1**.

Step 2: We prove the lemma for **Case 2**, *i.e.*, $\limsup_{\alpha \searrow 1} \frac{d_\alpha}{\lambda_\alpha} = \infty$.

The proof is similar to **Case 1**. In this case, since $\lim_{\alpha\searrow 1} d_\alpha = 0$ and $\lim_{\alpha\searrow 1} \frac{d_\alpha}{\lambda_\alpha} = \infty$, when $\frac{1}{\alpha-1}$ is large enough, it is easy to see that for any $2d_\alpha \le t \le \delta$, there holds

$$D^+_{2t}(x'_\alpha) \setminus D^+_t(x'_\alpha) \subset D^+_{4t}(x_\alpha) \setminus D^+_{t/2}(x_\alpha).$$

By assumption (3.9), we have

$$\int_{D^+_{2t}(x'_\alpha)\setminus D^+_t(x'_\alpha)} |\nabla u_\alpha|^2 dx \le \epsilon^2 \text{ for any } t \in (2d_\alpha, \delta). \tag{3.18}$$

Using the same argument as in the previous **Case 1**, we can get

$$\int_{D^+_\delta(x'_\alpha)\setminus D^+_{2d_\alpha}(x'_\alpha)} |x - x'_\alpha|^{-2} |\frac{\partial u_\alpha}{\partial \theta'}|^2 dx \le C(\alpha - 1 + \epsilon). \tag{3.19}$$

Finally, noting that $D^+_{d_\alpha}(x_\alpha) \setminus D^+_{\lambda_\alpha R}(x_\alpha) = D_{d_\alpha}(x_\alpha) \setminus D_{\lambda_\alpha R}(x_\alpha)$, by the interior blow-up analysis of α-harmonic maps (see Lemma 3.2 in [17]), there holds

$$\lim_{R\to\infty} \lim_{\alpha\searrow 1} \int_{D_{d_\alpha}(x_\alpha)\setminus D_{\lambda_\alpha R}(x_\alpha)} |x - x_\alpha|^{-2} |\frac{\partial u_\alpha}{\partial \theta}|^2 dx = 0. \tag{3.20}$$

We finished the proof of this lemma.

□

Combining Lemma 2.5 with Lemma 3.3, we have

Lemma 3.4. *Under the assumption of Lemma 3.2, there hold:*

(1) *If* $\lim_{\alpha\searrow 1} \frac{d_\alpha}{\lambda_\alpha} < \infty$, *then*

$$\lim_{\delta\to 0} \lim_{R\to\infty} \lim_{\alpha\searrow 1} \int_{D^+_\delta(x'_\alpha)\setminus D^+_{R\lambda_\alpha}(x'_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha} u_\alpha|^2)^{\alpha-1} |x - x'_\alpha|^{-2} |\frac{\partial u_\alpha}{\partial \theta'}|^2 dx = 0;$$

(2) *If* $\lim_{\alpha\searrow 1} \frac{d_\alpha}{\lambda_\alpha} = \infty$, *then*

$$\lim_{\delta\to 0} \lim_{\alpha\searrow 1} \int_{D^+_\delta(x'_\alpha)\setminus D^+_{2d_\alpha}(x'_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha} u_\alpha|^2)^{\alpha-1} |x - x'_\alpha|^{-2} |\frac{\partial u_\alpha}{\partial \theta'}|^2 dx = 0,$$

$$\lim_{R\to\infty} \lim_{\alpha\searrow 1} \int_{D_{d_\alpha}(x_\alpha)\setminus D_{R\lambda_\alpha}(x_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha} u_\alpha|^2)^{\alpha-1} |x - x_\alpha|^{-2} |\frac{\partial u_\alpha}{\partial \theta}|^2 dx = 0.$$

To precede, let $0 < t_0 \le 1$ and let us firstly consider a case that $\lim_{\alpha\searrow 1} \frac{\lambda_\alpha^{t_0}}{d_\alpha} = \infty$. Denote

$$F^+_\alpha(t) = \int_{D^+_{\lambda^t_\alpha}(x'_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha} u_\alpha|^2)^{\alpha-1} |\nabla u_\alpha|^2 dx,$$

$$F^+_{r',\alpha}(t) = \int_{D^+_{\lambda^t_\alpha}(x'_\alpha)\setminus D^+_{\lambda^{t_0}_\alpha}(x'_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha} u_\alpha|^2)^{\alpha-1} |\frac{\partial u_\alpha}{\partial r'}|^2 dx$$

and

$$F^+_{\theta',\alpha}(t) = \int_{D^+_{\lambda^t_\alpha}(x'_\alpha)\setminus D^+_{\lambda^{t_0}_\alpha}(x'_\alpha)} (\sigma_\alpha + |\nabla_{g_\alpha} u_\alpha|^2)^{\alpha-1} |x - x'_\alpha|^{-2} |\frac{\partial u_\alpha}{\partial \theta'}|^2 dx,\ 0 < t \le t_0 \le 1.$$

By Corollary 2.9, for $t \in [\epsilon, t_0]$, we have

$$(1 - \frac{1}{2\alpha})\frac{d}{dt} F^+_{r',\alpha}(t) - \frac{1}{2\alpha}\frac{d}{dt} F^+_{\theta',\alpha} = \frac{\alpha - 1}{\alpha} \log \lambda_\alpha F^+_\alpha(t) + O(\lambda^t_\alpha \log \lambda_\alpha).$$

Then

$$(1 - \frac{1}{2\alpha}) F^+_{r',\alpha}(t) - \frac{1}{2\alpha} F^+_{\theta',\alpha} = \frac{1}{2}\int_{t_0}^t [\frac{1}{\alpha} \log \lambda_\alpha^{2(\alpha-1)} F^+_\alpha(t) + O(\lambda^t_\alpha \log \lambda_\alpha)] dt. \tag{3.21}$$

On one hand, it is easy to see that the C^1 norm of $(1 - \frac{1}{2\alpha})F^+_{r',\alpha}(t) - \frac{1}{2\alpha}F^+_{\theta',\alpha}$ is uniformly bounded. On the other hand, Lemma 3.4 tells us $\{F^+_{\theta',\alpha}(t)\}$ converges to 0 uniformly on $[\epsilon, t_0]$ for any $\epsilon > 0$. Then we know the sequences $\{F^+_\alpha(t)\}$, $\{F^+_{r',\alpha}(t)\}$ and $\{F^+_{\theta',\alpha}(t)\}$ are compact in $C^0([\epsilon, t_0])$ topology for any $\epsilon > 0$. Thus, there exist two functions F^+ and $F^+_{r'}$ which belong to $C^0([\epsilon, t_0])$ such that

$$F^+_\alpha \to F^+ \quad \text{and} \quad F^+_{r',\alpha} \to F^+_{r'} \quad \text{in } C^0([\epsilon, t_0])$$

as $\alpha \searrow 1$.

According to Lemma 3.4, it is easy to see that $F^+_{\theta',\alpha} \to 0$ in $C^0([\epsilon, t_0])$.

To proceed, let us recall some notations concerning our new quantity mentioned in the introduction.

If $\lim_{\alpha\searrow 1} \frac{d_\alpha}{\lambda_\alpha} = \infty$, set

$$d_\alpha = \lambda_\alpha^{s_\alpha},$$

where $0 < s_\alpha < 1$. By passing to subsequences, we can assume that the following limit exists:

$$\lim_{\alpha\searrow 1} s_\alpha = s_0 \in [0, 1]. \tag{3.22}$$

Lemma 3.5. *If $\lim_{\alpha\searrow 1} \frac{\lambda_\alpha^{t_0}}{d_\alpha} = \infty$ for some $0 < t_0 < 1$, then the functionals $F^+(t)$ and $F^+_{r'}(t)$ satisfy the following relation*

$$F^+_{r'}(t) = \mu^{t_0 - t} F^+(t_0) - F^+(t_0) \ for\ any\ 0 < t \le t_0 < 1. \tag{3.23}$$

Moreover, we have:

(1) *If $\lim_{\alpha\searrow 1} \frac{d_\alpha}{\lambda_\alpha} = a$, then*

$$\lim_{t_0 \to 1^-} F^+(t_0) = \Lambda := \mu E(w^1(x)), \tag{3.24}$$

where $\mu := \lim_{\alpha\searrow 1}(\lambda_\alpha)^{2-2\alpha}$.

(2) *If* $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$ *and* $s_0\in(0,1]$, *then*

$$\lim_{t_0\to s_0^-}F^+(t_0)=\mu^{1-s_0}\Lambda. \tag{3.25}$$

Proof. Letting $\alpha\searrow 1$ in (3.21), we get

$$F^+_{r'}(t)=-\log\mu\int_{t_0}^{t}F^+(s)ds\ for\ any\ 0<t<t_0. \tag{3.26}$$

Since

$$F^+_\alpha(t)=F^+_{r',\alpha}(t)+F^+_{\theta',\alpha}(t)+F^+_\alpha(t_0),$$

letting $\alpha\searrow 1$ yields

$$F^+(t)=F^+_{r'}(t)+F^+(t_0).$$

Then,

$$F^+_{r'}(t)=-\log\mu\int_{t_0}^{t}F^+(s)ds=-\log\mu\int_{t_0}^{t}\left(F^+_{r'}(s)+F^+(t_0)\right)ds,$$

which implies $F^+_{r'}(t)\in C^1$ and

$$\frac{d}{dt}F^+_{r'}(t)=-\log\mu(F^+_{r'}(t)+F^+(t_0)).$$

Thus, there holds

$$F^+_{r'}(t)=\mu^{t_0-t}F^+(t_0)-F^+(t_0).$$

For equation (3.24), i.e., in the case of $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=a$, by Corollary 2.9, integrating (2.14) from $\lambda_\alpha R$ to $\lambda_\alpha^{t_0}$, we have

$$\begin{aligned}
&F^+_\alpha(t_0)-\int_{D^+_{\lambda_\alpha R}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx\\
&\leq C\int_{D^+_{\lambda_\alpha^{t_0}}(x'_\alpha)\setminus D^+_{\lambda_\alpha R}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2dx+C\int_{\lambda_\alpha R}^{\lambda_\alpha^{t_0}}\frac{\alpha-1}{r}dr+C(\lambda_\alpha^{t_0}-\lambda_\alpha R)\\
&\leq C\int_{D^+_{\lambda_\alpha^{t_0}}(x'_\alpha)\setminus D^+_{\lambda_\alpha R}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2dx\\
&\quad+C\left((t_0-1)(\alpha-1)\log\lambda_\alpha-(\alpha-1)\log R\right)+C(\lambda_\alpha^{t_0}-\lambda_\alpha R).
\end{aligned} \tag{3.27}$$

Combining this with Lemma 3.4 and (3.9), we get

$$\begin{aligned}
\lim_{t_0\to 1^-}F^+(t_0)&=\lim_{R\to\infty}\lim_{\alpha\searrow 1}\int_{D^+_{\lambda_\alpha R}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx\\
&=\lim_{R\to\infty}\lim_{\alpha\searrow 1}\int_{D^+_{\lambda_\alpha R}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx\\
&=\lim_{R\to\infty}\lim_{\alpha\searrow 1}\int_{D_R(0)\cap B_\alpha}(\sigma_\alpha\lambda_\alpha^2+|\nabla_{g_\alpha(x_\alpha+\lambda_\alpha x)}w_\alpha|^2)^{\alpha-1}(\lambda_\alpha)^{2-2\alpha}|\nabla w_\alpha|^2dx\\
&=\mu\lim_{R\to\infty}\int_{D_R(0)\cap\mathbb{R}^2_a}|\nabla w^1|^2dx=\mu E(w^1(x))=\Lambda.
\end{aligned}$$

For equation (3.25), i.e., in the case of $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$ and $s_0\in(0,1]$, for any $0<t_0<s_0$, we decompose $D^+_{\lambda_\alpha^{t_0}}(x'_\alpha)$ as follows

$$D^+_{\lambda_\alpha^{t_0}}(x'_\alpha)=D^+_{\lambda_\alpha^{t_0}}(x'_\alpha)\setminus D^+_{2d_\alpha}(x'_\alpha)\cup D^+_{2d_\alpha}(x'_\alpha)\setminus D^+_{d_\alpha}(x_\alpha)\cup D^+_{d_\alpha}(x_\alpha).$$

Firstly, by (3.9) and Lemma 2.5, it is easy to see that

$$\int_{D^+_{2d_\alpha}(x'_\alpha)\setminus D^+_{d_\alpha}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx\le\int_{D^+_{4d_\alpha}(x_\alpha)\setminus D^+_{d_\alpha}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx\to 0 \tag{3.28}$$

as $\alpha\searrow 1$.

Secondly, noting that $D^+_{d_\alpha}(x_\alpha)=D_{\lambda_\alpha^{s_\alpha}}(x_\alpha)$ is a interior ball, by Lemma 3.5 in [17], we get

$$\lim_{\alpha\searrow 1}\int_{D_{d_\alpha}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx=\mu^{1-s_0}\Lambda,$$

where $\Lambda=\mu E(w^1)$.

Finally, we need to show

$$\lim_{t_0\to s_0^-}\lim_{\alpha\searrow 1}\int_{D^+_{\lambda_\alpha^{t_0}}(x'_\alpha)\setminus D^+_{2d_\alpha}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx=0. \tag{3.29}$$

In fact, by Corollary 2.9, we have

$$\begin{aligned}
&\int_{D^+_{\lambda_\alpha^{t_0}}(x'_\alpha)\setminus D^+_{2d_\alpha}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx\\
&\le C\int_{D^+_{\lambda_\alpha^{t_0}(x'_\alpha)}\setminus D^+_{2d_\alpha}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2dx+C\int_{2d_\alpha}^{\lambda_\alpha^{t_0}}\frac{\alpha-1}{r}dr+C(\lambda_\alpha^{t_0}-2d_\alpha)\\
&\le C\int_{D^+_{\lambda_\alpha^{t_0}(x'_\alpha)}\setminus D^+_{2d_\alpha}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2dx\\
&\quad+C(t_0-s_\alpha-\log_{\lambda_\alpha}2)(\alpha-1)\log\lambda_\alpha+C(\lambda_\alpha^{t_0}-2d_\alpha).
\end{aligned}\tag{3.30}$$

Letting $\alpha\searrow 1$ and $t_0\to s_0^-$ in the above inequality, then (3.29) follows from Lemma 3.4 immediately. We finished the proof of this lemma. □

Lemma 3.6. *Under the assumption of Lemma 3.2, we have that for any $t\in(0,1)$, there holds*

$$\lim_{\alpha\searrow 1}\int_{D^+_{\lambda_\alpha^t}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx=\mu^{1-t}\Lambda.$$

Proof. We need to consider the following cases:

Case (a-1): $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=a$.

In this case, we know that for any $0<t_0<1$, there holds $\lim_{\alpha\searrow 1}\frac{\lambda_\alpha^{t_0}}{d_\alpha}=\infty$. Noting that

$$D^+_{\lambda_\alpha^t}(x_\alpha)\setminus D^+_{\lambda_\alpha^t}(x'_\alpha)\cup D^+_{\lambda_\alpha^t}(x'_\alpha)\setminus D^+_{\lambda_\alpha^t}(x_\alpha)\subset D^+_{\lambda_\alpha^t}(x_\alpha)\setminus D^+_{\frac12\lambda_\alpha^t}(x_\alpha)\cup D^+_{2\lambda_\alpha^t}(x_\alpha)\setminus D^+_{\lambda_\alpha^t}(x_\alpha), \tag{3.31}$$

when α is closed enough to 1, by Lemma 3.5 and (3.9), for any $t < t_0 < 1$, we have

$$\begin{aligned}
&\lim_{\alpha\searrow 1}\int_{D^+_{\lambda^t_\alpha}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx\\
&=\lim_{\alpha\searrow 1}\int_{D^+_{\lambda^t_\alpha}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx = F^+_{r'}(t)+F^+(t_0)=\mu^{t_0-t}F^+(t_0).
\end{aligned}$$

Then the conclusion of the lemma follows from letting $t_0 \nearrow 1$ and (3.24).

Case (a-2): $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$.

In this case, we have $s_0\in[0,1]$. If $0<t<s_0$, then for any $0<t_0<s_0$, there holds $\lim_{\alpha\searrow 1}\frac{\lambda_\alpha^{t_0}}{d_\alpha}=\infty$ and it is easy to see that

$$\int_{D^+_{\lambda^t_\alpha}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx = F^+_{r',\alpha}(t)+F^+_{\theta',\alpha}(t)+F^+_\alpha(t_0),$$

where $0<t\le t_0<s_0$.

Noting that (3.31) holds when α is closed enough to 1, by Lemma 3.5 and (3.9), we have

$$\begin{aligned}
&\lim_{\alpha\searrow 1}\int_{D^+_{\lambda^t_\alpha}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx\\
&=\lim_{\alpha\searrow 1}\int_{D^+_{\lambda^t_\alpha}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx = F^+_{r'}(t)+F^+(t_0)=\mu^{t_0-t}F^+(t_0).
\end{aligned}$$

Then the conclusion of the lemma follows from letting $t_0\nearrow s_0$ and (3.25).

If $s_0<t<1$, then $D^+_{\lambda^t_\alpha}(x_\alpha)=D_{\lambda^t_\alpha}(x_\alpha)$ is a interior ball and the conclusion of the lemma follows immediately from Lemma 3.5 in [17].

If $t=s_0$, we have

$$\begin{aligned}
&\lim_{\alpha\searrow 1}\int_{D^+_{\lambda^{s_0}_\alpha}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx\\
&=\lim_{s\to 0}\lim_{\alpha\searrow 1}\left(\int_{D^+_{\lambda^{s_0}_\alpha}(x_\alpha)\setminus D^+_{\lambda^{s_0+s}_\alpha}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx+\int_{D^+_{\lambda^{s_0+s}_\alpha}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx\right)\\
&=\lim_{s\to 0}\lim_{\alpha\searrow 1}\int_{D^+_{\lambda^{s_0}_\alpha}(x_\alpha)\setminus D^+_{\lambda^{s_0+s}_\alpha}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx+\mu^{1-s_0}\Lambda\\
&\le\lim_{s\to 0}\lim_{\alpha\searrow 1}\int_{D^+_{\lambda^{s_0-s}_\alpha}(x_\alpha)\setminus D^+_{\lambda^{s_0+s}_\alpha}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx+\mu^{1-s_0}\Lambda, \qquad (3.32)
\end{aligned}$$

where the last second equality follows from the interior case.

Similarly to the argument of deriving (3.30), we obtain

$$\begin{aligned}
&\lim_{s\to0}\lim_{\alpha\searrow1}\int_{D^+_{\lambda_\alpha^{s_0-s}}(x'_\alpha)\setminus D^+_{2d_\alpha}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx\\
&\le C\lim_{s\to0}\lim_{\alpha\searrow1}\int_{D^+_{\lambda_\alpha^{s_0-s}(x'_\alpha)}\setminus D^+_{2d_\alpha}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2dx\\
&\quad+C\lim_{s\to0}\lim_{\alpha\searrow1}\left\{(s_0-s-s_\alpha-\log_{\lambda_\alpha}2)(\alpha-1)\log\lambda_\alpha+\lambda_\alpha^{s_0-s}-2d_\alpha\right\}=0. \qquad (3.33)
\end{aligned}$$

Similarly, by Corollary 2.7, there holds

$$\lim_{s\to0}\lim_{\alpha\searrow1}\int_{D_{d_\alpha}(x_\alpha)\setminus D_{\lambda_\alpha^{s_0+s}}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx=0. \qquad (3.34)$$

Combining (3.32) with (3.28), (3.33), (3.34) and noting that

$$D^+_{\lambda_\alpha^{s_0}}(x_\alpha)\setminus D^+_{\lambda_\alpha^{s_0+s}}(x_\alpha)\subset D^+_{\lambda_\alpha^{s_0-s}}(x'_\alpha)\setminus D^+_{2d_\alpha}(x'_\alpha)\cup D^+_{2d_\alpha}(x'_\alpha)\setminus D^+_{d_\alpha}(x_\alpha)\cup D^+_{d_\alpha}(x_\alpha)\setminus D^+_{\lambda_\alpha^{s_0+s}}(x_\alpha),$$

the conclusion of the lemma holds for $t=s_0$. □

Now, we proceed to prove Theorem 3.1.

Proof of Theorem 3.1. We just need to consider the following two cases:

Case (b-1): If $\lim_{\alpha\searrow1}\frac{d_\alpha}{\lambda_\alpha}=a$ or $\lim_{\alpha\searrow1}\frac{d_\alpha}{\lambda_\alpha}=\infty$ and $s_0=0$.

First, we assume $\lim_{\alpha\searrow1}\frac{d_\alpha}{\lambda_\alpha}=\infty$ and $s_0=0$. By Corollary 2.9, we obtain

$$\begin{aligned}
&\int_{D^+_{2\delta}(x'_\alpha)\setminus D^+_{2d_\alpha}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2\\
&\le C\int_{D^+_{2\delta}(x'_\alpha)\setminus D^+_{2d_\alpha}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2+C\int_{2d_\alpha}^{2\delta}\frac{\alpha-1}{r}dr+C(2\delta-2d_\alpha). \qquad (3.35)
\end{aligned}$$

Similarly, by Corollary 2.7, for any $0<t<1$, there holds

$$\begin{aligned}
&\int_{D_{d_\alpha}(x_\alpha)\setminus D_{\lambda_\alpha^t}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2\\
&\le C\int_{D_{d_\alpha}(x_\alpha)\setminus D_{\lambda_\alpha^t}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|x-x_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta}|^2+C\int_{\lambda_\alpha^t}^{d_\alpha}\frac{\alpha-1}{r}dr+C(d_\alpha-\lambda_\alpha^t).
\end{aligned}$$

By (3.28) and Lemma 3.4, we have

$$\begin{aligned}
&\lim_{\delta\to0}\lim_{t\to0}\lim_{\alpha\searrow1}\int_{D^+_\delta(x_\alpha)\setminus D^+_{\lambda_\alpha^t}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2\\
&\le\lim_{\delta\to0}\lim_{\alpha\searrow1}\left(\int_{D^+_{2\delta}(x'_\alpha)\setminus D^+_{2d_\alpha}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2+\int_{D^+_{2d_\alpha}(x'_\alpha)\setminus D^+_{d_\alpha}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2\right)\\
&\quad+\lim_{t\to0}\lim_{\alpha\searrow1}\int_{D^+_{d_\alpha}(x_\alpha)\setminus D^+_{\lambda_\alpha^t}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2=0.
\end{aligned}$$

Similarly to deriving (3.35), it is easy to see that the above equality also holds for the case $\lim_{\alpha\searrow1}\frac{d_\alpha}{\lambda_\alpha}=a$.

Case (b-2): If $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$ and $s_0\in(0,1]$.

For any $0<t<s_0$, by Corollary 2.7, we get

$$\begin{aligned}&\int_{D^+_{2\delta}(x'_\alpha)\setminus D^+_{\frac{1}{2}\lambda^t_\alpha}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2\\ &\leq C\int_{D^+_{2\delta}(x'_\alpha)\setminus D^+_{\frac{1}{2}\lambda^t_\alpha}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2+C\int_{\frac{1}{2}\lambda^t_\alpha}^{2\delta}\frac{\alpha-1}{r}dr+C(2\delta-\frac{1}{2}\lambda^t_\alpha).\end{aligned}$$

Then,

$$\begin{aligned}&\lim_{\delta\to0}\lim_{t\to0}\lim_{\alpha\searrow1}\int_{D^+_\delta(x_\alpha)\setminus D^+_{\lambda^t_\alpha}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2\\ &\leq\lim_{\delta\to0}\lim_{t\to0}\lim_{\alpha\searrow1}\int_{D^+_{2\delta}(x'_\alpha)\setminus D^+_{\frac{1}{2}\lambda^t_\alpha}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2=0.\end{aligned}$$

Lemma 3.6 tells us that the following equality holds for both cases,

$$\lim_{t\to0}\lim_{\alpha\searrow1}\int_{D^+_{\lambda^t_\alpha}(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2=\mu\Lambda,$$

thus,

$$\lim_{\delta\to0}\lim_{t\to0}\lim_{\alpha\searrow1}\int_{D^+_\delta(x_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2=\mu\Lambda.$$

The proof of Theorem 3.1 is completed. □

Proof of Theorem 1.1. By a standard induction argument as in [6, 17], the conclusions of Theorem 1.1 follow immediately from Theorem 3.1. □

4. DETAILED NECK ANALYSIS

In this section, we shall study the asymptotic behaviors of the necks and prove our second main result Theorem 1.2.

We shall first consider the following simple model case.

Theorem 4.1. *Under the assumption of Theorem 3.1, if we assume there is only one bubble w^1, which is either a harmonic sphere or a harmonic disk with free boundary $w^1(\partial D_1(0))$ on K, let $\nu=\lim_{\alpha\searrow1}(\lambda_\alpha)^{-\sqrt{\alpha-1}}$, we have*

(1) *when $\nu=1$, the set $u_0(D^+_1(0))\cup w^1(S^2)$ or $u_0(D^+_1(0))\cup w^1(D_1(0))$ is a connected set in N;*

(2) *when $\nu\in(1,\infty)$, then*

(2-a) *if* $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$ *and* $s_0=0$*, where the quantity* s_0 *is defined as in* (3.22)*, then* w^1 *is a harmonic sphere and the set* $u_0(D_1^+(0))$ *and* $w(S^2)$ *are connected by a geodesic* Γ^2 *in* N *with length*

$$L(\Gamma^2)=\sqrt{\frac{E(w^1)}{\pi}}\log\nu;$$

(2-b) *if* $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=a<\infty$ *or* $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$ *and* $s_0=1$*, then* w^1 *is a harmonic disk with free boundary* $w^1(\partial D_1(0))$ *on* K *or a harmonic sphere respectively and the set* $u_0(D_1^+(0))$ *and* $w(S^2)$ *are connected by a geodesic-like curve* Γ^1 *in* K *with length*

$$L(\Gamma^1)=\sqrt{\frac{2E(w^1)}{\pi}}\log\nu;$$

Here, Γ^1 *is defined by* ω^1 *satisfying the following equation*

$$\frac{d^2\omega^1}{ds^2}+A(\omega^1)(\frac{d\omega^1}{ds},\frac{d\omega^1}{ds})+\frac{1}{2}D^2\sigma(\omega^1)(\frac{d\omega^1}{ds},\frac{d\omega^1}{ds})=0.$$

(2-c) *if* $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$ *and* $s_0\in(0,1)$*, then* w^1 *is a harmonic sphere and the set* $u_0(D_1^+(0))$ *and* $w(S^2)$ *are connected by a curve* $\Gamma=\Gamma^1\cup\Gamma^2$ *where* Γ^2 *is a geodesic in* N *with length*

$$L(\Gamma^2)=(1-s_0)\sqrt{\frac{E(w^1)}{\pi}}\log\nu,$$

and Γ^1 *is a geodesic-like curve in* K *as in (2-b) with length*

$$L(\Gamma^1)=s_0\sqrt{\frac{2E(w^1)}{\pi}}\log\nu.$$

Moreover, Γ *is a continuous curve,* Γ^1 *and* Γ^2 *are smooth curves which are connected at the point* $y\in K$ *defined by*

$$y=\lim_{\alpha\searrow 1}u_\alpha(\partial D_{d_\alpha}(x_\alpha)).$$

(3) *when* $\nu=\infty$*, the neck contains at least an infinite length curve which is either a geodesic in* N *or a geodesic-like curve in* K *as in (2-b);*

In particular, if $K\subset N$ *is a totally geodesic submanifold, then the geodesic-like curves in* K *in (2) and (3) are geodesics in* N*.*

To proceed, we make the following observation:

Lemma 4.2. *Under the assumption of Theorem 4.1, we have:*

$$\int_{D_\delta^+(x'_\alpha)}|\nabla^2 u_\alpha||x-x'_\alpha||\nabla u_\alpha|dx\le C(1+\frac{d_\alpha}{\lambda_\alpha})\Lambda_1,$$

when $\delta>0$ *and* $\alpha-1>0$ *are sufficiently small, where* $C>0$ *is a constant independent of* α*.*

Proof. Firstly, we prove the lemma for the case of $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=a$.

On one hand, by Lemma 3.2, we have

$$\int_{D_\delta^+(x'_\alpha)\setminus D^+_{\lambda_\alpha R}(x'_\alpha)}|\nabla^2 u_\alpha|r'\frac{\partial u_\alpha}{\partial r'}dx\le C\int_{D^+_{2\delta}(x'_\alpha)\setminus D^+_{\frac12\lambda_\alpha R}(x'_\alpha)}|\nabla u_\alpha|^2dx.$$

On the other hand, under the "one bubble" assumption, $w_\alpha(x) = u_\alpha(x_\alpha + \lambda_\alpha x)$ has no energy concentration points, by Lemma 2.4, we get

$$\begin{aligned}\lim_{\alpha\searrow 1}\int_{D^+_{\lambda_\alpha R}(x'_\alpha)}|\nabla^2 u_\alpha| r' \frac{\partial u_\alpha}{\partial r'}dx &\le \lim_{\alpha\searrow 1}\lambda_\alpha R\int_{D^+_{2\lambda_\alpha R}(x_\alpha)}|\nabla^2 u_\alpha||\nabla u_\alpha|dx = \lim_{\alpha\searrow 1} R\int_{D_{2R}(0)\cap B_\alpha}|\nabla^2 w_\alpha||\nabla w_\alpha|dx\\ &\le \lim_{\alpha\searrow 1} R(\int_{D_{2R}(0)\cap B_\alpha}|\nabla^2 w_\alpha|^2dx)^{1/2}(\int_{D_{2R}(0)\cap B_\alpha}|\nabla w_\alpha|^2dx)^{1/2}\\ &\le C\lim_{\alpha\searrow 1}\int_{D^+_{4R}(0)\cap B_\alpha}|\nabla w_\alpha|^2dx \le C\Lambda_1.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\int_{D^+_\delta(x'_\alpha)}|\nabla^2 u_\alpha| r'\frac{\partial u_\alpha}{\partial r'}dx &= \int_{D^+_\delta(x'_\alpha)\setminus D^+_{\lambda_\alpha R}(x'_\alpha)}|\nabla^2 u_\alpha| r'\frac{\partial u_\alpha}{\partial r'}dx + \int_{D^+_{\lambda_\alpha R}(x'_\alpha)}|\nabla^2 u_\alpha| r'\frac{\partial u_\alpha}{\partial r'}dx\\ &\le \int_{D^+_{2\delta}(x'_\alpha)\setminus D^+_{\frac{1}{2}\lambda_\alpha R}(x'_\alpha)}|\nabla u_\alpha|^2dx + C\Lambda_1 \le C\Lambda_1,\end{aligned}$$

and the conclusion of the lemma follows immediately.

Secondly, if $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha} = \infty$, we decompose the domain $D^+_\delta(x'_\alpha)$ as follows:

$$D^+_\delta(x'_\alpha) = D^+_\delta(x'_\alpha)\setminus D^+_{4d_\alpha}(x'_\alpha)\cup D^+_{4d_\alpha}(x'_\alpha)\setminus D^+_{\lambda_\alpha R}(x_\alpha)\cup D_{\lambda_\alpha R}(x_\alpha)$$

By Lemma 3.2, we have

$$\int_{D^+_\delta(x'_\alpha)\setminus D^+_{4d_\alpha}(x'_\alpha)}|\nabla^2 u_\alpha||x - x'_\alpha||\nabla u_\alpha|dx \le C\int_{D^+_{2\delta}(x'_\alpha)\setminus D^+_{2d_\alpha}(x'_\alpha)}|\nabla u_\alpha|^2dx$$

and

$$\begin{aligned}\int_{D^+_{4d_\alpha}(x'_\alpha)\setminus D^+_{\lambda_\alpha R}(x_\alpha)}|\nabla^2 u_\alpha||x - x'_\alpha||\nabla u_\alpha|dx &\le C\frac{d_\alpha}{\lambda_\alpha}\int_{D^+_{5d_\alpha}(x_\alpha)\setminus D^+_{\lambda_\alpha R}(x_\alpha)}|\nabla^2 u_\alpha||x - x_\alpha||\nabla u_\alpha|dx\\ &\le C\frac{d_\alpha}{\lambda_\alpha}\int_{D^+_{10d_\alpha}(x_\alpha)\setminus D^+_{\frac{1}{2}\lambda_\alpha R}(x_\alpha)}|\nabla u_\alpha|^2dx.\end{aligned}$$

Lastly,

$$\begin{aligned}\int_{D_{\lambda_\alpha R}(x_\alpha)}|\nabla^2 u_\alpha||x - x'_\alpha||\nabla u_\alpha|dx &\le Cd_\alpha\int_{D_{\lambda_\alpha R}(x_\alpha)}|\nabla^2 u_\alpha||\nabla u_\alpha|dx\\ &= C\frac{d_\alpha}{\lambda_\alpha}\int_{D_R(0)}|\nabla^2 w_\alpha||\nabla w_\alpha|dx\\ &\le C\frac{d_\alpha}{\lambda_\alpha}(\int_{D_R(0)}|\nabla^2 w_\alpha|^2dx)^{1/2}(\int_{D_R(0)}|\nabla w_\alpha|^2dx)^{1/2}\\ &\le C\frac{d_\alpha}{\lambda_\alpha}\int_{D^+_{2R}(0)}|\nabla w_\alpha|^2dx \le C\frac{d_\alpha}{\lambda_\alpha}\Lambda_1.\end{aligned}$$

Then we proved the lemma. □

Denote

$$H^+(r) := -\int_{D^+_r(x'_\alpha)}\frac{\nabla|\nabla_{g_\alpha}u_\alpha|^2\nabla u_\alpha}{\sigma_\alpha + |\nabla_{g_\alpha}u_\alpha|^2}(x - x'_\alpha)\cdot\nabla u_\alpha dx.$$

We shall consider some simpler situations in Theorem 4.1.

Proof of Theorem 4.1 when $\nu = 1$ and

$$\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha} = a < \infty \ \ or \ \lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha} = \infty \ and \ \liminf_{\alpha\searrow 1} -\sqrt{1+|H^+(2d_\alpha)|}\sqrt{\alpha-1}s_\alpha\log\lambda_\alpha = 0.$$

Since $\nu = 1$, then $\mu = 1$. By the energy identity

$$\lim_{\delta\to 0}\lim_{\alpha\searrow 1}\int_{D^+_\delta(x_\alpha)}(\sigma_\alpha + |\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx = \Lambda = \lim_{\delta\to 0}\lim_{\alpha\searrow 1}\int_{D^+_{\lambda_\alpha R}(x_\alpha)}(\sigma_\alpha + |\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx,$$

we have

$$\lim_{\delta\to 0}\lim_{\alpha\searrow 1}\int_{D^+_\delta(x_\alpha)\setminus D^+_{\lambda_\alpha R}(x_\alpha)}(\sigma_\alpha + |\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx = 0. \tag{4.1}$$

We divide the proof into two steps.

Step 1: We prove the lemma for the case that $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha} = a < \infty$.

Without loss of generality, we may assume $\delta = 2^{k_\alpha}\lambda_\alpha R$ for some positive integer k_α which goes to infinity as $\alpha \searrow 1$. Set

$$Q(t) := D^+_{2^{t_0+t}\lambda_\alpha R}(x'_\alpha)\setminus D^+_{2^{t_0-t}\lambda_\alpha R}(x'_\alpha) \quad and \quad \widehat{Q}(t) := D_{2^{t_0+t}\lambda_\alpha R}(x_\alpha)\setminus D_{2^{t_0-t}\lambda_\alpha R}(x_\alpha)$$

and

$$f(t) := \int_{Q(t)}|\nabla u_\alpha|^2dx$$

where $1 \le t_0 \le k_\alpha$ and $0 \le t \le \min\{t_0, k_\alpha - t_0\}$. We shall derive the energy decay on the neck domains in a similar manner as in [5].

By the computations in Lemma 3.3, similarly to deriving (3.12), we have

$$\begin{aligned}\int_{\widehat{Q}(t)}|\nabla\widehat{u_\alpha}|^2dx \le& \int_{\widehat{Q}(t)}|\frac{\partial\widehat{u_\alpha}}{\partial r'}|^2dx + \int_{\partial\widehat{Q}(t)}\frac{\partial\widehat{u_\alpha}}{\partial n}(\widehat{u_\alpha} - \widehat{u_\alpha^*}) + C\epsilon\int_{\widehat{Q}(t)}|\nabla\widehat{u_\alpha}|^2dx\\ &+ C\int_{\widehat{Q}(t)}|\widehat{F}(\nabla^2\widehat{u_\alpha},\nabla\widehat{u_\alpha})||\widehat{u_\alpha} - \widehat{u_\alpha^*}|dx.\end{aligned} \tag{4.2}$$

By the definition of $\widehat{u_\alpha}$ and computations in Lemma 3.3, there hold

$$\int_{\widehat{Q}(t)}|\frac{\partial\widehat{u_\alpha}}{\partial r'}|^2dx \le (2+\delta')\int_{Q(t)}|\frac{\partial u_\alpha}{\partial r'}|^2dx \tag{4.3}$$

and

$$(2-\delta')\int_{Q(t)}|\nabla u_\alpha|^2dx \le \int_{\widehat{Q}(t)}|\nabla\widehat{u_\alpha}|^2dx \le (2+\delta')\int_{Q(t)}|\nabla u_\alpha|^2dx. \tag{4.4}$$

According to Lemma 2.11, we have

$$\int_{\partial^+D^+_t(x'_\alpha)}|\frac{\partial u_\alpha}{\partial r'}|^2 \le \frac{1}{2}\int_{\partial^+D^+_t(x'_\alpha)}|\nabla u_\alpha|^2 + \frac{\alpha-1}{t}H^+(t). \tag{4.5}$$

By Lemma 4.2, we have $\sup_{t\le\delta}|H^+(t)|\le C$. Integrating (4.5) over $[2^{t_0-t}\lambda_\alpha R, 2^{t_0+t}\lambda_\alpha R]$, we obtain

$$\begin{aligned}\int_{Q(t)}|\frac{\partial u_\alpha}{\partial r'}|^2 &\le \frac{1}{2}\int_{Q(t)}|\nabla u_\alpha|^2 + C\int_{2^{t_0-t}\lambda_\alpha R}^{2^{t_0+t}\lambda_\alpha R}\frac{\alpha-1}{t}dt\\ &\le \frac{1}{2}\int_{Q(t)}|\nabla u_\alpha|^2 + C(\alpha-1)t.\end{aligned}\tag{4.6}$$

Combining (4.2), (4.3), (4.4), (4.6) with (3.16), we have

$$\left(1-\frac{3}{2}\delta' - C\epsilon\right)\int_{Q(t)}|\nabla u_\alpha|^2dx \le \int_{\partial\widehat{Q}(t)}\frac{\partial\widehat{u}_\alpha}{\partial n}(\widehat{u}_\alpha - \widehat{u}_\alpha^*) + C(\alpha-1)(t+1).\tag{4.7}$$

As for the boundary part, by Poincaré's inequality, we have

$$\begin{aligned}\int_{\partial(D_{2^{t_0+t}\lambda_\alpha R}(x_\alpha'))}\frac{\partial\widehat{u}_\alpha}{\partial n}(\widehat{u}_\alpha-\widehat{u}_\alpha^*) &\le (\int_{\partial(D_{2^{t_0+t}\lambda_\alpha R}(x_\alpha'))}|\frac{\partial\widehat{u}_\alpha}{\partial r'}|^2)^{\frac{1}{2}}(\int_{\partial(D_{2^{t_0+t}\lambda_\alpha R}(x_\alpha'))}|\widehat{u}_\alpha-\widehat{u}_\alpha^*|^2)^{\frac{1}{2}}\\ &\le (\int_{\partial(D_{2^{t_0+t}\lambda_\alpha R}(x_\alpha'))}|\frac{\partial\widehat{u}_\alpha}{\partial r'}|^2)^{\frac{1}{2}}(2^{t_0+t}\lambda_\alpha R\int_0^{2\pi}|\frac{\partial\widehat{u}_\alpha}{\partial\theta'}|^2)^{\frac{1}{2}}\\ &\le \frac{1}{2}(\int_{\partial(D_{2^{t_0+t}\lambda_\alpha R}(x_\alpha'))}|\frac{\partial\widehat{u}_\alpha}{\partial r'}|^2 + 2^{t_0+t}\lambda_\alpha R\int_0^{2\pi}|\frac{\partial\widehat{u}_\alpha}{\partial\theta'}|^2)\\ &= \frac{1}{2}2^{t_0+t}\lambda_\alpha R\int_{\partial(D_{2^{t_0+t}\lambda_\alpha R}(x_\alpha'))}|\nabla\widehat{u}_\alpha|^2\\ &\le (1+\frac{\delta'}{2})2^{t_0+t}\lambda_\alpha R\int_{\partial^+(D^+_{2^{t_0+t}\lambda_\alpha R}(x_\alpha'))}|\nabla u_\alpha|^2,\end{aligned}\tag{4.8}$$

where for the last inequality, we used (3.13).

Similarly, we get

$$\int_{\partial(D_{2^{t_0-t}\lambda_\alpha R}(x_\alpha'))}\frac{\partial\widehat{u}_\alpha}{\partial n}(\widehat{u}_\alpha-\widehat{u}_\alpha^*) \le (1+\frac{\delta'}{2})2^{t_0-t}\lambda_\alpha R\int_{\partial^+(D^+_{2^{t_0-t}\lambda_\alpha R}(x_\alpha'))}|\nabla u_\alpha|^2.\tag{4.9}$$

Taking $\epsilon>0$ small enough, then $\delta'>0$ is small and from (4.7), we have

$$f(t)\le\frac{C}{\log 2}f'(t)+C(\alpha-1)(t+1),$$

which implies

$$\left(2^{-\frac{1}{C}t}f(t)\right)'\ge -C(\alpha-1)2^{-\frac{1}{C}t}(t+1).$$

Integrating from 2 to L, we have

$$f(2)\le C2^{-\frac{1}{C}L}f(L)+C(\alpha-1).$$

Now, let $t_0=i$ and $L=L_i=\min\{i,k_\alpha-i\}$. Then we have

$$\int_{D^+_{2^{i+2}\lambda_\alpha R}(x_\alpha')\setminus D^+_{2^{i-2}\lambda_\alpha R}(x_\alpha')}|\nabla u_\alpha|^2dx\le C2^{-\frac{1}{C}L_i}\int_{D^+_{2\delta}(x_\alpha')\setminus D^+_{\frac{1}{2}\lambda_\alpha R}(x_\alpha')}|\nabla u_\alpha|^2dx+C(\alpha-1).$$

By Lemma 2.3 and Lemma 2.4, we obtain

$$Osc_{D^+_{2^{i+1}\lambda_\alpha R}(x'_\alpha)\setminus D^+_{2^{i-1}\lambda_\alpha R}(x'_\alpha)}u_\alpha \le C\left(\int_{D^+_{2^{i+2}\lambda_\alpha R}(x'_\alpha)\setminus D^+_{2^{i-2}\lambda_\alpha R}(x'_\alpha)}|\nabla u_\alpha|^2dx\right)^{\frac{1}{2}}$$

$$\le C2^{-\frac{L_i}{2C}}\left(\int_{D^+_{2\delta}(x'_\alpha)\setminus D^+_{\frac{1}{2}\lambda_\alpha R}(x'_\alpha)}|\nabla u_\alpha|^2dx\right)^{\frac{1}{2}} + C\sqrt{\alpha-1},$$

which implies

$$Osc_{D^+_\delta(x'_\alpha)\setminus D^+_{\lambda_\alpha R}(x'_\alpha)}u_\alpha \le \sum_{i=1}^{k_\alpha-1} Osc_{D^+_{2^{i+1}\lambda_\alpha R}(x'_\alpha)\setminus D^+_{2^{i-1}\lambda_\alpha R}(x'_\alpha)}u_\alpha$$

$$\le C\left(\int_{D^+_{2\delta}(x'_\alpha)\setminus D^+_{\frac{1}{2}\lambda_\alpha R}(x'_\alpha)}|\nabla u_\alpha|^2dx\right)^{\frac{1}{2}} + C\sqrt{\alpha-1}(\log\delta-\log R) - C\sqrt{\alpha-1}\log\lambda_\alpha.$$

Combining this with (3.9) and (4.1), we obtain

$$\lim_{\delta\to0}\lim_{R\to\infty}\lim_{\alpha\searrow1} Osc_{D^+_\delta(x_\alpha)\setminus D^+_{\lambda_\alpha R}(x_\alpha)}u_\alpha$$

$$\le \lim_{\delta\to0}\lim_{R\to\infty}\lim_{\alpha\searrow1}\left(Osc_{D^+_\delta(x_\alpha)\setminus D^+_{\frac{1}{2}\delta}(x_\alpha)}u_\alpha + Osc_{D^+_{\frac{1}{2}\delta}(x_\alpha)\setminus D^+_{2\lambda_\alpha R}(x_\alpha)}u_\alpha + Osc_{D^+_{2\lambda_\alpha R}(x_\alpha)\setminus D^+_{\lambda_\alpha R}(x_\alpha)}u_\alpha\right)$$

$$\le \lim_{\delta\to0}\lim_{R\to\infty}\lim_{\alpha\searrow1}\left(Osc_{D^+_\delta(x_\alpha)\setminus D^+_{\frac{1}{2}\delta}(x_\alpha)}u_\alpha + Osc_{D^+_\delta(x'_\alpha)\setminus D^+_{\lambda_\alpha R}(x'_\alpha)}u_\alpha + Osc_{D^+_{2\lambda_\alpha R}(x_\alpha)\setminus D^+_{\lambda_\alpha R}(x_\alpha)}u_\alpha\right) = 0.$$

Step 2: We prove the lemma for the case that $\limsup_{\alpha\searrow1}\frac{d_\alpha}{\lambda_\alpha} = \infty$ and

$$\liminf_{\alpha\searrow1} -\sqrt{1+|H^+(2d_\alpha)|}\sqrt{\alpha-1}s_\alpha\log\lambda_\alpha = 0.$$

In this case, we decompose the neck domain $D^+_\delta(x_\alpha)\setminus D^+_{\lambda_\alpha R}(x_\alpha)$ as follows

$$D^+_\delta(x_\alpha)\setminus D^+_{\lambda_\alpha R}(x_\alpha) = D^+_\delta(x_\alpha)\setminus D^+_{\frac{1}{2}\delta}(x_\alpha)\cup D^+_{\frac{1}{2}\delta}(x_\alpha)\setminus D^+_{4d_\alpha}(x_\alpha)\cup D^+_{4d_\alpha}(x_\alpha)\setminus D^+_{d_\alpha}(x_\alpha)\cup D^+_{d_\alpha}(x_\alpha)\setminus D^+_{\lambda_\alpha R}(x_\alpha).$$

Firstly, by (3.9), we have

$$\lim_{\delta\to0}\lim_{R\to\infty}\lim_{\alpha\searrow1}\left(Osc_{D^+_\delta(x_\alpha)\setminus D^+_{\frac{1}{2}\delta}(x_\alpha)}u_\alpha + Osc_{D^+_{4d_\alpha}(x_\alpha)\setminus D^+_{d_\alpha}(x_\alpha)}u_\alpha\right) = 0. \tag{4.10}$$

Secondly, since $D^+_{d_\alpha}(x_\alpha) = D_{d_\alpha}(x_\alpha)$ is a interior ball and $\nu = 1$, by the interior case (see the proof of Theorem 1.3 in [17]), we have

$$\lim_{R\to\infty}\lim_{\alpha\searrow1} Osc_{D^+_{d_\alpha}(x_\alpha)\setminus D^+_{\lambda_\alpha R}(x_\alpha)}u_\alpha = 0. \tag{4.11}$$

Finally, we estimate the oscillation on the domain $D^+_\delta(x'_\alpha)\setminus D^+_{3d_\alpha}(x'_\alpha)$. By Lemma 3.2, we know that for any $2d_\alpha \le t \le \delta$, there holds

$$|H^+(t) - H^+(2d_\alpha)| \le \int_{D^+_\delta(x'_\alpha)\setminus D^+_{2d_\alpha}(x'_\alpha)}|\nabla^2 u_\alpha||x-x'_\alpha||\nabla u_\alpha|dx \le C\int_{D^+_{2\delta}(x'_\alpha)\setminus D^+_{\frac{3}{2}d_\alpha}(x'_\alpha)}|\nabla u_\alpha|^2dx. \tag{4.12}$$

Similarly to the proof of **Step 1** (replace $\lambda_\alpha R$ with $3d_\alpha$) and noting that by (4.5) and Lemma 4.2, (4.6) becomes

$$\begin{aligned}\int_{Q(t)}|\frac{\partial u_\alpha}{\partial r'}|^2 &\leq \frac{1}{2}\int_{Q(t)}|\nabla u_\alpha|^2 + \int_{2^{t_0-t}3d_\alpha}^{2^{t_0+t}3d_\alpha}\frac{\alpha-1}{t}H^+(t)dt\\ &\leq \frac{1}{2}\int_{Q(t)}|\nabla u_\alpha|^2 + C(\alpha-1)(1+|H^+(2d_\alpha)|)t,\end{aligned} \tag{4.13}$$

where the last inequality follows from (4.12), then we can obtain

$$\begin{aligned}Osc_{D^+_\delta(x'_\alpha)\backslash D^+_{3d_\alpha}(x'_\alpha)}u_\alpha &\leq C\left(\int_{D^+_{2\delta}(x'_\alpha)\backslash D^+_{\frac{3}{2}d_\alpha}(x'_\alpha)}|\nabla u_\alpha|^2dx\right)^{\frac{1}{2}} + C\sqrt{1+|H^+(2d_\alpha)|}\sqrt{\alpha-1}\left(\log(2\delta)-\log(\frac{3}{2}d_\alpha)\right)\\ &= C\left(\int_{D^+_{2\delta}(x'_\alpha)\backslash D^+_{\frac{3}{2}d_\alpha}(x'_\alpha)}|\nabla u_\alpha|^2dx\right)^{\frac{1}{2}} + C\sqrt{1+|H^+(2d_\alpha)|}\sqrt{\alpha-1}\left(\log(2\delta)-\log(\frac{3}{2}\lambda_\alpha^{s_\alpha})\right)\end{aligned} \tag{4.14}$$

which implies

$$\lim_{\delta\to0}\lim_{R\to\infty}\lim_{\alpha\searrow1}Osc_{D^+_{\frac{1}{2}\delta}(x_\alpha)\backslash D^+_{4d_\alpha}(x_\alpha)}u_\alpha \leq \lim_{\delta\to0}\lim_{R\to\infty}\lim_{\alpha\searrow1}Osc_{D^+_\delta(x'_\alpha)\backslash D^+_{3d_\alpha}(x'_\alpha)}u_\alpha = 0.$$

Combining these together, we have

$$\lim_{\delta\to0}\lim_{R\to\infty}\lim_{\alpha\searrow1}Osc_{D^+_\delta(x_\alpha)\backslash D^+_{\lambda_\alpha R}(x_\alpha)}u_\alpha = 0.$$

This shows that the set $u_0(D^+_1(0))\cup w^1(S^2)$ or $u_0(D^+_1(0))\cup w^1(D_1(0))$ ia a connected set in N. □

To handle the more complicated situations in Theorem 4.1, we need a series of auxiliary lemmas.

Lemma 4.3. *Under the assumptions of Theorem 4.1, we have:*

(1) *If* $\lim_{\alpha\searrow1}\frac{d_\alpha}{\lambda_\alpha} = a$ *and* $\nu > 1$, *then for any sequence* $t_\alpha \in [t_1, t_2]$, $0 < t_1 \leq t_2 < 1$, *passing to a subsequence, for any* $R > 0$, *we have*

$$\lim_{\alpha\searrow1}\frac{1}{\alpha-1}\int_{D^+_{R\lambda_\alpha^{t_\alpha}}(x'_\alpha)\backslash D^+_{\frac{1}{R}\lambda_\alpha^{t_\alpha}}(x'_\alpha)}|x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2dx = 0. \tag{4.15}$$

(2) *If* $\lim_{\alpha\searrow1}\frac{d_\alpha}{\lambda_\alpha} = \infty$, $s_0 > 0$ *and*

$$-\liminf_{\alpha\searrow1}\sqrt{1+|H^+(2d_\alpha)|}\sqrt{\alpha-1}s_\alpha\log\lambda_\alpha > 0,$$

then for any sequence $t_\alpha \in [t_1, t_2]$, $0 < t_1 \leq t_2 < s_0$, *by passing to a subsequence, for any* $R > 0$, *we have*

$$\lim_{\alpha\searrow1}\frac{1}{(\alpha-1)(1+|H^+(\lambda_\alpha^{t_\alpha})|)}\int_{D^+_{R\lambda_\alpha^{t_\alpha}}(x'_\alpha)\backslash D^+_{\frac{1}{R}\lambda_\alpha^{t_\alpha}}(x'_\alpha)}|x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2dx = 0. \tag{4.16}$$

Proof. Without loss of generality, we prove the case (2). With the help of Lemma 4.2, the case (1) is similar to the case (2) and it is in fact easier.

Step 1: We prove that for any positive k, there holds

$$\lim_{\alpha\searrow 1}\frac{1}{(\alpha-1)(1+|H^+(\lambda_\alpha^{t_\alpha})|)}\int_{D^+_{2^k\lambda_\alpha^{t_\alpha}}(x'_\alpha)\setminus D^+_{\frac{1}{2^k}\lambda_\alpha^{t_\alpha}}(x'_\alpha)}|x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2dx\leq C, \tag{4.17}$$

where $C>0$ is a positive constant independent of k.

Firstly, we claim:

$$\lim_{\tau\to 0}\lim_{\alpha\searrow 1}\int_{D^+_{\lambda_\alpha^{t_\alpha-\tau}}(x'_\alpha)\setminus D^+_{\lambda_\alpha^{t_\alpha+\tau}}(x'_\alpha)}|\nabla u_\alpha|^2dx=0. \tag{4.18}$$

In fact, by Corollary 2.9, we have

$$\begin{aligned}&\int_{D^+_{\lambda_\alpha^{t_\alpha-\tau}}(x'_\alpha)\setminus D^+_{\lambda_\alpha^{t_\alpha+\tau}}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx\\ &\leq C\int_{D^+_{\lambda_\alpha^{t_\alpha-\tau}}(x'_\alpha)\setminus D^+_{\lambda_\alpha^{t_\alpha+\tau}}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2dx+C\tau(\alpha-1)\log\lambda_\alpha+O(\lambda_\alpha^{t_\alpha-\tau}).\end{aligned}$$

Letting $\alpha\searrow 1$ and $\tau\to 0$, then (4.18) follows immediately from Lemma 3.4.

Taking a small positive constant $\tau<\min\{t_1,s_0-t_2\}$, set $2^t\leq\lambda_\alpha^{-\tau}$ and

$$Q_2(t):=D^+_{2^t\lambda_\alpha^{t_\alpha}}(x'_\alpha)\setminus D^+_{2^{-t}\lambda_\alpha^{t_\alpha}}(x'_\alpha)\quad\textit{and}\quad\widehat{Q}_2(t):=D_{2^t\lambda_\alpha^{t_\alpha}}(x'_\alpha)\setminus D_{2^{-t}\lambda_\alpha^{t_\alpha}}(x'_\alpha).$$

Denote

$$f_2(t):=\int_{Q_2(t)}|\nabla u_\alpha|^2dx.$$

By (4.2), (4.3), (4.4) and Lemma 3.2, we have

$$\begin{aligned}(2-\delta')\int_{Q_2(t)}|\nabla u_\alpha|^2dx&\leq(2+\delta')\int_{Q_2(t)}|\frac{\partial u_\alpha}{\partial r'}|^2dx+\int_{\partial\widehat{Q}_2(t)}\frac{\partial\widehat{u_\alpha}}{\partial n}(\widehat{u_\alpha}-\widehat{u_\alpha^*})+C\epsilon\int_{Q_2(t)}|\nabla u_\alpha|^2dx\\ &\quad+C(\alpha-1)\int_{D^+_{2^{t+1}\lambda_\alpha^{t_\alpha}}(x'_\alpha)\setminus D^+_{2^{-t-1}\lambda_\alpha^{t_\alpha}}(x'_\alpha)}|\nabla u_\alpha|^2dx.\end{aligned} \tag{4.19}$$

Similar to deriving (4.8), (4.9) and (4.6), by (4.18), we have

$$(2-\delta'-C\epsilon)f_2(t)\leq(2+\delta')\int_{Q_2(t)}|\frac{\partial u_\alpha}{\partial r'}|^2dx+\frac{(1+\frac{\delta'}{2})}{\log 2}f_2'(t)+C(\alpha-1)o(\alpha,\tau) \tag{4.20}$$

where $\lim_{\tau\to 0}\lim_{\alpha\searrow 1}o(\alpha,\tau)\to 0$.

According to Lemma 2.11, we have

$$\int_{Q_2(t)}|\frac{\partial u_\alpha}{\partial r'}|^2dx=\frac{1}{2}\int_{Q_2(t)}|\nabla u_\alpha|^2dx+\int_{\frac{1}{2^t}\lambda_\alpha^{t_\alpha}}^{2^t\lambda_\alpha^{t_\alpha}}\frac{\alpha-1}{r}H^+(r)dr. \tag{4.21}$$

Noting that for any $\frac{1}{2^t}\lambda_\alpha^{t_\alpha}\leq r\leq 2^t\lambda_\alpha^{t_\alpha}$, there holds

$$\begin{aligned}|H^+(r)-H^+(\lambda_\alpha^{t_\alpha})|&\leq C\int_{D^+_{2^t\lambda_\alpha^{t_\alpha}}(x'_\alpha)\setminus D^+_{\frac{1}{2^t}\lambda_\alpha^{t_\alpha}}(x'_\alpha)}|\nabla^2u_\alpha||x-x'_\alpha||\frac{\partial u_\alpha}{\partial|x-x'_\alpha|}|dx\\ &\leq C\int_{D^+_{2^{t+1}\lambda_\alpha^{t_\alpha}}(x'_\alpha)\setminus D^+_{\frac{1}{2^{t+1}}\lambda_\alpha^{t_\alpha}}(x'_\alpha)}|\nabla u_\alpha|^2dx=o(\alpha,\tau),\end{aligned} \tag{4.22}$$

thus,

$$|\int_{\frac{1}{2^t}\lambda_\alpha^{t_\alpha}}^{2^t\lambda_\alpha^{t_\alpha}} \frac{\alpha-1}{r}H^+(r)dr - (\alpha-1)H^+(\lambda_\alpha^{t_\alpha})2t\log 2| \le C(\alpha-1)o(\alpha,\tau)t. \tag{4.23}$$

Therefore,

$$\left|\int_{Q_2(t)}|\frac{\partial u_\alpha}{\partial r'}|^2dx - \frac{1}{2}\int_{Q_2(t)}|\nabla u_\alpha|^2dx - (\alpha-1)H^+(\lambda_\alpha^{t_\alpha})2t\log 2\right| \le C(\alpha-1)o(\alpha,\tau)t. \tag{4.24}$$

Then we get

$$(1-\frac{3\delta'}{2}-C\epsilon)f_2(t) \le \frac{(1+\frac{\delta'}{2})}{\log 2}f_2'(t) + (2+\delta')(\alpha-1)H^+(\lambda_\alpha^{t_\alpha})2t\log 2 + C(\alpha-1)o(\alpha,\tau)t,$$

which implies

$$\left(2^{-\zeta t}f_2(t)\right)' \ge -(\alpha-1)H^+(\lambda_\alpha^{t_\alpha})(2\log 2)^2 2^{-\zeta t}t + C(\alpha-1)o(\alpha,\tau)2^{-\zeta t}t,$$

where

$$\zeta = \frac{1-\frac{3\delta'}{2}-C\epsilon}{1+\frac{\delta'}{2}} > 0 \tag{4.25}$$

is a constant.

Letting $2^L = \lambda_\alpha^{-\tau}$, then integrating above inequality from k to L and noting that

$$\begin{aligned}\int_k^L 2^{-\zeta t}tdt &= \frac{1}{\zeta\log 2}(k2^{-\zeta k} - L2^{-\zeta L}) + \frac{1}{\zeta\log 2}\int_k^L 2^{-\zeta t}dt\\ &\le \frac{k}{\zeta\log 2}2^{-\zeta k} + (\frac{1}{\zeta\log 2})^2 2^{-\zeta k},\end{aligned}$$

we obtain

$$f_2(k) \le 2^{\zeta k}2^{-\zeta L}f_2(L) + (\alpha-1)H^+(\lambda_\alpha^{t_\alpha})\frac{4k\log 2}{\zeta} + C(\alpha-1)\left(H^+(\lambda_\alpha^{t_\alpha}) + o(\alpha,\tau)(k+1)\right). \tag{4.26}$$

On the other hand, by (4.24), we have

$$\begin{aligned}\int_{Q_2(k)}|\frac{\partial u_\alpha}{\partial r'}|^2dx - \int_{Q_2(k)}|x-x_\alpha'|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2dx &= 2\int_{Q_2(k)}|\frac{\partial u_\alpha}{\partial r'}|^2dx - \int_{Q_2(k)}|\nabla u_\alpha|^2dx\\ &\ge (\alpha-1)H^+(\lambda_\alpha^{t_\alpha})4k\log 2 - C(\alpha-1)(\alpha,\tau)k.\end{aligned} \tag{4.27}$$

By (4.26) and (4.27), we obtain

$$\begin{aligned}&\int_{Q_2(k)}|x-x_\alpha'|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2dx\\ &\le 2^{\zeta k}\lambda_\alpha^{\zeta\tau} + (\alpha-1)H^+(\lambda_\alpha^{t_\alpha})4k\log 2(\frac{1}{\zeta}-1) + C(\alpha-1)\left(H^+(\lambda_\alpha^{t_\alpha}) + o(\alpha,\tau)(k+1)\right).\end{aligned} \tag{4.28}$$

Since

$$-\lim_{\alpha\searrow 1}\sqrt{1+|H^+(2d_\alpha)|}\sqrt{\alpha-1}s_\alpha\log\lambda_\alpha > 0,$$

it is easy to see that

$$C\sqrt{1+|H^+(\lambda_\alpha^{t_\alpha})|}\sqrt{\alpha-1} \ge \sqrt{1+|H^+(2d_\alpha)|}\sqrt{\alpha-1} \ge -\frac{C}{s_\alpha\log\lambda_\alpha},$$

where the first inequality follow from (4.12), then

$$\frac{\lambda_\alpha^{\zeta\tau}}{(\alpha-1)(1+|H^+(\lambda_\alpha^{t_\alpha})|)} \leq C\frac{\lambda_\alpha^{\zeta\tau}}{(s_\alpha \log\lambda_\alpha)^2} \to 0$$

as $\alpha \searrow 1$. Letting $\alpha \searrow 1$, $\epsilon \to 0$ and $\tau \to 0$, then $\zeta \to 1$ and we have

$$\lim_{\alpha\searrow 1}\frac{1}{(\alpha-1)(1+|H^+(\lambda_\alpha^{t_\alpha})|)}\int_{D^+_{2^k\lambda_\alpha^{t_\alpha}}(x'_\alpha)\setminus D^+_{2^{-k}\lambda_\alpha^{t_\alpha}}(x'_\alpha)} |x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2 dx \leq C, \tag{4.29}$$

where $C>0$ is independent of k.

Step 2: We prove the conclusion of the lemma.

By (4.29) and Fubini's theorem, for any $\epsilon > 0$, there exist $k_0 > 0$ which is independent of α, and $L_\alpha \in [2^{k_0}, 2^{k_0+1}]$ such that

$$\frac{1}{(\alpha-1)(1+|H^+(\lambda_\alpha^{t_\alpha})|)} L_\alpha\lambda_\alpha^{t_\alpha}\int_{\partial^+ D^+_{L_\alpha\lambda_\alpha^{t_\alpha}}(x'_\alpha)} |x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2 < \epsilon$$

and

$$\frac{1}{(\alpha-1)(1+|H^+(\lambda_\alpha^{t_\alpha})|)} \frac{1}{L_\alpha}\lambda_\alpha^{t_\alpha}\int_{\partial^+ D^+_{\frac{1}{L_\alpha}\lambda_\alpha^{t_\alpha}}(x'_\alpha)} |x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2 < \epsilon.$$

Then, by (4.8), we have

$$\begin{aligned}\int_{\partial(D_{L_\alpha\lambda_\alpha^{t_\alpha}}(x'_\alpha))}\frac{\partial\widehat{u_\alpha}}{\partial n}(\widehat{u_\alpha}-\widehat{u_\alpha^*}) &\leq (2+\delta')(L_\alpha\lambda_\alpha^{t_\alpha}\int_{\partial^+(D^+_{L_\alpha\lambda_\alpha^{t_\alpha}}(x'_\alpha))}|\frac{\partial u_\alpha}{\partial r'}|^2)^{\frac12}(\int_0^\pi|\frac{\partial u_\alpha}{\partial\theta'}|^2)^{\frac12}\\ &\leq (2+\delta')\sqrt{\epsilon(\alpha-1)(1+|H^+(\lambda_\alpha^{t_\alpha})|)}(L_\alpha\lambda_\alpha^{t_\alpha}\int_{\partial^+(D^+_{L_\alpha\lambda_\alpha^{t_\alpha}}(x'_\alpha))}|\frac{\partial u_\alpha}{\partial r'}|^2)^{\frac12}. \end{aligned}\tag{4.30}$$

By Corollary 2.9, we have

$$\begin{aligned}L_\alpha\lambda_\alpha^{t_\alpha}\int_{\partial^+(D^+_{L_\alpha\lambda_\alpha^{t_\alpha}}(x'_\alpha))}|\frac{\partial u_\alpha}{\partial r'}|^2 &\leq CL_\alpha\lambda_\alpha^{t_\alpha}\int_{\partial^+(D^+_{L_\alpha\lambda_\alpha^{t_\alpha}}(x'_\alpha))}|x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2 + C(\alpha-1) + C\lambda_\alpha^{t_\alpha}\\ &\leq C(\alpha-1)(1+|H^+(\lambda_\alpha^{t_\alpha})|).\end{aligned}$$

Thus,

$$\int_{\partial(D_{L_\alpha\lambda_\alpha^{t_\alpha}}(x'_\alpha))}\frac{\partial\widehat{u_\alpha}}{\partial n}(\widehat{u_\alpha}-\widehat{u_\alpha^*}) \leq C\sqrt{\epsilon}(\alpha-1)(1+|H^+(\lambda_\alpha^{t_\alpha})|). \tag{4.31}$$

Similarly,

$$\int_{\partial(D_{\frac{1}{L_\alpha}\lambda_\alpha^{t_\alpha}}(x'_\alpha))}\frac{\partial\widehat{u_\alpha}}{\partial n}(\widehat{u_\alpha}-\widehat{u_\alpha^*}) \leq C\sqrt{\epsilon}(\alpha-1)(1+|H^+(\lambda_\alpha^{t_\alpha})|). \tag{4.32}$$

Combining (4.31), (4.32) with (4.19), we get

$$\begin{aligned}&(2-\delta'-C\epsilon)\int_{Q_2(\frac{\log L_\alpha}{\log 2})}|\nabla u_\alpha|^2dx\\ &\leq (2+\delta')\int_{Q_2(\frac{\log L_\alpha}{\log 2})}|\frac{\partial u_\alpha}{\partial r}|^2dx + C(\alpha-1)(1+|H^+(\lambda_\alpha^{t_\alpha})|)(\sqrt{\epsilon}+o(\alpha,\tau)).\end{aligned}\tag{4.33}$$

By (4.24), we have

$$(1-\frac{3}{2}\delta'-C\epsilon)\int_{Q_2(\frac{\log L_\alpha}{\log 2})}|\nabla u_\alpha|^2dx$$
$$\le(2+\delta')(\alpha-1)H^+(\lambda_\alpha^{t_\alpha})2\log L_\alpha+C(\alpha-1)(1+|H^+(\lambda_\alpha^{t_\alpha})|)\log L_\alpha(\sqrt{\epsilon}+o(\alpha,\tau)). \tag{4.34}$$

Similarly to deriving (4.27), by (4.24), we have

$$\int_{Q_2(\frac{\log L_\alpha}{\log 2})}|\frac{\partial u_\alpha}{\partial r}|^2dx\ge\int_{Q_2(\frac{\log L_\alpha}{\log 2})}|x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta}|^2dx$$
$$+2(\alpha-1)H^+(\lambda_\alpha^{t_\alpha})2\log L_\alpha-C(\alpha-1)\log L_\alpha o(\alpha,\tau). \tag{4.35}$$

From (4.34) and (4.35), we get

$$\begin{aligned}&\frac{1}{(\alpha-1)(1+|H^+(\lambda_\alpha^{t_\alpha})|)}\int_{Q_2(\frac{\log L_\alpha}{\log 2})}|x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta}|^2dx\\&\le\left(\frac{2+\delta'}{1-\frac{3}{2}\delta'-C\epsilon}-2\right)2\log L_\alpha+C\log L_\alpha(\sqrt{\epsilon}+o(\alpha,\tau))\\&\le\delta'2\log L_\alpha+C\log L_\alpha(\sqrt{\epsilon}+o(\alpha,\tau))\end{aligned}$$

if we let $\alpha\searrow 1$ and then let $\epsilon>0$ and $\tau>0$ tend to zero, then $\delta'\to 0$ and the conclusion of the lemma follows immediately. □

Lemma 4.4. *Under the assumption of Theorem 4.1, if* $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$ *and*

$$-\liminf_{\alpha\searrow 1}\sqrt{1+|H^+(2d_\alpha)|}\sqrt{\alpha-1}s_\alpha\log\lambda_\alpha>0,$$

we have

$$\limsup_{\delta\to 0}\limsup_{\alpha\searrow 1}\sup_{4d_\alpha\le t\le\delta}|H^+(t)|\le C. \tag{4.36}$$

Proof. Under the assumption that $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$, we have two cases that $s_0\in(0,1]$ and $s_0=0$.

Step 1: We consider the case of $s_0\in(0,1]$.

For any $t_2\in(0,s_0)$, we claim:

$$\limsup_{\alpha\searrow 1}|H^+(\lambda_\alpha^{t_2})|\le C. \tag{4.37}$$

If not, we may assume $\lim_{\alpha\searrow 1}H^+(\lambda_\alpha^{t_2})=\infty$.

On one hand, by Corollary 2.9, Lemma 2.5 and Lemma 4.3, we have

$$\begin{aligned}
&\frac{1}{(\alpha-1)H^+(\lambda_\alpha^{t_2})}\int_{D^+_{2\lambda_\alpha^{t_2}}(x'_\alpha)\setminus D^+_{\lambda_\alpha^{t_2}}(x'_\alpha)}|\nabla u_\alpha|^2dx\\
&\le \frac{1}{\beta_0}\frac{1}{(\alpha-1)H^+(\lambda_\alpha^{t_2})}\int_{D^+_{2\lambda_\alpha^{t_2}}(x'_\alpha)\setminus D^+_{\lambda_\alpha^{t_2}}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx\\
&\le \frac{2\alpha}{2\alpha-1}\frac{1}{\beta_0}\frac{1}{(\alpha-1)H^+(\lambda_\alpha^{t_2})}\int_{D^+_{2\lambda_\alpha^{t_2}}(x'_\alpha)\setminus D^+_{\lambda_\alpha^{t_2}}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|x-x_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2dx\\
&\quad+\frac{1}{\beta_0}\frac{1}{H^+(\lambda_\alpha^{t_2})}\frac{2}{2\alpha-1}\int_{\lambda_\alpha^{t_2}}^{2\lambda_\alpha^{t_2}}\frac{1}{t}F^+_\alpha(\log_{\lambda_\alpha}t)dt+o(1)\to 0,
\end{aligned}$$

where we used the fact that

$$\frac{\lambda_\alpha^{t_2}}{(\alpha-1)(1+H^+(\lambda_\alpha^{t_\alpha}))}\le C\frac{\lambda_\alpha^{t_2}}{(s_\alpha\log\lambda_\alpha)^2}\to 0$$

as $\alpha\searrow 1$.

On the other hand, by Lemma 2.11, we have

$$\begin{aligned}
&\frac{1}{(\alpha-1)H^+(\lambda_\alpha^{t_2})}\int_{D^+_{2\lambda_\alpha^{t_2}}(x'_\alpha)\setminus D^+_{\lambda_\alpha^{t_2}}(x'_\alpha)}|\nabla u_\alpha|^2dx\\
&=\frac{2}{(\alpha-1)H^+(\lambda_\alpha^{t_2})}\int_{D^+_{2\lambda_\alpha^{t_2}}(x'_\alpha)\setminus D^+_{\lambda_\alpha^{t_2}}(x'_\alpha)}|x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2dx+\frac{2}{H^+(\lambda_\alpha^{t_2})}\int_{\lambda_\alpha^{t_2}}^{2\lambda_\alpha^{t_2}}\frac{1}{r}H^+(r)dr\to 2\log 2.
\end{aligned}$$

where we used (4.23). This is a contradiction.

Thus, for any $4d_\alpha\le t\le\delta$, by Lemma 3.2, we obtain

$$\begin{aligned}
|H^+(t)|&\le|H^+(t)-H^+(\lambda_\alpha^{t_2})|+|H^+(\lambda_\alpha^{t_2})|\\
&\le|H^+(\lambda_\alpha^{t_2})|+C\int_{D^+_\delta(x'_\alpha)\setminus D^+_{4d_\alpha}(x'_\alpha)}|\nabla^2u_\alpha||x-x'_\alpha||\nabla u_\alpha|dx\\
&\le|H^+(\lambda_\alpha^{t_2})|+C\int_{D^+_{2\delta}(x'_\alpha)\setminus D^+_{2d_\alpha}(x'_\alpha)}|\nabla u_\alpha|^2dx\le C.
\end{aligned}\tag{4.38}$$

Step 2: We consider the case of $s_0=0$.

Without loss of generality, we assume $\delta=2^{2k_\alpha}2d_\alpha$, where k_α is an integer which tends to infinity as $\alpha\searrow 1$. By argument in (4.38), we just need to prove that

$$\limsup_{\delta\to0}\limsup_{\alpha\searrow1}|H^+(2^{k_\alpha}2d_\alpha)|=\limsup_{\delta\to0}\limsup_{\alpha\searrow1}\left|H^+\left(\sqrt{\frac{\delta}{2d_\alpha}}\right)\right|\le C.\tag{4.39}$$

If not, we may assume $\lim_{\delta\to0}\lim_{\alpha\searrow1}H^+(2^{k_\alpha}2d_\alpha)=\infty$.

Firstly, we claim: for any $L>0$, there holds

$$\lim_{\delta\to0}\lim_{\alpha\searrow1}\frac{1}{(\alpha-1)(1+|H^+(2^{k_\alpha}2d_\alpha)|)}\int_{D^+_{L2^{k_\alpha}2d_\alpha}(x'_\alpha)\setminus D^+_{\frac{1}{L}2^{k_\alpha}2d_\alpha}(x'_\alpha)}|x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2dx=0.\tag{4.40}$$

The proof is similar to **Step 1** in the proof of Lemma 4.3. For example, the equation (4.18) becomes

$$\lim_{\delta\to 0}\lim_{\alpha\searrow 1}\int_{D^+_\delta(x'_\alpha)\setminus D^+_{2d_\alpha}(x'_\alpha)}|\nabla u_\alpha|^2dx=0.$$

Set

$$Q_2(t):=D^+_{2^t2^{k_\alpha}2d_\alpha}(x'_\alpha)\setminus D^+_{2^{-t}2^{k_\alpha}2d_\alpha}(x'_\alpha)\quad and\quad \widehat{Q_2}(t):=D_{2^t2^{k_\alpha}2d_\alpha}(x'_\alpha)\setminus D_{2^{-t}2^{k_\alpha}2d_\alpha}(x'_\alpha).$$

Denote

$$f_2(t):=\int_{Q_2(t)}|\nabla u_\alpha|^2dx,$$

where $0\le t\le k_\alpha$.

Similarly to deriving (4.28) (taking $L=k_\alpha$), we get

$$\begin{aligned}&\int_{Q_2(k)}|x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta}|^2dx\\ &\le 2^{\zeta k}(\frac{2d_\alpha}{\delta})^{\frac{\zeta}{2}}+(\alpha-1)H^+(2^{k_\alpha}2d_\alpha)4k\log 2(\frac{1}{\zeta}-1)+C(\alpha-1)\Big(H^+(2^{k_\alpha}2d_\alpha)+o(\alpha,\delta)k\Big),\end{aligned}\tag{4.41}$$

where $\zeta>0$ is the constant defined in (4.25).

Since $s_\alpha\to 0$, by (4.12), it is easy to see that

$$\frac{(d_\alpha)^{\frac{\zeta}{2}}}{(\alpha-1)(1+|H^+(2^{k_\alpha}2d_\alpha)|)}\le C\frac{(d_\alpha)^{\frac{\zeta}{2}}}{(\alpha-1)(1+|H^+(2d_\alpha)|)}\le\frac{(\lambda_\alpha)^{\frac{s_\alpha\zeta}{2}}}{(s_\alpha\log\lambda_\alpha)^2}\to 0.$$

Then, we have

$$\lim_{\delta\to 0}\lim_{\alpha\searrow 1}\frac{1}{(\alpha-1)(1+|H^+(2^{k_\alpha}2d_\alpha)|)}\int_{D^+_{2^k2^{k_\alpha}2d_\alpha}(x'_\alpha)\setminus D^+_{2^{-k}2^{k_\alpha}2d_\alpha}(x'_\alpha)}|x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2dx\le C,\tag{4.42}$$

where $C>0$ is independent of k. Then (4.40) follows from a similar argument of **Step 2** in Lemma 4.3 and the conclusion of the lemma follows from a similar argument of **Step 1** by replacing $\lambda_\alpha^{t_2}$ with $2^{k_\alpha}2d_\alpha$. □

Lemma 4.5. *Assume $\nu>1$. Let $0<t_1\le t_2<1$, then for any sequence $t_\alpha\in[t_1,t_2]$, by passing to a subsequence, for any $R>0$, we have:*

(1) *If $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=a$ or $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$ and $t_2<s_0$, then*

$$\lim_{\alpha\searrow 1}\frac{1}{\alpha-1}\int_{D^+_{R\lambda_\alpha^{t_\alpha}}(x'_\alpha)\setminus D^+_{\frac{1}{R}\lambda_\alpha^{t_\alpha}}(x'_\alpha)}|x-x'_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta'}|^2dx=0.\tag{4.43}$$

(2) *If $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$ and $s_0<t_1$, then*

$$\lim_{\alpha\searrow 1}\frac{1}{\alpha-1}\int_{D^+_{R\lambda_\alpha^{t_\alpha}}(x_\alpha)\setminus D^+_{\frac{1}{R}\lambda_\alpha^{t_\alpha}}(x_\alpha)}|x-x_\alpha|^{-2}|\frac{\partial u_\alpha}{\partial\theta}|^2dx=0.\tag{4.44}$$

Proof. For the case (1), noting that $\nu>1$ and $s_0>0$, then

$$-\lim_{\alpha\searrow 1}\sqrt{1+|H^+(2d_\alpha)|}\sqrt{\alpha-1}s_\alpha\log\lambda_\alpha>0$$

and the conclusion of the lemma is a consequence of Lemma 4.3 and Lemma 4.4.

Since $\nu>1$, the case (2) follows directly from the interior case, see Proposition 4.2 in [17]. □

Proposition 4.6. *Suppose that $\nu > 1$ and t_α is a positive number such that $0 < t_1 \le t_\alpha \le t_2 < 1$, then we have:*

(1) *If $\lim_{\alpha\searrow 1} \frac{d_\alpha}{\lambda_\alpha} = a$ or $\lim_{\alpha\searrow 1} \frac{d_\alpha}{\lambda_\alpha} = \infty$ and $t_2 < s_0$, assume $y = \lim_{\alpha\searrow 1} u_\alpha(\partial^+ D^+_{\lambda_\alpha^{t_\alpha}}(x'_\alpha))$, there holds*

$$\frac{1}{\sqrt{\alpha-1}}\left(\widehat{u}_\alpha(x'_\alpha + \lambda_\alpha^{t_\alpha} x) - \widehat{u}_\alpha(x'_\alpha + (\lambda_\alpha^{t_\alpha}, 0))\right) \to \overrightarrow{a}\log|x|$$

strongly in $C^1(D_R(0) \setminus D_{\frac{1}{R}}(0), \mathbb{R}^N)$ for any $R > 0$, where $\overrightarrow{a} \in T_y K$ is a vector in $\mathbb{R}^N$ with

$$|\overrightarrow{a}| = \mu^{1-\lim_{\alpha\searrow 1} t_\alpha}\sqrt{2\frac{E(w^1)}{\pi}}.$$

(2) *If $\lim_{\alpha\searrow 1} \frac{d_\alpha}{\lambda_\alpha} = \infty$ and $s_0 < t_1$, assume $y = \lim_{\alpha\searrow 1} u_\alpha(\partial D_{\lambda_\alpha^{t_\alpha}}(x_\alpha))$, there holds*

$$\frac{1}{\sqrt{\alpha-1}}\left(u_\alpha(x_\alpha + \lambda_\alpha^{t_\alpha} x) - u_\alpha(x_\alpha + (\lambda_\alpha^{t_\alpha}, 0))\right) \to \overrightarrow{a}\log|x|$$

strongly in $C^j(D_R(0) \setminus D_{\frac{1}{R}}(0), \mathbb{R}^N)$ for any $R > 0$ and integer j, where $\overrightarrow{a} \in T_y N$ is a vector in $\mathbb{R}^N$ with

$$|\overrightarrow{a}| = \mu^{1-\lim_{\alpha\searrow 1} t_\alpha}\sqrt{\frac{E(w^1)}{\pi}}.$$

Proof. We shall first prove the case (1).

Set

$$u'_\alpha(x) = u_\alpha(x'_\alpha + \lambda_\alpha^{t_\alpha} x), \quad v_\alpha(x) = \frac{1}{\sqrt{\alpha-1}}\left(u_\alpha(x'_\alpha + \lambda_\alpha^{t_\alpha} x) - u_\alpha(x'_\alpha + (\lambda_\alpha^{t_\alpha}, 0))\right)$$

and

$$\widehat{v}_\alpha(x) = \frac{1}{\sqrt{\alpha-1}}\left(\widehat{u}_\alpha(x'_\alpha + \lambda_\alpha^{t_\alpha} x) - \widehat{u}_\alpha(x'_\alpha + (\lambda_\alpha^{t_\alpha}, 0))\right).$$

By (4.26), Lemma 4.4 and Lemma 2.3, we have

$$\|\nabla u'_\alpha\|_{C^0(D^+_{2^k}(0)\setminus D^+_{2^{-k}}(0))} + \|\nabla^2 u'_\alpha\|_{C^0(D^+_{2^k}(0)\setminus D^+_{2^{-k}}(0))} \le C(k)\sqrt{\alpha-1}$$

where we used the fact

$$\lambda_\alpha = o((\alpha-1)^m)$$

for any $m > 0$ since $\nu > 1$.

Then,

$$\|\nabla\widehat{v}_\alpha\|_{C^0(D_{2^k}(0)\setminus D_{2^{-k}}(0))} + \|\nabla^2\widehat{v}_\alpha\|_{L^\infty(D_{2^k}(0)\setminus D_{2^{-k}}(0))} \le C(k).$$

Noting that $\widehat{v}_\alpha(1,0) = 0$, thus

$$\|\widehat{v}_\alpha\|_{C^0(D_{2^k}(0)\setminus D_{2^{-k}}(0))} \le C(k).$$

Since $\widehat{v}_\alpha$ satisfies the following equation

$$\Delta\widehat{v}_\alpha + \sqrt{\alpha-1}O(|\nabla\widehat{v}_\alpha|^2) + (\alpha-1)O(|\nabla^2\widehat{v}_\alpha|) = 0,$$

then there exists a subsequence of $\widehat{v}_\alpha$ (also denoted by itself) such that

$$\widehat{v}_\alpha \to v_0 \quad in \quad C^1_{loc}(\mathbb{R}^2 \setminus \{0\})$$

where v_0 satisfies

$$\Delta v_0 = 0.$$

Moreover, Lemma 4.5 tells us that $v_0(x) = v_0(|x|)$. Set

$$v_0 = \overrightarrow{a}\log r = (a_1, ..., a_N)\log r.$$

By Corollary 2.9 and Lemma 4.3, we have

$$\begin{aligned}\frac{1}{\alpha-1}\int_{D^+_{2\lambda_\alpha^{t_\alpha}}(x'_\alpha)\setminus D^+_{\lambda_\alpha^{t_\alpha}}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx &= \frac{2\alpha}{2\alpha-1}\int_{\lambda_\alpha^{t_\alpha}}^{2\lambda_\alpha^{t_\alpha}}\frac{1}{t}F^+_\alpha(\log_{\lambda_\alpha}t)dt+o(1)\\ &= \frac{2\alpha}{2\alpha-1}\log 2F^+_\alpha(t_\alpha)+o(1)\\ &\to 2\log 2F^+(\lim_{\alpha\searrow 1}t_\alpha),\end{aligned}$$

where we used the fact that

$$\lambda_\alpha^{t_\alpha}=o((\alpha-1)^m)$$

for any $m>0$ since $\nu>1$.

On the other hand, there also holds

$$\begin{aligned}\frac{1}{\alpha-1}\int_{D^+_{2\lambda_\alpha^{t_\alpha}}(x'_\alpha)\setminus D^+_{\lambda_\alpha^{t_\alpha}}(x'_\alpha)}(\sigma_\alpha+|\nabla_{g_\alpha}u_\alpha|^2)^{\alpha-1}|\nabla u_\alpha|^2dx &= \int_{D^+_2(0)\setminus D^+_1(0)}(\sigma_\alpha+|\nabla_{g_\alpha}v_\alpha|^2\frac{\alpha-1}{\lambda_\alpha^{2t_\alpha}})^{\alpha-1}|\nabla v_\alpha|^2dx\\ &\to \pi\log 2|\vec{a}|^2\mu^{\lim_{\alpha\searrow 1}t_\alpha}.\end{aligned}$$

Therefore,

$$|\vec{a}|^2=\frac{2}{\pi}\mu^{-\lim_{\alpha\searrow 1}t_\alpha}F^+(\lim_{\alpha\searrow 1}t_\alpha)=\frac{2\Lambda}{\pi}\mu^{1-2\lim_{\alpha\searrow 1}t_\alpha},$$

where the last equality follows from Lemma 3.6.

Since the case (2) is the interior case and $\nu>1$, the conclusion follows immediately from Proposition 4.3 in [17]. □

With the help of Proposition 4.6, we have the following corollary.

Corollary 4.7. *Under the assumption of Proposition 4.6, we have:*

(1) *If* $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=a$ *or* $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$ *and* $t_2<s_0$*, there holds*

$$\int_{\lambda_\alpha^t}^{2\lambda_\alpha^t}\frac{1}{\sqrt{\alpha-1}}|\frac{\partial\widehat{u_\alpha}}{\partial r'}|dr'\to\log 2\mu^{1-t}\sqrt{2\frac{E(w^1)}{\pi}}\quad in\ C^0([t_1,t_2])$$

and

$$\begin{aligned}\frac{1}{\sqrt{\alpha-1}}(r'|\frac{\partial\widehat{u_\alpha}}{\partial r'}|)(\lambda_\alpha^t,\theta')&\to\mu^{1-t}\sqrt{2\frac{E(w^1)}{\pi}}\quad in\ C^0([t_1,t_2]),\\ \frac{1}{\sqrt{\alpha-1}}(r'^{-1}|\frac{\partial\widehat{u_\alpha}}{\partial\theta'}|)(\lambda_\alpha^t,\theta')&\to 0\quad in\ C^0([t_1,t_2]).\end{aligned}$$

(2) *If* $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$ *and* $s_0<t_1$*, there holds*

$$\int_{\lambda_\alpha^t}^{2\lambda_\alpha^t}\frac{1}{\sqrt{\alpha-1}}|\frac{\partial u_\alpha}{\partial r}|dr\to\log 2\mu^{1-t}\sqrt{\frac{E(w^1)}{\pi}}\quad in\ C^0([t_1,t_2])$$

and

$$\begin{aligned}\frac{1}{\sqrt{\alpha-1}}(r|\frac{\partial u_\alpha}{\partial r}|)(\lambda_\alpha^t,\theta)&\to\mu^{1-t}\sqrt{\frac{E(w^1)}{\pi}}\quad in\ C^0([t_1,t_2]),\\ \frac{1}{\sqrt{\alpha-1}}(r^{-1}|\frac{\partial u_\alpha}{\partial\theta}|)(\lambda_\alpha^t,\theta)&\to 0\quad in\ C^0([t_1,t_2]).\end{aligned}$$

Proof. We shall only prove the second conclusion of the case (1) under the assumption $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=a$, since the other conclusions can be proved in a similar way.

In fact, if it was false, then there would exist $t_\alpha\in[t_1,t_2]$ and $\theta_\alpha\in[0,2\pi]$ such that

$$\left|\frac{1}{\sqrt{\alpha-1}}(r'|\frac{\partial u_\alpha}{\partial r'}|)(\lambda_\alpha^{t_\alpha},\theta_\alpha)-\mu^{1-t_\alpha}\sqrt{2\frac{E(w^1)}{\pi}}\right|\geq b>0. \tag{4.45}$$

However, by Proposition 4.6, we have

$$\frac{\lambda_\alpha^{t_\alpha}}{\sqrt{\alpha-1}}\frac{\partial u_\alpha}{\partial r'}(\lambda_\alpha^{t_\alpha},\theta_\alpha)\to\overrightarrow{a},$$

where $|\overrightarrow{a}|=\mu^{1-\lim_{\alpha\searrow 1}t_\alpha}\sqrt{2\frac{E(w^1)}{\pi}}$. This yields the following

$$\frac{1}{\sqrt{\alpha-1}}(r'|\frac{\partial u_\alpha}{\partial r'}|)(\lambda_\alpha^{t_\alpha},\theta_\alpha)\to\mu^{1-\lim_{\alpha\searrow 1}t_\alpha}\sqrt{2\frac{E(w^1)}{\pi}},$$

which contradicts to (4.45). We finished the proof of this corollary. □

For $0<t_1<t_2<1$, define the following curves:

if $\lim_{\alpha\searrow 1}\frac{\lambda_\alpha^{t_2}}{d_\alpha}=\infty$,

$$\omega_\alpha^1(r'):=\frac{1}{2\pi}\int_0^{2\pi}\widehat{u}_\alpha(r',\theta')d\theta':[\lambda_\alpha^{t_2},\lambda_\alpha^{t_1}]\to\mathbb{R}^K,$$

which is denoted by Γ_α^1, where (r',θ') is the polar coordinate around the point x'_α;

if $\lim_{\alpha\searrow 1}\frac{\lambda_\alpha^{t_1}}{d_\alpha}=0$,

$$\omega_\alpha^2(r):=\frac{1}{2\pi}\int_0^{2\pi}u_\alpha(r,\theta)d\theta:[\lambda_\alpha^{t_2},\lambda_\alpha^{t_1}]\to\mathbb{R}^K,$$

which is denoted by Γ_α^2, where (r,θ) is the polar coordinate around the point x_α.

Denote

$$\ddot{\omega}_\alpha^1:=(\frac{d}{dr'})^2\omega_\alpha^1,\ \ddot{\omega}_\alpha^2:=\frac{d^2}{dr^2}\omega_\alpha^2,\ \dot{\omega}_\alpha^1:=\frac{d}{dr'}\omega_\alpha^1\quad and\quad \dot{\omega}_\alpha^2:=\frac{d}{dr}\omega_\alpha^2.$$

By a direct computation, we have

$$\begin{aligned}\ddot{\omega}_\alpha^1&=\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial^2}{\partial r'^2}\widehat{u}_\alpha(r',\theta')d\theta'\\&=\frac{1}{2\pi}\int_0^{2\pi}\Delta\widehat{u}_\alpha(r',\theta')d\theta'-\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{r'}\frac{\partial}{\partial r'}\widehat{u}_\alpha(r',\theta')d\theta'\\&=-\frac{1}{2\pi}\int_0^{\pi}A(u_\alpha)(du_\alpha,du_\alpha)d\theta'-\frac{1}{2\pi}\int_\pi^{2\pi}D^2(\sigma\circ\Pi_N)|_{\sigma(\widehat{u}_\alpha)}(D\sigma|_{\widehat{u}_\alpha}\cdot d\widehat{u}_\alpha,D\sigma|_{\widehat{u}_\alpha}\cdot d\widehat{u}_\alpha)d\theta'\\&\quad-\frac{\alpha-1}{\pi}\int_0^{\pi}O(|\nabla^2u_\alpha|)d\theta'-\frac{1}{r'}\dot{\omega}_\alpha^1.\end{aligned}$$

Denote the induced metric of Γ_α^i in $\mathbb{R}^K$, $i=1,2$, by h_α and the second fundamental form by $A_{\Gamma_\alpha^i}$. We have

Lemma 4.8. *Passing to a subsequence, for any $\lambda_\alpha^{t_\alpha}\in[\lambda_\alpha^{t_2},\lambda_\alpha^{t_1}]$, there hold:*

(1) *If* $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=a$ *or* $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$ *and* $t_2<s_0$*, then*

$$\dot\omega_\alpha^1(\lambda_\alpha^{t_\alpha})=\frac{\sqrt{\alpha-1}}{\lambda_\alpha^{t_\alpha}}(\overrightarrow{a}+o(1)),$$

$$h_\alpha(\frac{d}{dr'},\frac{d}{dr'})=|\dot\omega_\alpha^1|^2=\frac{\alpha-1}{\lambda_\alpha^{2t_\alpha}}(|\overrightarrow{a}|^2+o(1)),$$

and

$$A_{\Gamma_\alpha^1}(d\omega_\alpha^1,d\omega_\alpha^1)=\frac{\alpha-1}{\lambda_\alpha^{2t_\alpha}}\left(A(y)(\overrightarrow{a},\overrightarrow{a})+\frac{1}{2}D^2\sigma(y)(\overrightarrow{a},\overrightarrow{a})+o(1)\right),$$

where σ *is defined by* (1.9)*. Moreover, for any* $t\in[t_1,t_2]$*, there exists a positive constant* $C>0$*, such that*

$$\|A_{\Gamma_\alpha^1}\|_{h_\alpha}(\lambda_\alpha^t)\le C.$$

(2) *If* $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$ *and* $s_0<t_1$*, then*

$$\dot\omega_\alpha^2(\lambda_\alpha^{t_\alpha})=\frac{\sqrt{\alpha-1}}{\lambda_\alpha^{t_\alpha}}(\overrightarrow{a}+o(1)),$$

$$h_\alpha(\frac{d}{dr},\frac{d}{dr})=|\dot\omega_\alpha^2|^2=\frac{\alpha-1}{\lambda_\alpha^{2t_\alpha}}(|\overrightarrow{a}|^2+o(1)),$$

and

$$A_{\Gamma_\alpha^2}(d\omega_\alpha^2,d\omega_\alpha^2)=\frac{\alpha-1}{\lambda_\alpha^{2t_\alpha}}\left(A(y)(\overrightarrow{a},\overrightarrow{a})+o(1)\right).$$

Moreover, for any $t\in[t_1,t_2]$*, there exists a positive constant* $C>0$*, such that*

$$\|A_{\Gamma_\alpha^2}\|_{h_\alpha}(\lambda_\alpha^t)\le C.$$

Proof. For the case (1), set

$$G_\alpha^1:=-\ddot\omega_\alpha^1-\frac{\dot\omega_\alpha^1}{r}.$$

For any $\lambda_\alpha^{t_\alpha}\in[\lambda_\alpha^{t_2},\lambda_\alpha^{t_1}]$, by Proposition 4.6, we have

$$\frac{1}{\sqrt{\alpha-1}}\left(\widehat{u}_\alpha(x_\alpha'+\lambda_\alpha^{t_\alpha}x)-\widehat{u}_\alpha(x_\alpha'+(\lambda_\alpha^{t_\alpha},0))\right)\to\overrightarrow{a}\log|x|, \tag{4.46}$$

where $\overrightarrow{a}\in T_yK$ and $y=\lim_{\alpha\searrow 1}u_\alpha(x_\alpha'+\lambda_\alpha^{t_\alpha}(\cos\theta',\sin\theta'))$. Then, we have

$$\dot\omega_\alpha^1(\lambda_\alpha^{t_\alpha})=\frac{\sqrt{\alpha-1}}{\lambda_\alpha^{t_\alpha}}(\overrightarrow{a}+o(1))$$

and

$$h_\alpha(\frac{d}{dr'},\frac{d}{dr'})=|\dot\omega_\alpha^1|^2=\frac{\alpha-1}{\lambda_\alpha^{2t_\alpha}}(|\overrightarrow{a}|^2+o(1)).$$

Similarly to the computation in Lemma 4.5 in [17], we have

$$\begin{aligned} G_\alpha^1 &= \frac{1}{2\pi}\int_0^\pi A(u_\alpha)(du_\alpha, du_\alpha)d\theta' + \frac{1}{2\pi}\int_\pi^{2\pi} D^2(\sigma\circ\Pi_N)|_{\sigma(\widehat{u_\alpha})}(D\sigma|_{\widehat{u_\alpha}}\cdot d\widehat{u_\alpha}, D\sigma|_{\widehat{u_\alpha}}\cdot d\widehat{u_\alpha})d\theta' \\ &\quad + \frac{\alpha-1}{\pi}\int_0^\pi O(|\nabla^2 u_\alpha|)d\theta' \\ &= \frac{\alpha-1}{\lambda_\alpha^{2t_\alpha}}\left(A(y)(\overrightarrow{a},\overrightarrow{a}) + \frac{1}{2}D^2\sigma(y)(\overrightarrow{a},\overrightarrow{a}) + o(1) + \sqrt{\alpha-1}\int_0^\pi O(\frac{\lambda_\alpha^{2t_\alpha}|\nabla^2 u_\alpha|}{\sqrt{\alpha-1}})d\theta'\right) \\ &= \frac{\alpha-1}{\lambda_\alpha^{2t_\alpha}}\left(A(y)(\overrightarrow{a},\overrightarrow{a}) + \frac{1}{2}D^2\sigma(y)(\overrightarrow{a},\overrightarrow{a}) + o(1)\right), \end{aligned}$$

where we used the fact that

$$O(\frac{\lambda_\alpha^{2t_\alpha}|\nabla^2 u_\alpha|}{\sqrt{\alpha-1}}) = O(|\nabla^2 v_\alpha|) = O(1)$$

and

$$D^2(\sigma\circ\Pi_N)(y)(\overrightarrow{a},\overrightarrow{a}) = D^2\sigma(\overrightarrow{a},\overrightarrow{a}) + A(y)(\overrightarrow{a},\overrightarrow{a})$$

for $y\in K$ and $\overrightarrow{a}\in T_yK$. Since

$$\langle A(y)(\overrightarrow{a},\overrightarrow{a}),\overrightarrow{a}\rangle = \langle D^2\sigma(y)(\overrightarrow{a},\overrightarrow{a}),\overrightarrow{a}\rangle = 0,$$

we have

$$\begin{aligned} -A_{\Gamma_\alpha^1}(d\omega_\alpha^1, d\omega_\alpha^1) &= \ddot{\omega}_\alpha^1 - \frac{\langle\ddot{\omega}_\alpha^1,\dot{\omega}_\alpha^1\rangle}{|\dot{\omega}_\alpha^1|^2}\dot{\omega}_\alpha^1 = -G_\alpha^1 + \frac{\langle G_\alpha^1,\dot{\omega}_\alpha^1\rangle}{|\dot{\omega}_\alpha^1|^2}\dot{\omega}_\alpha^1 \\ &= \frac{\alpha-1}{\lambda_\alpha^{2t_\alpha}}\left(A(y)(\overrightarrow{a},\overrightarrow{a}) + \frac{1}{2}D^2\sigma(y)(\overrightarrow{a},\overrightarrow{a}) + o(1)\right), \end{aligned}$$

which implies $\|A_{\Gamma_\alpha^1}\|_{h_\alpha}(\lambda_\alpha^{t_\alpha}) \le C$.

For the case (2), since $\nu > 1$ and $D_{\lambda_\alpha^{t_1}}(x_\alpha)$ is an interior ball and the conclusions of the lemma follows immediately from Lemma 4.5 in [17]. □

Proposition 4.9. *After passing to a subsequence, the sequence of curves $\{\Gamma_\alpha^i\}$ in $\mathbb{R}^N$, each of which is defined by ω_α^i, $i = 1,2$, and parameterized by its arc length, converges to a curve $\omega^i\subset N$ as α goes to 1, where w^2 is a geodesic on (N,h), i.e.*

$$\frac{d^2\omega^2}{ds^2} + A(\omega^2)(\frac{d\omega^2}{ds},\frac{d\omega^2}{ds}) = 0$$

and $\omega^1\subset K$ satisfies the following equation that

$$\frac{d^2\omega^1}{ds^2} + A(\omega^1)(\frac{d\omega^1}{ds},\frac{d\omega^1}{ds}) + \frac{1}{2}D^2\sigma(\omega^1)(\frac{d\omega^1}{ds},\frac{d\omega^1}{ds}) = 0.$$

In particular, if $K\subset N$ is a totally geodesic submanifold, then ω^1 is also a geodesic on (N,h).

Proof. We first derive the equation for ω^1. Let s be the arc length of $\omega_\alpha^1(r')$ with $s(\lambda_\alpha^{t_\alpha}) = 0$ where $t_\alpha\in[t_1,t_2]$. Set

$$y_\alpha^1 = \omega_\alpha^1(\lambda_\alpha^{t_\alpha}) \to y^1 \in K,$$

as $\alpha\searrow 1$. According to Lemma 4.8 and the fact that

$$\frac{d^2\omega_\alpha^1}{ds^2} = -A_{\Gamma_\alpha^1}(\omega_\alpha^1)(\frac{d\omega_\alpha^1}{ds},\frac{d\omega_\alpha^1}{ds}), \tag{4.47}$$

we know that $\omega^1_\alpha(s)$ converges in C^1 to a vector value function from $[0, s_1]$ to $\mathbb{R}^N$, denoted by $\omega^1(s)$, where s_1 is small enough. To proving that $\omega^1(s)$ satisfies

$$\frac{d^2\omega^1}{ds^2} + A(\omega^1)(\frac{d\omega^1}{ds}, \frac{d\omega^1}{ds}) + \frac{1}{2}D^2\sigma(\omega^1)(\frac{d\omega^1}{ds}, \frac{d\omega^1}{ds}) = 0,$$

we just need to show

$$A_{\Gamma^1_\alpha}(\omega^1_\alpha)(\frac{d\omega^1_\alpha}{ds}, \frac{d\omega^1_\alpha}{ds}) \to A(\omega^1)(\frac{d\omega^1}{ds}, \frac{d\omega^1}{ds}) + \frac{1}{2}D^2\sigma(\omega^1)(\frac{d\omega^1}{ds}, \frac{d\omega^1}{ds}), \tag{4.48}$$

strongly in $C^0([0, s_1], \mathbb{R}^N)$.

In fact, if not, then for any small s_1, passing to a subsequence, we can find

$$s'_\alpha = s(\lambda_\alpha^{t'_\alpha}) = \int_{\lambda_\alpha^{t_\alpha}}^{\lambda_\alpha^{t'_\alpha}} |\dot\omega^1_\alpha(r')|dr' \to s' \in (0, s_1)$$

and

$$\left|A_{\Gamma^1_\alpha}(\omega^1_\alpha)(\frac{d\omega^1_\alpha}{ds}, \frac{d\omega^1_\alpha}{ds}) - A(\omega^1)(\frac{d\omega^1}{ds}, \frac{d\omega^1}{ds}) - \frac{1}{2}D^2\sigma(\omega^1)(\frac{d\omega^1}{ds}, \frac{d\omega^1}{ds})\right|_{s=s'_\alpha} \geq \epsilon > 0.$$

Here, we can choose s_1 is small such that $t'_\alpha \in [\frac{t_1}{2}, t_2]$ when $\alpha - 1$ is small enough. In fact, without loss of generality, we may assume

$$\lambda_\alpha^{\frac{t_1}{2}} = 2^{l_\alpha}\lambda_\alpha^{t_\alpha},$$

where l_α is an integer which goes to infinity as $\alpha \searrow 1$. By Corollary 4.7, we have

$$\begin{aligned}\int_{\lambda_\alpha^{t_\alpha}}^{\lambda_\alpha^{\frac{t_1}{2}}} |\dot\omega^1_\alpha(r')|dr' = \sum_{i=1}^{l_\alpha}\int_{2^{i-1}\lambda_\alpha^{t_\alpha}}^{2^i\lambda_\alpha^{t_\alpha}} |\dot\omega^1_\alpha(r')|dr' &\geq \sqrt{\alpha-1}\,l_\alpha\left(\sqrt{2\frac{E(w^1)}{\pi}}\log 2 + o(1)\right)\\ &\geq C(t_\alpha - \frac{t_1}{2})\log\lambda_\alpha^{-\sqrt{\alpha-1}} \geq C\frac{t_1}{2}\log\nu > 0.\end{aligned}$$

Thus, we can take $s_1 \leq C\frac{t_1}{2}\log\nu$, then $t'_\alpha \in [\frac{t_1}{2}, t_2]$ when $\alpha - 1$ is small enough.

Combining Lemma 4.8 with the fact that

$$\frac{\dot\omega^1_\alpha(\lambda_\alpha^{t'_\alpha})}{|\dot\omega^1_\alpha(\lambda_\alpha^{t'_\alpha})|} = \frac{d\omega^1_\alpha}{ds}(s'_\alpha) \to \frac{d\omega^1}{ds}(s'),$$

we have

$$\begin{aligned}A_{\Gamma^1_\alpha}(\omega^1_\alpha)(\frac{d\omega^1_\alpha}{ds}, \frac{d\omega^1_\alpha}{ds})\Big|_{s=s'_\alpha} &= \frac{1}{|\dot\omega^1_\alpha(\lambda_\alpha^{t'_\alpha})|^2}A_{\Gamma^1_\alpha}(\omega^1_\alpha)(\dot\omega^1_\alpha(r'), \dot\omega^1_\alpha(r'))\Big|_{r'=\lambda_\alpha^{t'_\alpha}}\\ &\to \left(A(\omega^1)(\frac{d\omega^1}{ds}, \frac{d\omega^1}{ds}) - \frac{1}{2}D^2\sigma(\omega^1)(\frac{d\omega^1}{ds}, \frac{d\omega^1}{ds})\right)\Big|_{s=s'}\end{aligned}$$

which is a contradiction to the choice of s'_α, and (4.48) holds.

Therefore, by (4.47) and (4.48), we get

$$\frac{d\omega^1}{ds}(s) - \frac{d\omega^1}{ds}(0) = -\int_0^s A(\omega^1)(\frac{d\omega^1}{ds}, \frac{d\omega^1}{ds}) + \frac{1}{2}D^2\sigma(\omega^1)(\frac{d\omega^1}{ds}, \frac{d\omega^1}{ds})ds$$

for any $s \in [0, s_1]$. This implies the equation of ω^1 immediately. Since the process of deriving the equation of ω^2 is similar and easier, we omit the details. □

Now, we are able to complete the proof of the main Theorem 4.1.

Proof of Theorem 4.1 Without loss of generality, we assume $\lambda_\alpha^{t_1} = 2^{k_\alpha}\lambda_\alpha^{t_2}$ for some integer

$$k_\alpha = \frac{t_1 - t_2}{\log 2}\log \lambda_\alpha,$$

which tends to infinity as $\alpha \searrow 1$. Then it is sufficient to consider the cases listed in the statement of the theorem.

Case (3): $\nu = \infty$.

(3-a) If $\lim_{\alpha\searrow 1} \frac{d_\alpha}{\lambda_\alpha} = a$, for any $0 < t_1 < t_2 < 1$, by Corollary 4.7, there holds

$$L(\Gamma_\alpha^1|_{D_{2^{k+1}\lambda_\alpha^{t_2}}(x'_\alpha)\setminus D_{2^k\lambda_\alpha^{t_2}}(x'_\alpha)}) \geq \sqrt{\alpha-1}\left(\sqrt{2\frac{E(w^1)}{\pi}}\log 2 + o(1)\right).$$

Then

$$L(\Gamma_\alpha^1) \geq Ck_\alpha\sqrt{\alpha-1} \geq -C\sqrt{\alpha-1}\log\lambda_\alpha \to \infty.$$

(3-b) $\lim_{\alpha\searrow 1} \frac{d_\alpha}{\lambda_\alpha} = \infty$ and $s_0 = 0$.

Then for any $0 < t_1 < t_2 < 1$, by Corollary 4.7, there holds

$$L(\Gamma_\alpha^2|_{D_{2^{k+1}\lambda_\alpha^{t_2}}(x_\alpha)\setminus D_{2^k\lambda_\alpha^{t_2}}(x_\alpha)}) \geq \sqrt{\alpha-1}\left(\sqrt{\frac{E(w^1)}{\pi}}\log 2 + o(1)\right).$$

Then

$$L(\Gamma_\alpha^2) \geq Ck_\alpha\sqrt{\alpha-1} \geq -C\sqrt{\alpha-1}\log\lambda_\alpha \to \infty.$$

(3-c) If $\lim_{\alpha\searrow 1} \frac{d_\alpha}{\lambda_\alpha} = \infty$ and $0 < s_0 \leq 1$, then for any $0 < t_1 < t_2 < s_0$, by Corollary 4.7, there holds

$$L(\Gamma_\alpha^1|_{D_{2^{k+1}\lambda_\alpha^{t_2}}(x'_\alpha)\setminus D_{2^k\lambda_\alpha^{t_2}}(x'_\alpha)}) \geq \sqrt{\alpha-1}\left(\sqrt{2\frac{E(w^1)}{\pi}}\log 2 + o(1)\right).$$

Then

$$L(\Gamma_\alpha^1) \geq Ck_\alpha\sqrt{\alpha-1} \geq -C\sqrt{\alpha-1}\log\lambda_\alpha \to \infty.$$

Case (2): $\nu \in (1,\infty)$.

Since $\nu \in (1,\infty)$ implies $\mu = 1$, then we have

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{\alpha\searrow 1}\int_{D_\delta^+(x_\alpha)\setminus D^+_{\lambda_\alpha R}(x_\alpha)} |\nabla u_\alpha|^2 dx = 0$$

(2-a) $\lim_{\alpha\searrow 1} \frac{d_\alpha}{\lambda_\alpha} = \infty$ and $s_0 = 0$.

Firstly, in this case we may find that

$$-\lim_{\alpha\searrow 1}\sqrt{1+|H^+(2d_\alpha)|}\sqrt{\alpha-1}s_\alpha\log\lambda_\alpha = 0.$$

In fact, if not, by Lemma 4.4, we know $|H^+(2d_\alpha)|$ is uniformly bounded. Thus,

$$-\lim_{\alpha\searrow 1}\sqrt{1+|H^+(2d_\alpha)|}\sqrt{\alpha-1}s_\alpha\log\lambda_\alpha \le C\lim_{\alpha\searrow 1}s_\alpha\log\lambda_\alpha^{-\sqrt{\alpha-1}}=0,$$

which is a contradiction.

Secondly, we prove the following equalities

$$\lim_{\delta\to 0}\lim_{t\to 0}\lim_{\alpha\searrow 1} Osc_{D^+_\delta(x_\alpha)\setminus D^+_{\lambda^t_\alpha}(x_\alpha)}u_\alpha = 0 \tag{4.49}$$

and

$$\lim_{R\to\infty}\lim_{t\to 1}\lim_{\alpha\searrow 1} Osc_{D^+_{\lambda^t_\alpha}(x_\alpha)\setminus D^+_{\lambda_\alpha R}(x_\alpha)}u_\alpha = 0. \tag{4.50}$$

We decompose the domain $D^+_\delta(x_\alpha)\setminus D^+_{\lambda^t_\alpha}(x_\alpha)$ as follows:

$$D^+_\delta(x_\alpha)\setminus D^+_{\lambda^t_\alpha}(x_\alpha)=D^+_\delta(x_\alpha)\setminus D^+_{4d_\alpha}(x'_\alpha)\cup D^+_{4d_\alpha}(x'_\alpha)\setminus D^+_{d_\alpha}(x_\alpha)\cup D^+_{d_\alpha}(x_\alpha)\setminus D^+_{\lambda^t_\alpha}(x_\alpha).$$

By (4.14), we can get

$$\begin{aligned}Osc_{D^+_\delta(x_\alpha)\setminus D^+_{4d_\alpha}(x'_\alpha)}u_\alpha &\le Osc_{D^+_{2\delta}(x'_\alpha)\setminus D^+_{4d_\alpha}(x'_\alpha)}u_\alpha\\ &\le C\sqrt{E(u_\alpha;D^+_{4\delta}(x'_\alpha)\setminus D^+_{2d_\alpha}(x'_\alpha))}-C\sqrt{1+|H^+(2d_\alpha)|}\sqrt{\alpha-1}\log 2\lambda_\alpha^{s_\alpha}.\end{aligned}$$

Since $D^+_{d_\alpha}(x_\alpha)\setminus D^+_{\lambda^t_\alpha}(x_\alpha)=D_{d_\alpha}(x_\alpha)\setminus D_{\lambda^t_\alpha}(x_\alpha)$ and $\nu\in(1,\infty)$, by the proof in Section 4.1 in [17], we obtain

$$Osc_{D_{d_\alpha}(x_\alpha)\setminus D_{\lambda^t_\alpha}(x_\alpha)}u_\alpha \le C\sqrt{E(u_\alpha;D_{d_\alpha}(x_\alpha)\setminus D_{\lambda^t_\alpha}(x_\alpha)}+Cs_\alpha\log\lambda_\alpha^{-\sqrt{\alpha-1}}+Ct\log\lambda_\alpha^{-\sqrt{\alpha-1}}$$

and

$$Osc_{D^+_{\lambda^t_\alpha}(x_\alpha)\setminus D^+_{\lambda_\alpha R}(x_\alpha)}u_\alpha \le C\sqrt{E(u_\alpha;D^+_{\lambda^t_\alpha}(x_\alpha)\setminus D^+_{\lambda_\alpha R}(x_\alpha)}+C(1-t)\log\lambda_\alpha^{-\sqrt{\alpha-1}}+C\sqrt{\alpha-1}\log R.$$

Combining these together with (3.9) and noting that $D^+_{4d_\alpha}(x'_\alpha)\setminus D_{d_\alpha}(x_\alpha)\subset D_{5d_\alpha}(x_\alpha)\setminus D_{d_\alpha}(x_\alpha)$, we know (4.49) and (4.50) hold.

Finally, by Corollary 4.7, we have

$$L(\Gamma^2_\alpha|_{D_{2^{k+1}\lambda_\alpha^{t_2}}(x_\alpha)\setminus D_{2^k\lambda_\alpha^{t_2}}(x_\alpha)})=\sqrt{\alpha-1}\left(\sqrt{\frac{E(w^1)}{\pi}}\log 2+o(1)\right).$$

Then

$$L(\Gamma^2)=\lim_{\alpha\searrow 1}\sqrt{\alpha-1}k_\alpha\sqrt{\frac{E(v^1)}{\pi}}\log 2=(t_2-t_1)\sqrt{\alpha-1}\sqrt{\frac{E(w^1)}{\pi}}\log\nu.$$

(2-b) $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=a$ or $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$ and $s_0=1$.

If $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$ and $s_0=1$, it is easy to see that

$$-\lim_{\alpha\searrow 1}\sqrt{1+|H^+(2d_\alpha)|}\sqrt{\alpha-1}s_\alpha\log\lambda_\alpha>0.$$

According Lemma 4.2 and Lemma 4.4, we obtain that $|H^+(2d_\alpha)|$ is uniformly bounded for both cases.

Noting that

$$D^+_{\lambda^t_\alpha}(x_\alpha)\setminus D^+_{\lambda_\alpha R}(x_\alpha)\subset D^+_{2\lambda^t_\alpha}(x'_\alpha)\setminus D^+_{2d_\alpha}(x'_\alpha)$$

if $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=a$ and

$$D^+_{\lambda^t_\alpha}(x_\alpha)\setminus D^+_{\lambda_\alpha R}(x_\alpha)\subset D^+_{2\lambda^t_\alpha}(x'_\alpha)\setminus D^+_{4d_\alpha}(x'_\alpha)\cup D^+_{4d_\alpha}(x'_\alpha)\setminus D^+_{d_\alpha}(x_\alpha)\cup D^+_{d_\alpha}(x_\alpha)\setminus D^+_{\lambda_\alpha R}(x_\alpha)$$

if $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$ and $s_0=1$, by the argument in (2-a), we know (4.50) holds for both cases. Similarly, (4.49) holds since $D^+_\delta(x_\alpha)\setminus D^+_{\lambda^t_\alpha}(x_\alpha)\subset D^+_{2\delta}(x'_\alpha)\setminus D^+_{\frac{1}{2}\lambda^t_\alpha}(x'_\alpha)$.

By Corollary 4.7, we have

$$L(\Gamma^1_\alpha|_{D^+_{2^{k+1}\lambda^{t_2}_\alpha}(x'_\alpha)\setminus D^+_{2^k\lambda^{t_2}_\alpha}(x'_\alpha)})=\sqrt{\alpha-1}\left(\sqrt{2\frac{E(w^1)}{\pi}}\log 2+o(1)\right).$$

Then

$$L(\Gamma^1)=\lim_{\alpha\searrow 1}\sqrt{\alpha-1}k_\alpha\sqrt{2\frac{E(w^1)}{\pi}}\log 2=(t_2-t_1)\sqrt{2\frac{E(w^1)}{\pi}}\log\nu.$$

(2-c) $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$ and $s_0\in(0,1)$.

Firstly, similar to the argument in (2-a) and (2-b), we know (4.49) and (4.50) hold.

Secondly, by Corollary 4.7, for any $t_2<s_0$, we have

$$L(\Gamma^1_\alpha|_{D^+_{2^{k+1}\lambda^{t_2}_\alpha}(x'_\alpha)\setminus D^+_{2^k\lambda^{t_2}_\alpha}(x'_\alpha)})=\sqrt{\alpha-1}\left(\sqrt{\frac{2E(w^1)}{\pi}}\log 2+o(1)\right)$$

and for any $t_1>s_0$, we have

$$L(\Gamma^2_\alpha|_{D^+_{2^{k+1}\lambda^{t_2}_\alpha}(x'_\alpha)\setminus D^+_{2^k\lambda^{t_2}_\alpha}(x'_\alpha)})=\sqrt{\alpha-1}\left(\sqrt{\frac{E(w^1)}{\pi}}\log 2+o(1)\right).$$

Then,

$$L(\Gamma^1)=\lim_{t_1\to 0}\lim_{t_2\to s_0}\lim_{\alpha\searrow 1}\sqrt{\alpha-1}k_\alpha\sqrt{2\frac{E(w^1)}{\pi}}\log 2=s_0\sqrt{2\frac{E(w^1)}{\pi}}\log\nu$$

and

$$L(\Gamma^2)=\lim_{t_1\to s_0}\lim_{t_2\to 1}\lim_{\alpha\searrow 1}\sqrt{\alpha-1}k_\alpha\sqrt{2\frac{E(w^1)}{\pi}}\log 2=(1-s_0)\sqrt{\frac{E(w^1)}{\pi}}\log\nu.$$

The left thing we need to prove

$$\lim_{s\to 0}\lim_{\alpha\searrow 1}Osc_{D^+_{\lambda^{s_0-s}_\alpha}(x_\alpha)\setminus D^+_{\lambda^{s_0+s}_\alpha}(x_\alpha)}u_\alpha=0. \tag{4.51}$$

Noting that

$$D^+_{\lambda^{s_0-s}_\alpha}(x_\alpha)\setminus D^+_{\lambda^{s_0+s}_\alpha}(x_\alpha)\subset D^+_{2\lambda^{s_0-s}_\alpha}(x'_\alpha)\setminus D^+_{2d_\alpha}(x'_\alpha)\cup D^+_{2d_\alpha}(x'_\alpha)\setminus D^+_{d_\alpha}(x_\alpha)\cup D^+_{d_\alpha}(x_\alpha)\setminus D^+_{\lambda^{s_0+s}_\alpha}(x_\alpha),$$

similarly to the discussions in (2-a) and (2-b), we will obtain (4.51).

Case (1): $\nu=1$.

(1-a) $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=a$ or $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$ and $\lim_{\alpha\searrow 1}\sqrt{1+|H^+(2d_\alpha)|}\sqrt{\alpha-1}s_\alpha\log\lambda_\alpha=0$.

It is easy to see that the conclusion of (1) in Theorem 4.1 follows immediately from the proof at the beginning of this section.

(1-b) $\lim_{\alpha\searrow 1}\frac{d_\alpha}{\lambda_\alpha}=\infty$ and $\lim_{\alpha\searrow 1}\sqrt{1+|H^+(2d_\alpha)|}\sqrt{\alpha-1}s_\alpha\log\lambda_\alpha>0$.

By Lemma 4.4, we get that $|H^+(2d_\alpha)|$ is uniformly bounded. Then there must hold

$$\lim_{\alpha\searrow 1}\sqrt{1+|H^+(2d_\alpha)|}\sqrt{\alpha-1}s_\alpha\log\lambda_\alpha=0,$$

which is a contradiction. Thus, this case will not happen and we get the no neck result. We finished the proof of Theorem 4.1. □

Proof of Theorem 1.2 It is easy to see that Theorem 1.2 is a consequence of Theorem 4.1. □

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